A Generalized Nyquist Criterion and an Optimum Linear Receiver for a Pulse Modulation System

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A pulse modulation system is modeled with M waveforms $\{s_m(t)\}_1^M$, each of which is amplitude scaled and simultaneously transmitted over a single physical channel. An infinite pulse train is assumed with signal interval T, which is determined by bandwidth consideration of the channel. We restrict the receiver to be linear with M outputs, one for each signal waveform.

At a high signal-to-noise ratio the main sources of interference at the input to the receiver are the intersymbol interference and crosstalk; by crosstalk we mean the interference between the different waveforms. It is desirable, therefore, for the receiver to eliminate both types of interference and to minimize the remaining error due to additive noise in the channel. This constraint on the intersymbol interference and crosstalk is defined as the generalized Nyquist criterion.

The receiver which accomplishes the above is determined for a mean square error criterion. Finally, some examples are presented which demonstrate the ease with which the generalized Nyquist criterion can be used to design waveforms without intersymbol interference or crosstalk.

I. THE MATHEMATICAL MODEL

The mathematical model for a pulse modulation system is shown in Fig. 1. The M waveforms $\{s_m(t)\}_1^M$, which are assumed linearly independent and of equal energy, are simultaneously transmitted over a single physical channel. Information is carried on each waveform by amplitude scaling the waveforms $s_m(t)$ by the real numbers $\{a_m\}_1^M$ which are random variables. An infinite pulse train is assumed with signal interval T so that the resulting transmitted waveform is

$$\sum_{n=-\infty}^{\infty} \sum_{m=1}^{M} a_{nm} s_m(t - nT). \tag{1}$$

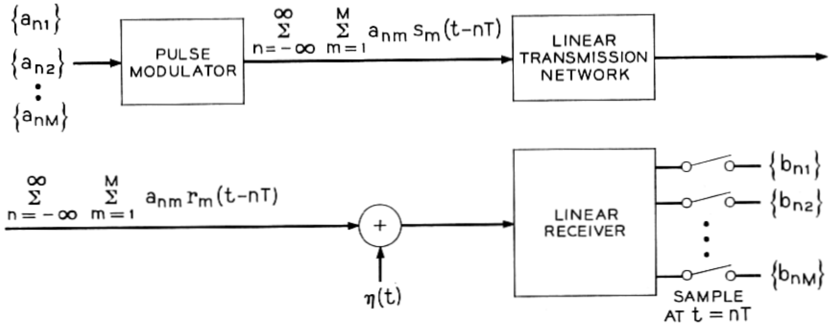

Fig. 1 — Model of the pulse modulation transmission system.

Characterizing the linear time invariant channel by its impulse response, $h(t)$, we define $r_m(t)$ as the convolution of $s_m(t)$ with $h(t)$ so that the received signal waveform is

$$\sum_{n=-\infty}^{\infty} \sum_{m=1}^{M} a_{nm} r_m(t - nT). \tag{2}$$

To this the channel adds stationary zero mean noise, $\eta(t)$, with correlation function $n(\tau)$ and spectral density $N(f)$. The received waveform is processed by a bank of receivers $\{w_k\}_1^M$ whose M outputs are sampled at times $t = nT$, $n = 0, \pm 1, \pm 2, \cdots$ to give b_{nm} which are the estimates of the a_{nm}.

If we consider the set $\{s_m(t)\}_1^M$ with our one physical channel as comprising M different channels then we can refer to the interference of the waveform due to $s_k(t)$ with that of $s_m(t)$ $(m \neq k)$ as crosstalk.

Restricting our attention to linear time-invariant receivers then we can characterize the receivers $\{w_k\}_1^M$ by impulse response $\{w_k(t)\}_1^M$ so that the output of the receivers can be expressed as

$$b_k(t) = \sum_{p=-\infty}^{\infty} \sum_{m=1}^{M} a_{pm} v_{mk}(t - pT) + \int_{-\infty}^{\infty} \eta(x) w_k(t - x)\, dx, \tag{3}$$

where

$$v_{mk}(t) = \int_{-\infty}^{\infty} w_k(t - x) r_m(x)\, dx. \tag{4}$$

The sampled outputs are designated by b_{nk},

$$b_{nk} = b_k(nT). \tag{5}$$

At high signal-to-noise ratio (S/N) where

$$S/N = \frac{\sum_{m=1}^{M} v_{mm}^2(0)}{\sum_{k=1}^{M} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} n(y - x) w_k(x) w_k(y)\, dx\, dy}, \tag{6}$$

