Some Properties of Power Sums of Truncated Normal Random Variables

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The power sum of P_n n components $X_1, X_2, \cdots, X_n$ is defined by the relation

$$P_n = 10 \log_{10} [10^{X_1/10} + \cdots + 10^{X_n/10}].$$

The distributions of such power sums are studied both analytically and by Monte Carlo simulation techniques for the case where the components are independent, identically distributed, truncated normal random variables. Results are given in terms of distributions and moments of P_n. The number of components varies from 2 to 256, and the standard deviation of the component variables before truncation ranges from 1 to 10 dB. The dependence of the results on the choice of truncation point is also investigated.

I. INTRODUCTION

It is common practice in communications engineering to express signal and noise powers on a logarithmic scale. As is well known, such a scale serves both to narrow the numerical range between large and small powers and to simplify some computations by replacing multiplication by addition. The decibel scale is most commonly used. Employing this scale, the power level x of a power w is defined by

$$x = 10 \log_{10} \frac{w}{w_0}, \tag{1}$$

where w_0 is a reference power, and x is expressed in decibels (dB) over the reference power w_0. Note from (1) that $w/w_0 = 10^{x/10}$.

In the situation where a number of uncorrelated signal sources feed into the same load, the power level p_n of a sum of powers $w_1, \ldots, w_n$ is given by

$$p_n = 10 \log_{10} [10^{x_1/10} + \cdots + 10^{x_n/10}], \tag{2}$$

where x_i is the power level of w_i. Examples of such sums arise in crosstalk computations, overload theory for multichannel amplifiers, noise calculations on carrier systems and multihop radio systems, and in the evaluations of noise distributions on built-up connections between telephone subscribers. Here, however, the power levels are in many situations random rather than deterministic variables. Thus, in analogy with (2), one is faced with the random variable

$$P_n = 10 \log_{10} [10^{X_1/10} + \cdots + 10^{X_n/10}], \tag{3}$$

where each X_i is a random variable with known distribution. The classical power sum problem consists of finding the distribution function and the moments of the power sum P_n defined in (3). This problem does not, however, possess a simple closed-form mathematical solution. As a result, the task of finding approximate solutions has received extensive attention, beginning at least 35 years ago and persisting till this date.

Among earlier contributions to the problem, we can distinguish those that give specific methods for numerical evaluation of the power sum distribution without introducing any other approximations than those that are directly related to the numerical technique that is being used.[1, 3, 4, 5, 6] Another approach is based on approximating the power sum with a normally distributed random variable.[2, 7] This approach, due to R. I. Wilkinson,[2] is quite appealing, since it leads to simple evaluation formulas. Moreover, it has now been put on a firm mathematical foundation with the development of a limit theorem by N. A. Marlow. In a companion paper,[8] he proves that power sums are asymptotically normally distributed, provided some mild conditions on the component variables are satisfied.

The present paper considers power sums of independent, identically distributed, truncated normal random variables, since this is a situation of considerable practical importance in transmission engineering work. Two approaches are being used. In the first one, asymptotic expressions are developed for the mean and variance of P_n. The second approach is based on Monte Carlo simulation.[9] This method has a number of distinct advantages over other numerical methods in that

(*i*) it can accept any number of component variables with arbitrarily specified distribution functions,

(*ii*) independence among the component variables is not required,

(*iii*) computation errors do not cumulate as more than two variables are added, and (*iv*) accuracy can be determined through the evaluation of confidence limits.

Our main results are numerical estimates of moments of P_n and selected graphs of its distribution function. A wide range of component distributions is covered with n ranging from 2 to 256. Most of the results are based on a nominal symmetric truncation of the component variables at ± 3.5 standard deviations from the mean. In addition, the effect on P_n of choosing other truncation points is discussed, and some general trends are developed.

