Minimal Synthesis of Two-Variable Reactance Matrices*

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A simple algebraic method stemming from ideas in minimal state-variable realization theory is developed for the synthesis of two-variable reactance matrices. The method rests mainly on the factorization of a one variable polynomial matrix which is para-Hermitian and positive semidefinite on the imaginary axis, and always yields a realization minimal in both variables.

I. INTRODUCTION

Two-variable reactance functions and matrices, originally introduced to represent the characteristics of lumped passive networks with variable elements,[1, 2] have become more important because of their application to the synthesis of lumped-distributed networks. Ansell first showed the two-variable reactance property of networks composed of lossless transmission lines and lumped reactances.[3] The two-variable theory has also been applied to the synthesis of networks consisting of lumped resistors capacitors and uniformly distributed RC lines,[4, 5] which are of importance in microelectronic structures.[6, 7] Besides the various applications, the two-variable reactance theory is of theoretical interest in itself since it can be shown that passive RLC synthesis is a special case of two-variable reactance synthesis.[2]

Koga[8] demonstrated that every $n \times n$ two-variable reactance matrix $W(p, s)$ can be realized as the impedance seen at the first n ports of a lossless $(n+qr)$-port network in the p-plane terminated at its last qr ports with unit inductors in the s-plane; q is the rank of $W(p, s)$, and r is the highest degree of s in the least common denominator of

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the elements of w. The method is quite complicated and rests heavily on the theory of algebraic functions and the structure of para unitary matrices. Also it does not guarantee the use of a minimum number of elements. Youla[9] solved the problem of synthesizing a lossless two-variable scattering matrix by adapting an earlier method for synthesizing one-variable scattering matrices.[10] The method could be adapted to the direct synthesis of an impedance matrix but appears to be unduly complicated because of the need to find the transformation required to transform a generally unrealizable coupling network into a realizable one.

A simple algebraic method stemming from ideas in minimal state-variable realization theory and having similar beginnings as that of Youla[9] is developed here for the synthesis of two-variable reactance matrices. The method rests mainly on the factorization of a one-variable polynominal matrix which is para Hermitian* and positive semidefinite on the imaginary axis. Such a factorization is well known in n-port network theory and once it is accomplished, the coupling network is obtained by simple matrix operations. Furthermore the method always yields a network minimal in both types of elements.

We first introduce some basic definitions and necessary theorems, and later we add more as the need arises. The synthesis procedure is developed in Section III. Since the various proofs involved are rather indirect and tend to cloud the simplicity of the actual procedure, the synthesis procedure is outlined in Section IV. The reader interested only in the procedure and not in the theory behind it may go directly to Section IV where step-by-step instructions are given for the synthesis of any two variable reactance matrix. In Section V an example is worked out. The notation used in this paper is almost the same as found in earlier work to assure easy reading for those familiar with it.[9] Capital letters indicate matrices; bold face letters indicate matrix transposition. A superscript dagger indicates the substitution of $-s$ or $-p$ for s and p respectively, in the case of two-variable functions.

II. BASIC DEFINITIONS AND THEOREMS

The basic notion in the two variable theory is that of a two variable positive real matrix, which is a straightforward extension of the same notion in the one variable theory (See p. 96 of Ref. 11 and p. 32 of Ref. 8).

* A matrix $A(p)$ is said to be para-Hermitian if $A(p) = \mathbf{A}^{\dagger}(p)$ where the bold face letter denotes matrix transposition and the superscript dagger denotes replacement of p by $-p$.

Definition 1: An $n \times n$ matrix $W(p, s)$ is said to be a two variable positive real matrix if

(*i*) W is real for real p and s.
(*ii*) W is analytic in the domain Re $p > 0$ and Re $s > 0$.
(*iii*) $W + \mathbf{W}^*$ is positive semidefinite in the domain Re $p > 0$ and Re $s > 0$.

By statements such as: "W is analytic" in the definition and in what follows, we mean, "each element of W is analytic." A two variable function is said to be analytic at a point if it has a total differential at the point. The bold face letter indicates matrix transposition, and the superscript star indicates the complex conjugation of each element.

If $W(p, s)$ satisfies conditions (ii) and (iii) of the definition and not necessarily condition (i), it will be called a two variable positive matrix.

Definition 2: An $n \times n$ matrix $W(p, s)$ is said to be a two variable reactance matrix if

(*i*) W is a two variable positive real matrix.
(*ii*) $W + \mathbf{W}^\dagger \equiv 0$.

The superscript dagger indicates the operation of substituting $-p$ and $-s$ for p and s in the original matrix. This definition of a two variable reactance matrix is similar to the corresponding one in the one variable theory. (See p. 102 of Ref. 11 and p. 32 of Ref. 8.) Analogously, as in the one variable case (p. 117 of Ref. 11), it is generally hard to check if condition (*iii*) of Definition 1, which involves the whole domain Re $p > 0$ and Re $s > 0$, is satisfied for a given two variable matrix; we would like to find an equivalent set of conditions that are easier to check. In the case of two variable reactance matrices, the following theorem proved by Ozaki and Kasami[2] in the scalar case, and extended to nonsymmetric matrices by Koga, (p. 33 of Ref. 8) serves this purpose.

Theorem 1: The necessary and sufficient conditions for an $n \times n$ matrix $W(p, s)$ to be a two variable reactance matrix are:

(*i*) *W is rational in p and s, and real for real p and s.*
(*ii*) *W is analytic in the domain Re $p > 0$; Re $s > 0$.*
(*iii*) $W \equiv -\mathbf{W}^\dagger$.
(*iv*) *For any (p_0, s_0) with Re p_0 = Re s_0 = 0, which is a regular*

point of W, poles of $W(p_0, s)$ and $W(p, s_0)$ are simple and restricted to the imaginary s and p axes respectively.

(v) $\partial W/\partial p$ and $\partial W/\partial s$ are positive semidefinite Hermitian for Re p = Re s = 0, except at poles.

The proof of this theorem can be found on p. 33 of Ref. 8. We will interpret the above conditions on a physical basis. Assuming that a network realization consisting of reactances in the p and s-planes exists for $W(p, s)$, condition *i* is fairly obvious, since the general loop impedance will be a real rational function in p and s. Condition *iii* is also an obvious consequence of this reason, since the substitution of $-p$ and $-s$ in $W(p, s)$ is equivalent to changing the sign of all element values and hence of every branch and loop impedance. Under the assumption of existence of a two element kind of reactance network corresponding to the given $W(p, s)$, condition *iv* is also clear, because p is fixed as a pure imaginary number, the p-type elements can be considered as "frequency insensitive reactances," and their presence in a network consisting of pure reactances in the s-plane cannot create poles off the imaginary s axis. Similar reasoning justifies condition v for s fixed at any imaginary number, the positive semidefiniteness of $\partial W/\partial p$ can be considered as an extension of the positive slope of a reactance function in the one variable theory.

The necessary and sufficient conditions for a two variable reactance function are not discussed separately, since scalars can be considered as a special case of a reactance matrix.

If $W(p, s)$ has a pole at $p = p_0$, independent of the value of s, p_0 is said to be an s-independent pole of W. The following theorem (see p. 34 of Ref. 8) concerning such poles is important for the synthesis method to be given.

Theorem 2: A two variable reactance matrix $W_0(p, s)$ can be decomposed as

$$W_0(p, s) = W_1(p) + W_2(s) + W(p, s)$$

where W_1 and W_2 are reactance matrices in p and s, respectively, and W is a two variable reactance matrix with no p-independent or s-independent poles.

Any given two variable reactance matrix $W_o(p, s)$, by virtue of the above theorem, can be realized as a series connection of networks having W_1, W_2, and W as their impedance matrices, as shown in Fig. 1. Since W_1 and W_2 can be realized by existing techniques (See chapter 7 of Ref. 11) the given W_o can be realized if a method of synthesis is

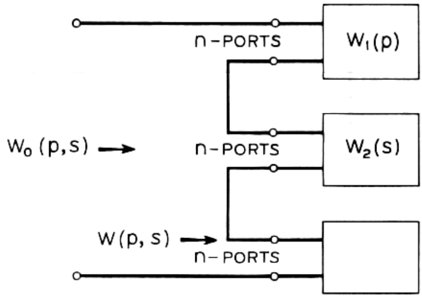

Fig. 1 — Interpretation of theorem 2.

found for W. Henceforth we assume that the given reactance matrix has no p-independent or s-independent poles.

III. SYNTHESIS OF TWO VARIABLE REACTANCE MATRICES

Let us assume that there is a passive n-port network representation, consisting of p- and s-type reactances, gyrators, and ideal transformers, for a given two variable $n \times n$ reactance matrix $W(p, s)$. In such a network it is always possible to replace each s-type capacitor by a gyrator-s-type inductor combination and then isolate all the s-type inductors, of which we assume there are k, as shown in Fig. 2, without changing the impedance seen at the prescribed ports. If we further assume that the $(n + k)$-port coupling network, consisting of p-type reactances, ideal transformers, and gyrators has a Z matrix, then the impedance matrix $W(p, s)$ seen at the first n ports is given by

$$W(p, s) = z_{11}(p) - z_{12}(p)[z_{22}(p) + sl_k]^{-1}z_{21}(p) \tag{1}$$

where $Z(p)$, the impedance matrix of the coupling network is given by

$$Z(p) = \begin{bmatrix} z_{11}(p) & z_{12}(p) \\ z_{21}(p) & z_{22}(p) \end{bmatrix}. \tag{2}$$

Since the coupling network is a lossless network in the p-plane

$$Z = -\mathbf{Z}^{\dagger} \tag{3}$$

and we have

$$W(p, s) = z_{11}(p) + z_{12}(p)[z_{22}(p) + sl_k]^{-1}\mathbf{z}_{12}^{\dagger}(p). \tag{4}$$

Next we show, by algebraic means, that every two variable reactance matrix can be decomposed into the form in equation (4), such that

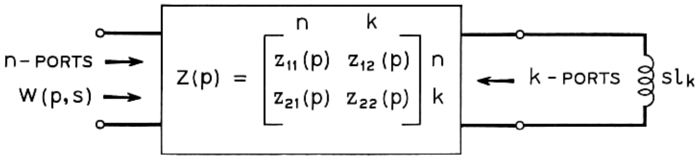

Fig. 2 — Extraction of S-type inductors.

$Z(p)$ of equation (2) describes a lossless network. Once such a decomposition is found, we can realize the given $W(p, s)$ by realizing $Z(p)$ by any of the existing techniques (see chapter 7 of Ref. 11) and terminating it at its last k ports with unit inductors in the s-plane.

To establish that any given two variable reactance matrix $W(p, s)$ can be decomposed as shown in equation (4), we first expand $W(p, s)$ and the expression on the right side of equation (4) about $s = \infty$ and find the expressions that relate z_{11}, z_{12}, and z_{22} with the expansion coefficients of $W(p, s)$. We then show that a set z_{11}, z_{12}, and z_{22}, which satisfies the above relations and at the same time guarantees that the $Z(p)$ of equation (2) is a reactance matrix in p, can always be found.