the main sources of interference at the input to the receivers are intersymbol interference and crosstalk. It is desirable, therefore, for the receiver to process its input so that the output eliminates intersymbol interference and crosstalk; i.e., that

$$\sum_{p=-\infty}^{\infty} \sum_{m=1}^{M} a_{pm} v_{mk}(nT) = a_{nk} \tag{7}$$

for all possible sequences of the a_{nm}. This is equivalent to requiring that

$$v_{mk}(nT) = \delta_{mk}\, \delta_{n0} \qquad \begin{matrix} m, k = 1, \cdots, M \\ n = 0, \pm 1, \pm 2, \cdots, \end{matrix} \tag{8}$$

where the δ_{ij} are Kronecker delta functions. Further justification for imposing this constraint at high S/N is provided in the Appendix.

We use as our error criterion the mean square error averaged over the receiver outputs

$$J_n = \frac{1}{M} \sum_{m=1}^{M} E\{(a_{nm} - b_{nm})^2\}, \tag{9}$$

where the expectation is with respect to the random variables a_{nk} and the noise.

We are now in a position to specify the problem concisely: to determine the linear receiver which minimizes the mean square error under the constraint that there be no intersymbol interference or crosstalk.

II. A GENERALIZED NYQUIST CRITERION

A waveform $v(t)$ is said to satisfy the Nyquist criterion[1] for the signal interval T, if

$$v(nT) = \delta_{n0} \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{10}$$

Denoting the Fourier transform of $v(t)$ by $V(f)$ (upper case letters will be used throughout to denote the Fourier transforms of the functions represented by lower case letters), we can state that (10) is true, if and only if,

$$\frac{1}{T} \sum_{\alpha=-\infty}^{\infty} V\left(f - \frac{\alpha}{T}\right) = 1. \tag{11}$$

This is easily shown using Poisson's sum formula (Papoulis)[3]

$$\frac{1}{T} \sum_{\alpha=-\infty}^{\infty} \Phi\left(f - \frac{\alpha}{T}\right) = \sum_{\alpha=-\infty}^{\infty} e^{-i\alpha 2\pi f T} \phi(\alpha T). \tag{12}$$

If we associate $\phi(t)$ with $v(t)$ then (10) implies and is implied by (11).

Our constraint that the $v_{mk}(t)$ satisfy (8) requires not only that the $v_{mm}(t)$ satisfy (10) but also that the $M(M-1)$ waveforms $v_{mk}(t)$ $(m \neq k)$ be zero at $t = nT$. We refer to (8) as the generalized Nyquist criterion. The equation analogous to (11) is

$$\frac{1}{T} \sum_{\alpha=-\infty}^{\infty} V_{mk}\left(f - \frac{\alpha}{T}\right) = \delta_{mk} . \tag{13}$$

This will be used interchangeably with (8) in solving the optimization problem. Since the $\{V_{mk}(f)\}$ can be checked almost by inspection to see if they satisfy (13), the equation is very simple to use.

III. THE CONSTRAINED OPTIMUM RECEIVER

The object of this section is to determine the linear receiver which, subject to the constraints of (8), minimizes the error expression (9). Because of the constraint of (8) we have

$$b_{nk} - a_{nk} = \int_{-\infty}^{\infty} \eta(x) w_k(t - x)\, dx \tag{14}$$

so that the error becomes

$$J = \frac{1}{M} \sum_{m=1}^{M} \int_{-\infty}^{\infty} W_m(f) W_m^*(f) N(f)\, df \tag{15}$$

which is independent of n.

