Minimum Cost Communication Networks

By E. N. GILBERT

(Manuscript received July 21, 1967)

Cities $A_1, \cdots, A_n$ in the plane are to be interconnected by two-way communication channels. $N(i, j)$ channels are to go between A_i and A_j. One could install the $N(i, j)$ channels along a straight line, for every pair i, j. However it is usually possible to save money by rerouting channels over longer paths in order to group channels together. In this way, large numbers of channels share such preliminary expenses as real estate, surveying, and trench digging.

The geometry of the least expensive network will depend on the numbers of channels $N(i, j)$ and on the function $f(N)$ which represents the cost per mile of installing N channels along a common route. If the preliminary expenses are the only expenses then $f(N)$ is a constant, independent of N. In that case the best network is obtained by routing channels along lines of the "Steiner minimal tree", a graph which has been studied extensively and which can be constructed by ruler and compass. In part, this paper generalizes Steiner minimal trees for the case of an arbitrary function $f(N)$. One again obtains a ruler and compass construction for a minimizing tree, which is likely to provide a best or good solution when preliminary costs are a significant part of the total cost. However the minimizing tree need not be the best solution in general because further cost reductions may now be possible by using graphs which have cycles. Other properties of Steiner minimal trees generalize only part way, and some examples illustrate the new complications.

The remainder of the paper considers functions $f(N)$ with special properties. A convexity property

$$f(N + 2) - 2f(N + 1) + f(N) \leqq 0, N = 1, 2, \cdots$$

ensures that there is a minimizing solution in which all $N(i, j)$ channels between A_i and A_j take the same path (no split routing). If $f(N)$ is a linear function ($f(N) = a + bN$), one can obtain simple bounds on the minimum cost. The lower bound is fairly accurate.

I. INTRODUCTION

Let points $A_1, \cdots, A_n$ in the plane represent n cities which require a communications network. Let $N(i, j)$ denote the number of channels which the network must supply between A_i and A_j. The network sought must provide these channels at minimum cost. In calculating costs suppose that a monotone function $f(N)$ represents the cost in dollars per mile to install N channels together along a common route.

One possible network just connects each pair A_i, A_j by $N(i, j)$ channels installed along a straight line path. This network will be called the *complete network* because the routes used form a complete graph. Fig. 1(a) is the complete network for a case with $n = 4$; the numbers on the lines are the $N(i, j)$.

The complete network makes each channel as short as possible; it is the cheapest network if $f(N) = N$. However, most situations have more complicated functions $f(N)$. In particular, there are usually some *preliminary costs* for surveying, obtaining the right-of-way, digging a trench, etc. These items have a non-zero cost $f(0)$ dollars per mile regardless of how many channels are to be installed.

In some cases preliminary costs may be so high that a network which merely minimizes preliminary costs is a reasonable choice. Such a network must minimize the total number of miles of right-of-way. For the example in Fig. 1(a), the network which minimizes preliminary costs is Fig. 1(b) [or, more simply, Fig. 1(c)]. Such networks can be drawn with a ruler and compass in a finite (possibly large) number of steps (see Ref. 1, 2).

When $f(N)$ is not constant, the cheapest network is harder to find. Still the methods which minimize only the preliminary costs generalize far enough to be useful. Sections III and IV develop these generalizations. In particular, if preliminary costs are a large fraction of the

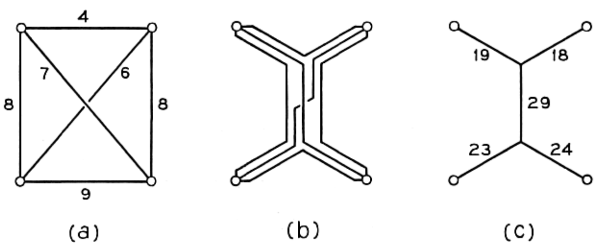


Fig. 1 — Networks.

total cost one has a good chance of constructing the cheapest network by these methods.

In many problems the cost function is linear, $f(N) = a + bN$. A linear cost function is obtained if the *incremental cost* $f(N) - f(N-1)$ of adding an Nth channel to a group of $N-1$ channels is a fixed amount b dollars per mile, independent of N. The cost of additional copper wires, channel filters, or repeaters usually does not depend on N. By contrast, consider waveguide systems. Each guide can supply thousands of channels. The incremental cost is small for most values of N but is large when adding channel N requires adding another guide; $f(N)$ is a staircase function. Section VI obtains some bounds on the cost of the cheapest network when $f(N)$ is linear. Section V finds a property of the minimal cost network when $f(N)$ is merely convex.

II. STEINER MINIMAL TREES

A network may be represented, as in Fig. 1(c), as a set of lines (the routes or right-of-ways) connecting $A_1, \cdots, A_n$ and perhaps some other points where lines join. This representation will be called the *graph* of the network. Figs. 1(b) and 1(c) illustrate the distinction between a network and its graph. A *Steiner point* is a junction point of the graph which is not one of $A_1, \cdots, A_n$. Fig. 1(c) has two Steiner points. The *minimal graph* is the graph of the cheapest network. A graph is *relatively minimal* if its Steiner points are located so that no small displacement of the Steiner points reduces the cost. If a graph is relatively minimal there is no guarantee that a more violent perturbation, altering the topology of the graph, may not secure a reduction; i.e., relatively minimal graphs need not be minimal.