II. ANALYTICAL RESULTS

Consider first the case where the X_i are independent, identically distributed random variables. Assume that the expectation

$$\theta = E[10^{X_1/10}]$$

and the central moments

$$\tau_j = E[10^{X_1/10} - \theta]^j,$$

exist and are finite for a sufficiently large range of j. We require $-1 \leqq j \leqq 8$ to derive the results for the mean of P_n, $-2 \leqq j \leqq 12$ for the variance and wider ranges for higher-order moments.

Rewrite the power sum P_n of $X_1, X_2, \cdots, X_n$, as

$$P_n = 10 \log_{10} S_n , \tag{4}$$

where

$$S_n = 10^{X_1/10} + \cdots + 10^{X_n/10}.$$

Now expand (4) in a finite Taylor series about the mean, $n\theta$, of S_n. This gives

$$P_n = \frac{1}{\lambda}\left[\log(n\theta) + \frac{S_n - n\theta}{n\theta} - \frac{1}{2}\left(\frac{S_n - n\theta}{n\theta}\right)^2 + \cdots + \frac{(-1)^{m+1}}{m}\left(\frac{S_n - n\theta}{n\theta}\right)^m + R_m\left(\frac{S_n - n\theta}{n\theta}\right)\right], \tag{5}$$

where

$$\lambda = \frac{1}{10 \log_{10} e} \approx 0.23026$$

and log stands for $\log_e$. The remainder term in (5) can be expressed in integral form as

$$R_m(x) = (-1)^m x^{m+1} \int_0^1 \frac{t^m\,dt}{1 + xt}, \qquad x > -1, \tag{6}$$

or, alternatively, as

$$R_m(x) = \frac{(-1)^m}{m+1}\left(\frac{x}{1 + \delta x}\right)^{m+1}, \qquad x > -1, \tag{7}$$

where $0 < \delta < 1$.

With $R_m(x)$ given by (7), one obtains

$$R_m\left(\frac{S_n - n\theta}{n\theta}\right) \leqq 0 \quad \text{for} \quad m \text{ odd},$$

so that from (5) we get our first result

$$E(P_n) \leqq LAP_n, \tag{8}$$

where

$$LAP_n = 10 \log_{10}(n\theta) = \frac{1}{\lambda} \log(n\theta) \tag{9}$$

is the level of average power.

To derive asymptotic expressions for the moments of P_n, we apply the Lemma in Appendix A and (6) to get

$$E\left[\left(\frac{S_n - n\theta}{n\theta}\right)^\alpha \left(R_m\left(\frac{S_n - n\theta}{n\theta}\right)\right)^\beta\right] = O(n^{-\frac{1}{2}(\alpha+\beta(m+1))}) \tag{10}$$

Next, to derive an asymptotic expression for $E(P_n)$, we take the expected value of both sides of (5) with $m = 3$. An application of (10) then gives

$$E(P_n) = LAP_n - \frac{\tau_2}{2\lambda\theta^2}\frac{1}{n} + O(1/n^2) \quad \text{as} \quad n \to \infty. \tag{11}$$

Here the independence of the component variables has been used to express the variance of S_n as $n\tau_2$, and the third central moment of S_n as $n\tau_3$. The term containing τ_3 is of order $1/n^2$.

To arrive at an asymptotic expression for the variance $\sigma^2(P_n)$, we use (5) with $m = 2$ and (11) to get

$$P_n - E(P_n) = \frac{1}{\lambda}\frac{S_n - n\theta}{n\theta} - \frac{1}{2\lambda}\left(\frac{S_n - n\theta}{n\theta}\right)^2 + \frac{1}{\lambda} R_2\left(\frac{S_n - n\theta}{n\theta}\right) + O(1/n) \quad \text{as} \quad n \to \infty. \tag{12}$$

Squaring (12), taking the expected value of both sides, and applying (10) to four of the resulting terms gives

$$\sigma^2(P_n) = \frac{\tau_2}{\lambda^2\theta^2}\frac{1}{n} + O(1/n^2) \quad \text{as} \quad n \to \infty. \tag{13}$$

