Computation of the Noncentral Chi-Square Distribution*

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(Manuscript received July 17, 1968)

This article gives a formula that allows accurate values of the cumulative noncentral chi-square distribution to be computed. Although this distribution has been recognized for a long time, none of the standard references give formulae that are suitable for computing accurate values over an extensive range of the parameters; approximations in terms of the chi-square distribution are usually recommended. A program written by the author, based on the formula given here, has been successful for computations involving more than 10,000 degrees of freedom. Since many steps are required when the degrees of freedom are as large as this, the program is not "fast" but it is believed to be accurate.

I. INTRODUCTION

The Non-Central Chi-Square Distribution is encountered in many statistical problems, one of the most important in communications studies being the detection of signals in noise using a square-law detector.[1] Marcum discussed this application but concluded that a satisfactory algorithm for computing system performance could not be based on the formula he used.[2] This article shows that a satisfactory algorithm can be based on the formula that Marcum derived if the expression is expanded in a power series and the terms are properly grouped before being evaluated.

More recently Urkowitz[3] discussed detection system performance in which the above distribution arose and recommended that approximations in terms of the chi-square distribution, given by Patnaik,[1] be used for computation. While these approximations are adequate for some purposes, it is desirable to have a reliable and accurate method of computing values, if only to check the approximations.

* This work was supported by the U. S. Navy under contract N00039-68-C-3584.

II. INTEGRATION OF NONCENTRAL CHI-SQUARE DISTRIBUTION

If the signal-to-noise power ratio is x for the sum of η independent samples of the output of a square-law detector, the following integral* gives the probability that the sum will be y or more. The variables are normalized to the variance of the individual noise samples, so the average signal-to-noise power ratio for one sample is x/η and the average output per sample is y/η. Considering one sample of noise to be the sum of the squares of two independent gaussian variables of unit variance, the integral is related to the noncentral chi-square distribution by the conversions given in equation (8).

$$Q = \int_y^\infty \left(\frac{z}{x}\right)^{(\eta-1)/2} \exp(-z - x) I_{\eta-1}[2(zx)^{\frac{1}{2}}]\, dz. \tag{1}$$

From (Ref. 4, Section 8.445) $I_{\eta-1}\ [2\ (zx)^{\frac{1}{2}}]$ is the modified Bessel function

$$I_m(t) = \sum_{k=0}^\infty \frac{(t/2)^{m+2k}}{k!\ \Gamma(m+k+1)}. \tag{2}$$

Thus

$$Q = \frac{\exp(-x)}{\Gamma(\eta)} \int_y^\infty \exp(-z) z^{\eta-1}\, dz + \frac{\exp(-x)}{\Gamma(\eta)} \int_y^\infty \exp(-z) z^{\eta-1} \cdot \left[\frac{xz}{1!\ \eta} + \frac{(xz)^2}{2!\ \eta(\eta+1)} + \frac{(xz)^3}{3!\ \eta(\eta+1)(\eta+2)} + \cdots\right] dz. \tag{3}$$

Notice that

$$\int_y^\infty \exp(-z) z^{\eta-1}\, dz = \Gamma(\eta, y) \tag{4}$$

the incomplete gamma function (Ref. 5, Section 6.5.3). Since

$$\int_y^\infty t^p \exp(-t)\, dt = -t^p \exp(-t) \Big|_y^\infty + p \int_y^\infty t^{p-1} \exp(-t)\, dt \tag{5}$$

equation (3) can be written

$$Q = \frac{\exp(-x)}{\Gamma(\eta)} \Bigg[\Gamma(\eta, y) + \frac{x}{1!} \left\{ \Gamma(\eta, y) + \left(\frac{y}{\eta}\right) y^{\eta-1} \exp(-y) \right\}$$

* Sometimes called the generalized Marcum Q-function. See Ref. 2.