The procedure to be described here finds relatively minimal graphs which are trees having exactly three lines incident at each Steiner point. The cheapest of these relatively minimal trees will be called the *Steiner minimal tree* for $A_1, \cdots, A_n$. The procedure in question is a modification of one which applies when the cost function is simply $f(N) = 1$. In order to have an easy terminology by which one may compare a given problem against the corresponding problem with $f(N) = 1$, I use the adjective *ordinary* freely to mean "having $f(N) = 1$". Thus, "ordinary minimal graph, ordinary relatively minimal graph, ordinary Steiner minimal tree, $\cdots$" mean "minimal graph, relatively minimal graph, Steiner minimal tree, $\cdots$ in the case $f(N) = 1$".

The ordinary case is a simpler one than the general case because the ordinary minimal graph is the ordinary Steiner minimal tree. In general, the minimal graph need not be a tree (recall that the complete graph is minimal if $f(N) = N$). Moreover, even the cheapest tree need not be a Steiner minimal tree. For example, consider four cities A_1, $\cdots$, A_4 at the corners of a unit square as shown in Fig. 2. For the demand matrix $N(i, j)$ take

$$|| N(i,j) || = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 10 \\ 1 & 1 & 0 & 1 \\ 1 & 10 & 1 & 0 \end{bmatrix}$$

and let N channels cost $f(N) = 1+N$ dollars per mile. Fig. 2(a) shows the cheapest tree. It is not a Steiner minimal tree because four lines meet at its Steiner point. Fig. 2(b) shows a typical tree in which three lines meet at each Steiner point. However, Fig. 2(b) is not relatively minimal; its cost decreases when the two Steiner points are displaced toward the center of the square. If one continues to displace these Steiner points, in hopes of finding a relatively minimal tree, they finally merge together as in Fig. 2(a).

III. GENERALIZATIONS FROM THE ORDINARY CASE

In Ref. 1 we gave some simple properties of ordinary relatively minimal trees and ordinary Steiner minimal trees. Some of these properties generalize directly while others do not. This section will discuss the simplest generalizations. In some cases the proofs are omitted because the arguments of Ref. 1 apply with only trivial changes.

3.1 *Mechanics*

A graph of a network may be interpreted as a mechanical system of elastic bands (the lines). A_1, $\cdots$, A_n are fixed supports for the

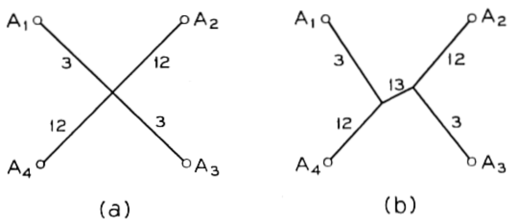


Fig. 2 — Four cities problem.

bands incident there but the bands at a Steiner point are merely joined together and left free to move. Let each band have a tension equal to the cost per mile of the channels in the corresponding line. Then the mechanical system has a potential function equal to the cost of the graph; the system is in stable equilibrium if and only if the graph is relatively minimal.

3.2 *Angles at a Steiner point*

At a Steiner point S let vectors $v, v', v'', \cdots$ denote the forces (tensions) exerted by the elastic bands. The condition for mechanical equilibrium (relatively minimal graph) is $v + v' + v'' + \cdots = 0$. The magnitudes $|v|, |v'|, |v''|, \cdots$ are the costs per mile of the lines at S. When S has only three lines, the law of cosines determines the angles between the lines. For instance,

$$\cos (v', v'') = (| v |^2 - | v' |^2 - | v'' |^2)/(2 | v' | | v'' |). \qquad (1)$$

The analogous condition on ordinary relatively minimal trees, which stated that three lines meet at 120° at S, is an instance of (1) with $|v| = |v'| = |v''|$. When four or more lines meet at S the equilibrium condition does not determine the angles at S.

3.3 *Number of Steiner points*

Consider any tree joining $A_1, \cdots, A_n$ and let s be the number of Steiner points. It is no restriction to assume that no Steiner point has less than three lines; for clearly such Steiner points can save no cost. Then (see Ref. 1, Section 3. 4)

$$s \leqq n - 2$$

with equality holding if and only if each Steiner point has three lines and each A_i has one line.

3.4 *Uniqueness*

Suppose a graph, not necessarily a tree, is given for a network connecting $A_1, \cdots, A_n$. The numbers of channels are also supposed prescribed for each line of the graph. Now perturb the positions of the Steiner points trying to reach a relative minimum cost for graphs with the same topology. As illustrated by Fig. 2, it can happen that a relative minimum may be only approached but not attained. In the ordinary case, when one does find a relatively minimal graph one can conclude that there are no others with the same topology.

In the general case, there is no such uniqueness. For example, suppose A_1, A_2, A_3 are at the vertices of an equilateral triangle and suppose $N(i, j) = 1$ for all pairs (i, j). Let $f(N) = 1 + (3^{\frac{1}{2}} - 1)(N - 1)$. Fig. 3 shows a possible graph and gives the angles, obtained from (1), which suffice for a relative minimum. These angles do not determine the locations of the Steiner points. It suffices to put each Steiner point S_i at the same distance from the center O of the triangle and on the line OA_i.

Fig. 4(a) shows that one may encounter non-uniqueness even when searching for a relatively minimal tree. To perturb S into a position of minimum cost, place S anywhere on the line segment A_2A_3. The individual channels $[N(i, j) = 1$ for all $i, j]$ appear in Fig. 4(b). Steiner points, such as S in Fig. 4(b), at which all incident lines meet at either 180° or 360° have no real interest. Any channel which makes a 180° turn at S can be rerouted away from S over a shorter path using only existing right-of-ways. After the shortening [Fig. 4(c)] the Steiner point is gone.