We are now left with the interesting variational problem of minimizing J with respect to all linear receivers $W_k(f)$ such that

$$\frac{1}{T} \sum_{\alpha=-\infty}^{\infty} R_m\left(f - \frac{\alpha}{T}\right) W_k\left(f - \frac{\alpha}{T}\right) = \delta_{mk} ; \tag{16}$$

i.e., which satisfy the generalized Nyquist constraint. In order to do this, we vary each $W_k(f)$ by an amount $\epsilon\, \Gamma_k(f)$, where the $\Gamma_k(f)$ must be such that (16) is still valid. We require

$$\frac{1}{T}\sum_{\alpha=-\infty}^{\infty} R_m\left(f-\frac{\alpha}{T}\right)\left[W_k\left(f-\frac{\alpha}{T}\right)+\epsilon\Gamma_k\left(f-\frac{\alpha}{T}\right)\right]$$

$$=\frac{1}{T}\sum_{\alpha=-\infty}^{\infty} R_m\left(f-\frac{\alpha}{T}\right)W_k\left(f-\frac{\alpha}{T}\right)+\epsilon\sum_{\alpha=-\infty}^{\infty} R_m\left(f-\frac{\alpha}{T}\right)\Gamma_k\left(f-\frac{\alpha}{T}\right)=\delta_{mk}$$

$$k,\ m=1,2,\cdots,M \tag{17}$$

so that $\Gamma_k(f)$ must satisfy the condition

$$\sum_{\alpha=-\infty}^{\infty} R_m\left(f-\frac{\alpha}{T}\right)\Gamma_k\left(f-\frac{\alpha}{T}\right)=0 \qquad m,\ k=1,2,\cdots,M. \tag{18}$$

The error with variations becomes

$$\begin{aligned}J(\epsilon) &= \frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty}[W_k(f)+\epsilon\Gamma_k(f)][W_k^*(f)+\epsilon\Gamma_k^*(f)]N(f)\,df\\ &= \frac{1}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty}W_k(f)W_k^*(f)N(f)\,df\\ &\qquad+\frac{\epsilon}{M}\sum_{k=1}^{M}\int^{\infty}[\Gamma_k(f)W_k^*(f)+\Gamma_k^*(f)W_k(f)]N(f)\,df\\ &\qquad+\frac{\epsilon^2}{M}\sum_{k=1}^{M}\int_{-\infty}^{\infty}\Gamma_k(f)\Gamma_k^*(f)N(f)\,df.\end{aligned} \tag{19}$$

$J(0)$ is minimum if (the $w_k(t)$ are constrained to be real)

$$\int_{-\infty}^{\infty}\Gamma_k(f)W_k^*(f)N(f)\,df=0 \qquad (\text{for each } k=1\cdots M), \tag{20}$$

where $\Gamma_k(f)$ must satisfy (18).

In order to solve for $W_k(f)$ we manipulate (20) as follows:

$$\begin{aligned}&\int_{-\infty}^{\infty}\Gamma_k(f)W_k^*(f)N(f)\,df\\ &\quad=\sum_{\alpha=-\infty}^{\infty}\int_{-1/2T+\alpha/T}^{1/2T+\alpha/T}\Gamma_k(f)W_k^*(f)N(f)\,df\\ &\quad=\sum_{\alpha=-\infty}^{\infty}\int_{-1/2T}^{1/2T}\Gamma_k\left(f-\frac{\alpha}{T}\right)W_k^*\left(f-\frac{\alpha}{T}\right)N\left(f-\frac{\alpha}{T}\right)df\\ &\quad=\int_{-1/2T}^{1/2T}\sum_{\alpha=-\infty}^{\infty}\left[W_k^*\left(f-\frac{\alpha}{T}\right)N\left(f-\frac{\alpha}{T}\right)\right]\Gamma_k\left(f-\frac{\alpha}{T}\right)df=0.\end{aligned} \tag{21}$$

Comparing (21) with (18) it can be recognized that (21) is satisfied by a $W_k(f)$ such that

$$W_k(f) = \sum_{c=1}^{M} \frac{R_c^*(f)}{N(f)} Z_{ck}(f) \qquad k = 1, \cdots, M, \tag{22}$$

where the $Z_{ck}(f)$ are arbitrary periodic functions of f with period $1/T$.