The given two variable reactance matrix $W(p, s)$ can be assumed to have no p-independent or s-independent poles by virtue of Theorem 2 and hence can be written in the form

$$W(p, s) = \frac{B_0(p)s^r + B_1(p)s^{r-1} + \cdots + B_r(p)}{a_0(p)s^r + a_1(p)s^{r-1} + \cdots + a_r(p)} \tag{5}$$

where the $B_i(p)$ are real polynomial matrices in p and the scalar

$$g(p, s) = a_0(p)s^r + a_1(p)s^{r-1} + \cdots + a_r(p) \tag{6}$$

is the least common denominator of the entries in $W(p, s)$. For any ordinary value of p, $W(p, s)$ can be expanded in the neighborhood of $s = \infty$ as[9]

$$W(p, s) = A_{-1}(p) + \sum_{l=0}^{\infty} \frac{A_l(p)}{s^{l+1}}. \tag{7}$$

Expanding the right side of equation (4) in the neighborhood of $s = \infty$

$$z_{11}(p) + z_{12}(p)[z_{22}(p) + s1_k]^{-1}z_{12}^{\dagger}(p) = z_{11} + (-1)^l \sum_{l=0}^{\infty} \frac{z_{12}z_{22}^{l}z_{12}^{\dagger}}{s^{l+1}}. \tag{8}$$

For the equality in equation (4) to hold, we identify

$$z_{11}(p) = A_{-1}(p) = W(p, \infty) \tag{9}$$

and

$$A_l(p) = (-1)^l z_{12} z_{22}^l \mathbf{z}_{12}^\dagger \qquad l = 0, 1, 2, \cdots . \tag{10}$$

Since the $Z(p)$ formed out of z_{11}, z_{12} and z_{22}

$$Z(p) = \begin{bmatrix} z_{11} & z_{12} \\ -\mathbf{z}_{12}^\dagger & z_{22} \end{bmatrix} \tag{11}$$

has to describe a lossless network in the p plane, we must have

$$Z = -\mathbf{Z}^\dagger$$

as given by equation (3), and hence

$$z_{11} = -\mathbf{z}_{11}^\dagger \tag{12}$$

and

$$z_{22} = -\mathbf{z}_{22}^\dagger . \tag{13}$$

With the identification in equation (9), equation (12) is always satisfied, since by equation (9)

$$z_{11} = W(p, \infty) = -\mathbf{W}(-p, -\infty),$$

and thus z_{11} is uniquely determined. The problem is to chose a pair z_{12}, z_{22} to satisfy equation (10) and at the same time guarantee that equation (11) describes a lossless network in the p-plane. For $Z(p)$ to describe a lossless network, it must be positive real and satisfy equation (3).

Before proceeding further, we would like to know more about $A_l(p)$, the expansion coefficients in equation (7). By equating the right sides of equations (5) and (7),

$$\begin{aligned} &B_0(p)s^r + B_1(p)s^{r-1} + \cdots + B_r(p) \\ &\quad = [a_0(p)s^r + a_1(p)s^{r-1} + \cdots + a_r(p)]\left[A_{-1}(p) + \sum_{l=0}^{\infty} \frac{A_l(p)}{s^{l+1}}\right]. \end{aligned} \tag{14}$$

Equating coefficients of like powers of s on both sides of equation (14), (see p. 207 of Ref. 12, Vol. II).

$$\begin{aligned} &a_0(p)A_{-1}(p) = B_0(p) \\ &a_1(p)A_{-1}(p) + a_0(p)A_0(p) = B_1(p) \\ &a_2(p)A_{-1}(p) + a_1(p)A_0(p) + a_0(p)A_1(p) = B_2(p) \\ &\vdots \end{aligned} \tag{15}$$

$$a_r(p)A_{-1}(p) + a_{r-1}(p)A_0(p) + \cdots + a_0(p)A_{r-1}(p) = B_r(p)$$

and

$$a_0(p)A_i(p) + a_1(p)A_{i-1}(p) + \cdots + a_r(p)A_{i-r}(p) = 0_n \text{ for } i \geqq r.$$

From equation (15) an expression for $A_l(p)$ can be written* in the convenient form (see p. 14 Ref. 9)

$$A_l(p) = \frac{(-1)^{l+1}}{a_0^{l+2}(p)} \begin{vmatrix} B_0(p) & a_0(p) & 0 & 0 & \cdots & 0 \\ B_1(p) & a_1(p) & a_0(p) & 0 & \cdots & 0 \\ B_2(p) & a_2(p) & a_1(p) & a_0(p) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ B_l(p) & a_l(p) & a_{l-1}(p) & a_{l-2}(p) & \cdots & a_0(p) \\ B_{l+1}(p) & a_{l+1}(p) & a_l(p) & a_{l-1}(p) & \cdots & a_1(p) \end{vmatrix} \tag{16}$$

$$l = -1, 0, 1, 2, \cdots$$

$$B_l = 0_n \qquad \text{for} \quad l > r$$

$$a_l = 0 \qquad \text{for} \quad l > r$$

where the $(l + 2) \times (l + 2)$ determinant is expanded formally in terms of its first column. In equation (16) the B_i are matrices, the a_i are scalars, and the determinant is not a determinant in the usual sense. From equation (16), it can be seen that $A_l(p)$ is of the form

$$A_l(p) = \frac{\text{real polynomial matrix in } p}{a_0^{l+2}(p)}. \tag{17}$$

Another important property of the $A_l(p)$'s is obtained from the relation

$$W(p, s) = -\mathbf{W}(-p, -s)$$

which implies

$$A_{-1}(p) + \sum_{l=0}^{\infty} \frac{A_l(p)}{s^{l+1}} = -\mathbf{A}_{-1}(-p) - \sum (-1)^{l+1} \frac{\mathbf{A}_l(-p)}{s^{l+1}}. \tag{18}$$

Hence by equating like powers of s

*Alternate methods of obtaining these $A_l(p)$'s are by diffe rentiation of $W(p, s)$

$$A_l(p) = \frac{\partial^{l+1} W(p, s)}{\partial s^{l+1}} \bigg|_{s=\infty}$$

or by long division.

$$A_l = (-1)^l \mathbf{A}_l^\dagger . \tag{19}$$

If, for the purpose of choosing a pair z_{12}, z_{22} that satisfies equation (10) and, at the same time, guarantees that the $Z(p)$ of equation (11) describes a lossless network in the p-plane, we define $P_l(p)$ as

$$P_l(p) = \begin{bmatrix} z_{12} \\ z_{12}z_{22} \\ z_{12}z_{22}^2 \\ \vdots \\ z_{12}z_{22}^l \end{bmatrix} . \tag{20}$$

Then

$$P_l\mathbf{P}_l^\dagger = \begin{bmatrix} z_{12}\mathbf{z}_{12}^\dagger & z_{12}\mathbf{z}_{22}^\dagger\mathbf{z}_{12}^\dagger & z_{12}\mathbf{z}_{22}^{2\dagger}\mathbf{z}_{12}^{2\dagger} & \cdots & z_{12}\mathbf{z}_{22}^{l\dagger}\mathbf{z}_{12}^\dagger \\ z_{12}z_{22}\mathbf{z}_{12}^\dagger & z_{12}z_{22}\mathbf{z}_{22}^\dagger\mathbf{z}_{12}^\dagger & z_{12}z_{22}\mathbf{z}_{22}^{2\dagger}\mathbf{z}_{12}^\dagger & \cdots & z_{12}z_{22}\mathbf{z}_{22}^{l\dagger}\mathbf{z}_{12}^\dagger \\ z_{12}z_{22}^2\mathbf{z}_{12}^\dagger & z_{12}z_{22}^2\mathbf{z}_{22}^\dagger\mathbf{z}_{12}^\dagger & z_{12}z_{22}^2\mathbf{z}_{22}^{2\dagger}\mathbf{z}_{22}^\dagger & \cdots & z_{12}z_{22}^2\mathbf{z}_{22}^{l\dagger}\mathbf{z}_{12}^\dagger \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ z_{12}z_{22}^l\mathbf{z}_{12}^\dagger & z_{12}z_{22}^l\mathbf{z}_{22}^\dagger\mathbf{z}_{12}^\dagger & z_{12}z_{22}^l\mathbf{z}_{22}^{2\dagger}\mathbf{z}_{22}^\dagger & \cdots & z_{12}z_{22}^l\mathbf{z}_{21}^{l\dagger}\mathbf{z}_{12}^\dagger \end{bmatrix} \tag{21}$$

In the above matrix, the entry in the ith row and jth column is $z_{12}z_{22}^i\mathbf{z}_{22}^{j\dagger}\mathbf{z}_{12}^\dagger$ and by equation (13)

$$z_{12}z_{22}^i\mathbf{z}_{22}^{j\dagger}\mathbf{z}_{12}^\dagger = (-1)^i z_{12}z_{22}^{i+j}\mathbf{z}_{12}^\dagger . \tag{22}$$

Since we wish the equality in equation (10) to hold

$$z_{12}z_{22}^i\mathbf{z}_{22}^{j\dagger}\mathbf{z}_{12}^\dagger = (-1)^i z_{12}z_{22}^{i+j}\mathbf{z}_{12}^\dagger = (-1)^i A_{i+j} . \tag{23}$$

If we define $T_l(p)$ as

$$T_l(p) = \begin{bmatrix} A_0(p) & A_1(p) & A_2(p) & \cdots & A_l(p) \\ -A_1(p) & -A_2(p) & -A_3(p) & \cdots & -A_{l+1}(p) \\ A_2(p) & A_3(p) & A_4(p) & \cdots & A_{l+2}(p) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^l A_l(p) & (-1)^l A_{l+1}(p) & (-1)^l A_{l+2}(p) & \cdots & (-1)^l A_{2l}(p) \end{bmatrix} \tag{24}$$

from equation (23), we can see that

$$T_l = P_l\mathbf{P}_l^\dagger . \tag{25}$$

Equation (25) suggests that a way of obtaining a pair z_{12}, z_{22} would be to form the matrix $T_l(p)$, factor it in the form of equation (25), and then try to identify z_{12} and z_{22} from these factors. We do not know in advance if the matrix $T_l(p)$ formed from the expansion coefficients of W about $s = \infty$ can always be factored as indicated in equation (25); hence we first study the properties of $T_l(p)$, to see if it can be factored in the desired form.