In spite of examples like Figs. 3 and 4, a weak kind of uniqueness holds even in the general case. Any relatively minimal tree is either the unique relatively minimal tree with the given topology or else it has a Steiner point, like S in Fig. 4(b), at which all lines meet at angles of either 180° or 360°. An outline of the proof follows. As in Ref. 1 the argument uses an "averaging" operation for graphs. If G and G' are two graphs with the same topology, the *averaged graph* $pG + qG'$ (where $p \geqq 0$, $q \geqq 0$, and $p + q = 1$) has vertices of the form $pV + qV'$ where V, V' are corresponding vertices, $V \in G$ and $V' \in G'$. For each line V_1V_2 of G (and correspondingly, $V_1'V_2'$ of G') $pG + qG'$ has the line joining $pV_1 + qV_1'$ to $pV_2 + qV_2'$. If L is a line V_1V_2 of G and L' the corresponding line of G', let $pL + qL'$ denote the corresponding line of $pG + qG'$. The lengths $|L|$, $|L'|$, $|pL + qL'|$ of these lines satisfy

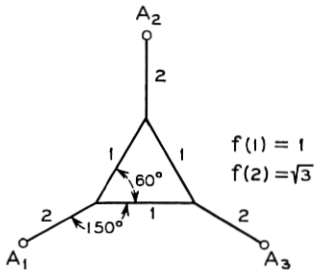


Fig. 3 — Example of non-uniqueness.

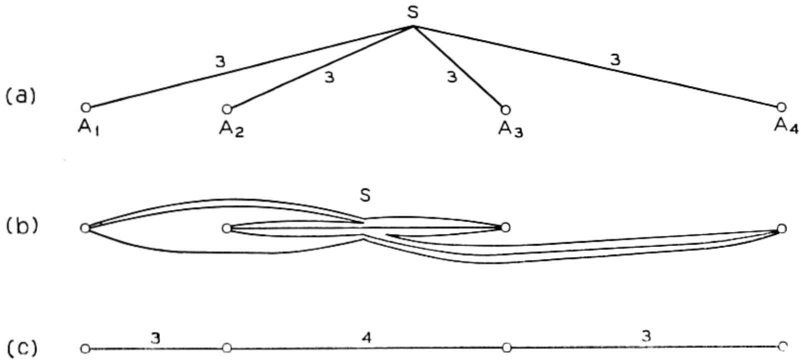


Fig. 4 — Non-uniqueness for trees.

$$| pL + qL' | \leqq p | L | + q | L' | \tag{2}$$

with equality holding only if the directions of the line segments V_1V_2 and $V_1'V_2'$ are the same.

One can now prove that all relatively minimal graphs with the same topology have the same cost. For suppose, on the contrary, that graphs G, G' have the same topology and have costs c, c' with $c < c'$. Because of (2) the cost of $pG + qG'$ is no greater than $pc + qc'$. Then, taking p to be small, $pG + qG'$ is a slight perturbation of G' and costs less than c'. Then G' cannot have been relatively minimal, a contradiction.

If G and G' are two different graphs which both attain the relative minimum cost, then (2) shows that an average graph $pG + qG'$ will cost even less (a contradiction) unless every line of G' is parallel to its corresponding line in G. Note that the graphs obtained from Figs. 3 and 4(b) all had that property. Now suppose G and G' are relatively minimal trees. If G and G' differ some Steiner point S in G is connected to vertices V_1 and V_2 such that $V_1' = V_1$, $V_2' = V_2$, but $S' \neq S$. For instance, V_1 and V_2 might be two of $A_1, \cdots, A_n$. But, to avoid the contradiction noted above, SV_1 and $S'V_1$ must be parallel, as must SV_2 and $S'V_2$. That can be true when $S \neq S'$ only if S, S', V_1, V_2 are colinear, whence V_1S makes a 360° angle with V_2S.

3.5 *Number of choices*

In Ref. 1 the solution to the ordinary case is found by constructing a relatively minimal tree, if one exists, for each of the topologically distinct ways of interconnecting $A_1, \cdots, A_n$. Because of 3.3 there are only a finite number of cases to consider. For $s = 0, 1, \cdots, n - 2$, the number of cases with s Steiner points turns out to be

$$2^{-s}\binom{n}{s+2}(n+s-2)!/s!.$$

In the general problem, each of these cases is again a candidate for the Steiner minimal tree. The total number of cases for $n = 3, 4, 5, 6, 7, \cdots$ are 4, 27, 270, 3645, 62370, $\cdots$. Of course the minimal graph may not be one of these trees; in general, there will be many more cases.

Fortunately, it seems to be easy to guess topologies which, if not actually best, cost only slightly more than the minimal cost. In Ref. 1, for example, we were unable to invent a problem in which the minimum cost was less than 86.6 percent of the cost of the (easily constructed) best tree having no Steiner points. The four cities in the unit square of Fig. 2 illustrate the same thing. Again let $f(N) = 1 + N$ and let $||N(i, j)||$ be the same as in Section II. Table I compares the cost of the cheapest graph, Fig. 2(a), with some other simple ones.

