Slope Overload Noise in Differential Pulse Code Modulation Systems

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In differential pulse code modulation (DPCM) systems, often referred to as predictive quantizing systems, the quantizing noise manifests itself in two forms, granular noise and slope overload noise. The study of overload noise in DPCM may be abstracted to the following stochastic processes problem. Let the input to the system be a Gaussian stochastic process $\{x(t)\}$ *with a bandlimited* $(0, f_0)$ *spectrum* $F(f)$. *Denote the output of the system by* $y(t)$. *Most of the time* $y(t)$ *is equal to* $x(t)$. *During time intervals of this kind, the absolute value of the derivative* $x'(t) = dx(t)/dt$ *is less than a given positive constant* x_0'. *(In a DPCM system,* $x_0' = kf_s$ *where* k *is the maximum level of the quantizer and* f_s *is the sampling frequency.) There are time intervals,* $I_i(t_0^{(i)}, t_1^{(i)})$ $(i = 0, \pm 1, \pm 2, \cdots)$, *for which* $y(t) \neq x(t)$. *These time intervals begin at time instants* $t_0^{(i)}$ *such that* $| x'(t_0^{(i)}) |$ *increases through the value* x_0'. *For* $t \in I_i$, $y(t) = x(t_0^{(i)}) + (t - t_0^{(i)})x_0'$. *The interval ends at* $t_1^{(i)}$, *when* $x(t)$ *and* $y(t)$ *become equal again. The overload noise in the DPCM system is defined to be* $n(t) = x(t) - y(t)$. *The problem is to study the random process* $\{n(t)\}$. *In the present paper, we will give an upper bound to the average noise power* $\langle n^2(t)\rangle_{av}$ *which at the same time is a very good approximation to the noise power itself.*

Two previous attempts have been made to find $\langle n^2(t)\rangle_{av}$. *One, due to Rice and O'Neal, involves an approximation valid only for very large* x_0'. *Another approach to the problem, due to Zetterberg, includes an ingenious way of avoiding the determination of* $t_1^{(i)}$. *A new approach is given here that combines the best features of the two methods. The present result is a better approximation for slope overload noise than has been previously obtained. The result differs from previous results but is asymptotically equal to that given by Rice and O'Neal for* $x_0' \to \infty$. *In the region where overload noise is important, the present result is in very good agreement* *with computer simulation and experiment. The technique used could be applied for the determination of other statistical characteristics of the error random process.*

I. INTRODUCTION

This paper is concerned with the slope overload noise in Differential Pulse Code Modulation (DPCM) systems, often referred to as predictive quantizing systems. Delta Modulation (ΔM), the simplest member of the DPCM family, is a European invention of the mid-forties.[1] DPCM was first revealed in a Phillips Company patent[2] in 1951 and as a predictive quantizing system in a patent by C. C. Cutler[3] of the Bell Telephone Laboratories in 1952. ΔM and DPCM are receiving renewed attention due to the present trend toward digital communications and general efforts aimed at redundancy reduction[4] in picture transmission. The present work was motivated, to a large extent, by the application of DPCM to Picturephone® signal transmission.

Work on ΔM and DPCM was reported in the early and mid-fifties. Most representative are the papers by (*i*) DeJager[5] on ΔM, mainly of introductory and descriptive nature, (*ii*) Van de Weg[6] on uniform DPCM—we will refer to it in the sequel, and (*iii*) Zetterberg[7] whose long paper on ΔM is the most detailed study of the subject to date. Recent publications note the beginning of a "renaissance" period for ΔM and DPCM.[8,9,10,4]

In DPCM systems the quantization noise manifests itself in two forms, the granular noise and the slope overload noise. The granular noise is essentially uncorrelated with the input signal and has a more or less flat power spectrum and an approximately uniform amplitude probability distribution, resembling the granular noise in standard PCM. The granular noise for single integration DPCM systems with a uniform quantizer has been studied by Van de Weg.[6]

In contrast with a straight PCM system, which overloads in amplitude, a differential PCM system overloads in slope. Consider a DPCM system (Fig. 1) with a single integrator in the feedback path and a symmetric quantizer which is not necessarily uniform. Practical DPCM systems have leaky integrators. For simplicity, we are considering only perfect integrators here. Let k be the maximum level of the quantizer and f_s the sampling frequency. Then the maximum slope that the system can follow is $x_0' = kf_s$, corresponding to the emission of a string of impulses of strength k by the quantizer of Fig. 1. For a fixed value of $x_0' = kf_s$ and for $k \rightarrow 0$ the granular noise tends to zero, and the total

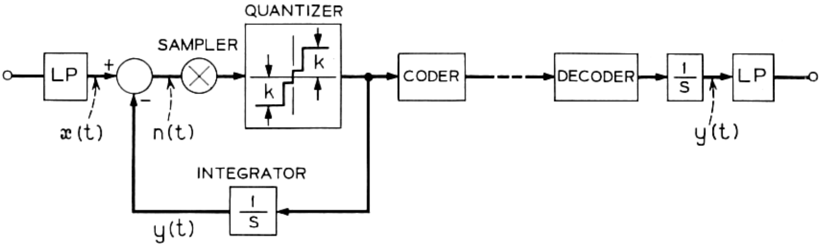

Fig. 1 — Single integration DPCM with a symmetric quantizer.

noise is due to slope overload alone. In this paper, we concentrate on certain statistics of the overload noise defined precisely in Section III.

II. SUMMARY OF RESULTS AND COMPARISON WITH PREVIOUS WORK

There exist two previous papers concerning overload noise in DPCM systems. Approximate results are given for the slope overload noise N_0 in terms of the slope capability x_0' of the DPCM system and the power spectrum of the input signal, assumed to be Gaussian. The result due to Zetterberg[7] (with some corrections) is as follows

$$N_{0,Z} = \frac{4\sqrt{2}}{35\pi^{\frac{3}{2}}}\left(\frac{b_1^2}{b_2}\right)\left(\frac{3b_1^{\frac{1}{2}}}{x_0'}\right)^5 A(\lambda) \exp\left(-\frac{x_0'^2}{2b_1}\right),$$

where b_1 and b_2 are the variances of the first and second derivatives of the input signal, respectively, and they are given in terms of the spectrum in (1) of the following section. The quantity λ and the function $A(\lambda)$ are defined in (31) and (32), respectively. The second result is due to Rice and O'Neal.[8] Their basic approximations are: (*i*) a truncation of the Taylor series for $x(t)$, around a transition point, including terms through the third derivative; and (*ii*) the assumption that the third derivative of $x(t)$ at the transition points has, as a random variable, a very small variance compared to its mean value. Therefore, the third derivative is taken to be a deterministic constant with value equal to its mean. With these assumptions, (22) of Ref. 8 results in

$$N_{0,R} = \frac{1}{4\sqrt{2\pi}}\left(\frac{b_1^2}{b_2}\right)\left(\frac{3b_1^{\frac{1}{2}}}{x_0'}\right)^5 \exp\left(-\frac{x_0'^2}{2b_1}\right).$$

There are two points that we want to make here:

(*i*) When the formula above together with an expression for the granular noise given in Ref. 8 are used to compute S/N we see that the agreement with computer simulation is not very satisfactory in the region of severe slope overload. This formula does, however, identify the peak of the S/N ratio quite successfully (see Fig. 11).

(*ii*) When we compare Zetterberg's and Rice's results by considering the ratio $N_{0,Z}/N_{0,R}$ we get

$$\frac{N_{0,Z}}{N_{0,R}} = \frac{32}{35\pi} A(\lambda) = \frac{1}{3.44} A(\lambda)$$

$$10 \log_{10} \frac{N_{0,Z}}{N_{0,R}} = -5.36 + 10 \log_{10} A(\lambda) \text{ dB}.$$

Thus, we see that the two results differ substantially.

Hence, the question of the average slope overload noise power cannot be considered settled since the two results above are different and they both differ from computer simulation and experiment. The present paper sheds further light on the question of the slope overload noise. Our principle result is the approximation

$$N_0 = \frac{1}{4\sqrt{2\pi}} \left(\frac{b_1^2}{x_0'}\right)\left(\frac{3b_1^{\frac{1}{2}}}{x_0'}\right)^5 \exp\left(-\frac{x_0'^2}{2b_1}\right) A(\chi),$$

where the quantity χ and the function $A(\chi)$ are defined in (64) and (66), respectively. This expression, like the previous ones, is a function of only two things—the maximum slope capability x_0' of the DPCM system and the power spectrum of the input signal. Indeed all the variables appearing in this formula are calculated directly from these two quantities only [see (1) and (64)]. The present formula gives better agreement with computer simulation than the one by Rice and O'Neal, when used to compute S/N (see Fig. 11).

We might also point out here that the present work applies to any system which is slope limited, not just to DPCM or digital encoding systems.

III. PROBLEM DEFINITION

With reference to Fig. 1 let the input $\{x(t)\}$ be a stationary band-limited Gaussian random process. Let $\psi(\tau)$ be the autocorrelation function of $x(t)$ and $F(f)$ the one-sided power spectrum. Let f_0 be the bandwidth of $x(t)$ and $F_s = f_s/f_0$ the normalized sampling frequency. The random process $\{x(t)\}$ is assumed to be zero mean. Let b_n be the variance of the nth derivative of $x(t)$ ($n = 1, 2, \cdots$). These numbers (b_n) will be extensively used in the sequel. They are given by the relation

$$b_n = \int_0^{f_0} (2\pi f)^{2n} F(f)\, df. \tag{1}$$

The output signal $y(t)$ follows the input signal $x(t)$ during certain time intervals. Within these time intervals

$$\left| \frac{dx(t)}{dt} \right| < x_0' .$$

The rest of the time $y(t)$ follows segments of straight lines having slope x_0' or $-x_0'$. If t_0 is a time instant at which a transition from the input signal to the straight line segment takes place, we have

$$x'(t_0) = \frac{dx(t_0)}{dt} = x_0' , \qquad x''(t_0) > 0$$

or

$$x'(t_0) = -x_0' , \qquad x''(t_0) < 0. \tag{2}$$

For

$$\begin{aligned} x'(t_0) &= x_0' \\ y(t) &= x(t_0) + (t - t_0)x_0' \qquad t \,\varepsilon\, (t_0 , t_1) \end{aligned} \tag{3}$$

and for

$$\begin{aligned} x'(t_0) &= -\, x_0' \\ y(t) &= x(t_0) - (t - t_0)x_0' \qquad t \,\varepsilon\, (t_0 , t_1), \end{aligned} \tag{4}$$

where t_1 is the smallest time $t_1 > t_0$ for which

$$x(t_1) = y(t_1) = x(t_0) + (t_1 - t_0)x'(t_0). \tag{5}$$

Since the overload noise is defined to be

$$n(t) = x(t) - y(t), \tag{6}$$

the problem boils down to the study of the random process $\{n(t)\}$. We will concentrate on the derivation of an upper bound to the average noise power $\langle n^2(t)\rangle_{av}$ which at the same time is a very good approximation to the noise power itself. Other statistical properties of $n(t)$ can be obtained, but we will only mention them at the conclusion of the paper.

In contrast with straight PCM the evaluation of the overload noise in DPCM systems is not easy. The beginning of a slope overload burst can be defined statistically in a clear manner. Difficulties arise in defining a valid tractable procedure for determining the duration of the burst and its end point (t_1).

As pointed out before two previous attempts have been made to find $\langle n^2(t) \rangle_{av}$.[7,8] One, due to Rice and O'Neal,[8] involves a Taylor series approximation for determining the end point t_1 of the burst valid only for very large x_0', i.e., in a region where slope overload noise is not dominant since it is over-shadowed by the granular part of the quantization error. Another approach to the problem is due to Zetterberg.[7] His approach includes an ingenious way of avoiding the determination of t_1. Unfortunately, his work contains a conceptual error in the averaging procedure. The error resides in his interpretation of continuous conditional probability density functions in the vertical window sense.

A new approach is given here that combines the best features of the two methods. The result is asymptotically equal to that given by Rice and O'Neal for $x_0' \to \infty$. In the region where overload noise is important, the present result is in very good agreement with computer simulation and experiment. As noted above, the technique can also be applied to the determination of other statistical properties of the error random process.