$$+ \frac{x^2}{2!}\left\{\Gamma(\eta, y) + \left(\frac{y}{\eta} + \frac{y^2}{\eta(\eta+1)}\right)y^{\eta-1}\exp(-y)\right\}$$

$$+ \frac{x^3}{3!}\left\{\Gamma(\eta, y) + \left(\frac{y}{\eta} + \frac{y^2}{\eta(\eta+1)} + \frac{y^3}{\eta(\eta+1)(\eta+2)}\right)y^{\eta-1}\exp(-y)\right\}$$

$$+ \cdots \text{ and so on}\Bigg]. \tag{6}$$

Summing the terms by columns gives

$$Q = \frac{\Gamma(\eta, y)}{\Gamma(\eta)} + \frac{y^{\eta-1}\exp(-y)}{\Gamma(\eta)}\left[\frac{y}{\eta}\exp(-x)\sum_{r=1}^{\infty}\frac{x^r}{r!} + \frac{y^2}{\eta(\eta+1)}\exp(-x)\sum_{r=2}^{\infty}\frac{x^r}{r!} + \frac{y^3}{\eta(\eta+1)(\eta+2)}\exp(-x)\sum_{r=3}^{\infty}\frac{x^r}{r!} + \cdots \text{ and so on}\right] \tag{7}$$

A satisfactory computing algorithm can be based on equation (7) where we notice that Q can be expressed as the sum of two parts, $Q_1 = \Gamma(\eta, y)/\Gamma(\eta)$ which is independent of x, and another part which we call Q_2.

III. DISCUSSION

The noncentral chi-square cumulative distribution can be written $Q(\chi'^2|\nu, \lambda)$ (see Ref. 5, Section 26.4.25), where the distribution is integrated from χ'^2 to infinity, the number of degrees of freedom is ν, and the noncentral parameter is λ. This integral is the same as that given in equation (7) if we put

$$\begin{aligned} \nu &= 2\eta \\ \chi'^2 &= 2y \\ \lambda &= 2x \end{aligned} \tag{8}$$

so that

$$\begin{aligned} Q(2y \mid 2\eta, 2x) &= Q_1 + Q_2 \\ &= Q(2y \mid 2\eta) + Q_2 \end{aligned} \tag{9}$$

where $Q(2y|2\eta) = Q(\chi^2|\nu)$, the cumulative chi-square distribution (see Ref. 4, Section 26.4.2).

If M independent samples of the output of a square-law detector are averaged, when the input is narrowband gaussian noise plus a CW signal at the center of the band, Q can be used to find the probability that a threshold value will be exceeded. Expressing all parameters in units of the narrowband noise power, the desired threshold is y/η, x/η is the signal-to-noise power ratio, and $M = \eta$.

It is interesting that the Rayleigh distribution, and the Rice distribution, are equivalent to the chi-square and non-central chi-square distributions respectively, when the latter are expressed in terms of a parameter χ equal to the square root of χ^2, and $\eta = 1$.

Marcum[2] gave an expression of the form shown in equation (3) for the output of a square-law detector. He stated that it could only be used satisfactorily for values of η up to about 10. More recently Urkowitz[3] has discussed the integration of a square-law detector output and recommends that the noncentral chi-square distribution be computed using an approximation given by Patnaik[1] in terms of the chi-square distribution. Patnaik compares with exact values some results computed using the approximation and finds errors of the order of 1% around $Q = 0.5$. The accuracy is much less for values around unity and for values less than 0.01.

Brennan and Reed have shown that, when the order of the Bessel function in equation (1) is zero, corresponding to one sample, a straightforward recursive method applied to the resulting equation (6) can be used to compute the integral.[6] They suggested that a similar procedure could be used even on the form of equation (1) given here. However, as pointed out by Marcum, such a technique rapidly becomes useless as η increases above about 10.

A program written by the author, based on equation (7), has been used satisfactorily for η as large as 8192, and simultaneously for values of x/η up to 0.1. The exact values given by Patnaik were checked. Further checks were made possible by the development of a uniform asymptotic expansion by S. O. Rice, with which it is possible to get results outside the useful range of the algorithm given here.[7]

Table 1 compares values obtained with the author's program (CHISQ) and corresponding values supplied by S. O. Rice using his uniform asymptotic expansion (UAE), with results obtained using the Patnaik[1] and Gauss approximations (Ref. 5, Section 26.4.29).