A similar approach can be used to derive asymptotic expressions for higher-order moments. The measures of skewness and excess, denoted by $\gamma_1(P_n)$ and $\gamma_2(P_n)$, respectively, are defined by

$$\gamma_1(P_n) = \frac{E(P_n - EP_n)^3}{\sigma^3(P_n)}$$

and

$$\gamma_2(P_n) = \frac{E(P_n - EP_n)^4}{\sigma^4(P_n)} - 3.$$

They are found to satisfy the expressions

$$\gamma_1(P_n) = \left[\frac{\tau_3}{\tau_2^{\frac{3}{2}}} - \frac{3\tau_2^{\frac{1}{2}}}{\theta}\right]\frac{1}{n^{\frac{1}{2}}} + O(1/n^{\frac{3}{2}}) \quad \text{as} \quad n \to \infty \tag{14}$$

and

$$\gamma_2(P_n) = \left[\frac{\tau_4}{\tau_2^2} - \frac{12\tau_3}{\theta\tau_2} + \frac{20\tau_2}{\theta^2} - 3\right]\frac{1}{n} + O(1/n^2) \quad \text{as} \quad n \to \infty. \tag{15}$$

The asymptotic results given in (11), (13), (14), and (15) are all consistent with Marlow's normal limit theorem.[8] The main virtue of the asymptotic results above is that they indicate the rate of convergence of the four quantities considered. This is of practical interest since engineering applications often involve a finite and fairly small number of component variables.

In the particular case where the X_i are truncated normal with mean 0 dB, standard deviation before truncation of σ dB, and symmetric truncation at $\pm c\sigma$ dB, the results contained in Appendix B can be used to express (11), (13), (14), and (15) in terms of σ, c, and n. For the mean and the variance we get, respectively,

$$\mu(P_n) = LAP_n - \frac{\exp(\lambda^2\sigma^2)U_c(\sigma) - 1}{2\lambda n} + O(1/n^2) \tag{16}$$

and

$$\sigma^2(P_n) = \frac{\exp(\lambda^2\sigma^2)U_c(\sigma) - 1}{\lambda^2 n} + O(1/n^2), \tag{17}$$

where the truncation factor $U_c(\sigma)$ is defined by

$$U_c(\sigma) = \frac{T_c(2\sigma)}{T_c^2(\sigma)},$$

with

$$T_c(\sigma) = \frac{\Phi(c - \lambda\sigma) - \Phi(-c - \lambda\sigma)}{\Phi(c) - \Phi(-c)}$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)\, dt.$$

Derivation of the Wilkinson estimates for the mean and variance of the power sum P_n is given in Appendix B. This derivation uses the same ideas employed by R. I. Wilkinson in 1934.[2] Thus, P_n is approximated by a normally distributed random variable P_{nw}. As above, the components are independent, identically distributed truncated normal with mean 0 dB, standard deviation before truncation of σ dB, and truncation at $\pm c\sigma$ dB. From Appendix B we then have

$$\mu(P_{nw}) = LAP_n - 5 \log_{10}\left[1 + \frac{\exp(\lambda^2\sigma^2)U_c(\sigma) - 1}{n}\right] \tag{18}$$

$$\sigma^2(P_{nw}) = \frac{10}{\lambda} \log_{10}\left[1 + \frac{\exp(\lambda^2\sigma^2)U_c(\sigma) - 1}{n}\right]. \tag{19}$$

The first terms in the asymptotic expansion of (18) and (19), respectively, agree exactly with the results in (16) and (17). This agreement establishes the important result that expressions (18) and (19) are asymptotically correct to the order of n included in (16) and (17). Finally, we note that the actual result due to Wilkinson is contained in (18) and (19); the case with nontruncated component variables is obtained by putting the truncation factor $U_c(\sigma) = 1$.