In Section IV, we give a critique of Zetterberg's work. It must be emphasized that Zetterberg's valuable work contains concepts and techniques on which our improved results are based. The wedding of the best in the methods of Rice and Zetterberg is accomplished in our Section V. Theoretical results are compared with computer simulation in Section VI and agreement is seen to be excellent.* Finally, in Section VII we indicate how other statistical properties of $n(t)$ may be obtained by utilizing some of the approaches developed herein.

IV. CRITIQUE OF ZETTERBERG'S APPROACH

Using an argument based on the ergodicity of the random process $\{x(t)\}$ Zetterberg[7] states that

$$\langle n^2(t) \rangle = \langle n^2(t) \rangle_{av} = S_{x_0'} \left\langle \int_0^{s_1} n^2(t_0 + s) \, ds \right\rangle^{\dagger}, \tag{7}$$

where

$$\begin{aligned} s &= t - t_0 \\ s_1 &= t_1 - t_0 \end{aligned} \tag{8}$$

and $S_{x'_0}$ is the average number of points of transition per second. In what follows, we summarize his procedure deviating slightly from his notation and arguments to clarify a few points. Consider the ensemble

* Comparison with experiments will be given in another paper.[11]

† $\langle \ \rangle$ denotes ensemble average and $\langle \ \rangle_{av}$ time average.

of the sequences $\{t_i(\zeta)\}$,† $i = 0, \pm 1, \pm 2, \cdots$, of time instants such that $x'(t_i(\zeta), \zeta) = x_0'$ and $x''(t_i(\zeta), \zeta) > 0$ or $x'(t_i(\zeta), \zeta) = -x_0'$ and $x''(t_i(\zeta), \zeta) \pm 0$ for $i = 0, \pm 1, \pm 2, \cdots$. Zetterberg avoids the definition of the end point of the burst by defining a sequence of random processes $\{m_i(s, \zeta)\}$ (see Fig. 2), with index corresponding to the above time instants, in the following way:

$$m_i(s, \zeta) = [x(t_i(\zeta) + s, \zeta) - x(t_i(\zeta), \zeta) - x_0's]\cdot\mu(s) \cdot\mu(x(t_i(\zeta) + s, \zeta) - x(t_i(\zeta), \zeta) - x_0's) \quad (9)$$

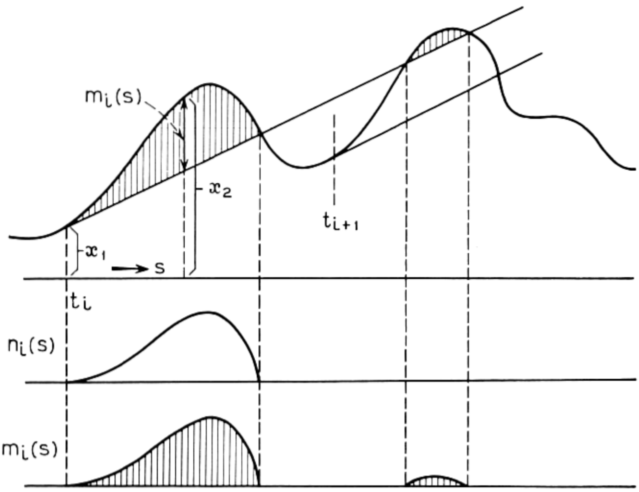

Fig. 2 — An overload noise "burst" $n_i(s)$ and the approximating function $m_i(s)$.

for

$$x'(t_i(\zeta), \zeta) = x_0' \quad \text{and} \quad x''(t_i(\zeta), \zeta) > 0,$$

and

$$m_i(s, \zeta) = [x(t_i + s, \zeta) - x(t_i, \zeta) + x_0's]\mu(s) \cdot\mu(-x(t_i + s, \zeta) + x(t_i, \zeta) - x_0's) \quad (10)$$

for

$$x'(t_i, \zeta) = -x_0', \qquad x''(t_i, \zeta) < 0.$$

($\mu(s)$ is the unit step.)

† For clarity we show in this paragraph the input random process as generated by an experiment with outcome ζ.

For brevity, we drop the index i and the argument ζ. We denote, as before, the beginning of a burst by t_0, the end by t_1 and by s_1 its duration, such that for a "positive burst," i.e., $x'(t_0) > 0$ we have

$$m(s) = [x(t_0 + s) - x(t_0) - x_0's]\mu(s)\mu[x(t_0 + s) - x(t_0) - x_0's]. \quad (11)$$

In general, as shown in Fig. 2, $m(s)$ contains not only noise burst corresponding to the transition point t_0 but also some additional "bursts" cut from the function $x(t)$ by the straight line starting at the point $(t_0, x(t_0))$ and having slope x_0'. This makes

$$\int_0^\infty m^2(s)\, ds \geqq \int_0^{s_1} n^2(t_0 + s)\, ds \quad (12)$$

and

$$\left\langle \int_0^\infty m^2(s)\, ds \right\rangle \geqq \left\langle \int_0^{s_1} n^2(t_0 + s)\, ds \right\rangle. \quad (13)$$

For sufficiently large values of $x_0'/\sqrt{b_1}$, (b_1 is the variance of $x'(t)$), however, the probability is small that the situation depicted in Fig. 2 will occur. Also, generally the additional sections in $m(s)$ occur in reduced amplitude and the squaring reduces the introduced error still further. Denote by $R_{x'_0}$ the average number of points for which

$$x'(t_i) = x_0', \qquad x''(t_i) > 0$$

or

$$x'(t_i) = -x_0', \qquad x''(t_i) < 0.$$

It is seen from Fig. 3 that $R_{x'_0} \geqq S_{x'_0}$, since a burst cannot start when another is taking place even if the conditions on the first and second derivative are satisfied. But again, for sufficiently large $x_0'/\sqrt{b_1}$, $R_{x'_0}$ is a good estimate of $S_{x'_0}$. It follows from the discussion above that the quantity

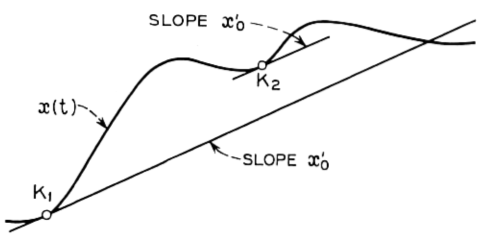

Fig. 3 — $K_1, K_2 \subset R_{x'_0}$ whereas $K_1 \subset S_{x'_0}$ but $K_2 \not\subset S_{x_0'}$.

$$R_{x'_0}\left\langle \int_0^\infty m^2(s)\, ds \right\rangle \tag{14}$$

is an upper bound and actually under certain conditions a good estimate of $\langle n^2(t) \rangle_{av}$.

When one defines

$$Q_{x'_0} \equiv \left\langle \int_0^\infty m^2(s)\, ds \right\rangle = \int_0^\infty \langle m^2(s) \rangle\, ds, \tag{15}$$

where the equality above holds provided that the integrals exist, then

$$N_0 = R_{x'_0} Q_{x'_0} \tag{16}$$

is Zetterberg's upper bound to the overload noise.

At this point Zetterberg takes the ensemble average $\langle m^2(s) \rangle$ in the following way:

$$\langle m^2(s) \rangle = \int_{-\infty}^{\infty} \int_{x_1 + x'_0 s}^{\infty} (x_2 - x_1 - x'_0 s)^2 p(x_1, x_2 \mid \dot{x}_1 = x'_0 ; s)\, dx_2\, dx_1$$

$$+ \int_{-\infty}^{\infty} \int_{-\infty}^{x_1 - x'_0 s} (x_2 - x_1 + x'_0 s)^2 p(x_1, x_2 \mid \dot{x}_1 = -x'_0 ; s)\, dx_2\, dx_1, \tag{17}$$

where $p(x_1, x_2 \mid \dot{x}_1 ; s)$ is the conditional joint probability density function of the random variables $X_1 = x(t_0) X_2 = x(t_0 + s)$ given the value of the random variable $\dot{X}_1 = dx(t_0)/dt$, understood in the vectical window sense. It turns out that the averaging procedure as described by (17) is wrong for two reasons:

(*i*) The joint probability density of X_1 and X_2 should be subject not only to the condition $\dot{X}_1 = dx(t_0)/dt = \pm x'_0$, but also to the condition $\ddot{X}_1 = d^2x(t_0)/dt^2 \gtrless 0$. If we do not impose the above condition on the second derivative at the beginning of the burst, then an $m(s)$ of the form depicted in Fig. 4 would erroneously add to the approximation of the average slope overload noise power per burst.

(*ii*) It is known[12] that conditional probability densities must be treated with great caution. M. Kac and D. Slepian in Ref. 12 have illustrated with examples how different the expression for conditional probability densities might be, depending on the way we understand them. From the ensemble viewpoint quantities like the *conditional joint probability density for the rv* $X_1 = x(t_0)$ *and* $X_2 = x(t_0 + s)$ *given that* $\dot{X}_1 = dx(t_0)/dt = x'_0$ are not clearly defined since the set of sample functions with $dx(t_0)/dt = x'_0$ has probability zero. We can of course give meaning to the conditional densities by means of limiting proce-

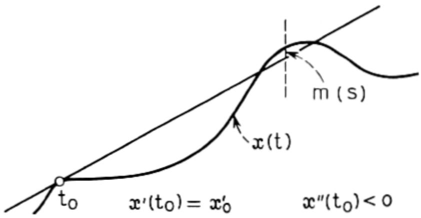

Fig. 4 — Consequences of not requiring $x''(t_o)$ to be positive.

dures. As Kac and Slepian point out, a condition like $dx(t_0)/dt = x_0'$ would be replaced by a condition, A, with nonzero probability, depending on parameters, such that when these parameters tend to limiting values A becomes the condition $dx(t_0)/dt = x_0'$. It turns out that, in general, the resulting conditional probability density function depends on the manner in which A approaches the condition $dx(t_0)/dt = x_0'$. Two window conditions are considered below.

(*i*) A *vertical window* condition is a condition of the form

$$x_0' < \frac{dx(t_0)}{dt} < x_0' + \delta. \tag{18}$$

Then, with reference to Fig. 5(a),

$$p(x_1, x_2 \mid \dot{x}(t_0) = x_0'; s)_{vw}$$

$$= \lim_{\delta \to 0} \frac{\int_{x'_0}^{x'_0+\delta} p(x_1, x_2, \dot{x}_1; s)\, d\dot{x}_1}{\int_{x'_0}^{x'_0+\delta} p(\dot{x}_1)\, d\dot{x}_1} = \frac{p(x_1, x_2, x_0'; s)}{p(x_0')}, \tag{19}$$

where $p(x_1, x_2, \dot{x}_1; s)$ is the joint probability density function of the random variables $X_1 = x(t_0)$ $X_2 = x(t_0 + s)$ and $\dot{X}_1 = dx_1(t_0)/dt$ and $p(\dot{x}_1)$ is the probability density function of the derivative $\dot{X}_1 = dx(t_0)/dt$. Note that the time argument of the density functions above are written taking into account the stationarity of the input process $\{x(t)\}$.