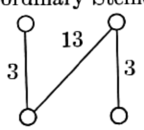
These comparisons suggest that one should be willing to accept a good network (perhaps the best relatively minimal network obtained for several reasonable topologies) even though it is not proved to have absolutely minimum cost. There are usually too many cases to find the best network by exhaustion; also the saving in cost is apt to be slight.

IV. CONSTRUCTION ALGORITHMS

The ruler and compass construction of relatively minimal trees is similar to the construction in the ordinary case.

Consider first the case $n = 3$. Fig. 5(a) shows a typical case with given points A_1, A_2, A_3 to be joined to a Steiner point S. The costs c_i per mile of the three lines SA_i are supposed known. The angles α_1, α_2,

TABLE I

Graph	Cost (in dollars)
Fig. 2(a)	24.04
complete graph	26.38
ordinary Steiner min. tree	25.55
	27.80

α_3 at which lines meet at S are now determined from the equilibrium condition,* e.g., by (1). A ruler and compass construction for α_1, α_2, α_3 is easy because these angles are the exterior angles to a triangle with sides c_1, c_2, c_3 [Fig. 5(b)].

In general, c_1, c_2, c_3 might have any values, including some which are not constructable by ruler and compass (e.g., perhaps $c_1 = 2^{1/3}$, $c_2 = \pi$, $c_3 = e$). Then Fig. 5(b) is itself not constructable without first using the ruler as a "scale" to lay off segments of lengths c_1, c_2, c_3. I assume that these segments have already been drawn. Then all other

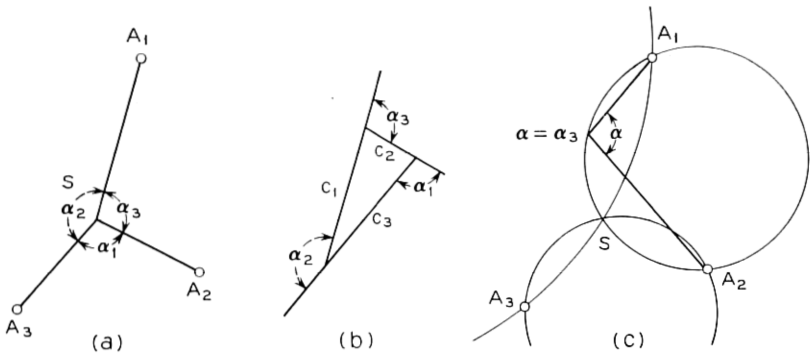


Fig. 5 — First construction with $n = 3$.

constructions, such as the one for α_1, α_2, α_3, can use the ruler and compass in the manner intended by Euclid.

Since angle $A_1SA_2 = \alpha_3$, S lies on a circular arc of angle $2\pi - 2\alpha_3$ through A_1 and A_2. By constructing this arc, and a similar arc for A_2A_3 or A_3A_1, one constructs S as an intersection of circular arcs [Fig. 5(c)]. The same construction appears in Ref. 4.

In Fig. 5(c) consider the line A_3S extended to meet the circle through A_1 and A_2 again. Let $A_{1,2}$ denote this new point of intersection [Fig. 6(a)]. The point $A_{1,2}$ has interesting properties which are needed for solving cases with $n \geqq 4$.

First note [Fig. 6(b)] that the exterior angles of the triangle $A_1A_2A_{1,2}$ are α_1, α_2, α_3. Then this triangle is similar to the triangle of Fig. 5(b) and so can be constructed by ruler and compass (if $|A_1A_2|$

* If one of the c_i exceeds the sum of the other two, say $c_1 + c_2 < c_3$, no choice of angles satisfy the equilibrium condition. The minimal tree consists just of two lines (A_1A_3 and A_2A_3 in the case cited). In many cases the function $f(N)$ is convex, as defined in (3), and then $c_1 + c_2 < c_3$ cannot happen.

$=d$, then $|A_1A_{1,2}| = dc_2/c_3$). The important fact used later on is that this construction will produce $A_{1,2}$ from c_1, c_2, c_3, A_1, and A_2, without using A_3.

Another construction for the case $n = 3$ proceeds as follows. With A_1A_2 as a base erect a triangle* with sides $|A_1A_2| = d$, dc_2/c_3, and dc_1/c_3 to construct $A_{1,2}$. Circumscribe this triangle in a circle C_{12}. If A_3 lies inside C_{12} there is no Steiner point (the cheapest solution consists of two lines A_3A_1 and A_3A_2). If A_3 lies outside C_{12} draw the line segment $A_{1,2}A_3$. Observe whether this segment crosses the arc A_1A_2 of C_{12} which does not contain $A_{1,2}$. If there is a crossing point

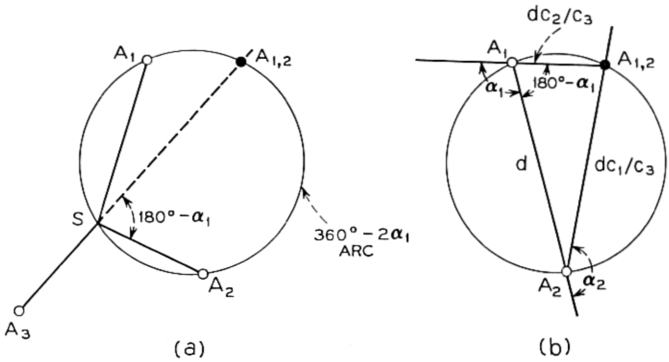


Fig. 6 — Construction of $A_{1,2}$.