Consider the matrix $T_l(p)$ when $l = r$, r being the s-degree of $g(p, s)$, as given in equation (6),

$$T_r = \begin{bmatrix} A_0 & A_1 & A_2 & \cdots & A_{r-1} & A_r \\ -A_1 & -A_2 & -A_3 & \cdots & -A_r & -A_{r+1} \\ A_2 & A_3 & A_4 & \cdots & A_{r+1} & A_{r+2} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ (-1)^{r-1}A_{r-1} & (-1)^{r-1}A_r & (-1)^{r-1}A_{r+1} & \cdots & (-1)^{r-1}A_{2r-1} & (-1)^{r-1}A_{2r-1} \\ (-1)^r A_r & (-1)^r A_{r+1} & (-1)^r A_{r+2} & \cdots & (-1)^r A_{2r-1} & (-1)^r A_{2r} \end{bmatrix}. \quad (26)$$

The matrix obtained by deleting the last column and row in equation (26) is T_{r-1}, and by equation (15) it is easy to see that the last column is a linear combination of the first r columns. Hence*

$$\text{rank } T_r = \text{rank } T_{r-1}$$

and

$$\text{rank } T_l = \text{rank } T_{r-1} \qquad \text{for} \quad l \geqq r - 1.$$

The rank of T_{r-1} is connected with the s-degree $\delta_s[W(p, s)]$ of $W(p, s)$ which is defined in Definition 3 (see p. 10 of Ref. 9).

Definition 3: The s-degree of a rational two variable matrix $W(p, s)$ is obtained from the rule

$$s = \text{degree of} \quad W(p, s) = \delta_s[W(p, s)] = \max_{p_0} \delta[W(p_0, s)]$$

where $\delta[W(p_o, s)]$ is the McMillan degree (see part II of Ref. 13) of $W(p_0, s)$. For any fixed p_o, $W(p_o, s)$ is a matrix of rational functions in s with its McMillan degree uniquely specified; hence the above definition uniquely specifies the s-degree of $W(p, s)$. The relationship between the s-degree of $W(p, s)$ and the rank of T_{r-1} is stated formally in the following lemma.

Lemma 1: The rank of $T_{r-1}(p)$ is equal to the s-degree of $W(p, s)$.

* By the rank of rational or polynomial matrix we mean the "normal rank," which is defined to be the rank everywhere except at a finite number of values of the variable.

The proof of this lemma for the one variable case can be found in Ref. 14 and on p. 200 of Ref. 10, and for the two variable case on p. 17 of Ref. 9.

To show that the matrix $T_{r-1}(p)$ can always be factored in the form of equation (25), we need the following lemma.

Lemma 2: The matrix $T_{r-1}(p)$ *defined by equation* (24) *for* $l = r - 1$ *satisfies*

(*i*) $T_{r-1} = \mathbf{T}_{r-1}^{\dagger}$
(*ii*) $T_{r-1}(j\omega)$ *is Hermitian and positive semidefinite.*

Proof:

Since $A_l = (-1)^l \mathbf{A}_l^{\dagger}$ by equation (19), the proof of i is readily seen from equation (26)

$$T_{r-1} = \mathbf{T}_{r-1}^{\dagger}. \tag{27}$$

To prove (*ii*), we first notice that by Theorem 1, for any real ω, $W(j\omega, s)$ has only simple poles, which are restricted to the imaginary axis in the s-plane. Hence $W(j\omega, s)$ can be expressed in the partial fraction form

$$W(j\omega, s) = A_{-1}(j\omega) + \sum_{i=1}^{r} \frac{R_i(j\omega)}{s - j\alpha_i(\omega)} \tag{28}$$

where, $R_i(j\omega)$ are the residue matrices at the poles $j\alpha_i(\omega)$, and the $\alpha_i(\omega)$ are real.

It is shown in Appendix A that the $R_i(j\omega)$ are Hermitian and positive semidefinite for each ω. Now, if each term in the sum on the right side of equation (28) is expanded about $s = \infty$, we have

$$W(j\omega, s) = A_{-1}(j\omega) + \sum_{i=1}^{r} \sum_{q=0}^{\infty} \frac{(j\alpha_i)^q}{s^{q+1}} R_i(j\omega). \tag{29}$$

For the purpose of comparison, equation (7), written with $p = j\omega$, is

$$W(j\omega, s) = A_{-1}(j\omega) + \sum_{q=0}^{\infty} \frac{A_q(j\omega)}{s^{q+1}}. \tag{30}$$

The right sides of equation (29) and (30) are expansions of $W(j\omega, s)$ about $s = \infty$, and because of the uniqueness of a power series expansion

$$A_q(j\omega) = \sum_{i=1}^{r} (j\alpha_i)^q R_i(j\omega). \tag{31}$$

By noting that the α_i are real and the $R_i(j\omega)$ are Hermitian and positive semidefinite for each ω, we have

$$A_0(j\omega) = \sum_{i=1}^{r} R_i(j\omega) \qquad \geqq 0 \tag{32}^*$$

$$A_1(j\omega) = j \sum_{i=1}^{r} \alpha_i R_i(j\omega) \tag{33}$$

$$A_2(j\omega) = -\sum_{i=1}^{r} \alpha_i^2 R_i(j\omega) \qquad \leqq 0 \tag{34}$$

$$\vdots$$

$$A_{4m-3}(j\omega) = -j \sum_{i=1}^{r} \alpha_i^{4m-3} R_i(j\omega) \tag{35a}$$

$$A_{4m-2}(j\omega) = -\sum_{i=1}^{r} \alpha_i^{4m-2} R_i(j\omega) \qquad \leqq 0 \tag{35b}$$

$$A_{4m-1}(j\omega) = -j \sum_{i=1}^{r} \alpha_i^{4m-1} R_i(j\omega) \tag{35c}$$

$$A_{4m}(j\omega) = \sum_{i=1}^{r} \alpha_i^{4m} R_i(j\omega) \qquad \geqq 0. \tag{35d}$$

By direct substitution of equation (33) into equation (24), $T_{r-1}(j\omega)$ can be written as

$$T_{r-1}(j\omega) = \sum_{i=1}^{r} \begin{bmatrix} R_i & j\alpha_i R_i & -\alpha_i^2 R_i & \cdots & (j\alpha_i)^{r-1} R_i \\ -j\alpha_i R_i & \alpha_i^2 R_i & j\alpha_i^3 R_i & \cdots & -(j\alpha_i)^r R_i \\ -\alpha^2 R_i & -j\alpha_i^3 R_i & \alpha_i^4 R_i & \cdots & (j\alpha)^{r+1} R_i \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{r-1}(j\alpha)^{r-1} R_i & (-1)^{r-1}(j\alpha)^r R_i & (-1)^{r-1}(j\alpha)^{r+1} R_i & \cdots & (-1)^{r-1}(j\alpha)^{2r-2} R_i \end{bmatrix}. \tag{36}$$

The matrix sum on the right side of equation (36) can be written

$$T_{r-1}(j\omega) = \sum_{i=1}^{r} L_i \begin{bmatrix} R_i & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \mathbf{L}_i^* \tag{37}$$

where

---

* By the notation $A \geqq 0$ or $A \leqq 0$, we mean that the associated Hermitian form of A is positive semidefinite or negative semidefinite.

$$L_i = \begin{bmatrix} 1_n & 0 & 0 & \cdots & 0 \\ j\alpha_i 1_n & 1_n & 0 & \cdots & 0 \\ \alpha_i^2 1_n & 0 & 1_n & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^r (j\alpha)^{r-1} 1_n & 0 & 0 & \cdots & 1_n \end{bmatrix}^{-1} . \tag{38}$$

Since each $R_i(j\omega)$ is Hermitian and positive semidefinite for each ω, the sum on the right side of equation (37) is also Hermitian and positive semidefinite. Hence, we have proved the lemma.

We have shown that T_{r-1}, a matrix of rational functions, is para-Hermitian and positive semidefinite on the imaginary axis. Such a matrix can always be factored in the form shown in equation (25), (see p. 133 of Ref. 15). It is tempting to factor T_{r-1} at this stage and find z_{12}, z_{22} to satisfy the required conditions, but we will factor $a_0^{2r}\ T_{r-1}$ instead of T_{r-1} for the reason that the factors would be polynomial matrices.

From equation (17) we can see that $a_0^{2r}\ T_{r-1}$ is a polynomial matrix in p. To be able to factor $a_0^{2r}\ T_{r-1}$ in the required fashion, we have to show that $\hat{T} = a_0^{2r}\ T_{r-1}$ is para Hermitian and positive semidefinite on the $j\omega$ axis. To do this, we obtain the required additional information about the polynomial $a_0(p)$ from the following theorem. Since the theorem contains more information than we need at this point, we will only state it here; a proof is given in Appendix B.

Theorem 3: If

$$W(p, s) = \frac{B_0(p)s^r + B_1(p)s^{r-1} + \cdots + B_r(p)}{a_0(p)s^r + a_1(p)s^{r-1} + \cdots + a_r(p)}$$

is a two variable reactance matrix, then for all $i = 0, 1, \cdots, r$

(i)	$\dfrac{B_i}{a_i}$	is a reactance matrix in p
(ii)	$\dfrac{a_i}{a_{i+1}}$	is a reactance function in p
(iii)	a_i	has all its zeros on the $j\omega$ axis and these are simple
(iv)	$\dfrac{\mathbf{X}B_iX}{\mathbf{X}B_{i+1}X}$	for all constant real $n \times 1$ vectors, X, is a reactance function in p

From this theorem, $a_o(p)$ can be represented as

$$a_0(p) = p^{\nu} \prod_i (p^2 + \omega_i^2) = \pm a_0(-p) \tag{39}$$

where $\nu = 0$ or 1. Hence

$$\hat{T}_{r-1} = a_0^{2r} T_{r-1} = \hat{\mathbf{T}}_{r-1}^{\dagger} . \tag{40}$$

From the form of a_o shown in equation (39) and Lemma 1, it can be seen that

$$\hat{T}_{r-1}(j\omega) \geqq 0 \tag{41}$$

except when simultaneously, $\nu = 1$ and r is odd; in which case

$$\hat{T}_{r-1}(j\omega) \leqq 0. \tag{42}$$

We will assume that $\hat{T}_{r-1}(j\omega) \geqq 0$ in developing the synthesis procedure and discuss the needed modification when $\hat{T}_{r-1}(j\omega) \leqq 0$ later.

If the s-degree of $W(p, s)$ is equal to k, by Lemma 1 the rank of $T_{r-1}(p)$ and hence of $\hat{T}_{r-1}(p)$ is k. Since $\hat{T}_{r-1} = \hat{\mathbf{T}}_{r-1}^{\dagger}$ and $\hat{T}_{r-1}(j\omega) \geqq 0$ there exists a factorization [16,17]

$$\hat{T}_{r-1}(p) = M(p)\mathbf{M}^{\dagger}(p) \tag{43}$$

where $M(p)$ is an $nr \times k$ polynomial matrix and has a left inverse $M^{-1}(p)$ which is analytic in Re $p > 0$.