In order to completely specify $W_k(f)$ we must determine the $Z_{ck}(f)$. Substituting (22) into (16), we obtain

$$\frac{1}{T} \sum_{\alpha=-\infty}^{\infty} R_m\left(f - \frac{\alpha}{T}\right) \sum_{c=1}^{M} \frac{R_c^*\left(f - \frac{\alpha}{T}\right)}{N\left(f - \frac{\alpha}{T}\right)} Z_{ck}\left(f - \frac{\alpha}{T}\right)$$

$$= \frac{1}{T} \sum_{c=1}^{M} Z_{ck}(f) \sum_{\alpha=-\infty}^{\infty} \frac{R_m\left(f - \frac{\alpha}{T}\right) R_c^*\left(f - \frac{\alpha}{T}\right)}{N\left(f - \frac{\alpha}{T}\right)} = \delta_{mk} \tag{23}$$

$$m, k = 1, 2, \cdots, m$$

since $Z_{ck}(f)$ is periodic with period $1/T$. Let

$$L_{mc}(f) = \frac{1}{T} \sum_{\alpha=-\infty}^{\infty} R_m\left(f - \frac{\alpha}{T}\right) R_c^*\left(f - \frac{\alpha}{T}\right) \Big/ N\left(f - \frac{\alpha}{T}\right) \tag{24}$$

then (23) becomes

$$\frac{1}{T} \sum_{c=1}^{M} L_{mc}(f) Z_{ck}(f) = \delta_{mk} \qquad m, k = 1, 2 \cdots m \tag{25}$$

or in matrix form

$$L(f)Z(f) = I, \tag{26}$$

where

$$L(f) = [L_{ij}(f)]$$

and

$$Z(f) = [Z_{ij}(f)]$$

are M by M matrices.

Thus, we have, if L is nonsingular for all f, that

$$Z(f) = [L(f)]^{-1} \tag{27}$$

so that $|L| \neq 0$ is a necessary and sufficient condition for a solution to exist. $W_k(f)$ is now completely specified and a realization of the optimum constrained receiver is shown in Fig. 2.

A simple expression for the resulting mean square error is obtained from a manipulation similar to that of (21):

$$J_{\text{opt}} = \frac{T}{M} \sum_{m=1}^{M} \int_{-1/2T}^{1/2T} Z_{mm}(f)\, df. \tag{28}$$

IV. EXAMPLES

In this section examples are presented which demonstrate the ease with which the generalized Nyquist criterion can be used to design waveforms without intersymbol interference or crosstalk.

4.1 *Example*

We start out by making the simplifying assumption that $N(f) = 1$. In addition, if the transmitted waveforms $\{S_m(f)\}_1^M$ are chosen such that $R_m(f)$, where $R_m(f) = S_m(f)H(f)$ satisfy the equation

$$\sum_{\alpha=-\infty}^{\infty} R_m\left(f - \frac{\alpha}{T}\right) R_k^*\left(f - \frac{\alpha}{T}\right) = d_m\, \delta_{mk}\,, \tag{29}$$

then a solution exists since the L matrix becomes a diagonal matrix

$$L = I\underline{d}; \qquad d = \begin{bmatrix} d_1 \\ \vdots \\ d_m \end{bmatrix} \tag{30}$$

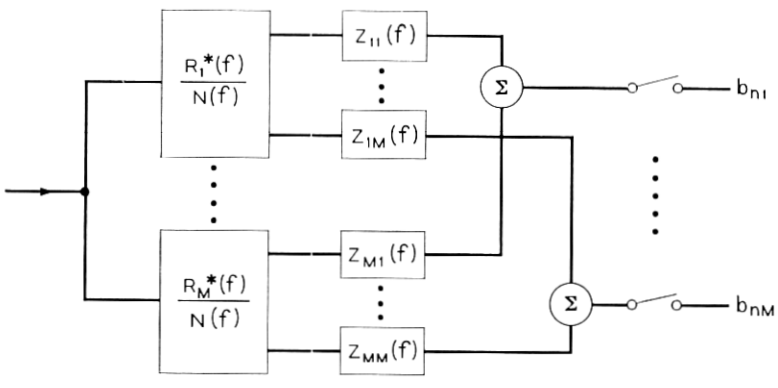

Fig. 2 — A realization of the optimum constrained receiver.

and

$$Z = L^{-1} = \begin{bmatrix} 1/d_1 & & 0 \\ & \ddots & \\ 0 & & 1/d_m \end{bmatrix} \tag{31}$$

with the resulting error

$$J = \frac{1}{M} \sum_{m=1}^{M} \frac{1}{d_m}. \tag{32}$$

Under these conditions the outputs of the matched filters satisfy the generalized Nyquist constraint except for scale factors and the $Z_{cm}(f)$ functions need only perform the appropriate scaling. We consider next two cases where (29) is satisfied.