S, then S is the desired Steiner point. If not, then there is no relatively minimal tree with the given topology. The best solution consists of A_1A_2 and A_1A_3 if A_2 and A_3 are on opposite sides of the line $A_1A_{1,2}$; use A_1A_2 and A_2A_3 if $A_2A_{1,2}$ separates A_1 from A_3. Fig. 7 shows how the cheapest tree depends on the location of A_3.

When the construction produces a legitimate Steiner point [Fig. 7(d)], Ref. 1 showed, in the ordinary case, that the length $|SA_1| + |SA_2| + |SA_3|$ of the tree is just $|A_3A_{1,2}|$. The appropriate generalization here is that the cost of the tree is the same as that of $|A_3A_{1,2}|$ miles of circuit costing c_3 dollars per mile, i.e.,

$$c_1 \mid SA_1 \mid + c_2 \mid SA_2 \mid + c_3 \mid SA_3 \mid = c_3 \mid A_3A_{1,2} \mid . \tag{3}$$

The proof of (3) will follow from a theorem in Ptolemy's Μεγάλη Σύνταξις stating that the product of the diagonals of a quadrilateral

* In general, there are two such triangles. Construct the one which places $A_{1,2}$ and A_3 on opposite sides of the line A_1A_2.

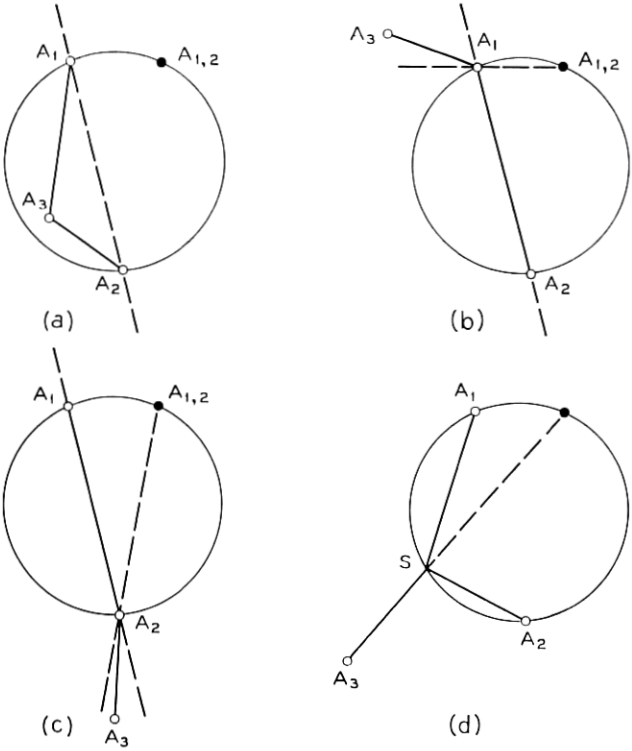


Fig. 7 — Second construction with $n = 3$.

equals the sum of the products of opposite sides.[3] When applied to the quadrilateral $A_1SA_2A_{1,2}$ in Fig. 6 the theorem becomes

$$| SA_{1,2} | \cdot d = | SA_1 | dc_1/c_3 + | SA_2 | dc_2/c_3$$

or

$$c_1 | SA_1 | + c_2 | SA_2 | = c_3 | SA_{1,2} |.$$

Add $c_3 | SA_3 |$ to both sides to get (3).

The construction of Fig. 7 may be used iteratively to find relatively minimal trees with $n \geqq 3$ when each Steiner point is restricted to have only three incident lines.

The details are similar to the ordinary case[1] and so it suffices here to give an illustrative example. Fig. 8 shows cities $A_1, \cdots, A_5$ to be interconnected by a graph having Steiner points S_1, S_2, S_3. To locate

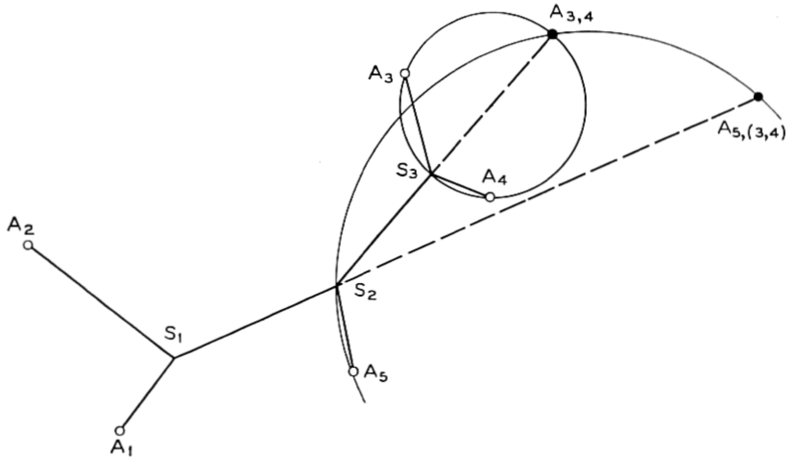


Fig. 8 — A construction with $n = 5$.

the S_i one begins by finding a pair of cities which are to be connected to a common Steiner point; A_3 and A_4 will serve in Fig. 8. Construct $A_{3,4}$ as in Fig. 6(b), and draw the circle circumscribing A_3, A_4 and $A_{3,4}$. S_3 will be obtained ultimately by intersecting this circle with the line $S_2A_{3,4}$ [compare Fig. 7(d)] but at the moment the position of S_2 is unknown. Nevertheless, the problem is now reduced to drawing a new tree for A_1, A_2, $A_{3,4}$, and A_5 with Steiner points S_1 and S_2 (the cost per mile for the new line $S_2A_{3,4}$ is taken to be the same as the original cost per mile of S_2S_3). Again pick a pair of cities with a common Steiner point, say A_5 and $A_{3,4}$; draw the triangle with base $A_5A_{3,4}$ to construct a new point $A_{5,(3,4)}$. Now the problem reduces to drawing a tree for A_1, A_2, and $A_{5,(3,4)}$. This is a case with $n = 3$ which is solved as described above. The solution locates S_1. One can then locate S_2 on the line $S_1A_{(3,4),5}$ and finally, S_3 on the line $S_2A_{(3,4)}$.