(*ii*) A *horizontal window* condition is a condition of the form $dx(t)/dt = x_0'$ for some t such that

$$t_0 \leqq t \leqq t_0 + \delta.$$

Then,

$$p(x_1, x_2 \mid x'(t_0) = x_0'; s)_{hw}$$

$$= \lim_{\delta \to 0} \left\{ \int_0^\infty dx_1'' \int_{x'_0 - x''_1 \delta}^{x'_0} p(x_1, x_2, x_1', x_1''; s)\, dx_1' \right.$$
$$\left. + \int_{-\infty}^0 dx_1'' \int_{x'_0}^{x'_0 - x''_1 \delta} p(x_1, x_2, x_1', x_1''; s)\, dx_1' \right\}$$
$$\cdot \left\{ \int_0^\infty dx_1'' \int_{x'_0 - x''_1 \delta}^{x'_0} p(x_1', x_1'')\, dx_1' \right.$$
$$\left. + \int_{-\infty}^0 dx_1'' \int_{x'_0}^{x'_0 - x''_1 \delta} p(x_1', x_1'')\, dx_1' \right\}^{-1}$$
$$= \frac{\displaystyle\int_{-\infty}^{\infty} | x_1'' | \, p(x_1, x_2, x_0', x_1''; s)\, dx_1''}{\displaystyle\int_{-\infty}^{\infty} | x_1'' | \, p(x_0', x_1'')\, dx_1''}, \tag{20}$$

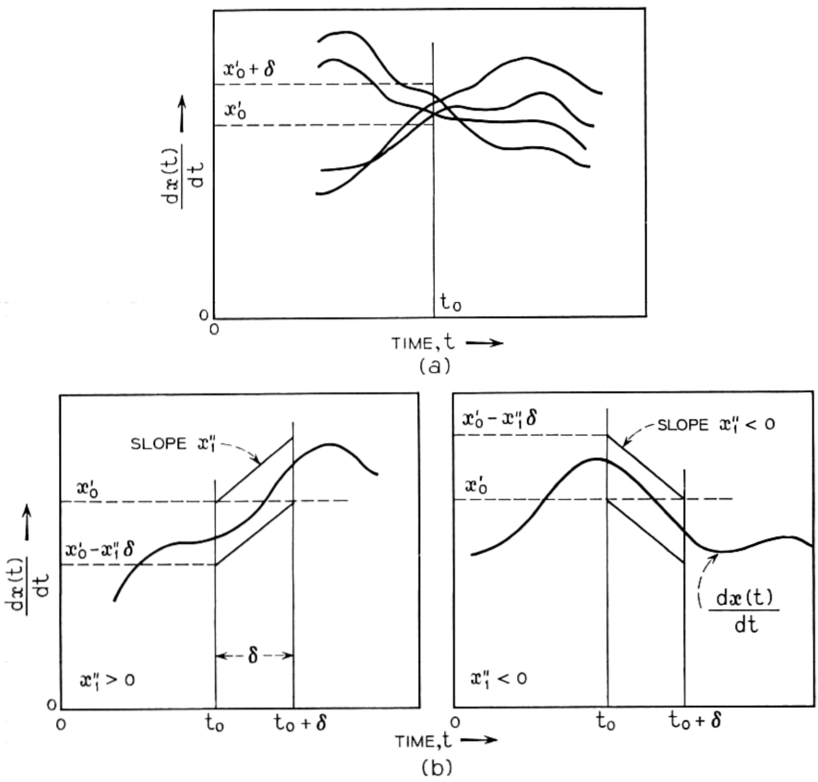

Fig. 5 — (a) A "vertical window" condition. (b) A "horizontal window" condition.

where $p(x_1, x_2, x_1', x_1''; s)$ is the joint probability density function of the random variables $X_1 = x(t_0)$, $X_2 = x(t_0 + s)$, $X_1' = dx(t_0)/dt$ and $X_1'' = d^2x(t_0)/dt^2$ and $p(x_1', x_1'')$ the joint probability density of the random variables X_1' and X_1''. Equation (20) follows from the fact that the "horizontal window" condition is equivalent (within first order in small quantities and for a given second derivative, say $x_1'' > 0$) to $x_0' - x_1'' \delta \leqq dx(t_0)/dt \leqq x_0'$. For $x_1'' < 0$ condition A is satisfied only if $x_0' \leqq dx(t_0)/dt \leqq x_0' - x_1'' \delta$ [see Fig. 5(b)].

Consider now according to Kac and Slepian an "empirical or time derived joint probability density for x_1 and x_2 given that $x'(t_0) = x_0'$" resulting from taking one sample function of the process and observing the values of $x(t)$ and $x(t + s)$ at each value of t for which $dx(t)/dt = x_0'$ (s is of course a given number). It turns out that the empirical or time derived density thus obtained is equal to the conditional density defined in the horizontal window sense.

Note that if we impose the additional condition $x''(t_0) > 0$ we have

$$p(x_1, x_2 \mid x'(t_0) = x_0', x''(t_0) > 0; s)_{hw} = \frac{\int_0^\infty x_1'' p(x_1, x_2, x_0', x_1''; s)\, dx_1''}{\int_0^\infty x_1'' p(x_0', x_1'')\, dx_1''}. \tag{20a}$$

It will become clearer in a later section where the averaging is done carefully that one should interpret the conditional probability densities in the horizontal window sense.

Zetterberg defines the conditional densities in the integrals of (17) in the vertical window sense; this follows from the way that he computes them.

But let us overlook for a moment these shortcomings of Ref. 7 and continue with the approach presented there. For a Gaussian input process $\{x(t)\}$ Zetterberg derives the following expression for $Q_{x'_0}$.

$$Q_{x'_0} = \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty k(s) u^2 \exp\{-\tfrac{1}{2}(u + g(s))^2\}\, du\, ds, \tag{21}$$

where

$$k(s) = 2(\psi_0 - \psi(s)) - \frac{1}{b_1}\left\{\frac{d\psi(s)}{ds}\right\}^2 \tag{22}$$

$$g(s) = \frac{x_0' s}{\sqrt{k(s)}}\left\{1 + \frac{1}{b_1 s}\frac{d\psi(s)}{ds}\right\}. \tag{23}$$

$b_n (n = 1, 2, \cdots)$ is defined in (1) and

$$\psi_0 = \psi(0).$$

The following asymptotic expressions are valid (noted in Ref. 7). For

$$s \to 0$$

$$k(s) \approx \frac{b_2 s^4}{4} \tag{24}$$

$$g(s) \approx x_0' s \frac{\sqrt{b_2}}{3b_1}. \tag{25}$$

For

$$s \to \infty$$

$$k(s) \approx 2\psi_0 \tag{26}$$

$$g(s) \approx \frac{x_0' s}{\sqrt{2\psi_0}}. \tag{27}$$

(Note the meaning of the symbol $\approx$ as used here:

$$x(s) \approx y(s) \quad \text{for} \quad s \to s_0$$

if

$$\lim_{s \to s_0} \frac{x(s)}{y(s)} = 1.)$$

An approximate calculation of the integral for $Q_{x'_0}$ as given by (21) is based by Zetterberg on the following simplifications. He uses the asymptotic formula for $g(s)$ for small s. This is a justificable approximation since the smaller values of $g(s)$ are more important in the evaluation of the integral (21) and in any case the slopes of $g(s)$ for $s \to 0$ and $s \to \infty$ do not differ drastically.

For $k(s)$ he sets

$$k(s) = \begin{cases} \dfrac{b_2 s^4}{4}, & \text{for} \quad s < s_1 \\ 2\psi_0 , & \text{for} \quad s > s_1 , \end{cases} \tag{28}$$

where s_1 is determined such that

$$\frac{b_2 s_1^4}{4} = 2\psi_0 ,$$

i.e.,

$$s_1 = \sqrt[4]{\frac{8\psi_0}{b_2}}. \tag{29}$$

The evaluation of the integrals (21) for $Q_{x'_0}$ are not correct as reported in Ref. 7.* In Appendix A the evaluation of the integral is made and the result is [see (90)]

$$Q_{x'_0} = \sqrt{\frac{2}{\pi}} \cdot \frac{4}{35} \cdot \frac{b_1^{5/2}}{b_2^{3/2}} \left(\frac{3b_1^{1/2}}{x'_0} \right)^5 A(\lambda), \tag{30}$$

where

$$\lambda = \frac{2}{3} \frac{x'_0}{b_1} \sqrt[4]{\frac{b_2\psi_0}{2}} \tag{31}$$

and

$$A(\lambda) = 1 + P(\lambda)e^{-\lambda^2/2} - Q(\lambda)\Phi(\lambda) \tag{32}$$

with

$$P(\lambda) = \frac{17}{24}\lambda^6 + \frac{4}{3}\lambda^4 - \frac{\lambda^2}{2} - 1 \tag{33}$$

$$Q(\lambda) = \frac{17}{24}\lambda^7 + \frac{35}{16}\lambda^5 \tag{34}$$

$$\Phi(\lambda) = \int_\lambda^\infty e^{-v^2/2}\, dv. \tag{35}$$

For the number $R_{x'_0}$ both Rice and Zetterberg agree since the formula comes from one of Rice's classic papers;[13] namely

$$R_{x'_0} = \frac{1}{\pi} \left(\frac{b_2}{b_1} \right)^{\frac{1}{2}} \exp\left(-\frac{x_0'^2}{2b_1} \right). \tag{36}$$

Therefore, the overload noise according to Zetterberg is

$$N_{0,Z} = R_{x'_0} Q_{x'_0} = \frac{4\sqrt{2}}{35\pi^{\frac{3}{2}}} \left(\frac{b_1^2}{b_2} \right) \left(\frac{3b_1^{\frac{1}{2}}}{x'_0} \right)^5 A(\lambda) \exp\left(-\frac{x_0'^2}{2b_1} \right). \tag{37}$$

V. OVERLOAD NOISE—THE NEW APPROACH†

In this section we will determine the overload noise using an approach which combines the more accurate model of Zetterberg with the correct averaging procedure given by Rice.

* Zetterberg's expression corresponding to the $A(\lambda)$ given in (32) was not positive for all values of λ — clearly a nonphysical situation.
† In the present section we assume, without loss of generality, $\psi(0) = \sigma^2 = 1$.

This formulation proceeds as follows:

(*i*) The average noise energy per burst is approximated by ave $\{\int_0^\infty m^2(s)\, ds\}$, as per Zetterberg. This approach avoids Rice's approximation of $n(t)$ during a burst with a third-order polynomial and does not refer to the end point of the burst. On the other hand it yields clearly an upper bound on the overload noise, whereas in Rice's approach the sense of approximation is not clear.

(*ii*) The averaging process is done the "correct" physical way in the following paragraph. This paragraph is a paraphrasing of the lucid lecture given to us by Rice.

Consider a very long record of the input signal (Fig. 6) of time duration NT, where N is a very large positive integer and T is an extremely long time interval compared with the time unit. Mark on this time record of the input signal all points for which a positive burst begins—all points for which the derivative $dx(t)/dt$ increases through x_0'. Mark on the record of the signal all time instants s time units following the beginnings of the bursts and measure the value of $m(s)$. Let K be the average number of "positive" bursts per unit time. Then the total number of "positive" bursts in the time interval NT will be: NTK. The average value of $m^2(s)$ over all these positive bursts will be

$$\text{ave}\,\{m^2(s)\} = \frac{\sum_{i=1}^{NTK} \{m_i^2(s)\}}{NTK}. \tag{38}$$

Now break up the total signal record into N equal records of duration T and imagine them placed one below the other such that their beginnings lie on the same vertical line as shown in Fig. 6. Divide the time interval into $T/\Delta t$ equal small time intervals of length Δt and imagine vertical lines drawn at the dividing points. Consider a vertical strip of width Δt around time t and sum up the values of $m^2(s)$ over all members of the ensemble that have a "positive" burst which began in the time interval of duration Δt and around the time point $t - s = t_0$, i.e., s time units before t. This sum is independent of the vertical strip we consider and it is denoted by $\sum_{\Delta t} m^2(s)$.

If follows that

$$\sum_{i=1}^{NTK} m_i^2(s) = \frac{T}{\Delta t} \sum_{\Delta t} m^2(s). \tag{39}$$

When a member $x(t)$ is picked at random from the ensemble of the N $x(t)$'s we denote by p the chance that the following three things happen:

(*i*) A "positive" burst begins in the interval $(t - s,\ t - s + \Delta t)$ or equivalently the derivative $dx(t)/dt$ increases through x_0' during $t - s,\ t - s + \Delta t$.

(*ii*) The slope of $dx(t)/dt$ at $t_0 = t - s$ lies between x_1'' and $x_1'' + dx_1''$.

(*iii*) In the time interval $(t,\ t + \Delta t)$, $m(s)$ lies between $m(s)$ and $m(s) + d(m(s))$. Since $m(s) = x(t) - x(t - s) - x_0's$, this is equivalent to asking that $X_1 = x(t - s)$ lie between x_1 and $x_1 + dx_1$ where x_1 is any real number and $X_2 = x(t)$ lie between x_2 and $x_2 + dx_2$ where $x_2 \,\varepsilon\, (x_1 + x_0's,\ \infty)$.