The accuracy of the algorithm given in equation (7) decreases as x/η increases in the table, and the value in the last entry depended quite sensitively on the last digit of a 18 digit double precision con-

TABLE I — COMPARISON OF COMPUTATION METHODS

x/η	y/η	η	1-CHISQ	UAE	PATNAIK	GAUSS
0.01	1.05	8192	0.999801547E-00	0.9998015E-00	0.999801544E-00	0.999809E-00
0.05	1.05	8192	0.501464546E-00	0.5014645E-00	0.501467E-00	0.50110E-00
0.08	1.05	8192	0.552623909E-02	0.5526235E-02	0.552472E-02	0.56050E-02
0.1	1.05	8192	0.138627645E-04	0.1386275E-04	0.138700E-04	0.14803E-04
0.11	1.05	8192	0.281186446E-06	0.2811860E-06	0.280145E-06	0.31438E-06
0.12	1.05	8192	0.316387190E-08	0.3163860E-08	0.314279E-08	0.37651E-08
0.13	1.05	8192	0.20004E-10	0.199969E-10	0.197750E-10	0.25799E-10

stant used in the program. Notice that even for the last entry, the value actually computed, (CHISQ), appears to be correct to 14 places after the decimal point.

IV. EXTENSION TO A MORE GENERAL INTEGRAL

A more general integral is obtained by writing, for example, the βth moment of the partial noncentral chi-square distribution,

$$Q_\beta = \int_y^\infty z^\beta \left(\frac{z}{x}\right)^{(\eta-1)/2} \exp(-z - x) I_{\eta-1}[2(zx)^{\frac{1}{2}}]\, dz. \tag{11}$$

The corresponding form of equation (4) is

$$\int_y^\infty z^\beta \exp(-z) z^{\eta-1}\, dz = \Gamma(\xi, y) \tag{12}$$

where

$$\xi = \eta + \beta, \tag{13}$$

and the corresponding form of equation (7) becomes

$$\begin{aligned} Q = {} & \frac{\Gamma(\xi, y)}{\Gamma(\eta)} \exp(-x)\,{}_1F_1(\xi;\eta;x) \\ & + \frac{y^{\xi-1}\exp(-y)}{\Gamma(\eta)} \left[\frac{y}{\eta} \exp(-x) \sum_{r=1}^\infty \frac{x^r}{r!} \frac{(\xi+1)_{r-1}}{(\eta+1)_{r-1}} \right. \\ & + \frac{y^2}{\eta(\eta+1)} \exp(-x) \sum_{r=2}^\infty \frac{x^r}{r!} \frac{(\xi+2)_{r-1}}{(\eta+2)_{r-1}} \\ & + \frac{y^3}{\eta(\eta+1)(\eta+2)} \exp(-x) \sum_{r=3}^\infty \frac{x^r}{r!} \frac{(\xi+3)_{r-1}}{(\eta+3)_{r-1}} \\ & \left. + \cdots \text{ and so on} \right]. \end{aligned} \tag{14}$$

The confluent hypergeometric function ${}_1F_1(a;\, b;\, x)$ (Ref. 5, Section 13.1.10) is defined by

$$\begin{aligned} {}_1F_1(a;b;x) &= 1 + \frac{ax}{b1!} + \frac{a(a+1)x^2}{b(b+1)2!} + \frac{a(a+1)(a+2)x^3}{b(b+1)(b+2)3!} + \cdots \\ &= \sum_{r=0}^\infty \frac{(a)_r x^r}{(b)_r r!}. \end{aligned} \tag{15}$$

Equation (15) conveniently gives an example of Pochhammer's symbol $(\alpha)_r$ (Ref. 5, Section 6.1.22), also used in equation (14).

The structure of equation (14) is closely related to that of equation (7), so it can form the basis for a useful algorithm to compute the integral given in equation (11).

V. ACKNOWLEGEMENT

The interest of Mr. Stephen O. Rice in this work and his encouragement to publish the results, is gratefully acknowledged.

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