4.2 *Case I*

Only the case of $M = 2$ is presented here in detail although other values of M can similarly be handled.

First, note that since matched filters are used the actual phases of the $R_i(f)$ are not important since the output depends only on phase difference between $R_i(f)$ and $R_j(f)$. We use the phase of $R_1(f)$ as a reference phase.

$$R_1(f) = \begin{cases} ce^{j\phi_1(f)}, & |f| \leqq 1/T \\ 0, & |f| > 1/T \end{cases}$$

$$R_2(f) = \begin{cases} ce^{j[\phi_1(f)+\Delta\phi(f)]}, & |f| \leqq 1/T \\ 0, & |f| > 1/T \end{cases},$$

where $\Delta\phi(f) = \Delta\phi(-f) \pm \pi$ for $|f| \leqq 1/T$. The sign is chosen so that $|\Delta\phi(f)| \leqq \pi$.

To simplify matters we can choose

$$\Delta\phi(f) = \begin{cases} \pi/2, & f > 0 \\ -\pi/2, & f < 0 \end{cases}$$

so

$$R_1(f)R_2^*(f) = \begin{cases} c^2 e^{-j\Delta\phi(f)}, & |f| \leqq 1/T \\ 0, & f| > 1/T \end{cases}$$

$$= \begin{cases} -jc^2, & 0 \leqq f \leqq 1/T \\ jc^2, & -1/T \leqq f < 0 \\ 0, & \text{elsewhere.} \end{cases}$$

Therefore,

$$\sum_{\alpha=-\infty}^{\infty} R_1\left(f - \frac{\alpha}{T}\right) R_2^*\left(f - \frac{\alpha}{T}\right) = jc^2 - jc^2 = 0 \qquad \text{(Fig. 3).}$$

Similarly,

$$\sum_{\alpha=-\infty}^{\infty} R_2\left(f - \frac{\alpha}{T}\right) R_1^*\left(f - \frac{\alpha}{T}\right) = 0$$

and

$$\sum_{\alpha=-\infty}^{\infty} \left| R_m\left(f - \frac{\alpha}{T}\right) \right|^2 = 2c^2 \qquad m = 1, 2$$

so

$$L = \begin{bmatrix} 2c^2 & 0 \\ 0 & 2c^2 \end{bmatrix} = 2c^2 I$$

and

$$Z = \frac{1}{2c^2} I$$

$$J_0 = \frac{1}{2c^2}.$$

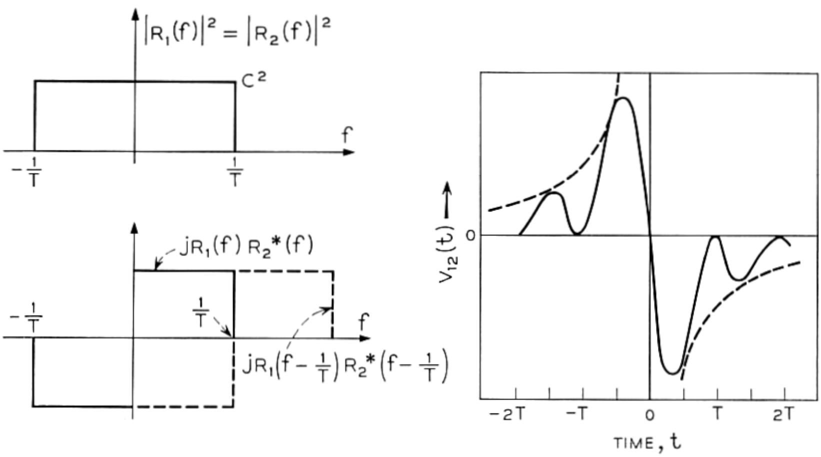

Fig. 3 — Case I transforms at the output of the matched filter.