Then we have

$$p = x_1''p(x_1\ ,\ x_2\ ,\ x_0'\ ,\ x_1''\ ;\ s)\ dx_1\ dx_2\ dx_1''\ \Delta t,$$

where

$$p(x_1\ ,\ x_2\ ,\ x_0'\ ,\ x_1''\ ;\ s)$$

is the joint probability density function of the random variables

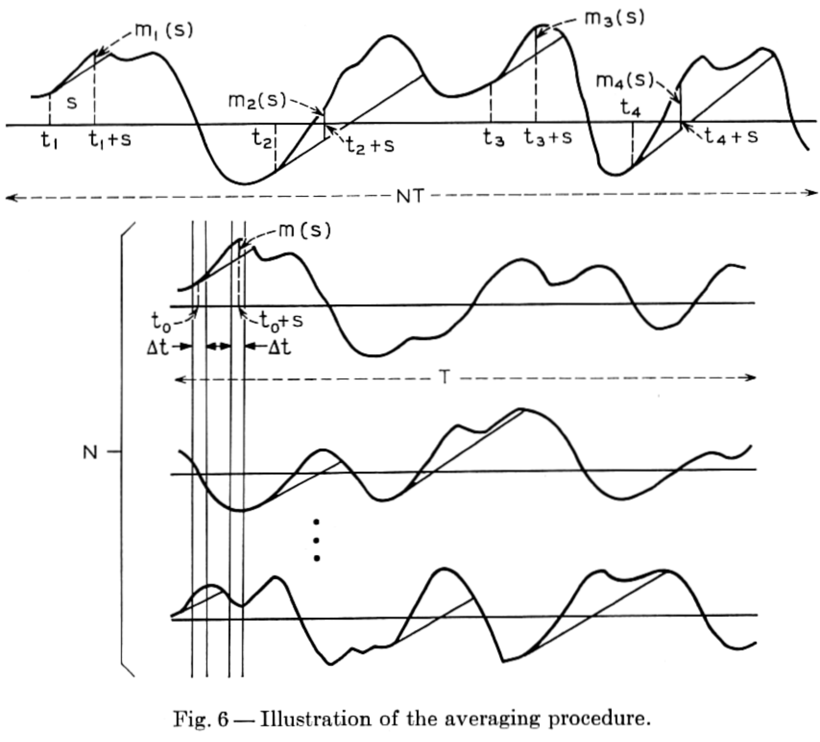

Fig. 6 — Illustration of the averaging procedure.

$$\begin{aligned} X_1 &= x(t - s) = x(t_0) \\ X_2 &= x(t) \\ X_1' &= \left.\frac{dx(t)}{dt}\right|_{t=t_0} \\ X_1'' &= \left.\frac{d^2x(t)}{dt^2}\right|_{t=t_0}. \end{aligned} \tag{40}$$

For an extremely large number N of members of the ensemble of $x(t)$'s the number of members satisfying the three conditions above will be

$$pN = (N\Delta t)x_1''p(x_1, x_2, x_0', x_1''; s)\, dx_2\, dx_1\, dx_1'' \tag{41}$$

and therefore,

$$\sum_{\Delta t} m^2(s) = \int_0^\infty \int_{-\infty}^\infty \int_{x_1+x_0's}^\infty (x_2 - x_1 - x_0's)^2(N\Delta t)x_1'' \cdot p(x_1, x_2, x_0', x_1''; s)\, dx_2\, dx_1\, dx_1''. \tag{42}$$

Consequently,

$$\begin{aligned} \text{ave}\,\{m^2(s)\} &= \frac{\frac{T}{\Delta t}\sum_{\Delta t} m^2(s)}{NTK} \\ &= \frac{1}{K}\int_0^\infty \int_{-\infty}^\infty \int_{x_1+x_0's}^\infty (x_2 - x_1 - x_0's)^2 x_1'' \cdot p(x_1, x_2, x_0', x_1''; s)\, dx_2\, dx_1\, dx_1''. \end{aligned} \tag{43}$$

Make the change of variables

$$x_2 = x_1 + x_0's + u.$$

Then

$$\text{ave}\,\{m^2(s)\} = \frac{1}{K}\int_0^\infty \int_0^\infty \int_{-\infty}^\infty x_1''u^2p(x_1, x_1 + x_0's + u, x_0', x_1''; s) \cdot dx_1\, dx_1''\, du. \tag{44}$$

Remark:

Note that[13]

$$K = \int_0^\infty x''p(x_0', x'')\, dx'', \tag{45}$$

where $p(x', x'')$ is the joint probability density function of the random variables $X' = dx(t)/dt$ and $X'' = d^2x(t)/dt^2$. Using (20a), therefore, and substituting into (44) we can write

$$\text{ave } \{m^2(s)\}$$
$$= \int_0^\infty \int_{-\infty}^\infty u^2 p_{hw}(x_1 , x_1 + x_0's + u \mid x'(t_0) = x_0' , x''(t_0) > 0; s)$$
$$\cdot dx_1 \, du. \qquad (46)$$

Hence, the present "physical" averaging procedure amounts to taking conditional densities in the horizontal window sense.

$$K = \frac{R_{x'_0}}{2} = \frac{1}{2\pi} \left(\frac{b_2}{b_1}\right)^{\frac{1}{2}} \exp \left\{ -\frac{x_0'^2}{2b_1} \right\}. \qquad (47)$$

On the other hand,

$$p(x_1 , x_1 + x_0's + u, x_0' , x_1'' ; s) = \frac{1}{(2\pi)^2 \mid M \mid^{\frac{1}{2}}} \exp \{ -\tfrac{1}{2}\mathbf{x}^t M^{-1} \mathbf{x} \}, \qquad (48)$$

where

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_1 + x_0's + u \\ x_0' \\ x_1'' \end{bmatrix}$$

and $\mathbf{x}^t$ is the transposed vector. M is the 4×4 cross-correlation matrix: $\{\mu_{ij}\}$, $i, j = 1, 2, 3, 4$ and it is given in Appendix B. $\mid M \mid$ is the determinant of M.

After some very lengthy algebraic manipulations which are summarized in Appendix B we find [see (137)]

$$\text{ave } \{m^2(s)\} = \frac{k_1(s)}{\sqrt{2\pi}} \int_0^\infty z^2 \exp \left[-\tfrac{1}{2}(z + g_1(s))^2 \right] \frac{\sqrt{1 - \lambda^2(s)}}{\lambda(s)} \varphi(\xi) \, dz \qquad (49)$$

where $k_1(s)$, $g_1(s)$, and $\lambda(s)$ are complicated functions of s, expressed in terms of the signal autocorrelation function and its derivatives. They are given in Appendix B, (138), (135), and (128), respectively. Note that they do not coincide with Zetterberg's $k(s)$ and $g(s)$ as given by (22) and (23). Other symbols in (49) are defined below.

$$\xi = \frac{\lambda(s)}{\sqrt{1 - \lambda^2(s)}} (z + g_1(s)) \qquad (50)$$

$$\varphi(\xi) = e^{-\xi^2/2} + \xi\Phi(-\xi) \tag{51}$$

with

$$\Phi(x) = \int_x^\infty e^{-z^2/2}\, dz. \tag{52}$$

Consequently,

$$Q_{x'_0} = \text{ave}\left\{\int_0^\infty m^2(s)\, ds\right\}$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^\infty k_1(s) \int_0^\infty z^2 \exp\left[-\tfrac{1}{2}(z+g_1(s))^2\right] \frac{\sqrt{1-\lambda^2(s)}}{\lambda(s)}\, \varphi(\xi)\, dz\, ds. \tag{53}$$

Up to this point we have made no approximations beyond those inherent in the initial model. In the following, additional approximations are required to evaluate (53). In Appendix C it is seen that at $s = 0$, $z = 0$

$$\xi = \xi_0 = \frac{b_2}{\sqrt{b_1 b_3 - b_2^2}} \cdot \frac{x_0'}{\sqrt{b_1}} \tag{54}$$

and for s and z large

$$\xi \cong \gamma_0 z + \delta_0 s,$$

where γ_0 and δ_0 are positive constants defined in Appendix C. The function $\varphi(\xi)$ is plotted in Fig. 7. It is easily seen that, for $\xi > 0$

$$\frac{\varphi(\xi)}{\sqrt{2\pi}\,\xi} = 1 + \frac{e^{-\xi^2/2}}{\xi\sqrt{2\pi}} \{1 - \xi e^{\xi^2/2}\Phi(\xi)\}.$$

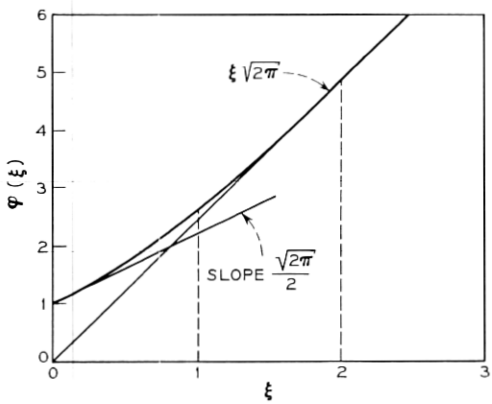

Fig. 7 — The function $\varphi(\xi)$.

For $\xi > 0$ we have

$$\xi e^{\xi^2/2}\Phi(\xi) < 1$$

and for ξ large

$$1 - \xi e^{\xi^2/2}\Phi(\xi) = \frac{1}{\xi^2}\left(1 - \frac{1}{\xi^2} + \cdots\right).$$

Hence, for ξ large

$$\frac{\varphi(\xi)}{\sqrt{2\pi}\,\xi} = 1 + \frac{e^{-\xi^2/2}}{\xi^3\sqrt{2\pi}}\left(1 - \frac{1}{\xi^2} + \cdots\right).$$

The derivative of $\varphi(\xi)$ is very close to $\sqrt{2\pi}$ for large ξ. Namely,

$$\frac{\varphi'(\xi)}{\sqrt{2\pi}} = 1 - \frac{\Phi(\xi)}{\sqrt{2\pi}}.$$

Note also that

$$\frac{\varphi(1)}{\sqrt{2\pi}} \cong 1.08, \qquad \frac{\varphi(2)}{2\sqrt{2\pi}} = 1.004, \qquad \frac{\varphi(3)}{3\sqrt{2\pi}} \cong 1.0002.$$

Hence, for $\xi \geqq 2$ the approximation

$$\varphi(\xi) \cong \xi\sqrt{2\pi} \tag{55}$$

is very good (error less than 1 percent). The approximations hold good even for ξ somewhat larger than 1, as seen in the calculations above. So that if $\xi_0 > 1$, as given by (54), it is justifiable, for the sake of simplicity, to substitute $\xi\sqrt{2\pi}$ for $\varphi(\xi)$ in the integral (53).

Another interesting comment here is that ξ_0, as given in (54), is equal to the ratio of the absolute value of the mean of the third derivative of the input process $x(t)$ over its standard deviation. Indeed, the mean of $x''(t_0)$, where t_0 is the beginning of a positive burst, is $-b_2x_0'/b_1$ and the standard deviation $\sqrt{\mathcal{B}/b_1}$, where

$$\mathcal{B} = \sqrt{b_1b_3 - b_2^2}$$

[see Rice's comments above (18) of Ref. 8]. Rice assumed that this ratio is large compared to unity. Here the approximation is good even with ξ_0 close to 1. With the approximation introduced in (55) and using (50) we get

$$\frac{\sqrt{1 - \lambda^2(s)}}{\lambda(s)}\,\varphi(\xi) \cong (z + g_1(s))\sqrt{2\pi}$$

and consequently,

$$Q_{x'_0} = \int_0^\infty k_1(s) \int_0^\infty z^2(z + g_1(s)) \exp\left[-\tfrac{1}{2}(z + g_1(s))^2\right] dz\, ds. \tag{56}$$

Integrating in the inside integral by parts we get the simplified expression

$$Q_{x'_0} = 2 \int_0^\infty k_1(s) \int_0^\infty z \exp\left[-\tfrac{1}{2}(z + g_1(s))^2\right] dz\, ds. \tag{57}$$

5.1 *Approximate Evaluation of the Noise Energy per Burst*

The following asymptotic expressions for $k_1(s)$ and $g_1(s)$ are found in Appendix C, for s small

$$k_1(s) \cong \frac{b_2 s^4}{4} \tag{58}$$

$$g_1(s) \cong \frac{\sqrt{b_2}}{3b_1} x'_0 s \tag{59}$$

and for large s

$$k_1(s) \approx k_\infty = \frac{b_1\sqrt{2}}{\sqrt{b_2}} \tag{60}$$

$$g_1(s) \approx \frac{x'_0 s}{\sqrt{2}}. \tag{61}$$

The function $g(s)$ has an approximately linear variation for small and large values of s.