From the definition of $\hat{T}_{r-1}$, we have

$$T_{r-1}(p) = \frac{M(p)\mathbf{M}^{\dagger}(p)}{a_0^{2r}}. \tag{44}$$

$M(p)$ can be partitioned into $n \times k$ blocks $M_i(p)$

$$M(p) = \begin{bmatrix} M_0(p) \\ --- \\ M_1(p) \\ --- \\ \vdots \\ --- \\ M_{r-1}(p) \end{bmatrix} \tag{45}$$

and hence

$$\mathbf{M}^{\dagger}(p) = [\mathbf{M}_0^{\dagger}(p) \vdots \mathbf{M}_1^{\dagger}(p) \vdots \cdots \vdots \mathbf{M}_{r-1}^{\dagger}(p)]. \tag{46}$$

Now by comparison of equation (45) with equation (20), we can immediately identify a suitable z_{12} as

$$z_{12} = \frac{M_0}{a_0^r}. \tag{47}$$

To find a suitable z_{22}, if we define T_d as

$$T_d(p) = \begin{bmatrix} -A_1 & -A_2 & -A_3 & \cdots & -A_r \\ A_2 & A_3 & A_4 & \cdots & A_{r+1} \\ -A_3 & -A_4 & -A_5 & \cdots & -A_{r+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^r A_r & (-1)^r A_{r+1} & (-1)^r A_{r+2} & \cdots & (-1)^r A_{2r} \end{bmatrix} \tag{48}$$

from equations (20), (21), and (25) we see that z_{22} must satisfy

$$\frac{1}{a_0^{2r}} M z_{22} \mathbf{M}^{\dagger} = T_d \,. \tag{49}$$

Even though equation (49) does not uniquely specify z_{22}, we can choose for z_{22}

$$z_{22} = a_0^{2r} M^{-1} T_d \mathbf{M}^{-1\dagger}. \tag{50}$$

From equation (19) and the definition of T_d, we see that $T_d = -\mathbf{T}_d^{\dagger}$ and hence

$$z_{22} = -\mathbf{z}_{22}^{\dagger}. \tag{51}$$

We now notice that by construction, the pair z_{12}, z_{22} defined by equations (47) and (50) satisfies

$$(-1)^l z_{12} z_{22}^l \mathbf{z}_{12}^{\dagger} = A_l \tag{10}$$

for all $0 \leqq l \leqq 2r - 2$. Our aim is to find z_{12} and z_{22} that satisfy equation (10) for all $l \geqq 0$. It is not immediately clear that the pair z_{12}, z_{22} defined by equations (47) and (50) satisfy equation (10) for all $l \geqq 0$.

To see that the chosen pair z_{12}, z_{22} does indeed satisfy equation (10) for all $l \geqq 0$ and not just for $0 \leqq l \leqq 2r - 2$, we introduce the generalized companion matrix $\Omega(p)$ defined by[10]

$$\Omega(p) = \begin{bmatrix} 0_n & 1_n & 0_n & \cdots & \cdots & 0_n \\ 0_n & 0_n & 1_n & \cdots & \cdots & 0_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0_n & 0_n & 0_n & \cdots & 0_n & 1_n \\ -\frac{a_r}{a_0} 1_n & -\frac{a_{r-1}}{a_0} 1_n & -\frac{a_{r-2}}{a_0} 1_n & \cdots & -\frac{a_2}{a_0} 1_n & -\frac{a_1}{a_0} 1_n \end{bmatrix}. \tag{52}$$

From equation (15) it can be seen that

$$T_d = -T_{r-1}\mathbf{\Omega}. \tag{53}$$

Hence, by equation (43)

$$\begin{aligned} z_{22} &= -a_0^{2r}M^{-1}T_{r-1}\mathbf{\Omega}\mathbf{M}^{-1\dagger} \\ &= -M^{-1}M\mathbf{M}^{\dagger}\mathbf{\Omega}\mathbf{M}^{-1\dagger} \\ &= -\mathbf{M}^{\dagger}\mathbf{\Omega}\mathbf{M}^{-1\dagger}, \end{aligned}$$

and by equation (51)

$$z_{22} = -\mathbf{M}^{\dagger}\mathbf{\Omega}\mathbf{M}^{-1\dagger} = M^{-1}\Omega^{\dagger}M. \tag{54}$$

Hence

$$\begin{aligned} z_{22}^2 &= -M^{-1}\Omega^{\dagger}M\mathbf{M}^{\dagger}\mathbf{\Omega}\mathbf{M}^{-1\dagger} \\ &= -a_0^{2r}M^{-1}\Omega^{\dagger}T_{r-1}\mathbf{\Omega}\mathbf{M}^{-1\dagger} \\ &= -a_0^{2r}M^{-1}\Omega^{2\dagger}T_{r-1}\mathbf{M}^{-1\dagger} \\ &= -M^{-1}\Omega^{2\dagger}M \end{aligned}$$

and

$$z_{22}^l = (-1)^{l-1}M^{-1}\Omega^{l\dagger}M; \qquad l > 0. \tag{55}$$

From the definition of Ω, we see that $g(p, \xi)$ is its minimal polynomial, and hence the matrix polynomial

$$g(p, \Omega) = a_o\Omega^r + a_1\Omega^{r-1} + \cdots + a_r1_{nr} \equiv 0_{nr} \tag{56}$$

and hence

$$g(-p, \Omega^{\dagger}) = 0_{nr}. \tag{57}$$

By equation (55)

$$g(p, z_{22}) = M^{-1}\ (-1)^{r-1}a_o\Omega^{r\dagger} + (-1)^{r-2}a_1\Omega^{r-1\dagger} + \cdots + a_r1_{nr}\ M,$$

and by equation (57) and Theorem 3

$$g(p, z_{22}) = \pm\ M^{-1}[g(-p, \Omega^{\dagger})]M \equiv 0_k. \tag{58}$$

From the last equation in equation (15), from equation (58), and from equation (10), which holds for $0 \leqq l \leqq 2r - 2$,

$$\begin{aligned} a_0A_{2r-1} &= -a_1A_{2r-2} - a_2A_{2r-3} - \cdots - a_rA_{r-1} \\ &= -z_{12}[a_1z_{22}^{2r-2} + a_2z_{22}^{2r-2} + \cdots + a_rz_{22}^{r-1}]\mathbf{z}_{12}^{\dagger} \\ &= -a_0z_{12}z_{22}^{2r-1}\mathbf{z}_{12}^{\dagger}\ . \end{aligned}$$

Hence

$$A_{2r-1} = (-1)^{2r-1} z_{12} z_{22}^{2r-1} \mathbf{z}_{12}^{\dagger} ,$$

and by induction

$$A_l = (-1)^l z_{12} z_{22}^l \mathbf{z}_{12}^{\dagger}$$

for all $l \geqq 0$, which is the same as equation (10).

We thus have a set of three matrices z_{11}, z_{12}, z_{22} such that the infinite set of equations obtained by equating the right sides of equations (7) and (8) are satisfied. Hence the right side of equation (4) and $W(p, s)$ have the same Taylor's series expansion in the neighborhood of $s = \infty$. By analytic continuation, for all p and s

$$W(p, s) = z_{11}(p) + z_{12}(p)[z_{22}(p) + s1_k]^{-1} \mathbf{z}_{12}^{\dagger}(p),$$

where z_{11}, z_{12}, and z_{22} are defined by equations (9), (47), and (50), respectively.

We have thus succeeded in decomposing $W(p, s)$ as shown in equation (4). It now remains to show that $Z(p)$ formed from the chosen z_{11}, z_{12}, and z_{22}

$$Z(p) = \begin{bmatrix} z_{11}(p) & z_{12}(p) \\ -\mathbf{z}_{12}^{\dagger}(p) & z_{22}(p) \end{bmatrix} = \begin{bmatrix} W(p, \infty) & \dfrac{M_0(p)}{a_0^r(p)} \\ -\dfrac{\mathbf{M}_0^{\dagger}(p)}{a_0^{r\dagger}(p)} & a_0^{2r}(p) M^{-1}(p) T_d(p) \mathbf{M}^{-1\dagger}(p) \end{bmatrix} \tag{59}$$

is a reactance matrix.

To show that the $Z(p)$ in equation (59) is a reactance matrix, we may choose any standard test, but we will choose the one given below since it is particularly suited for the problem at hand (see pp. 117 and 123 of Ref. 11):

Lemma 3: The necessary and sufficient conditions for a square matrix $Z(p)$ to be a reactance matrix are:

(*i*) *Z is rational and real for real p.*
(*ii*) *Poles of $Z(p)$ are simple and restricted to the imaginary axis.*
(*iii*) $Z + \mathbf{Z}^{\dagger} \equiv 0$.
(*iv*) *Residue matrices are positive semidefinite Hermitian.*

Since all the entries of $Z(p)$ in equation (59) are real and rational, condition i is satisfied.

From equations (13a) and (15), $z_{11} = A_{-1} = B_0/a_0$ is a reactance matrix by Theorem 3; hence its poles are simple and restricted to the imaginary axis. Also, the pole at $p = \infty$, if any exists, is simple for this block. Since M_0 is a polynomial matrix, it is clear that the poles of the off diagonal blocks z_{12} and $-\mathbf{z}_{12}^{\dagger}$ are in the zeros of a_0 and hence by Theorem 3 the poles of z_{12} are restricted to the $j\omega$ axis. However, it is not clear that these poles are simple. To show that they are indeed simple we will use the fact that $\hat{A}_0$ defined by

$$\hat{A}_0 = a_0^2 A_0 \tag{60}$$

is a polynomial matrix. From equations (10) and (47)

$$A_0 = \frac{M_0 \mathbf{M}_0^{\dagger}}{a_0^{2r}} \tag{61}$$

and from equation (39) $a_0 = \pm a_0^{\dagger}$. We first consider $a_0 = a_0^{\dagger}$ in which case

$$\hat{A}_0 = \left[\frac{M_0}{a_0^{r-1}}\right]\left[\frac{\mathbf{M}_0}{a_0^{r-1}}\right]^{\dagger}. \tag{62}$$

Equation (54) then shows that $\hat{A}_0 = \hat{\mathbf{A}}_0^{\dagger}$ and $\hat{A}_0(j\omega) \geqq 0$. Hence there exists an $n \times q$ polynomial matrix, Q, such that

$$\hat{A}_0 = Q\mathbf{Q}^{\dagger} \tag{63}$$

where q is the rank of $\hat{A}_0$. Equations (62) and (63) are two different factorizations of $\hat{A}_0$, hence:[17]

$$\frac{M_0}{a_0^{r-1}} = Q[1_q \vdots 0_{k \times (r-q)}]V \tag{64}$$

where $V(p)$ is a $k \times k$ para unitary matrix, that is, $V\mathbf{V}^{\dagger} = 1_k$. Since Q is a polynomial matrix, and $V(p)$ being para unitary can have no poles on the imaginary axis (see p. 186 of Ref. 11), the left side of equation (64) can have no poles on the imaginary axis. Hence a_0^{r-1}, which has all its zeros on the $j\omega$ axis, must divide M_0. Thus z_{12} has all its finite poles in the zeros of a_0. By Theorem 3, the zeros of a_0 are simple and restricted to the $j\omega$ axis. In the above, we have assumed that $a_0 = a_0^{\dagger}$; if $a_0 = -a_0^{\dagger}$ and r is odd, the same proof holds; if r is even we can construct a similar proof by factoring $-\hat{A}_0$ instead of $\hat{A}_0$.