In general, one has n cities $A_1, \cdots, A_n$ and at most $n-2$ Steiner points. By iterating the construction of Fig. 6(b) at most $n-2$ times one ultimately reduces the problem to a solvable case. There are three cautions to observe.

First of all, there are two triangles having a given base A_iA_j and given sides. The correct choice of triangle, and hence the correct $A_{i,j}$, is clear if one knows the location of the third point which shares the Steiner point of A_i and A_j. If this third point is itself a Steiner point and not yet located, one may have to try both possibilities for $A_{i,j}$.

However, if one can guess the correct choice of $A_{i,j}$ and then find a relatively minimal tree, the uniqueness result of Section III shows that one need not try the other choice.

Secondly, at some stage in the construction, one may find the situation shown in Fig. 7(a), (b), or (c) and so be unable to locate a Steiner point. This can happen either because no relatively minimal tree exists with the topology sought or because one of the $A_{i,j}$ was chosen wrong.

Thirdly, the construction described here produces only trees which have three lines at each Steiner point. A tree having Steiner points with four or more lines or a graph which is not a tree may be cheaper than the Steiner minimal tree in some cases.

V. SPLIT ROUTING

Unlike trees, which provide just one path between each pair of points, graphs with cycles offer a choice of paths. Then the $N(i, j)$ channels from i to j may be distributed over two or more paths (*split routing*). The example in Fig. 9 shows that split routing is sometimes economical. The three cities are at the corners of a unit equilateral triangle and the demands are $N(1, 2) = 13$, $N(1, 3) = N(2, 3) = 1$. The cost per mile for N channels is

$$f(N) = [(N + 2)/3].$$

Such a cost function might be encountered if channels are available only in cables containing 3 channels each; then $f(1) = f(2) = f(3)$, $f(4) = f(5) = f(6)$, etc. In Fig. 9(a) all channels follow direct paths in the complete graph. In Fig. 9(b) one of the channels from A_1 to A_2 has been rerouted through A_3. This reduces the cost of the line A_1A_2;

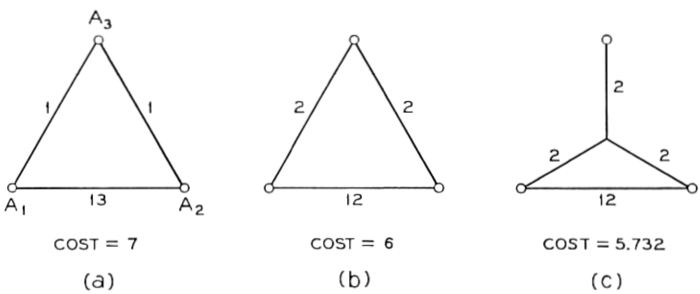


Fig. 9 — Split routing.

it increases the number of channels in the other lines but does not increase their cost. Fig. 9(c) shows the minimal graph, which also uses split routing.

The remainder of this section will show that split routing gains nothing if $f(N)$ is a convex function, i.e., if

$$f(N+2) - 2f(N+1) + f(N) \leqq 0 \tag{4}$$

for all N. Suppose $f(N)$ is convex and consider a network which uses split routing. Then one can find two channels, say α and β, which join cities A_i, A_j by different routes. To make cost comparisons easy, suppose that all other channels of the network have been installed and that the two channels for α and β have been installed on those lines of the graph which belong to both α and β. Now for $n = 0,1,2$ let $I_\alpha(n)$ be the incremental cost of installing n channels in each of the remaining lines of α and let $I_\beta(n)$ be a similar incremental cost for β. The cost to finish constructing the network is

$$\text{cost} = I_\alpha(1) + I_\beta(1). \tag{5}$$

However, $I_\alpha(n)$ is the sum of incremental costs of adding n channels to certain existing lines. If the kth line has $N_k[\alpha]$ channels

$$I_\alpha(n) = \sum_k \{f(N_k[\alpha] + n) - f(N_k[\alpha])\}.$$

Then (4) shows $f(N+2) - f(N) \leqq 2\{f(N+1) - f(N)\}$, so

$$I_\alpha(2) \leqq 2I_\alpha(1),$$

and similarly,

$$I_\beta(2) \leqq 2I_\beta(1).$$

Now (5) shows

$$\begin{aligned}\text{cost} &\geqq \tfrac{1}{2}\{I_\alpha(2) + I_\beta(2)\}\\ &\geqq \text{Min}\,\{I_\alpha(2), I_\beta(2)\}.\end{aligned}$$

The last inequality shows that it would be as cheap to complete two copies of one of the channels α or β as to complete one of each.