To calculate $Q_{x'_0}$ according to (57) we will use essentially the same approach used by Zetterberg; namely, use the asymptotic expression for $g_1(s)$ near 0 [see (59)] and for $k(s)$ the expression (58) when $s \leqq s_2$ and (60) when $s \geqq s_2$. Here, s_2 is the value of s for which the two expressions are equal; namely,

$$s_2 = \sqrt[4]{\frac{4k_\infty}{b_2}} = \frac{2^{5/8} b_1^{1/4}}{b_2^{3/8}} \tag{62}$$

and

$$g_1(s) = \alpha s,$$

where

$$\alpha = \frac{\sqrt{b_2}}{3b_1} x'_0 \tag{63}$$

set

$$\chi = \alpha s_2 = \frac{\sqrt{2}}{3}\left(\frac{2b_2}{b_1^2}\right)^{1/8} \frac{x_0'}{\sqrt{b_1}}. \tag{64}$$

The nature of approximation of the functions $k_1(s)$ and $g_1(s)$ by their values for large and small s is indicated in the Figs. 8 and 9.

The evaluation of the integral (57) for $Q_{x'_0}$ is done in Appendix D and the result is

$$Q_{x_0'} = \frac{\sqrt{2\pi}}{8}\left(\frac{3b_1}{x_0'}\right)^5 b_2^{-\frac{3}{2}} A(\chi), \tag{65}$$

where

$$A(\chi) = 1 - \frac{e^{-\chi^2/2}}{\sqrt{2\pi}} P(\chi) + \frac{1}{\sqrt{2\pi}} \Phi(\chi)Q(\chi) \tag{66}$$

$$P(\chi) = 2\left(\frac{16}{15}\chi^5 + \frac{1}{3}\chi^3 + \chi\right)$$

$$Q(\chi) = 2\left(\frac{16}{15}\chi^6 + \chi^4 - 1\right) \tag{67}$$

$$\Phi(\chi) = \int_\chi^\infty e^{-z^2/2}\, dz.$$

The average overload noise power is obtained by multiplying $Q_{x'_0}$ by the average number of bursts per unit time, given approximately in (36).

The average overload noise is, therefore,

$$N_0 = \frac{1}{4\sqrt{2\pi}}\left(\frac{b_1^2}{b_2}\right)\left(\frac{3b_1^{\frac{1}{2}}}{x_0'}\right)^5 \exp\left(-\frac{x_0'^2}{2b_1}\right) A(\chi), \tag{68}$$

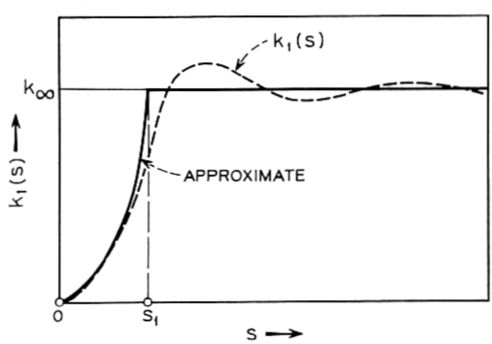

Fig. 8—$k_1(s)$ and the approximation used for the evaluation of $Q_{x'_0}$.

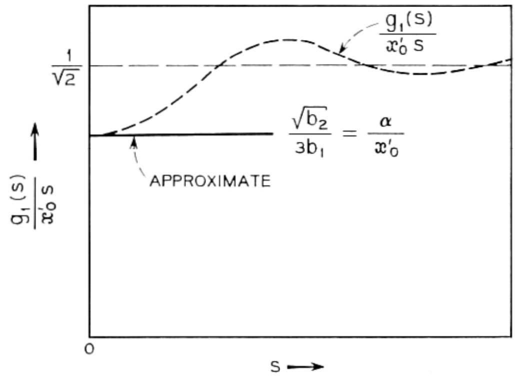

Fig. 9 — Approximation to $g_1(s)$ used for the evaluation of $Q_{x'_0}$.

where χ and $A(\chi)$ are given in (64) and (66), respectively. This result is equal to Rice's result [see (22), Ref. 8] times $A(\chi)$. For χ large compared to unity $A(\chi)$ is very close to 1 and thus, in this case (equivalent to x_0' being large compared to $\sqrt{b_1}$) the two results are identical. This is very interesting when we note that the route taken in the two approaches differ markedly.

The factor $A(\chi)$, for $\chi > 0$ is a positive monotonically increasing function of χ varying between 0 and 1. The function $A(\chi)$ is studied in Appendix E and $-10 \log_{10} A(\chi)$ is plotted in Fig. 10.

VI. COMPARISON WITH COMPUTER SIMULATION AND EXPERIMENTS

The new formula for the average slope overload noise power gives results for both, flat low-pass Gaussian, and band-limited RC Gaussian input signals, that agree in a very satisfactory manner with O'Neal's[8] computer simulation. For flat low-pass Gaussian input signals we have

$$b_1 = \frac{(2\pi f_0)^2}{3}$$

$$b_2 = \frac{(2\pi f_0)^4}{5}.$$

Using (64) we get, in this case,

$$\chi = \frac{(3.6)^{\frac{1}{2}}}{\pi\sqrt{6}}(kF_s) \cong 0.153(kF_s)$$

so that for $kF_s = 2$, 4, and 8 we have, respectively,

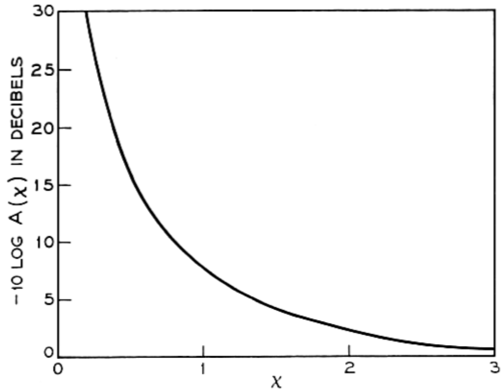

Fig. 10 — The function $-10 \log_{10} A(\chi)$.

$$\chi_1 = 0.306 \qquad \chi_2 = 0.612 \qquad \chi_3 = 1.224$$

$$A(\chi_1) \cong 5.3 \times 10^{-3} \qquad A(\chi_2) = 4.95 \times 10^{-2} \qquad A(\chi_3) \cong 0.270$$

and the corresponding corrections in O'Neal's curves (Fig. 4 of Ref. 8) would be

$$-10 \log_{10} A(\chi_1) \cong 23 \text{ dB}$$

$$-10 \log_{10} A(\chi_2) \cong 13 \text{ dB}$$

$$-10 \log_{10} A(\chi_3) \cong 5.7 \text{ dB}.$$

With these significant corrections the present analytical points pass through the computer simulation points, as seen in Fig. 11. Note that the slope overload noise as defined depends only on (kF_s) and not F_s.

Excellent agreement with computer simulation also occurs for RC shaped bandlimited input signals. For RC-shaped signals with spectrum given by (6) of Ref. 8 we have

$$b_1 = \frac{2\pi f_0 \alpha}{\tan^{-1}\left(\frac{2\pi f_0}{\alpha}\right)} - \alpha^2$$

$$b_2 = \frac{(2\pi f_0)^3 \alpha - 6\pi f_0 \alpha^3}{3 \tan^{-1}\left(\frac{2\pi f_0}{\alpha}\right)} + \alpha^4$$

so that for $\alpha = 0.25 f_0 (=1/RC)$

$$b_1 \cong 0.94 f_0^2$$

$$b_2 \cong 13.2 f_0^4 .$$

And from (64)

$$\chi \cong 0.744(kF_s)$$

so that for $kF_s = 1$ and 2 we have, respectively, $\chi_1 = 0.744$ and $\chi_2 = 1.488$ yielding a correction to Rice's result of about 10.6 and 4.2 dB, respectively. A comparison with Fig. 5 of Ref. 8, reveals the agreement with computer simulation.

For RC-shaped signals (Gaussian and bandlimited) with $\alpha = 0.068$ [corresponding roughly to the envelope of a black and white entertainment TV signal (FCC standard)]

$$b_1 = 0.267 f_0^2$$

$$b_2 = 3.57 f_0^4$$

$$\chi = 1.62(kF_s)$$

so that for $kF_s = \frac{1}{4}, \frac{1}{2}$, and 1 the corrections are, respectively, 18.6, 9.7, and 3.4 dB. Good agreement with computer simulation in this case may be noted by applying these corrections to Fig. 6 of Ref. 8.

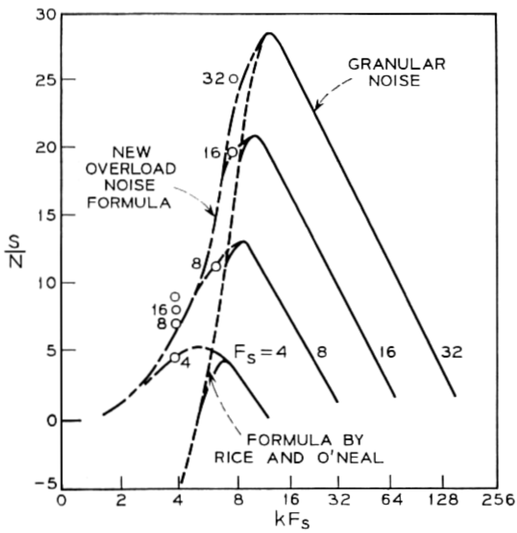

Fig. 11 — Flat bandlimited Gaussian signals—comparison of the new results with previous analytic results and computer simulation.

Comparison of the new analytical result with experiment will be covered elsewhere.

VII. OTHER STATISTICAL CHARACTERISTICS OF THE OVERLOAD NOISE

7.1 *Probability Density*

The technique used in the present paper, i.e., the substitution of $m(s)$ for $n(t)$ and the application of the averaging procedure presented in Section V, can be used for the determination of other statistical characteristics of the slope overload noise.

For example, let $q(m, s, x_0')$ be the probability density of $m(s)$, where s is a given number, i.e., a parameter, taking on nonnegative values. Let us define the following auxiliary probability functions $q^{\pm}(m, s, x_0')\, ds$, the conditional probability that $x(t) - x(t - s) \mp x_0's$ lies between m and $m + dm$ given that the derivative of $x(\cdot)$ increases (decreases) through x_0' between $t - s$ and $t - s + ds$, where $m \gtrless 0$ and $s > 0$. Clearly,

$$q^{-}(m, s, x_0') = q^{+}(-m, s, x_0') \quad \text{for} \quad m < 0.$$

Also using the same averaging procedure as in Section V and the definition of conditional probability densities in the horizontal window sense, we find that [see (20a) and (46)]

$$\begin{aligned} q^{+}&(m, s, x_0') \\ &= \int_{-\infty}^{\infty} p_{hw}(x_1\,, x_2 = x_1 + x_0's + m \mid x' = x_0'\,, x'' > 0)\, dx_1 \\ &= \frac{\displaystyle\int_{-\infty}^{\infty}\int_0^{\infty} x_1''p(x_1\,, x_2 = x_1 + x_0's + m, x_0'\,, x_1''\,; s)\, dx_1''\, dx_1}{\displaystyle\int_0^{\infty} x_1''p(x_0'\,, x_1'')\, dx_1''} \\ &= \frac{1}{K}\int_{-\infty}^{\infty}\int_0^{\infty} x_1''p(x_1\,, x_2 = x_1 + x_0's + m, x_0'\,, x_1''\,; s)\, dx_1''\, dx_1\,. \end{aligned}$$

From (100) and (101) we see that

$$q^{+}(m, s, x_0') = \frac{1}{(2\pi)^{\frac{3}{2}} K\sqrt{a(s)}}\, P(m, s).$$

$P(\cdot, \cdot)$ is defined in (101) and is determined in (116), Appendix B.