To show that the pole of z_{12} at $p = \infty$, if any, is simple. Consider the following representation for A_0 obtained from equations (15) and (17)

$$A_0 = \frac{a_0 B_1 - a_1 B_0}{a_0^2} = \frac{B_1}{a_1}\cdot\frac{a_1}{a_0} - \frac{B_0}{a_0}\cdot\frac{a_1}{a_0}. \tag{65}$$

Since B_1/a_1 and B_0/a_0 are reactance matrices and a_1/a_0 is a reactance function, according to Theorem 3, the right side of equation (65) behaves as Kp^ν near $p = \infty$, where K is a constant matrix and ν is an integer such that $-2 \leqq \nu \leqq 2$. But z_{12} satisfies

$$A_0 = z_{12}\mathbf{z}_{12}^\dagger$$

and hence the pole of z_{12} at $p = \infty$, if any, must be simple.

We now have to show that the z_{22} block also satisfies condition (*ii*) of the lemma. By equation (54)

$$z_{22} = -\mathbf{M}^\dagger\mathbf{\Omega}\mathbf{M}^{\dagger-1} = M^{-1}\Omega^\dagger M. \tag{54}$$

Since M^{-1} is analytic in the open right-half plane and Ω has all its poles in the zeros of a_0, by equation (54) the poles of z_{22} are restricted to the $j\omega$ axis. To show that these poles are simple we will prove by contradiction that a_0z_{22} is polynomial.

From equation (52), the definition of Ω, and equation (54) we see that if a_0z_{22} has a pole of multiplicity α at $p = j\omega_0$. In the neighborhood of this pole, we have the approximation

$$a_0z_{22} \approx \frac{K}{(p - j\omega_0)^\alpha} \tag{66}$$

where K is a constant matrix and α is a positive integer, and

$$a_0^2z_{22}^2 \approx \frac{K^2}{(p - j\omega_0)^{2\alpha}}. \tag{67a}$$

Now by equation (55) $z_{22}^2 = -M^{-1}\Omega^{2\dagger}M$, and hence in the neighborhood of $p = j\omega_0$

$$a_0^2z_{22}^2 \approx \frac{K_1}{(p - j\omega_0)^\beta} \tag{67b}$$

where K_1 is a constant matrix and β is a positive integer. Since the poles of a_0z_{22} are contained in the poles of M^{-1}, $\beta \leqq 2\alpha$. By comparison of equations (67a) and (67b), which must be equal, it is clear that either $\alpha = \beta = 0$ or $K_1 = K^2 = 0$. Since $z_{22} = -\mathbf{z}_{22}^\dagger$, $K = \mathbf{K}^*$, and hence $K^2 = K\mathbf{K}^* = 0$ implies that $K = 0$. Thus a_0z_{22} can have no poles on the $j\omega$ axis and this, coupled with the fact that z_{22} can have poles only on the $j\omega$ axis, guarantees that a_0z_{22} is always polynomial. We therefore conclude that all the finite poles of z_{22} are in the zeros of a_0, and their multiplicity cannot exceed that of the corresponding zeros of a_0. Hence, again by Theorem 3, all the finite poles of z_{22} are simple and restricted to the $j\omega$ axis.

To show that the pole at $p = \infty$ of z_{22}, if any, is simple, consider equation (15) written in this form:

$$
\begin{aligned}
A_{-1} &= \frac{B_0}{a_0} \\
A_0 &= \frac{B_1}{a_1} \cdot \frac{a_1}{a_0} - \frac{B_0}{a_0} \cdot \frac{a_1}{a_0} \\
A_1 &= \frac{B_2}{a_2} - \frac{a_2}{a_1} \cdot \frac{a_1}{a_0} - \frac{a_1}{a_0}\left[\frac{B_1}{a_1} \cdot \frac{a_1}{a_0} - \frac{B_0}{a_0} \cdot \frac{a_1}{a_0}\right] - \frac{a_2}{a_1} \cdot \frac{a_1}{a_0} \cdot \frac{B_0}{a_0} \cdot \\
&\vdots
\end{aligned}
\tag{68}
$$

Owing to the reactance nature of B_i/a_i and a_i/a_{i+1} by Theorem 3, and from the form of A_i shown in equation (68), near $p = \infty$, A_i behaves as

$$A_i \approx K_i p^{\nu_i} \tag{69}$$

where K_i is a constant matrix and ν_i is an integer such that

$$i + 2 \geqq \nu_i \geqq -(i + 2). \tag{70}$$

Also from equation (10)

$$z_{12} z_{22}^{i} \mathbf{z}_{12}^{\dagger} = (-1)^i A_i \approx \pm K_i p^{\nu_i}. \tag{71}$$

Since z_{12} has at most a simple pole at $p = \infty$, in the neighborhood of $p = \infty$

$$z_{12} \approx K p^{l} \tag{72}$$

where K is a constant matrix and l is an integer such that $l \leqq 1$. If z_{22} behaves as $K_{22} p^m$ near $p = \infty$, where K_{22} is a constant matrix and m an integer, then by equation (70), (71), and (72), $(i + 2) \geqq im + 2l \geqq -(i + 2)$. For such to be true for any fixed l and all integral $i \geqq 0$, m has to be less than or equal to unity. Hence the pole of z_{22} at $p = \infty$, if any, is simple.

We have thus shown that condition *ii* of Lemma 1 is satisfied for each block in $Z(p)$, and hence $Z(p)$ also satisfies it.

Since z_{11} is a reactance matrix, $z_{11} = -\mathbf{z}_{11}^{\dagger}$ and $z_{22} = -\mathbf{z}_{22}^{\dagger}$ by equation (51), we have $Z = -\mathbf{Z}^{\dagger}$ and thus condition (*iii*) of the lemma is also satisfied.

Now to complete the proof that $Z(p)$ is a reactance matrix, we have to show that the residue matrices at the poles are positive semidefinite Hermitian. To do this we need Lemma 4, which follows from the definitions of a two variable positive real and two variable reactance matrices (see p. 34 of Ref. 8).

Lemma 4. If $W(p, s)$ is a two variable reactance matrix with no p-independent or s-independent poles, $W[p, s(p)]$ is a reactance matrix in p for any reactance function $s(p)$.

To prove that $Z(p)$ satisfies condition *iv* of Lemma 1, which requires that the residue matrix of $Z(p)$ at any of its simple poles on the $j\omega$ axis is positive semidefinite Hermitian, we note that at any pole, $p = j\omega$, of $Z(p)$, if we set

$$s(p) = \frac{2lp}{p^2 + \omega_0^2} \qquad \text{for} \quad |\omega_0| < \infty$$

$$= lp \qquad \text{for} \quad \omega_0 = \infty$$

in

$$W(p, s) = z_{11} + z_{12}(z_{22} + sl_k)^{-1}\mathbf{z}_{12}^{\dagger}$$

[which is equation (4)] then by Lemma 4, $W[p, s(p)]$ is a reactance matrix in p for all positive l. Since $Z(p)$ is real for real p and $Z = -\mathbf{Z}^{\dagger}$, the residue matrix H at the pole $p = j\omega$ is Hermitian; if we write it as

$$H = \begin{bmatrix} H_{11} & H_{12} \\ \mathbf{H}_{12}^* & H_{22} \end{bmatrix} \tag{73}$$

then, K, the residue matrix of $W[p, s(p)]$ at $p = j\omega_0$ is given by

$$K = H_{11} - H_{12}(H_{22} + l1_k)^{-1}\mathbf{H}_{12}^* . \tag{74}$$

Since H, H_{11}, and H_{22} are Hermitian, there exist unitary matrices U_1 and U_2 such that

$$\Lambda_{11} = \mathbf{U}_1^* H_{11} U_1 = \text{diag}\,[d_1, d_2, \cdots, d_n] \tag{75}$$

and

$$\Lambda_{22} = \mathbf{U}_2^* H_{22} U_2 = \text{diag}\,[\lambda_1, \lambda_2, \cdots, \lambda_k]. \tag{76}$$

Hence

$$\mathbf{U}_1^* K U_1 = \Lambda_{11} - J_{12}(\Lambda_{22} + l1_k)^{-1}\mathbf{J}_{12}^* \tag{77}$$

where

$$J_{12} = \mathbf{U}_1^* H_{12} U_2 . \tag{78}$$

If J_{12i} denotes the ith column of J_{12}, the right side of equation (77) can be rewritten as

$$\mathbf{U}_1^* K U_1 = \Lambda_{11} - \sum_{i=1}^{k} \frac{1}{\lambda_i + l} J_{12i}\mathbf{J}_{12i}^* . \tag{79}$$

Since K is the residue matrix of a reactance matrix, for all $l > 0$, K is positive semidefinite. Λ_{11} is also positive semidefinite, since H_{11} is the residue matrix of the reactance matrix z_{11} . $J_{12i}\mathbf{J}_{12i}^*$ is obviously positive semidefinite and the left side of (79) can be positive semidefinite for all positive l only if all the λ_i are nonnegative. Hence Λ_{22} and H_{22} are positive semidefinite.

To show that H is positive semidefinite, we will show that H' defined by

$$H' = (\mathbf{U}_1^* \dotplus \mathbf{U}_2^*)H(U_1 \dotplus U_2) = \begin{bmatrix} \Lambda_{11} & J_{12} \\ \mathbf{J}_{12}^* & \Lambda_{22} \end{bmatrix} \tag{80}$$

is positive semidefinite. For this purpose, consider the Hermitian form

$$[\mathbf{X}_1^* \ \mathbf{X}_2^*] \begin{bmatrix} \Lambda_{11} & J_{12} \\ \mathbf{J}_{12}^* & D_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$
$$= \mathbf{X}_1^*\Lambda_{11}X_1 + \mathbf{X}_2^*\mathbf{J}_{12}^*X_1 + \mathbf{X}_1^*J_{12}X_2 + \mathbf{X}_2^*D_{22}X_2 \tag{81}$$

where

$$D_{22} = \Lambda_{22} + l1_k ; \qquad l > 0.$$

Since

$$\mathbf{U}_1^*KU_1 = \Lambda_{11} - J_{12}D_{22}^{-1}\mathbf{J}_{12}^*$$

is positive semidefinite, we obtain from equation (81) the following inequality:

$$[\mathbf{X}_1^* \ \mathbf{X}_2^*] \begin{bmatrix} \Lambda_{11} & J_{12} \\ \mathbf{J}_{12}^* & D_{22} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$
$$\geqq \mathbf{X}_1^*J_{12}D_{22}^{-1}\mathbf{J}_{12}^*X_1 + \mathbf{X}_1^*J_{12}X_2 + \mathbf{X}_2^*\mathbf{J}_{12}^*X_1 + \mathbf{X}_2^*\Lambda_{22}X_2 . \tag{82}$$

Since the right side of equation (82) can be expressed as $\mathbf{G}^*G$, where $G = [D_{22}^{1/2}J_{12}^*X_1 + \Lambda_{22}^{1/2}X_2]$, the Hermitian form in equation (81) is positive semidefinite for all $l > 0$; by a continuity argument we can see that H' and consequently H are positive semidefinite.

We have thus shown that $Z(p)$ does indeed describe a lossless network in the p-plane and thus $W(p, s)$ has the network representation shown in Fig. 2.