III. MONTE CARLO RESULTS FOR C = 3.5

Having established analytical estimates for the mean and variance of power sums of truncated normal random variables, let us now turn to estimation using the Monte Carlo technique. The power sum problem is basically solved by estimating the distribution function of P_n. Using the Monte Carlo method, one obtains an estimate of this function by random sampling. Each sample of the power sum is obtained by selecting

n independent samples, one from each of the component distributions on the dB scale. The corresponding sample value of the power sum is then directly computed from (2). For the results presented here, the component samples have been selected via computer generation of so-called pseudo-random numbers. These have approximately a uniform distribution over the unit interval. Using the inverse error function together with nominal truncation at $\pm 3.5\sigma$ gave a random variable with truncated normal distribution. Because of requirements of computing speed, this transformation has been achieved via a table look-up scheme with values of the transformation stored in the computer memory.

Table I summarizes Monte Carlo results in terms of estimates of the mean $\mu(P_n)$, the standard deviation $\sigma(P_n)$, and the measures of skewness and excess $\gamma_1(P_n)$ and $\gamma_2(P_n)$. Monte Carlo estimates of these quantities are denoted by the corresponding latin letters $m(P_n)$, $s(P_n)$, $g_1(P_n)$, and $g_2(P_n)$. The standard deviation and the measures of skewness and excess are estimated directly by the corresponding characteristics of the sample distribution. The mean is estimated through the formula

$$m(P_n) = LAP_n - (LAP_{MC} - m_{MC}). \tag{20}$$

The value of LAP_n is computed exactly from relation (9), while LAP_{MC} and m_{MC} are the LAP and the mean, respectively, of the sample distribution. The mean $\mu(P_n)$ could also be estimated by m_{MC}. However, $m(P_n)$ from (20) is preferred over m_{MC} because the Monte Carlo results show that it has a smaller sampling variance.

An indication of the accuracy of the results in Table I is given by the number of decimals included. The half-width of the 99 percent confidence interval that represents the sampling uncertainty is between one and five times the unit in the least significant digit. For the mean, the confidence interval width has, however, been computed for m_{MC} instead of for $m(P_n)$. The computation of these confidence intervals has been based on the asymptotic normality of the corresponding statistics.

Table I shows that the mean of the power sum increases by somewhat more than 3 dB when the number of component variables is doubled for a fixed σ. This effect is illustrated in Fig. 1, where the mean is plotted as a function of the number of components n. This figure shows that the increase in the mean is substantially more than 3 dB for a doubling of the number of components n in case n is small and σ is large. On the other hand, Fig. 1 indicates that the slope of the

TABLE I—MONTE CARLO ESTIMATES $m(P_n)$, $s(P_n)$, $g_1(P_n)$, $g_2(P_n)$ OF MEAN, STANDARD DEVIATION, MEASURE OF SKEWNESS, AND MEASURE OF EXCESS OF P_n. THE COMPONENTS ARE TRUNCATED NORMAL WITH MEAN $\mu = 0$, STANDARD DEVIATION BEFORE TRUNCATION σ, TRUNCATION AT $\pm 3.5\sigma$.