It is easy to verify that

$$q(m, s, x_0') = \begin{cases} q^+(m, s, x_0'), & \text{for} \quad m > 0 \\ q^-(m, s, x_0'), & \text{for} \quad m < 0. \end{cases}$$

Note also that there is a finite probability that $m(s) = 0$. Hence, the density $q(m, s, x_0')$ contains an impulse at $m = 0$ with strength $p(s)$

$$p(s) = 1 - 2 \int_0^\infty q^+(m, s, x_0')\, dm.$$

The probability density of m, i.e., without specified s, is clearly

$$P_M(m, x_0') = \int_0^\infty q(m, s, x_0')\, ds.$$

Clearly, $P_M(m, x_0')$ contains an impulse at $m = 0$ of strength

$$\int_0^\infty p(s)\, ds.$$

7.2 *Other Statistical Characteristics*

Another useful attribute of the noise is its covariance $\langle x(t)n(t)\rangle_{av}$ with the input random process. This quantity is of interest in comparing results obtained by a particular measured procedure with those obtained analytically. This will be discussed further in the paper referred to previously. The evaluation of $\langle xn\rangle_{av}$ has been performed applying the method presented in Section V. The calculations are even more complicated than the ones employed in the evaluation of $\langle n^2(t)\rangle_{av}$ and we will not consider them here. Moments of any order could be worked out. The expected value of $| n(t) |$ has also been determined. There are many statistical problems that may be generated by the study of slope overload noise in DPCM. These problems have their counter-part in the theory of level-crossings of random processes, but they are even more complicated.

VIII. ACKNOWLEDGMENT

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APPENDIX A

Correction of Zetterberg's $Q_{x'_0}$

The evaluation of $Q_{x'_0}$ (Q_γ with the notation in equation 4.26 of Ref. 7) is not done correctly in Ref. 7 since $A(x)$ in Equation (4.32) attains negative values. The integral to be evaluated is

$$Q_{x_0'} = \sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^\infty k(s)u^2 \exp\left[-\tfrac{1}{2}(u + g(s))^2\right] du\, ds \tag{69}$$

with

(*i*)
$$g(s) = as, \tag{70}$$

where

$$a = \frac{x_0'\sqrt{b_2}}{3b_1}$$

[see (25) and the following comments] and

(*ii*)
$$k(s) = \begin{cases} \dfrac{b_2 s^4}{4}, & \text{for } s \leqq s_1 \\ 2\psi_0, & \text{for } s > s_1, \end{cases} \tag{71}$$

where

$$s_1 = \sqrt[4]{\frac{8\psi_0}{b_2}}. \tag{72}$$

Make the change of variable

$$u + as = v.$$

Then

$$Q_{x_0'} = \frac{1}{a}\sqrt{\frac{2}{\pi}} \int_0^\infty \int_0^v k\left(\frac{v-u}{a}\right) u^2 e^{-v^2/2}\, du\, dv. \tag{73}$$

Set

$$X(v) = \int_0^v k\left(\frac{v-u}{a}\right) u^2\, du = \int_0^v k\left(\frac{z}{a}\right)(v-z)^2\, dz \quad \text{for } v \leqq as_1 = \lambda^* \tag{74}$$

$$\frac{X(v)}{2\psi_0/\lambda^4} = Y_1 = \int_0^v (v-z)^2 z^4\, dz = \frac{v^7}{105} \tag{75}$$

---

* λ here corresponds to Zetterberg's x. (λ is introduced to avoid confusion with the input $x(t)$.)

For $v \geqq \lambda$

$$\frac{X(v)}{2\psi_0/\lambda^4} = Y_2 = \int_0^\lambda (v - z)^2 z^4 \, dz + \lambda^4 \int_\lambda^v (v - z)^2 \, dz$$

$$= -\frac{34}{105}\lambda^7 + \frac{\lambda^4 v^3}{3} - \lambda^5 v^2 + \lambda^6 v. \tag{76}$$

Hence,

$$Q_{x'_0} = \sqrt{\frac{2}{\pi}} \frac{1}{a} \frac{2\psi_0}{\lambda^4} \left[\int_0^\lambda e^{-v^2/2} \frac{v^7}{105} \, dv \right.$$

$$\left. + \int_\lambda^\infty e^{-v^2/2} \left(\frac{\lambda^4 v^3}{3} - \lambda^5 v^2 + \lambda^6 v - \frac{34}{105}\lambda^7 \right) dv \right].$$

Consequently,

$$Q_{x'_0} = \sqrt{\frac{2}{\pi}} \frac{1}{a} \frac{2\psi_0}{35\lambda^4} [\tfrac{1}{3} J_7(\lambda) + \tfrac{35}{3}\lambda^4 I_3(\lambda)$$

$$- 35\lambda^5 I_2(\lambda) + 35\lambda^6 I_1(\lambda) - \tfrac{34}{3}\lambda^7 \Phi(\lambda)], \tag{77}$$

where

$$\Phi(\lambda) = \int_\lambda^\infty e^{-z^2/2} \, dz \tag{78}$$

and

$$I_n(\lambda) = \int_\lambda^\infty z^n e^{-z^2/2} \, dz \tag{79}$$

$$J_n(\lambda) = \int_0^\lambda z^n e^{-z^2/2} \, dz. \tag{80}$$

Integrating by parts we find the following recursive relations for I_n and J_n, respectively,

$$I_n(\lambda) = \lambda^{n-1} e^{-\lambda^2/2} + (n - 1) I_{n-2}(\lambda). \tag{81}$$

Clearly,

$$I_0(\lambda) = \Phi(\lambda) \tag{82}$$

and

$$I_1(\lambda) = e^{-\lambda^2/2} \tag{83}$$

$$J_n(\lambda) = -\lambda^{n-1} e^{-\lambda^2/2} + (n - 1) J_{n-2}(\lambda) \tag{84}$$

with

$$J_0(\lambda) = \int_0^\lambda e^{-z^2/2}\,dz = \frac{\sqrt{2\pi}}{2} - \Phi(\lambda) \tag{85}$$

$$J_1(\lambda) = 1 - e^{-\lambda^2/2}. \tag{86}$$

Applying these recursive relations we find

$$I_2(\lambda) = \lambda e^{-\lambda^2/2} + \Phi(\lambda), \tag{87}$$

$$I_3 = \lambda^2 e^{-\lambda^2/2} + 2e^{-\lambda^2/2}, \tag{88}$$

and

$$J_7 = 48\left[1 - e^{-\lambda^2/2}\left(\frac{\lambda^6}{48} + \frac{\lambda^4}{8} + \frac{\lambda^2}{2} + 1\right)\right]. \tag{89}$$

Substituting in (77) we get

$$Q_{x_0'} = \sqrt{\frac{2}{\pi}}\,\frac{1}{a}\,\frac{32}{35}\,\frac{\psi_0}{\lambda^4}\,[1 + P_1(\lambda)e^{-\lambda^2/2} - Q_1(\lambda)\Phi(\lambda)], \tag{90}$$

where

$$P_1(\lambda) = \tfrac{17}{24}\lambda^6 + \tfrac{4}{3}\lambda^4 - \tfrac{1}{2}\lambda^2 - 1$$

$$Q_1(\lambda) = \tfrac{17}{24}\lambda^7 + \tfrac{35}{16}\lambda^5.$$

APPENDIX B

Algebraic Manipulations with the Statistical Parameters

Denote by

$$M = \{\mu_{ij}\}(i, j = 1, \cdots, 4)$$

the cross-correlation matrix of the random variables

$$X_1 = x(t_0)$$

$$X_2 = x(t_0 + s)$$

$$X_1' = \frac{dx(t_0)}{dt}$$

$$X_1'' = \frac{d^2x(t_0)}{dt^2}.$$

Then we have

$$\mu_{11} = E(X_1^2) = 1$$

$$\begin{aligned}
\mu_{12} &= \mu_{21} = E(X_1X_2) = E(x(t_0 - s)x(t_0)) = \psi(s) \\
\mu_{13} &= \mu_{31} = E(X_1X_1') = 0 \\
\mu_{14} &= \mu_{41} = E(X_1X_1'') = (-1)^2 \frac{d^2\psi(\tau)}{d\tau^2}\bigg|_{\tau=0} = -b_1 \\
\mu_{22} &= E(X_2^2) = 1 \\
\mu_{23} &= \mu_{32} = E(X_2X_1') = -\frac{d\psi(s)}{ds} = -\dot{\psi}(s) \\
\mu_{24} &= \mu_{42} = E(X_2X_1'') = (-1)^2 \frac{d^2\psi(s)}{ds^2} = \ddot{\psi}(s) \\
\mu_{33} &= E(X_1'^2) = b_1 \\
\mu_{34} &= \mu_{43} = E(X_1'X_1'') = 0 \\
\mu_{44} &= E(X_1''^2) = b_2\ .
\end{aligned} \tag{91}$$

Therefore,

$$M = \begin{bmatrix} 1 & \psi(s) & 0 & -b_1 \\ \psi(s) & 1 & -\dot{\psi}(s) & \ddot{\psi}(s) \\ 0 & -\dot{\psi}(s) & b_1 & 0 \\ -b_1 & \ddot{\psi}(s) & 0 & b_2 \end{bmatrix}. \tag{92}$$

Call $| M |$ the determinant of M.

It turns out that

$$| M | = (b_2 - b_1^2)\{b_1(1 - \psi^2(s)) - \dot{\psi}^2(s)\} - b_1\{\psi(s) + b_1\ddot{\psi}(s)\}^2. \tag{93}$$

Denote by $M_{ij}(i, j = 1, \cdots, 4)$ the co-factors of the matrix M. Since M is a symmetric matrix, M^{-1} is also symmetric and $M_{ij} = M_{ji}$ and

$$M^{-1} = \frac{1}{| M |}\{M_{ij}\}. \tag{94}$$

These co-factors are given in terms of the statistics of the input process as follows:

$$\begin{aligned}
M_{11} &= b_1b_2 - b_1\ddot{\psi}^2(s) - b_2\dot{\psi}^2(s) \\
M_{12} &= -b_1(b_2\psi(s) + b_1\ddot{\psi}(s)) \\
M_{13} &= -\dot{\psi}(s)(b_2\psi(s) + b_1\ddot{\psi}(s)) \\
M_{14} &= b_1(b_1 + \psi(s)\ddot{\psi}(s) - \dot{\psi}^2(s))
\end{aligned}$$

$$\begin{aligned} M_{22} &= b_1(b_2 - b_1^2) \\ M_{23} &= \dot{\psi}(s)(b_2 - b_1^2) \\ M_{24} &= -b_1(\ddot{\psi}(s) + b_1\psi(s)) \\ M_{33} &= (1 - \psi^2(s))(b_2 - b_1^2) - (\ddot{\psi}(s) + b_1\psi(s))^2 \\ M_{34} &= -\dot{\psi}(s)(\ddot{\psi}(s) + b_1\psi(s)) \\ M_{44} &= b_1(1 - \psi^2(s)) - \dot{\psi}^2(s). \end{aligned} \tag{95}$$