In the development of the synthesis procedure we assumed that $a_0^{2r}(j\omega)T_{r-1}(j\omega) = \hat{T}_{r-1}(j\omega) \geqq 0$. If simultaneously, $a_0(p)$ is an odd function of p [in other words $\nu = 1$ in equation (39)] and r, the s-degree of the least common denominator of $W(p, s)$ is odd, then $\hat{T}_{r-1}(j\omega) \leqq 0$. In this case we factor $-\hat{T}_{r-1}(p)$ which is para Hermitian and positive semidefinite on the $j\omega$ axis. We will then have

$$-\hat{T}_{r-1} = M\,\mathbf{M}^{\dagger} \tag{83}$$

and hence, as before equation (44),

$$T_{r-1} = \frac{M}{a_0^r}\frac{\mathbf{M}^{\dagger}}{a_0^{r\dagger}}.$$

It is then clear that the identification of z_{12} and z_{22} can be done in exactly the same way as when $\hat{T}_{r-1}(j\omega) \geqq 0$.

It is of importance to notice that the number of s-plane inductors used in the realization of Fig. 2 is equal to the s-degree, $\delta_s[W(p, s)]$ which in general is smaller than the number required in Koga's technique. Appendix C shows that $\delta_s[W(p, s)]$ is the minimum number of s-plane inductors required in any realization, and that if a realization is minimal in the variable s it is automatically minimal in the variable p, the minimum number of p-type reactances needed in any realization being the p-degree, $\delta_p[W(p, s)]$.[18]

The main result of this section can be conveniently put in the form of a theorem:

Theorem 4: Every two variable reactance matrix $W(p, s)$ can be realized as the impedance seen at the first n-ports of a lossless $(n + k)$-port consisting of $\delta_p[W(p, s)]$ reactances in the p-plane, terminated at its last k ports with $\delta_s[W(p, s)]$ unit inductors in the s-plane. Furthermore, such a realization uses the minimum possible number of reactances of each kind. (The roles of p and s are completely interchangeable.)

Since several of the proofs involved in establishing Theorem 4 were rather indirect and lengthy, while the procedure for synthesis, summarized in Section IV, is itself rather simple.

IV. SUMMARY OF SYNTHESIS PROCEDURE

Given an $(n \times n)$ two variable reactance matrix $W_o(p, s)$, decompose it as

$$W_0(p, s) = W_1(p) + W_2(s) + W(p, s)$$

where W_1 and W_2 are reactance matrices in p and s, and $W(p, s)$ is a two variable reactance matrix with no p-independent or s-independent poles. Such a decomposition is always possible by Theorem 2.

Expand $W(p, s)$ as

$$W(p, s) = A_{-1}(p) + \sum_{l=0}^{\infty} \frac{A_l(p)}{s^{l+1}}$$

[which is the same as equation (7)] where the $A(p)$'s may be obtained by equations (16) or (16a) or by long division.

Find $g(p, s)$, the least common denominator of the entries in $W(p, s)$ and express it in the form

$$g(p, s) = a_0(p)s^r + a_1(p)s^{r-1} + \cdots + a_0(p).$$

[which is the same as equation (6)].

Form the $(nr \times nr)$ matrix $T_{r-1}(p)$, defined by

$$T_{r-1}(p) = \begin{bmatrix} A_0(p) & A_1(p) & A_2(p) & \cdots & A_{r-1}(p) \\ -A_1(p) & -A_2(p) & -A_3(p) & \cdots & -A_r(p) \\ A_2(p) & A_3(p) & A_4(p) & \cdots & A_{r+1}(p) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (-1)^{r-1}A_{r-1}(p) & (-1)^{r-1}A_r(p) & (-1)^{r-1}A_{r+1}(p) & \cdots & (-1)^{r-1}A_{2r-2}(p) \end{bmatrix}$$

which is equation (24).

Factor $\hat{T}_{r-1}(p) = a_0^{2r} T_{r-1}(p)$, a polynomial matrix, as

$$\hat{T}_{r-1}(p) = M\mathbf{M}^{\dagger} \tag{43}$$

unless simultaneously, $a_0 = -a_0^{\dagger}$ and r in equation (6) is odd, in which case factor $-\hat{T}_{r-1}(p)$. The factorization must be such that M is a $(k \times nr)$ polynomial matrix with k = rank of $T_{r-1}(p)$ and M^{-1}, the left inverse of M analytic in the open right plane. The existence of such a factorization is guaranteed by Lemmas 1 and 2.

Partition $M(p)$ into $(n \times k)$ blocks of equation (45)

$$M(p) = \begin{bmatrix} M_0(p) \\ \hline M_1(p) \\ \hline \vdots \\ \hline M_{r-1}(p) \end{bmatrix}.$$

Form the $(nr \times nr)$ matrix $\Omega(p)$ defined by equation (52)

With the identification of equations (9), (45), and (54)*

$$z_{11}(p) = A_{-1}(p),$$

and $$z_{12} = \frac{M_0(p)}{a_0(p)},$$

and $$z_{22} = M^{-1}(p)\Omega^{\dagger}(p)M(p),$$

the decomposition

$$W(p, s) = z_{11}(p) + z_{12}(p)[z_{22}(p) + s1_k]^{-1}\mathbf{z}_{12}^{\dagger} \tag{4}$$

is obtained. Notice that this is equation (4). It should also be noticed that $W(p, s)$ can be decomposed as in equation (4) even if it has s-independent poles, since the assumption that $W(p, \infty)$ is finite is enough to guarantee the validity of the procedure. For network realization it is usually more convenient to remove both p-independent and s-independent poles; we therefore removed them at the start of the procedure.

To realize $W(p, s)$ as the impedance of a passive network, we perform the following operations.

Form the $(n+k \times n+k)$ impedance matrix $Z(p)$ of the coupling network

$$Z(p) = \begin{bmatrix} A_{-1}(p) & \dfrac{M_0(p)}{a_0(p)} \\ -\dfrac{\mathbf{M}_0^{\dagger}(p)}{a_0^{\dagger}(p)} & M^{-1}(p)\Omega^{\dagger}(p)M(p) \end{bmatrix}. \tag{84}\dagger$$

Realize $Z(p)$ as a lossless $(n+k)$ port network in the p-plane and terminate its last k-ports with unit inductors in the s-plane. Also realize the reactance matrices $W_1(p)$ and $W_2(s)$ as lossless p-plane and s-plane n-ports, and connect all three networks in series as shown in Fig. 1. The given $W_o(p, s)$ is thus realized as a passive network.

V. AN EXAMPLE

It is desired to synthesize the two variable reactance matrix‡

* Equation (54) is used to determine $z_{22}(p)$, in preference to equation (50) since equation (54) is easier to compute.

† Equation (84) is the same as equation (59) except that for the z_{22} block equation (54) is used instead of equation (50) for the reason mentioned in the previous note.

‡ This example was given by Koga, (see p. 50 of Ref. 8).

$$W_0(p, s) = \begin{bmatrix} \dfrac{(p^2+1)(s^2+1)}{(p+s)(ps+1)} & \dfrac{ps-1}{p+s} \\ \dfrac{ps-1}{p+s} & \dfrac{ps+1}{p+s} \end{bmatrix}.$$

Since $W_o(p, s)$ has no p-independent or s-independent poles the first step 1 of Section IV need not be performed, and $W_o(p, s) = W(p, s)$. The least common denominator of the elements of $W(p, s)$ is

$$g(p, s) = ps^2 + (p^2 + 1)s + p$$

[which is equation (6)], and hence

$$a_0(p) = p, \quad a_1(p) = (p^2 + 1), a_2(p) = p, \quad \text{and} \quad r = 2.$$

The least common denominator of the minors of $W(p, s)$ is also $g(p, s)$ and hence

$$k = \delta_s[W(p, s)] = 2.$$

In the expansion, equation (7),

$$W(p, s) = A_{-1}(p) + \sum_{l=0}^{\infty} \frac{A_l(p)}{s^{l+1}}$$

by the formula of equation (16) or by long division

$$A_{-1}(p) = \frac{1}{p}\begin{bmatrix} p^2+1 & p^2 \\ p^2 & p^2 \end{bmatrix},$$

$$A_0(p) = -\frac{1}{p^2}\begin{bmatrix} (p^2+1)^2 & p^2(p^2+1) \\ p^2(p^2+1) & p^2(p^2-1) \end{bmatrix},$$

$$A_1(p) = \frac{1}{p^3}\begin{bmatrix} (p^2+1)^3 & p^4(p^2+1) \\ p^4(p^2+1) & p^4(p^2-1) \end{bmatrix},$$

$$A_2(p) = -\frac{1}{p^4}\begin{bmatrix} p^8+3p^6+4p^4+3p^2+1 & p^6(p^2+1) \\ p^6(p^2+1) & p^6(p^2-1) \end{bmatrix}.$$

$T_{r-1}(p) = T_1(p)$ defined by equation (24) is

$$T_{r-1}(p) = \frac{1}{p^4}\left[\begin{array}{cc|cc} -p^2(p^2+1)^2 & -p^4(p^2+1) & p(p^2+1)^3 & p^5(p^2+1) \\ -p^4(p^2+1) & -p^4(p^2-1) & p^5(p^2+1) & p^5(p^2-1) \\ \hline -p(p^2+1)^3 & -p^5(p^2+1) & p^8+3p^6+4p^4+3p^2+1 & p^6(p^2+1) \\ -p^5(p^2+1) & -p^5(p^2-1) & p^6(p^2+1) & p^6(p^2-1) \end{array}\right].$$

The polynomial matrix $T_1(p) = p_0^4 T_1(p)$ is factored by the method in Ref. 16 as equation (43)

$$T_1(p) = M(p)\mathbf{M}^{\dagger}(p) = \frac{1}{(2)^{\frac{1}{2}}}\begin{bmatrix} p(p^2+1) & -p(p^2+1) \\ p^2(p-1) & -p^2(p+1) \\ p^4-p^3+2p^2-p+1 & -(p^4+p^3+2p^2+p+1) \\ p^4-p^3 & -(p^4+p^3) \end{bmatrix}$$

$$\times \frac{1}{(2)^{\frac{1}{2}}}\begin{bmatrix} -p(p^2+1) & -p^2(p+1) & p^4+p^3+2p^2+p+1 & p^4+p^3 \\ p(p^2+1) & p^2(p-1) & -(p^4-p^3+2p^2-p+1) & -(p^4-p^3) \end{bmatrix}.$$

The (4×2) matrix $M(p)$ is partitioned as equation (45)

$$M(p) = \begin{bmatrix} M_0(p) \\ \hline M_1(p) \end{bmatrix}$$

$$= \frac{1}{2}\left[\begin{array}{cc} p^2(p^2+1) & -p^2(p^2+1) \\ p^2(p-1) & -p^2(p+1) \\ \hline p^4-p^3+2p^2-p+1 & -(p^4+p^3+2p^2+p+1) \\ p^4-p^3 & -(p^4+p^3) \end{array}\right].$$