	σ:	1	2	3	4	5	6	7	8	9	10
$n = 2$	m	3.07	3.23	3.5	3.8	4.1	4.5	5.1	5.5	6.1	6.6
	s	0.70	1.43	2.20	3.0	3.8	4.6	5.4	6.2	7.1	7.9
	g_1	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1
	g_2	0.0	0.0	0.0	0.0	0.0	0.1	0.0	0.0	0.0	0.0
$n = 4$	m	6.11	6.36	6.75	7.3	7.9	8.6	9.4	10.2	11.1	12.1
	s	0.50	1.03	1.60	2.25	2.9	3.5	4.2	4.9	5.6	6.3
	g_1	0.0	0.0	0.0	0.0	0.1	0.2	0.2	0.2	0.2	0.2
	g_2	0.0	0.0	0.0	0.0	0.1	0.1	0.1	0.0	0.0	0.0
$n = 8$	m	9.13	9.43	9.90	10.54	11.3	12.2	13.2	14.3	15.4	16.7
	s	0.36	0.73	1.15	1.64	2.14	2.7	3.3	3.9	5.4	5.0
	g_1	0.0	0.0	0.0	0.1	0.2	0.2	0.2	0.3	0.3	0.3
	g_2	0.0	0.0	0.0	0.1	0.1	0.1	0.2	0.2	0.2	0.1
$n = 16$	m	12.15	12.47	12.99	13.70	14.59	15.6	16.8	18.0	19.4	20.9
	s	0.252	0.52	0.82	1.19	1.56	2.01	2.51	3.1	3.6	4.0
	g_1	0.0	0.0	0.0	0.1	0.2	0.2	0.3	0.3	0.4	0.4
	g_2	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.1	0.1	0.2
$n = 32$	m	15.162	15.49	16.04	16.79	17.74	18.86	20.12	21.5	23.0	24.7
	s	0.178	0.37	0.59	0.84	1.14	1.48	1.91	2.36	2.8	3.3
	g_1	0.0	0.0	0.0	0.1	0.2	0.2	0.3	0.3	0.4	0.4
	g_2	0.0	0.0	0.0	0.1	0.2	0.1	0.0	0.0	0.1	0.2
$n = 64$	m	18.174	18.51	19.07	19.84	20.82	21.99	23.34	24.85	26.5	28.2
	s	0.126	0.260	0.42	0.61	0.82	1.10	1.42	1.76	2.18	2.6
	g_1	0.0	0.0	0.1	0.1	0.2	0.2	0.3	0.3	0.4	0.4
	g_2	0.0	0.0	0.1	0.1	0.2	0.0	0.0	−0.1	0.0	0.0
$n = 128$	m	21.185	21.525	22.09	22.87	23.87	25.07	26.46	28.02	29.73	31.6
	s	0.089	0.184	0.29	0.43	0.59	0.79	1.04	1.30	1.63	1.98
	g_1	0.0	0.0	0.0	0.1	0.1	0.1	0.2	0.2	0.3	0.3
	g_2	0.0	0.0	0.0	0.0	0.1	0.0	−0.1	−0.2	−0.1	−0.1
$n = 256$	m	24.196	24.537	25.10	25.89	26.90	28.12	29.53	31.13	32.89	34.81
	s	0.063	0.132	0.208	0.30	0.42	0.55	0.74	0.95	1.20	1.48
	g_1	0.0	0.0	0.0	0.1	0.1	0.1	0.1	0.2	0.2	0.2
	g_2	0.0	0.0	0.0	0.0	0.0	−0.1	0.0	−0.1	−0.2	−0.2

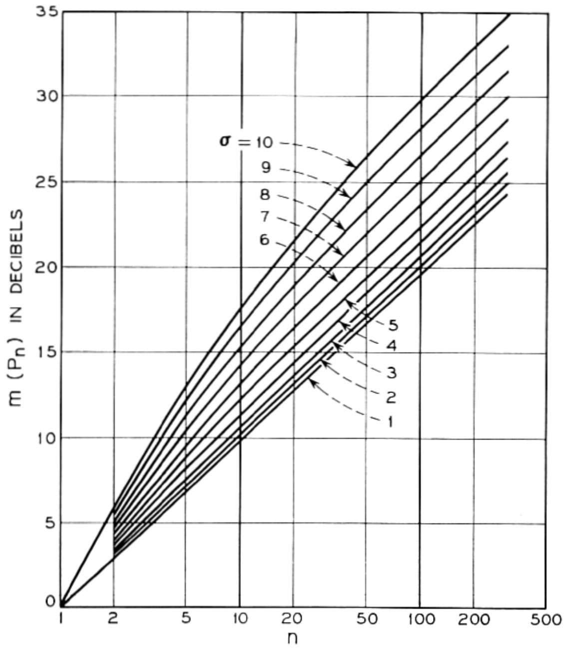

Fig. 1 — Monte Carlo estimates of $\mu(P_n)$. The components are truncated normal; $\mu = 0$, truncation at $\pm 3.5\sigma$.

graph of the mean levels off at approximately 3 dB for each doubling of the number of components at all values of σ for n large enough.