VI. LINEAR COST FUNCTIONS

Suppose $f(N)$ is linear, $f(N) = a + bN$. Consider any graph. Let L_i denote the length of the ith line of the graph and N_i the number of channels along that line. The cost of the network is

$$\text{cost} = aL + \sum_i N_i L_i b, \tag{6}$$

where $L = L_1 + L_2 + \ldots$ is the total length of the graph.

A simple lower bound on the cost of networks which satisfy a given demand for channels may be obtained by bounding the two terms in (6) separately. The preliminary cost term aL is at least as large as aL_0, where L_0 is the total length of the ordinary Steiner minimal tree connecting the given cities. The remaining cost in (6) would have been the cost of building the network if $f(N)$ had been bN. This cost is minimized by the complete network. Then

$$\text{cost} \geqq aL_0 + b \sum_{i<j} | A_i A_j | N(i, j). \tag{7}$$

Another way of writing (7) uses two new quantities,

$$L_c = \sum_{i<j} | A_i A_j |,$$

(the length of the complete graph) and

$$\nu = L_c^{-1} \sum_{i<j} | A_i A_j | N(i, j)$$

(the average of the numbers of channels required between pairs of cities with the distance between cities as a weighting factor). Then (7), combined with the observation that the cost of the complete graph is an upper bound, becomes

$$aL_0 + b\nu L_c \leqq \text{cost} \leqq aL_c + b\nu L_c\,. \tag{8}$$

The form (8) of (7) is useful when numbers of channels which will be required between cities can be predicted only relatively but not absolutely. Then ν is a convenient measure of "traffic level".

The lower bound (8) is an instance of a more general inequality expressing a convexity property of the minimum cost function $c(\nu)$:

$$c(\nu) \geqq \{(\nu_2 - \nu)c(\nu_1) + (\nu - \nu_1)c(\nu_2)\}/(\nu_2 - \nu_1) \tag{9}$$

for $\nu_1 \leqq \nu \leqq \nu_2$. According to (9) linear interpolation between known values $c(\nu_1)$, $c(\nu_2)$ gives a lower bound on $c(\nu)$. In particular, (9) becomes the left half of (8) in the limiting case $\nu_1 = 0$, $\nu_2 \to \infty$.

In the proof of (9) which follows it is convenient to extend the definition of $c(\nu)$ from a discrete set of ν values [at which all $N(i, j)$ are integers] to all positive real values. Although a line may require a nonintegral number N of channels to satisfy traffic level ν exactly, its cost will be computed still at $a + bN$ dollars per mile. Now let $c(G, \nu)$ be the cost of providing channels for traffic level ν using graph G. In specifying G I intend that the location of any Steiner points be specified and not to depend on ν. Then $c(G, \nu)$ is a linear function of ν. Since

$$c(\nu) = \operatorname*{Min}_{G} c(G, \nu), \tag{10}$$

the region below the curve $c = c(\nu)$ is an intersection of the half-spaces lying below the lines $c = c(G, \nu)$. Then the region in question is convex and (9) follows.

The lower bound (8) is asymptotic to the minimum cost both for small ν and large ν. Even at intermediate values of ν the lower bound is reasonably accurate. For example, when there are three cities at the vertices of an equilateral triangle and ν channels are required between each pair of cities, the lower bound stays within 11.3 percent of the true minimum for all ν. The worst disagreement occurs when $\nu = (1+3^{\frac{1}{2}}) a/b$.

For a more realistic illustration, I took the four cities New York, Chicago, Houston, and Los Angeles and the numbers of channels given in Table II.

Table II—Number of Channels Between Cities

Cities	Separation (miles)	Number of channels
Houst.—L.A.	1374	x
Houst.—Chi.	940	2x
Houst.—N.Y.	1420	4x
L.A. —Chi.	1745	5x
L.A. —N.Y.	2451	10x
Chi. —N.Y.	713	20x

Here x is another parameter to specify traffic level; the average number of channels per pair of cities turns out to be $\nu = 6.52x$. The number of channels listed is nearly proportional to the product of the populations of the cities.* The cost function was $f(N) = 17{,}000 + 7N$ dollars per mile. The complete graph and ordinary Steiner minimal tree have lengths

$$L_C = 8{,}643 \text{ miles}$$

$$L_0 = 2{,}980 \text{ miles}$$

* N. Y. population includes Philadelphia; Chicago population includes Detroit.

so the lower bound is

$$50{,}660{,}000 + 394{,}400x$$

dollars. Table III compares this bound with the true minimum cost. Fig. 10 shows some of the minimum graphs. The upper bound in (7) differs from the lower bound by

TABLE III—COST OF MINIMUM GRAPHS (MILLIONS OF DOLLARS)

x	ν	Minimum cost	Lower bound	Discrepancy (percent)
30	195.6	63.2	62.5	1.1
50	326	72.0	70.0	2.2
100	652	93.2	90.1	3.4
200	1,304	135.2	129.5	4.2
500	3,260	260.4	247.9	4.8
1000	6,520	466.0	445.0	4.5
5000	32,600	2096.0	2022.7	3.5

$a(L_C - L_0)$, which in this example is about 240 million dollars. Then, for values of x larger than those shown in Table III the two bounds will agree to better than 4.6 percent.

Suppose one kind of technology, say coaxial cable, provides channels with a linear cost function

$$f(N) = a + bN$$

and suppose that a competing technology, say waveguide or microwave relay, has another linear cost function

$$F(N) = A + BN.$$

Suppose that $a < A$ but $B < b$ so that the first technology is the more economical one to use if ν is small but the second is the more economical if ν is large. It is interesting to compare the two costs at various traffic levels and to find a value $\nu = \nu_0$ at which the two technologies are equally expensive.