In the time domain we have

$$v_{12}(t) = c^2\left(\cos\frac{2\pi t}{T} - 1\right)\Big/\pi t$$

and it is easily seen that

$$v_{12}(nT) = 0 \quad \text{for} \quad n = 0, \pm 1, \pm 2, \cdots .$$

4.3 *Case II*

Consider a set of band-limited frequency multiplexed signals $\{R_m(f)\}_1^M$. The bandwidths are $(1 + \gamma_m + \gamma_{m-1})/T$ where the $\gamma_m(0 \leqq \gamma_m \leqq 1)$ are parameters associated with the excess rolloff bandwidth, and the signals are separated in frequency by $1/T$ hertz so that the waveforms overlap the adjacent signals only. As in Case I, the actual phases are unimportant because of the matched filters so only phase differences $\phi_m(f)$ from a reference phase $\phi(f)$ will be important.

$$R_m(f) = |R_m(f)| \exp j[\phi(f) + \phi_m(f)]$$

$$R_m(f)R^*_{m+1}(f) = |R_m(f)R_{m+1}(f)| \exp j[\phi_m(f) - \phi_{m+1}(f)].$$

We define roll-off characteristics as a real function $Q_m(f)$ such that

$$Q_m(f) = 0 \quad \text{for} \quad |f| > \frac{\gamma_m}{2T}$$

and

$$Q_m(f) = -Q_m(-f); \qquad \text{for} \quad |f| \leqq \frac{\gamma_m}{2T}.$$

We can specify the $R_m(f)$ as follows:

$$|R_m(f)|$$
$$= c_m\sqrt{\text{rect}\left(\frac{|f| - m}{T}\right) + Q_m\left(|f| + \frac{1}{2T} - \frac{m}{T}\right) + Q_{m+1}\left(-|f| - \frac{1}{2T} - \frac{m}{T}\right)}$$

$$\Delta_m(f) = \phi_m(f) - \phi_{m-1}(f)$$

and

$$\Delta_m(f) = \Delta_m(-f) \pm \pi.$$

With the $R_m(f)$ specified it is easily checked that

$$\sum_{\alpha=-\infty}^{\infty} \left| R_m\left(f - \frac{\alpha}{T}\right) \right|^2 = c_m^2$$

$$R_m(f)R_{m-1}(f) = |R_m(f)|\,|R_{m-1}(f)| \exp[-j\Delta_m(f)]$$

$$= c_m c_{m-1}\left\{B_m\left(f - \frac{m-\frac{1}{2}}{T}\right)\exp[j\Delta_m(f)] + B_m\left(f + \frac{m-\frac{1}{2}}{T}\right)\exp[-j\Delta_m(f)]\right\},$$

where $B_m(f)$ is an even real function with bandwidth $2\gamma_{m-1}/T$.

$$B_m(f) = \sqrt{Q_m(|f|)[1 - Q_m(-|f|)]}.$$

We can specify $\Delta_m(f)$ as

$$\Delta_m(f) = \begin{cases} \dfrac{\pi}{2}, & f > 0 \\ -\dfrac{\pi}{2}, & f < 0 \end{cases}$$

without really restricting ourselves. The resulting $R_m R_{m-1}^*$ is shown in Fig. 4. Looking at Fig. 4, we see by inspection that the $\{R_m\}_1^M$ satisfy (29).

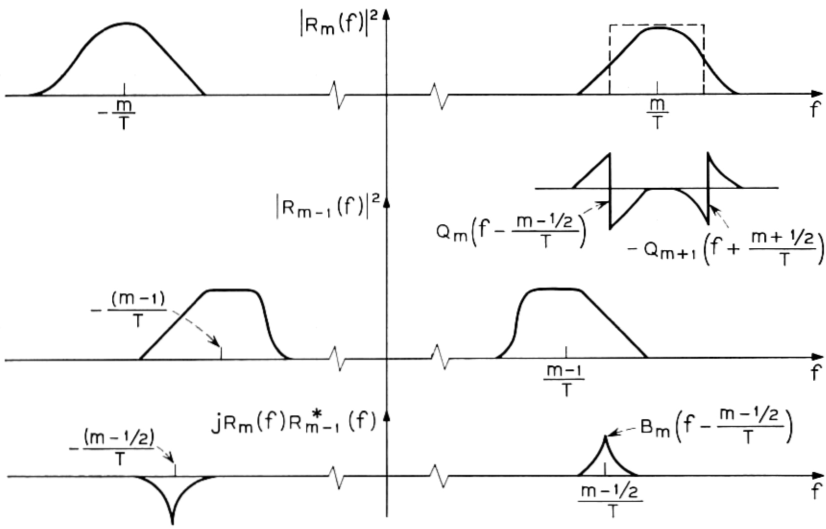

Fig. 4 — Case II transforms at the output of the matched filter.