It is easily seen that

$$\mathbf{x}^t M^{-1}\mathbf{x} = \frac{1}{|M|}(ax_1^2 + 2bx_1 + c), \tag{96}$$

where

$$a = a(s) = M_{11} + 2M_{12} + M_{22}, \tag{97}$$

a function of s only

$$b = (M_{14} + M_{24})x_1'' + (M_{12} + M_{22})(u + x_0's) + (M_{13} + M_{23})x_0', \tag{98}$$

a linear function in x_1'' and $(u + x_0's)$

$$c = M_{44}x_1''^2 + 2[M_{24}(u + x_0's) + M_{34}x_0'] + M_{22}(u + x_0's)^2 + 2M_{23}x_0'(u + x_0's) + M_{33}x_0'^2 \tag{99}$$

quadratic in x_1'' and $(u + x_0's)$.
Integrating with respect to x_1 in (44) we get

$$\begin{aligned} &K \text{ ave } \{m^2(s)\} \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{a(s)}} \int_0^\infty u^2 \int_0^\infty x_1'' \exp\left\{-\frac{1}{2|M|}\left(c - \frac{b^2}{a}\right)\right\} dx_1''\, du \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}\sqrt{a(s)}} \int_0^\infty u^2 P(u, s)\, du, \end{aligned} \tag{100}$$

where

$$P(u, s) = \int_0^\infty x_1'' \exp\left\{-\frac{1}{2|M|}\left(c - \frac{b^2}{a}\right)\right\} dx_1''. \tag{101}$$

It is seen that

$$\frac{1}{|M|}\left(c - \frac{b^2}{a}\right) = A(s)x_1''^2 + 2B(s, u, x_0')x_1'' + C(s, u, x_0'), \tag{102}$$

where

$$A(s) = \frac{1}{|M|}\left\{M_{44} - \frac{(M_{14} + M_{24})^2}{a(s)}\right\} \tag{103}$$

is a function of s only and

$$\begin{aligned} B(s) &= B_1(s)(u + x_0's) + B_2(s)x_0' \\ &= B_1(s)u + (sB_1(s) + B_2(s))x_0' \end{aligned} \tag{104}$$

is a linear function in u, with

$$B_1(s) = \frac{1}{|M|}\left\{M_{24} - \frac{(M_{14} + M_{24})(M_{12} + M_{22})}{a(s)}\right\} \tag{105}$$

$$B_2(s) = \frac{1}{|M|}\left\{M_{34} - \frac{(M_{14} + M_{24})(M_{13} + M_{23})}{a(s)}\right\}. \tag{106}$$

Further,

$$C(s, u, x_0') = C_1(s)(u + x_0's)^2 + 2C_2(s)x_0'(u + x_0's) + C_3(s)x_0'^2 \tag{107}$$

is quadratic in u, where

$$C_1(s) = \frac{1}{|M|}\left\{M_{22} - \frac{(M_{12} + M_{22})^2}{a(s)}\right\} \tag{108}$$

$$C_2(s) = \frac{1}{|M|}\left\{M_{23} - \frac{(M_{12} + M_{22})(M_{13} + M_{23})}{a(s)}\right\} \tag{109}$$

$$C_3(s) = \frac{1}{|M|}\left\{M_{33} - \frac{(M_{13} + M_{23})^2}{a(s)}\right\}. \tag{110}$$

Substituting in (101) we get

$$P(u, s) = \exp\left\{-\frac{1}{2}\left(C - \frac{B^2}{A}\right)\right\} \cdot \int_0^\infty x_1'' \exp\left\{-\frac{A}{2}\left(x_1'' + \frac{B}{A}\right)^2\right\} dx_1'' . \tag{111}$$

Make the change of variables

$$\sqrt{A}\left(x_1'' + \frac{B}{A}\right) = \eta. \tag{112}$$

Then

$$P(u, s) = \frac{1}{A(s)} \exp\left\{-\frac{1}{2}\left(C - \frac{B^2}{A}\right)\right\} \int_{B/\sqrt{A}}^{\infty} \left(\eta - \frac{B}{\sqrt{A}}\right) e^{-\eta^2/2}\, d\eta$$

$$= \frac{1}{A(s)} \exp\left\{-\frac{1}{2}\left(C - \frac{B^2}{A}\right)\right\}\left[e^{-B^2/2A} - \frac{1}{\sqrt{A}} \int_{B/\sqrt{A}}^{\infty} e^{-\eta^2/2}\, d\eta\right]. \tag{113}$$

Set

$$\xi = -\frac{B(u, s)}{\sqrt{A(s)}} = -\frac{B_1(s)}{\sqrt{A(s)}}\, u - \frac{sB_1(s) + B_2(s)}{\sqrt{A(s)}}\, x_0' \tag{114}$$

and

$$\varphi(\xi) = e^{-\xi^2/2} + \xi\Phi(-\xi), \tag{115}$$

where

$$\Phi(x) = \int_x^{\infty} e^{-z^2/2}\, dz.$$

Hence,

$$P(u, s) = \frac{1}{A(s)} \exp\left\{-\frac{1}{2}\left(C - \frac{B^2}{A}\right)\right\}\varphi(\xi). \tag{116}$$

Clearly,

$$C(u, s) = C_1(s)(u + x_0's)^2 + 2C_2(s)(u + x_0's) + C_3(s)x_0'^2$$

$$= (u\sqrt{C_1(s)} + g^*(s))^2 + x_0'^2\left[C_3(s) - \frac{C_2^2(s)}{C_1(s)}\right], \tag{117}$$

where

$$g^*(s) = x_0' \frac{sC_1 + C_2}{\sqrt{C_1}} \tag{118}$$

and C_1, C_2, and C_3 are given in (108), (109), and (110), respectively. Using these equations and the definition of $a(s)$ in (97) we find

$$| M | \left[C_3(s) - \frac{C_2^2(s)}{C_1(s)}\right] = M_{33} - \frac{(M_{13} + M_{23})^2}{a(s)}$$

$$- \frac{\{M_{23}(M_{11} + M_{12}) - M_{13}(M_{12} + M_{22})\}^2}{(M_{11}M_{22} - M_{12}^2)a(s)}. \tag{119}$$

We also note that

$$\frac{M_{13}}{M_{12}} = \frac{M_{23}}{M_{22}} = \frac{\dot{\psi}(s)}{b_1} \tag{120}$$

[use (95)].

From (119) and (120) it follows easily that

$$|M| \left[C_3(s) - \frac{C_2^2(s)}{C_1(s)} \right] = M_{33} - \frac{M_{22}\dot{\psi}^2}{b_1^2}.$$

Substituting the expressions for M_{22} and M_{33} from (95) we find

$$M_{33} - \frac{M_{22}\dot{\psi}^2}{b_1^2} = \frac{|M|}{b_1}.$$

Hence,

$$C_3(s) - \frac{C_2^2(s)}{C_1(s)} = \frac{1}{b_1}. \tag{121}$$

Set

$$v = u\sqrt{C_1(s)}. \tag{122}$$

Then we have

$$C = (v + g^*(s))^2 + \frac{x_0'^2}{b_1} \tag{123}$$

and from (114)

$$\xi = -\frac{B_1(s)}{\sqrt{A(s)C_1(s)}} \left\{ v + \sqrt{C_1(s)} \left(s + \frac{B_2(s)}{B_1(s)} \right) x_0' \right\}. \tag{124}$$

Using (120), the definitions of $B_1(s)$, $B_2(s)$, $C_1(s)$, and $C_2(s)$ in (105), (106), (108), and (109), respectively, and the relation

$$\frac{M_{34}}{M_{24}} = \frac{\dot{\psi}(s)}{b_1} \tag{125}$$

resulting from (95) we find

$$\frac{B_2(s)}{B_1(s)} = \frac{C_2(s)}{C_1(s)} = \frac{\dot{\psi}(s)}{b_1}. \tag{126}$$

Hence, (124) becomes

$$\xi = -\frac{B}{\sqrt{A}} = \lambda(s)(v + g^*(s)), \tag{127}$$

where

$$\lambda(s) = -\frac{B_1(s)}{\sqrt{A(s)C_1(s)}} \tag{128}$$

and g^* is defined in (118). Note also that

$$g^*(s) = \sqrt{C_1(s)}\left(s + \frac{\dot{\psi}(s)}{b_1}\right)x_0' . \tag{129}$$

From (123) and (125) we get

$$C - \frac{B^2}{A} = (1 - \lambda^2(s))(v + g^*(s))^2 + \frac{x_0'^2}{b_1}. \tag{130}$$

Using the value of K given in (47) we find for the quantity P as given in (116) that

$$\frac{P}{K} = \frac{2\pi}{A(s)}\left(\frac{b_1}{b_2}\right)^{\frac{1}{2}} \exp\left[-\tfrac{1}{2}(1 - \lambda^2(s))(v + g^*(s))^2\right]\varphi(\xi). \tag{131}$$

Substituting in (100) and using the change of variable as given in (122) we get

$$\text{ave}\,\{m^2(s)\} = \frac{(b_1/b_2)^{\frac{1}{2}}}{\sqrt{2\pi}\, A(s)\sqrt{a(s)}\,(\sqrt{C_1(s)})^3} \cdot \int_0^\infty v^2 \exp\left[-\tfrac{1}{2}(1 - \lambda^2(s))(v + g^*(s))^2\right]\varphi(\xi)\, dv, \tag{132}$$

where ξ and $\varphi(\xi)$ are given in (127) and (115), respectively.

Now make the following change of variables:

$$v\sqrt{1 - \lambda^2(s)} = z \tag{133}$$

and set

$$g^*(s)\sqrt{1 - \lambda^2(s)} = g_1(s) \tag{134}$$

so that from (129)

$$g_1(s) = \frac{x_0'}{b_1}(b_1 s + \dot{\psi}(s))\,\frac{\sqrt{A(s)C_1(s) - B_1^2(s)}}{\sqrt{A(s)}}. \tag{135}$$

Then

$$\xi = \frac{\lambda(s)}{\sqrt{1 - \lambda^2(s)}}(z + g_1(s)) \tag{136}$$

and

$$\text{ave}\,\{m^2(s)\} = \frac{k_1(s)}{\sqrt{2\pi}} \int_0^\infty z^2 \exp\left[-\tfrac{1}{2}(z + g_1(s))^2\right] \frac{\sqrt{1 - \lambda^2(s)}}{\lambda(s)}\, \varphi(\xi)\, dz, \tag{137}$$

where

$$k_1(s) = \left(\frac{b_1}{b_2}\right)^{\frac{1}{2}} \frac{-B_1(s)\sqrt{A(s)}}{\sqrt{A(s)}\,\{A(s)C_1(s) - B_1^2(s)\}^2}. \tag{138}$$

Finally, the average energy per burst becomes

$$Q_{x'_0} = \frac{1}{\sqrt{2\pi}} \int_0^\infty k_1(s) \int_0^\infty z^2 \exp\left[-\tfrac{1}{2}(z + g_1(s))^2\right] \frac{\sqrt{1-\lambda^2(s)}}{\lambda(s)}\, \varphi(\xi)\, dz\, ds. \tag{139}$$

APPENDIX C

Asymptotic Behavior of Several Functions of s for $s \to 0$ and $s \to \infty$

Assume that $f^n F(f)$ is integrable for n less than or equal to 8. For bandlimited signals, the usual case in practice, this requirement is automatically satisfied.

For small s the following Taylor expansions hold:

$$\psi(s) = 1 - b_1 \frac{s^2}{2!} + b_2 \frac{s^4}{4!} - b_3 \frac{s^6}{6!} + \frac{s^8}{8!} \psi^{(8)}(\theta s),$$

where θ is a number such that $0 < \theta < 1$ and $\psi^{(8)}(\theta s)$ is the 8th derivative of $\psi(s)$ evaluated at θs.

For a signal bandlimited to the band $(0, f_0)$ we have

$$\mid \psi^{(8)}(\theta s) \mid \leqq b_4 \leqq (2\pi f_0)^2 b_3 .$$

Therefore, the absolute value of the remainder term satisfies the following inequality:

$$\left| \frac{s^8}{8!} \psi^{(8)}(\theta s) \right| \leqq b_3 \frac{s^6}{6!} \cdot \frac{(2\pi f_0 s)^2}{56}$$

so that this term will be negligible if $(2\pi f_0 s)^2 \ll 56$, i.e., if $f_0 s \ll 1.2$. In the expansion for the first and second derivatives of $\psi(s)$ the first three terms are included and the remainder terms may be disregarded for the same values of s.