To find $M^{-1}(p)$, a left inverse of $M(p)$, it is enough to find a left inverse of M_0 if it exists, since

$$[M_0^{-1} \mid 0]\begin{bmatrix} M_0 \\ \hline M_1 \end{bmatrix} = 1_k .$$

In our example $k = 2$ and M_o is a nonsingular matrix and hence $M^{-1}(p)$ is given by

$$M^{-1}(p) = \frac{-1}{2p^3(p^2+1)}\left[\begin{array}{cc|cc} -p^2(p+1) & p(p^2+1) & 0 & 0 \\ -p^2(p-1) & p(p^2+1) & 0 & 0 \end{array}\right].$$

From the definition of Ω, equation (52)

$$\Omega(p) = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline -1 & 0 & -\dfrac{p^2+1}{p} & 0 \\ 0 & -1 & 0 & -\dfrac{p^2+1}{p} \end{array}\right].$$

Using equation (54)

$$z_{22} = M^{-1}\Omega^{\dagger}M$$

$$= \begin{bmatrix} \dfrac{p^2+1}{2p} & -\dfrac{(p+1)^2}{2p} \\[2ex] -\dfrac{(p-1)^2}{2p} & \dfrac{p^2+1}{2p} \end{bmatrix}.$$

Hence the coupling network formed by p-type elements has the (4×4) matrix of equation (84)

$$Z(p) = \begin{bmatrix} A_{-1} & \dfrac{M_0}{a_0} \\[2ex] -\dfrac{\mathbf{M}_0^{\dagger}}{a_0^{\dagger}} & M^{-1}\Omega^{\dagger}M \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{p^2+1}{2p} & p & \dfrac{p^2+1}{(2)^{\frac{1}{2}}p} & -\dfrac{p^2+1}{(2)^{\frac{1}{2}}p} \\[2ex] p & p & \dfrac{p-1}{(2)^{\frac{1}{2}}} & \dfrac{p+1}{(2)^{\frac{1}{2}}} \\[2ex] \dfrac{p^2+1}{(2)^{\frac{1}{2}}p} & \dfrac{p+1}{(2)^{\frac{1}{2}}} & \dfrac{p^2+1}{2p} & -\dfrac{(p+1)^2}{2p} \\[2ex] -\dfrac{p^2+1}{(2)^{\frac{1}{2}}p} & -\dfrac{p-1}{(2)^{\frac{1}{2}}} & -\dfrac{(p-1)^2}{2p} & \dfrac{p^2+1}{2p} \end{bmatrix}$$

$Z(p)$ can be verified to be lossless, and the given $W_o(p, s)$ can of course be realized as the impedance seen at the first two ports of $Z(p)$ when it is terminated at its last two ports by unit s-plane inductors.

VI. CONCLUSIONS

The synthesis method for two-variable reactance matrices developed here, in general yields a nonreciprocal coupling network even when the given two-variable reactance matrix is symmetric, and if a reciprocal coupling network is desired, Koga's method for generating a reciprocal network from the nonreciprocal one can be used.[8] This procedure generally yields a reciprocal network at the cost of increased numbers of elements of both kinds.

This method of synthesis of two-variable reactance matrices has been successfully applied to the synthesis of lumped-distributed RC networks which are important in microelectronics circuits.[7] In practice, the only laborious step in the synthesis procedure is the factorization of polynomial matrix in the desired form. Of great importance is the approximation of desired characteristics by rational functions in two-variables; any work in this area would greatly enhance the usefulness of the two-variable theory. The synthesis problem of n-variable positive real functions, for which many applications can be found,[7] can be reduced to the synthesis of $(n+1)$-variable reactance matrices.[21, 22] when $n = 1$ the two-variable method developed here gives rise to a new method of passive RLC synthesis, which is no more complex than the existing methods.

VII. ACKNOWLEDGEMENTS

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APPENDIX A

Partial Fraction Expansion of $W(j\omega, s)$

To guarantee the factorization of $T_{r-1}(j\omega)$ as $M\mathbf{M}^\dagger$ we needed Lemma 2, which asserts that T_{r-1} is para Hermitian and that $T_{r-1}(j\omega) \geqq 0$. In the proof of Lemma 2 we used the fact that $R_i(j\omega)$, the residue matrices of $W(j\omega, s)$, are positive semidefinite. The proof is given below.

Under the assumption that $W(p, s)$ has no p-independent of s-independent poles, for each real ω the s-plane poles of $W(j\omega, s)$ are simple and restricted to the imaginary s-axis by Theorem 1. Hence, for any fixed ω, we can write $W(j\omega, s)$ as

$$W(j\omega, s) = A_{-1}(j\omega) + \sum_{i=1}^{r} \frac{R_i(\omega)}{s - j\alpha_i(\omega)} \tag{85}$$

where the $\alpha_i(\omega)$ are real and the $R_i(\omega)$ are the residue matrices at the poles $j\alpha_i(\omega)$. As in equation (9), r is the s-degree of $g(p, s)$, the least common denominator of the elements of W.

By complex conjugation on both sides of equation (85)

$$W^*(j\omega, s) = A^*_{-1}(j\omega) + \sum_{i=1}^{r} \frac{R^*_i(\omega)}{s^* + j\alpha_i(\omega)}. \tag{86}$$

Since W and A_{-1} are rational in $j\omega$,

$$W^*(j\omega, s) = W(-j\omega, s^*)$$

and

$$A^*_{-1}(j\omega) = A_{-1}(-j\omega).$$

Hence equation (86) becomes

$$W(-j\omega, s^*) = A_{-1}(-j\omega) + \sum_{i=1}^{r} \frac{R^*_i(\omega)}{s^* + j\alpha_i(\omega)} \tag{87}$$

and

$$-\mathbf{W}(-j\omega, -s^*) = -\mathbf{A}_{-1}(-j\omega) + \sum_{i=1}^{r} \frac{\mathbf{R}^*_i(\omega)}{s^* - j\alpha_i(\omega)}. \tag{88}$$

Since equation (88) is an identity for s^*, we have

$$-\mathbf{W}(-j\omega, -s) = -\mathbf{A}_{-1}(-j\omega) + \sum_{i=1}^{r} \frac{\mathbf{R}^*_i(\omega)}{s - j\alpha_i(\omega)}. \tag{89}$$

From the definition of a two variable reactance matrix,

$$W(j\omega, s) = -\mathbf{W}(-j\omega, -s)$$

and by equation (19)

$$A_{-1}(j\omega) = -\mathbf{A}_{-1}(-j\omega).$$

Hence, by comparison of equations (85) and (89) we have the desired result

$$R_i(\omega) = \mathbf{R}^*_i(\omega). \tag{90}$$

To show that the $R_i(\omega)$ are positive semidefinite for each ω, we first notice that if

$$W(p, s) = \frac{\psi(p, s)}{g(p, s)}$$

where $\psi(p, s)$ is a polynomial matrix and $g(p, s)$ is the least common denominator of the entries in W, $R_i(\omega)$ in equation (85) is given by (see p. 308 of Ref. 19)

$$R_i(\omega) = \left. \frac{\psi(p, s)}{\dfrac{\partial g(p, s)}{\partial s}} \right|_{\substack{p=j\omega \\ s=j\alpha_i(\omega)}} . \tag{91}$$

Denoting $\partial g/\partial s$ by g_s and $\partial \psi/\partial s$ by ψ_s, for any $n \times 1$ constant matrix X, (p. 39 of Ref. 8)

$$\mathbf{X}^*R_iX = \left.\frac{\mathbf{X}^*\psi X}{g_s}\right|_{\substack{p=j\omega \\ s=j\alpha_i(\omega)}}$$

$$= \left[\frac{(\mathbf{X}^*\psi X)g_s - g(\mathbf{X}^*\psi_s X)}{(\mathbf{X}^*\psi X)^2}\right]^{-1}_{\substack{p=j\omega \\ s=j\alpha_i(\omega)}}$$

$$= \left[\frac{\partial}{\partial s}\left(\frac{g}{\mathbf{X}^*\psi X}\right)\right]^{-1}_{\substack{p=j\omega \\ s=j\alpha_i(\omega)}} .$$

Hence, if $X^*WX \not\equiv 0$

$$\mathbf{X}^*R_i(\omega)X = \left[\frac{\partial}{\partial s}(\mathbf{X}^*WX)^{-1}\right]^{-1}_{\substack{p=j\omega \\ s=j\alpha_i(\omega)}} . \tag{92}$$

From definitions 1 and 2, and Theorem 1, $\mathbf{X}^*WX$ is a two variable positive function and for $\text{Re}\, p = \text{Re}\, s = 0$

$$\text{Re}\,[\mathbf{X}^*WX] \equiv 0$$

and

$$\frac{\partial}{\partial s}(\mathbf{X}^*WX) \geqq 0.$$

Hence

$$\frac{\partial}{\partial s}(\mathbf{X}^*WX)^{-1} = -\frac{1}{(\mathbf{X}^*WX)^2}\cdot\frac{\partial}{\partial s}(\mathbf{X}^*WX) \geqq 0$$

for

$$\text{Re}\, p = \text{Re}\, s = 0$$

and consequently the left side of equation (90) is nonnegative.

Thus we have proved that the residue matrices, $R_i(\omega)$, are positive semidefinite Hermitian for each ω.

APPENDIX B

Proof of Theorem 3

Theorem 3: If

$$W(p, s) = \frac{B_0(p)s^r + B_1(p)s^{r-1} + \cdots + B_r(p)}{a_0(p)s^r + a_1(p)s^{r-1} + \cdots + a_r(p)}$$

is a two variable reactance matrix, then for all $i = 0, 1, \cdots r$

B_i/a_i is a reactance matrix in p.

a_i/a_{i+1} is a reactance function in p.

a_i has all its zeros on the j axis, and these are simple.

$\mathbf{X}B_iX/\mathbf{X}B_{i+1}X$ for all constant $n \times 1$ vectors, X, is a reactance function in p.

Proof: For any constant $n \times 1$ matrix X,

$$\mathbf{X}^*WX = \frac{(\mathbf{X}^*B_0X)s^r + (\mathbf{X}^*B_ix)s^{r-1} + \cdots + (\mathbf{X}^*B_rX)}{a_0s^r + a_1s^{r-1} + \cdots + a_r} \tag{93}$$

is a rational function in p and s with possible complex coefficients. For convenience, if we define

$$b_i = \mathbf{X}^*B_iX$$

$$w(p, s) = \mathbf{X}^*WX$$

$$f(p, s) = b_0s^r + b_is^{r-1} + \cdots + b_r$$

and as before

$$g(p, s) = a_0s^r + a_1s^{r-1} + \cdots + a_r$$

equation (93) can be written as

$$w(p, s) = \frac{f(p, s)}{g(p, s)}. \tag{94}$$

From the definition of a two variable reactance matrix, $w(p, s)$ is a two variable positive function, and hence for any p_0 with Re $p_0 > 0$, $w(p_0, s)$ is a positive function of s.[20] Consequently, for all s with Re $s > 0$

$$\text{Re}\,\frac{f(p_0, s)}{g(p_0, s)} \geqq 0. \tag{95}$$

Since equation (95) has to be satisfied for all s with Re $s > 0$ and hence for arbitrarily small s, it can be seen from equation (93) that

$$\text{Re}\,\frac{b_r(p_0)}{a_r(p_0)} \geqq 0$$

for all p_0 with Re $p_0 > 0$. Hence, $B_r(p)/a_r(p)$ is a positive real matrix and since $W = -\mathbf{W}^\dagger$

$$\left[\frac{B_r}{a_r}\right] = -\left[\frac{\mathbf{B}_r}{a_r}\right]^\dagger$$

and thus B_r/a_r is a reactance matrix in p.