V. ACKNOWLEDGMENT

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APPENDIX

The Optimum Mean Square Error Receiver

In this appendix the optimum mean square error receiver is obtained and it is shown that as S/N → ∞ this receiver and the optimum constrained receiver of Section III converge to the same receiver when the $\{a_{nk}\}$ are stationary.

The general expression for the mean square error is

$$\begin{aligned} J_n = \frac{1}{M} \sum_{m=1}^{M} E\Big\{ & a_{nm}^2 - 2 \sum_{p=-\infty}^{\infty} \sum_{k=1}^{M} a_{nm} a_{pk} v_{km}(nT - pT) \\ & + \sum_{p=-\infty}^{\infty} \sum_{k=1}^{M} \sum_{r=-\infty}^{\infty} \sum_{i=1}^{M} a_{pk} a_{ri} v_{km}(nT - pT) v_{im}(nT - rT) \\ & - 2a_{nm} \int_{-\infty}^{\infty} \eta(x) w_m(nT - x)\, dx \\ & + 2 \sum_{p=-\infty}^{\infty} \sum_{b=1}^{M} a_{pk} v_{km}(nT - pT) \int_{-\infty}^{\infty} \eta(x) w_m(nT - x)\, dx \\ & + \iint_{-\infty}^{\infty} \eta(x)\eta(y) w_m(nT - x) w_m(nT - y)\, dx\, dy \Big\} \end{aligned} \tag{33}$$

$$\begin{aligned} = \frac{1}{M} \sum_{m=1}^{M} \Big[& \rho_{mm}^{nn} - 2 \sum_{p=-\infty}^{\infty} \sum_{k=1}^{M} \rho_{mk}^{np} v_{km}(nT - pT) \\ & + \sum_{p=-\infty}^{\infty} \sum_{k=1}^{M} \sum_{r=-\infty}^{\infty} \sum_{i=1}^{M} \rho_{ki}^{pr} v_{km}(nT - pT) v_{im}(nT - rT) \\ & + \int_{-\infty}^{\infty} N(f) W_m(f) W_m^*(f)\, df \Big], \end{aligned}$$

where

$$\rho_{mk}^{np} = E\{a_{nm} a_{pk}\}. \tag{34}$$

Since the $\{a_{nk}\}$ are stationary we can write

$$\rho_{mk}^{n,p} \equiv \rho_{mk}^{(n-p)}. \tag{35}$$

Defining

$$M_{mk}(f) = \sum_{n=-\infty}^{\infty} \rho_{mk}^{(n)} \exp(-j2\pi fnT), \tag{36}$$

then J_n can be written as

$$J = \frac{1}{M} \sum_{m=1}^{M} \left[\rho_{mm}^{(0)} - 2 \sum_{k=1}^{M} \int_{-\infty}^{\infty} M_{mk}(f) V_{km}^*(f)\, df \right.$$
$$+ \sum_{k=1}^{M} \sum_{i=1}^{M} \int_{-\infty}^{\infty} M_{ki}(f) V_{km}^*(f) \frac{1}{T} \sum_{\alpha=-\infty}^{\infty} V_{im}\left(f - \frac{\alpha}{T}\right) df$$
$$\left. + \int_{-\infty}^{\infty} W_m(f) W_m^*(f) N(f)\, df \right] \tag{37}$$

which is independent of n so the index has been dropped.

Using variational calculus we obtain as a necessary condition on the optimum $\{W_m(f)\}_1^M$ that they satisfy the equations

$$\sum_{m=1}^{M} R_m^*(f) \left[\sum_{i=1}^{M} M_{mk}(f) \frac{1}{T} \sum_{\alpha=-\infty}^{\infty} R_i\left(f - \frac{\alpha}{T}\right) W_k\left(f - \frac{\alpha}{T}\right) - M_{mk}(f) \right]$$
$$+ W_k(f) N(f) = 0 \qquad k = 1, 2, \cdots, M. \tag{38}$$