Consequently, we have

$$\dot{\psi}(s) = -b_1 s + b_2 \frac{s^3}{3!} - b_3 \frac{s^5}{5!} + \frac{s^7}{7!} \psi^{(8)}(\theta_1 s)$$

$$\ddot{\psi}(s) = -b_1 + b_2 \frac{s^2}{2!} - b_3 \frac{s^4}{4!} + \frac{s^6}{6!} \psi^{(8)}(\theta_2 s)$$

with

$$0 \leqq \theta_1 , \theta_2 \leqq 1.$$

We, now, obtain the asymptotic behavior of some expressions of s which appear in the functions of s involved in the integral for $Q_{x'_0}$. Note that

(i) $$1 - \psi(s) = b_1 \frac{s^2}{2!} - b_2 \frac{s^4}{4!} + b_3 \frac{s^6}{6!} + 0(s^8)$$

(ii) $$1 - \psi^2(s) = b_1 s^2 - \left(\frac{b_2}{12} + \frac{b_1^2}{4}\right)s^4 + \left(\frac{b_1 b_2}{4!} + \frac{2b_3}{6!}\right)s^6 + 0(s^8)$$

(iii) $$\dot{\psi}^2(s) = b_1^2 s^2 - \frac{b_1 b_2 s^4}{3} + \left(\frac{2b_1 b_3}{5!} + \frac{b_2^2}{(3!)^2}\right)s^6 + 0(s^8)$$

(iv) $$\ddot{\psi}(s) + b_1 = \frac{b_2 s^2}{2} - \frac{b_3 s^4}{24} + 0(s^6)$$

$$(\ddot{\psi}(s) + b_1)^2 = \frac{b_2 s^4}{4} - \frac{b_2 b_3 s^6}{24} + 0(s^8)$$

(v) $$\ddot{\psi}(s) + b_1 \psi(s) = \frac{b_2 - b_1^2}{2} s^2 + \frac{b_1 b_2 - b_3}{4!} s^4 + 0(s^6)$$

(vi) $$(\ddot{\psi}(s) + b_1 \psi(s))^2 = \frac{(b_2 - b_1^2)^2}{4} s^4 + \frac{(b_2 - b_1^2)(b_1 b_2 - b_3)}{4!} s^6 + 0(s^8).$$

Using these formulas and the formulas of definition of the different functions of s we find after a considerable amount of algebraic manipulations, the following asymptotic expressions for $s \to 0$. Set

$$\mathcal{B} = b_1 b_3 - b_2^2 .$$

(i) $$a(s) \approx \frac{b_2 \mathcal{B}}{36} s^6$$

(ii) $$| M | \approx \frac{(b_2 - b_1^2)\mathcal{B}}{36} s^6$$

(iii) $$A(s) \approx \frac{9b_1}{\mathcal{B} s^2}$$

(iv) $$B_1(s) \approx -\frac{18b_1}{\mathcal{B} s^4}$$

(*v*) $$B_2(s) = \frac{\dot{\psi}(s)}{b_1} B_1(s) \approx \frac{18b_1}{\mathcal{B}s^3}$$

(*vi*) $$C_1(s) \approx \frac{36b_1}{\mathcal{B}s^6}$$

(*vii*) $$C_2(s) = \frac{\dot{\psi}(s)}{b_1} C_1(s) \approx -\frac{36b_1}{\mathcal{B}s^5}$$

(*viii*) $$A(s)C_1(s) - B_1^2(s) \approx \frac{36b_1}{b_2\mathcal{B}} \frac{1}{s^6}$$

(*ix*) $$\lambda(s) = -\frac{B_1(s)}{\sqrt{A(s)C_1(s)}} \approx 1$$

(*x*) $$\frac{\sqrt{1 - \lambda^2(s)}}{\lambda(s)} \approx \frac{1}{3} \sqrt{\frac{\mathcal{B}}{b_1 b_2}} \cdot s.$$

Using the formulas above we find that

$$k_1(s) = -\sqrt{\frac{b_1}{b_2}} \frac{B_1(s)\sqrt{A(s)}}{\sqrt{a(s)}\ \{A(s)C_1(s) - B_1^2(s)\}^2} \approx \frac{b_2 s^4}{4} \quad \text{for} \quad s \to 0$$

and

$$g_1(s) = x_0'\left(s + \frac{C_2(s)}{C_1(s)}\right) \frac{\sqrt{A(s)C_1(s) - B_1^2(s)}}{\sqrt{A(s)}} \approx x_0' \frac{\sqrt{b_2}}{3b_1} s \quad \text{for} \quad s \to 0,$$

i.e.,

$$g_1(s) \cong \alpha s \quad \text{for} \quad s \text{ small}$$

with

$$\alpha = x_0' \frac{b_2^{\frac{1}{2}}}{3b_1}.$$

For $s \to \infty$, $\psi(s)$, $\dot{\psi}(s)$, and $\ddot{\psi}(s)$ approach zero.

The following asymptotic expressions are easily derived for $s \to \infty$:

(*i*) $$a(\infty) = b_1(2b_2 - b_1^2)$$

(*ii*) $$|M| = b_1(b_2 - b_1^2)$$

(*iii*) $$A(\infty) = \frac{2}{2b_2 - b_1^2}$$

(*iv*) $$B_1(\infty) = -\frac{b_1}{2b_2 - b_1^2}$$

(v) $$B_2(\infty) = 0$$

(vi) $$C_1(\infty) = \frac{b_2}{2b_2 - b_1^2}$$

(vii) $$C_2(\infty) = 0$$

(viii) $$A(\infty)C_1(\infty) - B_1^2(\infty) = \frac{1}{2b_2 - b_1^2}$$

(ix) $$\lambda(\infty) = \frac{b_1}{\sqrt{2b_2}}$$

(x) $$\frac{\sqrt{1 - \lambda^2(\infty)}}{\lambda(\infty)} = \frac{\sqrt{2b_2 - b_1^2}}{b_1}.$$

Using these expressions we find

$$k_\infty = \lim_{s\to\infty} k(s) = b_1\sqrt{\frac{2}{b_2}}$$

and

$$g_1(s) \approx \frac{x_0' s}{\sqrt{2}} \quad \text{for} \quad s \to \infty .$$

Note also that for ξ as defined in (50) we have

(i) For $z = 0$ and s small

$$\xi = \xi_0 = \frac{\lambda(s)}{\sqrt{1 - \lambda^2(s)}}\, g_1(s) \cong 3\sqrt{\frac{b_1 b_2}{\mathcal{B}}}\, \frac{1}{s} \cdot x_0' \frac{\sqrt{b_2}}{3b_1}\, s.$$

Hence,

$$\xi_0 = \frac{b_2 x_0'}{\sqrt{\mathcal{B} b_1}}.$$

(ii) For s large

$$\xi \cong \frac{\sqrt{2b_2 - b_1^2}}{b_1}\, z + \frac{\sqrt{2b_2 - b_1^2}}{b_1\sqrt{2}}\, x_0' s \triangleq \gamma_0 z + \delta_0 s.$$

APPENDIX D

Approximate Evaluation of $Q_{x'_0}$

From (57) we have

$$Q_{x'_0} = 2\int_0^\infty k_1(s) \int_0^\infty z \exp\,[-\tfrac{1}{2}(z + g_1(s))^2]\, dz\, ds, \tag{140}$$

where

$$g_1(s) = \alpha s \qquad 0 < s < \infty$$

$$k_1(s) = \begin{cases} k_\infty \left(\dfrac{s}{s_2}\right)^4, & \text{for} \quad s \,\varepsilon\, (0, s_2) \\ k_\infty, & \text{for} \quad s \,\varepsilon\, (s_2, \infty). \end{cases} \tag{141}$$

The symbols α, s_2, and k_∞ are defined in the relations (63), (62), and (60), respectively.

Make the change of variables

$$y = z + \alpha s.$$

We then have

$$Q_{x'_0} = \frac{2}{\alpha} \int_0^\infty e^{-y^2/2} \int_0^y k_1\left(\frac{y - z}{\alpha}\right) z \, dz \, dy.$$

Set

$$X(y) = \int_0^y z k_1\left(\frac{y - z}{\alpha}\right) dz = \int_0^y (y - \eta) k_1\left(\frac{\eta}{\alpha}\right) d\eta.$$

Then

$$Q_{x'_0} = \frac{2}{\alpha} \int_0^\infty X(y) e^{-y^2/2} \, dy.$$

For $y \leqq \chi = \alpha s_1$

$$\frac{X(y)}{k_\infty/\chi^4} = Y_1 = \int_0^y \eta^4 (y - \eta) \, d\eta = \frac{1}{30} y^6.$$

For $y \geqq \chi$

$$\frac{X(y)}{k_\infty/\chi^4} = Y_2 = \frac{\chi^6}{30} + \chi^4 \int_\chi^y (y - \eta) \, d\eta = \frac{8\chi^6}{15} - \chi^5 y + \frac{\chi^4 v^2}{2}.$$

Consequently,

$$\frac{Q_{x'_0}}{2k_\infty/\alpha\chi^4} = \frac{1}{30} J_6 + \frac{1}{2}\chi^4 I_2 - \chi^5 I_1 + \frac{8}{15}\chi^6 \Phi(\chi), \tag{142}$$

where $\Phi(\chi)$, $I_n(\chi)$, and $J_n(\chi)$ are defined in (78), (79), and (80), respectively.

Applying the recursive relations (81) and (84), we find

$$I_1(\chi) = e^{-\chi^2/2}$$

$$I_2(\chi) = \chi e^{-\chi^2/2} + \Phi(\chi)$$

$$J_6 = \frac{15\sqrt{2\pi}}{2} - e^{-\chi^2/2}(\chi^5 + 5\chi^3 + 15\chi) - 15\Phi(\chi).$$

Substituting in (142) the values of J_6, I_2, I_1, we get

$$\frac{Q_{x'_0}}{2k_\infty/\alpha\chi^4} = \frac{\sqrt{2\pi}}{4}\left\{1 - \frac{e^{-\chi^2/2}}{\sqrt{2\pi}} P(\chi) + \frac{1}{\sqrt{2\pi}} \Phi(\chi)Q(\chi)\right\}, \qquad (143)$$

where

$$P(\chi) = 2(\tfrac{16}{15}\chi^5 + \tfrac{1}{3}\chi^3 + \chi) \qquad (144)$$

and

$$Q(\chi) = 2(\tfrac{16}{15}\chi^6 + \chi^4 - 1), \qquad (145)$$

i.e.,

$$Q_{x'_0} = \frac{\sqrt{2\pi}}{2} \frac{k_\infty}{\chi^4\alpha} A(\chi),$$

where

$$A(\chi) = 1 - \frac{e^{-\chi^2/2}}{\sqrt{2\pi}} P(\chi) + \frac{1}{\sqrt{2\pi}} \Phi(\chi)Q(\chi). \qquad (146)$$

Using the expressions (63), (64), and (60) for α, χ, and k_∞, respectively, we get

$$Q_{x_0'} = \frac{\sqrt{2\pi}}{8}\left(\frac{3b_1}{x_0'}\right)^5 b_2^{-\frac{3}{2}} A(\chi) = (\text{Rice's Results}) \cdot A(\chi). \qquad (147)$$

APPENDIX E

The Function $A(\chi)$

The function $A(\chi)$ as defined in (66) is a monotonically increasing function of χ in the interval $(0, \infty)$ with $A(0) = 0$ and $A(\infty) = 1$.

The computation of $A(\chi)$ for different values of χ was performed using the computer and $10 \log_{10} 1/A(\chi)$, the correcting factors of Rice's result, is shown in Fig. 10.

Expanding into Taylor series we can find that for χ small

$$A(\chi) \approx \chi^4 - 2\sqrt{\frac{2}{\pi}}\chi^5 + \left(\frac{113}{120\sqrt{2\pi}} + \frac{16}{15}\right)\chi^6 \cong \chi^4(1 - 1.6\chi + 1.44\chi^2);$$

whereas, for χ large, using the asymptotic expansion for

$$\frac{1}{\sqrt{2\pi}} \Phi(\chi) = \frac{1}{2} \operatorname{erfc}\left(\frac{\chi}{\sqrt{2}}\right),$$

we get

$$A(\chi) \approx 1 - \frac{4\chi^3 e^{-\chi^2/2}}{5\sqrt{2\pi}} \left(1 - \frac{3}{\chi^2}\right).$$

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