It is illuminating to compare these properties of the mean with the properties of LAP_n. According to relation (9), LAP_n increases by $10 \log_{10} 2 \approx 3$ dB for each doubling of the number of components n, similar to the increase of the mean noted above. Furthermore, relations (8) and (11) imply that $LAP_n - \mu(P_n)$ is nonnegative and approaches 0 as n increases toward infinity. The rate of decrease of $LAP_n - \mu(P_n)$ is illustrated by the Monte Carlo results plotted in Fig. 2.

Table I also shows that the standard deviation of the power sum decreases as the number of component variables is increased for fixed σ. This is illustrated in Fig. 3, where Monte Carlo estimates of $\sigma(P_n)$ are plotted as a function of the number of components n.

The measures of skewness and excess in Table I can be taken as an indication of the deviation from normality of the distribution of the power sum. These measures are zero for the normal distribution and they have low values for distributions that deviate only slightly from

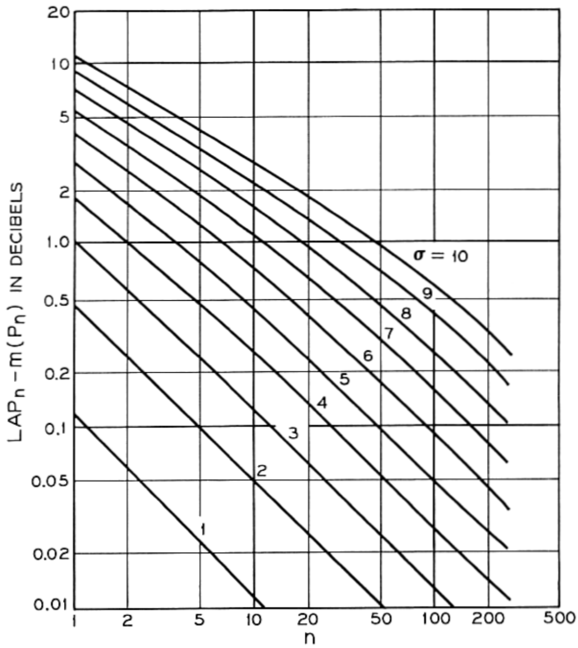

Fig. 2—Monte Carlo estimates of LAP$_n$-$\mu(P_n)$. The components are truncated normal; $\mu = 0$, truncation at $\pm 3.5\sigma$.

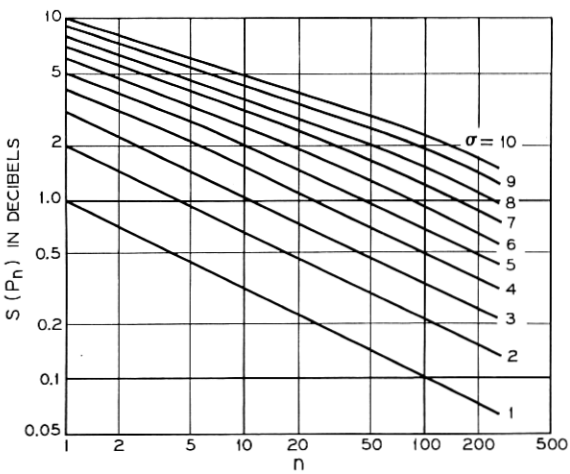

Fig. 3—Monte Carlo estimates of $\sigma(P_n)$. The components are truncated normal; $\mu = 0$, truncation at $\pm 3.5\sigma$.

normality. The table shows that g_1 and g_2 are very small for σ-values up to four over the range of n-values considered. The table also shows that g_1 is, in general, positive. This indicates that the power sum distribution is positively skewed. Moreover, g_1 considered as a function of the number of components n has definite maxima around $n = 32$ for all sufficiently large values of σ. In particular, this means that the magnitude of g_1 decreases as n becomes large enough. This behavior is consistent with the asymptotic behavior of the measure of skewness as expressed by relation (14).