The solutions for the $\{W_k(f)\}_1^M$ are

$$W_k(f) = \sum_{c=1}^{M} \frac{R_c^*(f)}{N(f)} Y_{ck}(f) \qquad k = 1, 2, \cdots, M, \tag{39}$$

where the $Y_{ck}(f)$ are periodic functions of f with period $1/T$. In order to see that the $\{W_k(f)\}_1^M$ of (39) satisfy (38) for the appropriate determination of the $\{Y_{ck}(f)\}$, substitute (39) for $W_k(f)$ in (38) to obtain

$$\sum_{m=1}^{M} R_m^*(f) \left[\sum_{i=1}^{M} M_{mi}(f) \frac{1}{T} \sum_{\alpha=-\infty}^{\infty} R_i[f - (\alpha/T)] \right.$$
$$\left. \cdot \sum_{c=1}^{M} \frac{R_c^*[f - (\alpha/T)]}{N[f - (\alpha/T)]} Y_{cn}[f - (\alpha/T)] - M_{mk}(f) \right]$$
$$+ \sum_{c=1}^{M} R_c^*(f) Y_{ck}(f) = 0 \qquad k = 1, 2, \cdots, M \tag{40}$$

or since the $Y_{ck}(f)$ are periodic

$$\sum_{m=1}^{M} R_m^*(f) \left[Y_{mk}(f) - M_{mn}(f) + \sum_{i=1}^{M} M_{mi}(f) \sum_{c=1}^{M} L_{ic}(f) Y_{ck}(f) \right] = 0$$
$$k = 1, 2, \cdots, M, \tag{41}$$

where $L_{ic}(f)$ is as defined in (24).

Defining $M(f)$ and $Y(f)$ as matrices whose elements are, respectively, $M_{ij}(f)$ and $Y_{ij}(f)$ and a column vector $R(f)$ whose elements are $R_m(f)$ (41) can be written as

$$R(f)^T(Y(f) - M(f) + M(f)L(f)Y(f)) = 0, \tag{42}$$

where $L(f)$ is as previously defined. Unless $R(f) = 0$ we require that

$$(I + ML)Y = M \tag{43}$$

$$Y = (I + ML)^{-1}M. \tag{44}$$

With Y so specified (39) satisfies (38) and the resulting mean square error is

$$J_{\text{opt}} = \frac{1}{M} \sum_{m=1}^{M} \left[\rho_{nm}^{(0)} - \sum_{k=1}^{M} \int_{-\infty}^{\infty} M_{mk}(f) R_k^*(f) \sum_{c=1}^{M} \frac{R_c(f)}{N(f)} Y_{cm}^*(f) \, df \right]. \tag{45}$$

Manipulating as in (21) and using the periodicity of $M(f)$ and $Y(f)$ we then obtain

$$J_{\text{opt}} = \frac{T}{M} \sum_{m=1}^{M} \int_{-1/2T}^{1/2T} \left[M_{mm}(f) - \sum_{k=1}^{M} \sum_{c=1}^{M} M_{mk}(f) L_{kc}^*(f) Y_{cm}^*(f) \right] df. \tag{46}$$

Lastly, recognizing that the integrand is $Y_{mm}^*(f)$ we get

$$J_{\text{opt}} = \frac{T}{M} \sum_{m=1}^{M} \int_{-1/2T}^{1/2T} Y_{mm}^*(f) \, df. \tag{47}$$

Finally, we wish to show that the optimum and constrained optimum receivers approach the same limit as S/N $\to \infty$.

We define U to be the resulting L matrix when the S/N is unity, and we write for any other S/N

$$L = aU, \tag{48}$$

where a is proportional to the signal energy. Since both receivers are of the same form, we need only show that $Y \to Z$ as $a \to \infty$.

$$\begin{aligned} Y &= (ML + I)^{-1}M \\ &= (aMU + I)^{-1}M \\ &= [(1/a)U^{-1}M^{-1} - (1/a^2)(U^{-1}M^{-1})^2 + (1/a^3)(U^{-1}M^{-1})^3 - \cdots]M \\ &= (1/a)U^{-1} + O(1/a^2), \end{aligned} \tag{49}$$

where $O(1/a^2)$ indicates terms dropping off at least as fast as $1/a.^2$ As $a \to \infty$ the terms of order $1/a^2$ become negligible with respect to the

$1/a$ term. Using the fact that $L^{-1} = 1/a\ U^{-1}$, we obtain the result

$$\lim_{a\to\infty} Y = L^{-1} = Z, \tag{50}$$

and the two receivers converge and the constrained optimum is optimum.

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