If instead of starting from the positive function $f(p_0, s)/g(p_0, s)$, we start from $\partial^\nu f(p_0, s)/\partial s^\nu / \partial^\nu g(p_0, s)/\partial s^\nu$, which is also a positive function[20] for all $0 \leqq \nu \leqq r$, the same arguments used in proving that B_r/a_i is a reactance matrix can be repeated to show that B_i/a_i is a reactance matrix in p for all $0 \leqq i \leqq r$.

Now to show that a_{i+1}/a_i is a reactance function in p, we can use a similar proof based on the fact that $\partial^{\nu-1} g(p_0, s)/\partial s^{\nu-1} / \partial^\nu g(p_0, s)/\partial s^\nu$ is a positive function.[20]

Again, it can be seen from the fact that

$$\partial^{\nu-1} g(p_0, s)/\partial s^{\nu-1} / \partial^\nu g(p_0, s)/\partial s^\nu$$

is a positive function.[20] that b_{i+1}/b_i is a positive function satisfying

$$\left[\frac{b_{i+1}}{b_i}\right] = -\left[\frac{b_{i+1}}{b_i}\right]^\dagger.$$

If X in equation (93) is chosen real b_{i+1}/b_i will be real for real p and hence $\mathbf{X}B_{i+1}X/\mathbf{X}B_iX$ for any real $n \times 1$ matrix, X, is a reactance function.

To see that the zeros of a_i are all simple and restricted to the imaginary axis: if any one of the a_i has a double zero on the $j\omega$ axis or a zero off the $j\omega$ axis, from the reactance nature of a_{i+1}/a_i for all $0 \leqq i \leqq r$, all the a_i must have the same zero, and, consequently, $W(p, s)$ will have an s-independent pole contradicting our original assumption that W has no such poles.

We have thus proved all the claims of Theorem 3.

APPENDIX C

Proof of the Minimality of the Realization of $W(p, s)$ in Both Variables

In this appendix we show that the realization of $W(p, s)$ that Section III gives is minimal in both the p and s variables. From the definitions of $\delta_s[W(p, s)]$ and $\delta_p[W(p, s)]$, it can be shown that if $W(p, s)$ is finite at $p = \infty$ and $s = \infty$,

$$\delta_s[W(p, s)] = \delta_s[\eta(p, s)]$$

$$\delta_p[W(p, s)] = \delta_p[\eta(p, s)]$$

where the two variable real polynomial

$$\eta(p, s) = d_0(p)s^k + d_1(p)s^{k-1} + \cdots + d_k(p) \tag{96}$$

is the least common denominator of all the minors of $W(p, s)$. The form in which $\eta(p, s)$ is written in equation (96) immediately reveals that

$$\delta_s[W(p, s)] = k.$$

And if $\eta(p, s)$ is written as

$$\eta(p, s) = c_0(p)p^m + c_1(p)p^{m-1} + \cdots + c_m(p), \tag{97}$$

it can be seen that

$$\delta_p[W(p, s)] = m.$$

C.1 *Minimum Elements*

Next we would like to find the minimum number of elements of each kind needed in the realization of $W(p, s)$.

Lemma 1 states that k, the rank of $T_{r-1}(p)$, is equal to the s-degree of $W(p, s)$, and the realization obtained there uses exactly k s-type elements. By equation (4)

$$W(p, s) = z_{11}(p) + z_{12}(p)[z_{22}(p) + s\mathbf{1}_k]^{-1}\mathbf{z}_{12}^{\dagger}(p).$$

Suppose that there exists a realization with k_0 s-type elements, where $k_0 < k = \text{rank } T_{r-1}(p)$. Then,

$$W(p, s) = z_{11}(p) + \underline{z}_{12}(p)[\underline{z}_{22}(p) + s\mathbf{1}_{k_0}]^{-1}\underline{\mathbf{z}}_{12}^{\dagger}(p)$$

where the matrices $\underline{z}_{12}(p)$ and $\underline{z}_{22}(p)$ are $n \times k_0$ and $k_0 \times k_0$, respectively. Then by equation (25), $T_{r-1}(p) = N(p)\mathbf{N}^{\dagger}(p)$ where

$$N(p) = \begin{bmatrix} \underline{z}_{12}(p) \\ \underline{z}_{12}(p)\underline{z}_{22}(p) \\ \vdots \\ \underline{z}_{12}(p)\underline{z}_{22}^{r-1}(p) \end{bmatrix}$$

is an $nr \times k_0$ matrix and hence, rank $N(p) \leqq k_0$. Also, we have

$$\text{rank } T_{r-1}(p) \leqq \text{rank } N(p) \leqq k_0 < k = \text{rank } T_{r-1}(p)$$

which is a contradiction, and hence $k = \text{rank } T_{r-1}(p) = \delta_s[W(p, s)]$ is the minimum number of s-type elements required in any realization. Now by repeating the same argument with a realization of $W(p, s)$ where p-type elements are extracted instead of s-type elements, we can see that any realization must contain at least m p-type elements where $m = \delta_p[(p, s)]$.

c.2 *Minimality of the Realization in Section III*

We next discuss the minimality of the realization in both p-type and s-type elements. For the purpose of realization, the reactance matrix $W(p, s)$ was decomposed as

$$W(p, s) = z_{11}(p) + z_{12}(p)[z_{22}(p) + sl_k]^{-1}\mathbf{z}_{12}^{\dagger}(p) \tag{4}$$

where

$$Z(p) = \begin{bmatrix} z_{11}(p) & z_{12}(p) \\ -\mathbf{z}_{12}^{\dagger}(p) & z_{22}(p) \end{bmatrix} \tag{11}$$

can be realized as the impedance matrix of a lossless $(n + k)$ port in the *p-plane*. $W(p, s)$ is the impedance seen at the first n ports when the above $(n + k)$ port network is terminated with unit s-plane inductors at its last k-ports. Since k is the s-degree of $W(p, s)$, the realization uses the minimum number of s-type elements. To show that the realization uses the minimum number of p-type elements, we have to show that $\delta[Z(p)] = \delta_p[W(p, s)]$. For this we need a relationship that exists between the least common denominator of the minors of $W(p, s)$ and the determinant $|z_{22}(p) + s1_k|$.

Every minor of $[z_{22}(p) + s1_k]^{-1}$ can be expressed as $\mu(p, s)/\varphi(p, s)$ (See p. 21, of Ref. 12, Vol. 1) where $\mu(p, s)$ and $\varphi(p, s)$ are polynomials in s with coefficients from the field of rational functions in p. Furthermore,

$$\varphi(p, s) = | z_{22}(p) + s1_k |$$

is a monic polynomial in s of degree k.

Since $W(p, s)$ has no p-independent or s-independent poles, every zero of $\eta(p, s)$ is a zero of $\varphi(p, s)$, and since $k = \delta_s[\varphi(p, s)] = \delta_s[\eta(p, s)]$, $\varphi(p, s)$ and $\eta(p, s)/d_o(p)$, which are monic polynomials in s with rational functions of p as coefficients, must be identical. Hence

$$| z_{22}(p) + s1_k | = \frac{\eta(p, s)}{d_0(p)}. \tag{98}$$

To show that $\delta[Z(p)] = \delta_p[W(p, s)]$ (since we already know that $\delta[Z(p)] \geqq \delta_p[W(p, s)]$) it is sufficient to show that $\delta[Z(p)] \leqq \delta_p[W(p, s)]$. To establish this inequality, consider the matrix $S(p, s)$ defined by

$$S(p, s) = [Z(p) - s1_{n+k}][Z(p) + s1_{n+k}]^{-1}. \tag{99}$$

When $s = 1$, $S(p, s)$ is the scattering matrix of a lossless network, since $Z(p)$ describes a lossless network and (see p. 184 of Ref. 11)

$$\delta[Z(p)] = \delta[S(p, 1)]. \tag{100}$$

Since $S(p, 1)$ is para unitary (see p. 131 of Ref. 15)

$$\delta[S(p, 1)] = \delta[| S(p, 1) |]. \tag{101}$$

Equating the determinants of matrices on both sides of equation (99)

$$| S(p, s) | = \frac{| Z(p) - s\mathbf{1}_{n+k} |}{| Z(p) + s\mathbf{1}_{n+k} |}.$$

Using a formula from the theory of determinants (see p. 46 of Ref. 12, Vol. I)

$$\begin{aligned} | S(p, s) | &= \frac{| (z_{11} - s\mathbf{1}_n) + z_{12}(z_{22} - s\mathbf{1}_k)^{-1}\mathbf{z}_{12}^{\dagger} |\cdot| z_{22} - s\mathbf{1}_k |}{| (z_{11} + s\mathbf{1}_n) + z_{12}(z_{22} + s\mathbf{1}_k)^{-1}\mathbf{z}_{12}^{\dagger} |\cdot| z_{22} + s\mathbf{1}_k |} \\ &= \frac{| W(p, -s) - s\mathbf{1}_n |}{| W(p, s) + s\mathbf{1}_n |}\cdot\frac{| z_{22} - s\mathbf{1}_k |}{| z_{22} + s\mathbf{1}_k |}. \end{aligned} \tag{102}$$

Now if $|W(p, s) + s\mathbf{1}_n|$ is written as

$$| W(p, s) + s\mathbf{1}_n | = \frac{h(p, s)}{\eta(p, s)} \tag{103}$$

where $h(p, s)$ is a real polynomial in p and s, since the left side of equation (102) is finite at $p = \infty$

$$\delta_p[h(p, s)] \leqq \delta_p[\eta(p, s)] = \delta_p[W(p, s)]. \tag{104}$$

Substituting equations (98) and (103) in equation (102), we have

$$\begin{aligned} | S(p, s) | &= \frac{h(p, -s)}{\eta(p, -s)}\cdot\frac{\eta(p, s)}{h(p, s)}\cdot\frac{\eta(p, -s)}{d_0(p)}\cdot\frac{d_0(p)}{\eta(p, s)} \\ &= \frac{h(p, -s)}{h(p, s)} \end{aligned} \tag{105}$$

and by equations (100), (101), and (104)

$$\delta[Z(p)] = \delta[S(p, 1)] \leqq \delta_p[W(p, s)].$$

We have thus shown that $\delta[Z(p)] = \delta_p[W(p, s)]$.

It should be noted that $Z(p)$ is the impedance matrix of any lossless coupling network in a realization of $W(p, s)$, minimal in s, and hence we come to the important conclusion that if a realization of $W(p, s)$ is minimal in one of the variables it is automatically minimal in the other variable.

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