Suppose one computes minimal graphs and minimal costs $c(\nu)$, as in Table III, using the function $f(N)$. The corresponding minimal graphs and costs $C(\nu)$ *for* $F(N)$ may be obtained immediately by the following "scaling" argument. First, note that if $F(N)$ were just a multiple $\lambda f(N)$ of $f(N)$, the minimal networks in the two technologies would be identical and the costs would satisfy $C(\nu) = \lambda c(\nu)$. Secondly, note that if $F(N) = f(\mu N)$ for some multiplier μ, then the minimal

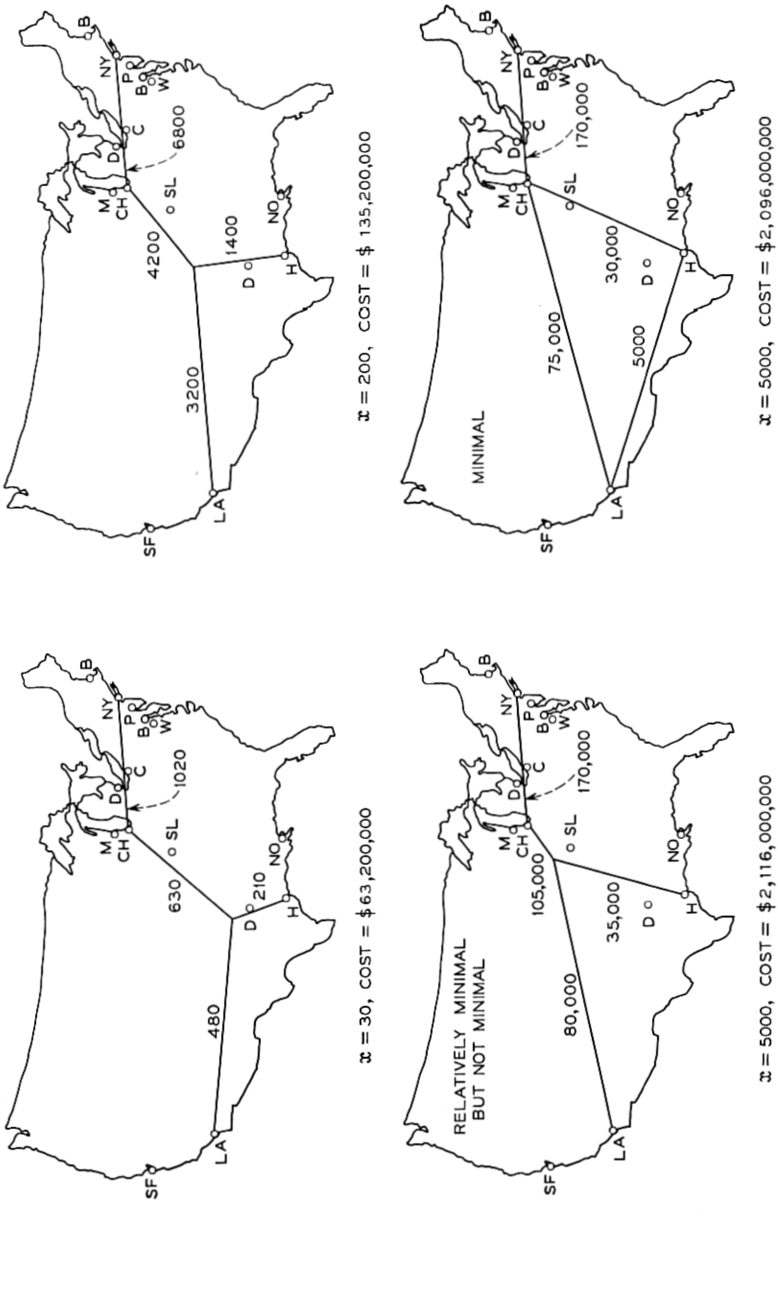


Fig. 10 — Minimal graphs for three traffic levels (cf Table III).

network in the second technology is the same as the one which the first technology had at the traffic level $\mu\nu$; also $C(\nu) = c(\mu\nu)$. Since, in general,

$$F(N) = \lambda f(\mu N),$$

where $\lambda = A/a$, and $\mu = aB/(Ab)$, the two observations above combine to show that

$$C(\nu) = (A/a)c(aB\nu/(Ab)).$$

Moreover, the minimal graph for the second technology is the one found for the first at traffic level $aB\nu/(Ab)$.

To get a very rough estimate of the traffic level ν_0 at which the two technologies are equally expensive one might use the lower bound in (8) as an approximation to the minimal cost. Doing this provides the estimate

$$\nu_0 = (A - a)L_0/\{(b - B)L_C\}.$$

REFERENCES

1. Gilbert, E. N. and Pollak, H. O., Steiner Minimal Trees, to appear in SIAM J. Appl. Math.
2. Melzak, Z. A., On the Problem of Steiner, Canadian Math. Bu., *4,* 1961, pp. 143–148.
3. Todhunter, Isaac, editor, *The Elements of Euclid,* Everyman, *851,* London 1939. Book VI, Prop. D.
4. van de Lindt, W. J., A Geometrical Solution of the Three Factory Problem, Mathematics Magazine *39,* 1966, pp. 162–165.