The results of the previous section show that both $LAP_n - \mu(P_n)$ and $\sigma(P_n)$ converge to 0 as n becomes infinite. From these two facts it follows that the distribution of $P_n - LAP_n$ converges to a distribution degenerate at 0. Fig. 4 illustrates this convergence by plots of the Monte Carlo estimates of the distribution function of P_n for $n = 1, 4, 16, 64$, and 256. This convergence is also illustrated in Fig. 5 where the 1 percent and 99 percent points of the distribution function of P_n are plotted in addition to the mean $m(P_n)$ and the level of average power LAP_n, for $\sigma = 10$. It is seen that the slope of the 1 percent point with a doubling of the number of components can be considerably larger than 3 dB, while the 99 percent point changes by somewhat less than 3 dB whenever the number of components is doubled. LAP_n does not represent a fixed percentage point on the distribution function as n is changing. It is, therefore, seen that the plots in Fig. 5 of some

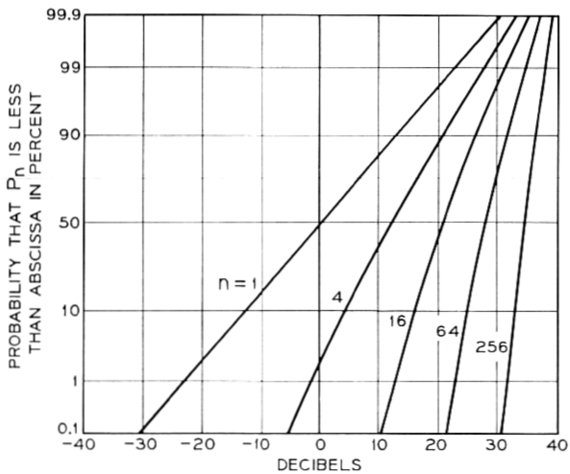

Fig. 4 — Monte Carlo estimates of distribution function of P_n. The components are truncated normal; $\mu = 0$, $\sigma = 10$, truncation at $\pm 3.5\sigma$.

percentage points would actually cross their asymptote LAP_n from below before approaching it asymptotically from above. This is true for all percentage points of the component distribution that lie between 0 and LAP_1. In other words, the fact that all percentage points approach LAP_n asymptotically does not imply that the approach is monotone.

IV. COMPARISON BETWEEN ANALYTICAL AND MONTE CARLO RESULTS

At this point it is natural to examine the relative agreement between the various analytical approximations and the Monte Carlo estimates. Figs. 6 and 7 contain plots of the asymptote (16), the Wilkinson approximation (18), and the Monte Carlo estimates of $LAP_n - \mu(P_n)$ for $\sigma = 6$ and 10, respectively. Both figures show the asymptote as an upper bound for $LAP_n - \mu(P_n)$. The plots also indicate that the Wilkinson expression gives a better agreement with the Monte Carlo results than the asymptote, and they illustrate the degree of agreement between the Monte Carlo results and the analytical expressions for various values of n. Finally, a comparison between the two figures shows that the analytical approximations are better for low values of σ than for high values. Figs. 8 and 9 present similar comparisons between Monte Carlo results and analytical approximations for $\sigma(N_n)$. The figures contain plots of the asymptotic expression (17), the Wilkinson expression (19), and the Monte Carlo estimate

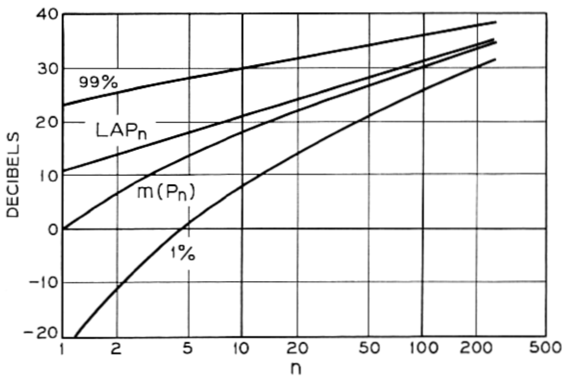

Fig. 5—LAP_n and Monte Carlo estimates of $\mu(P_n)$ and of two points on the distribution function of P_n. The components are truncated normal; $\mu = 0$, $\sigma = 10$, truncation at $\pm 3.5\sigma$.

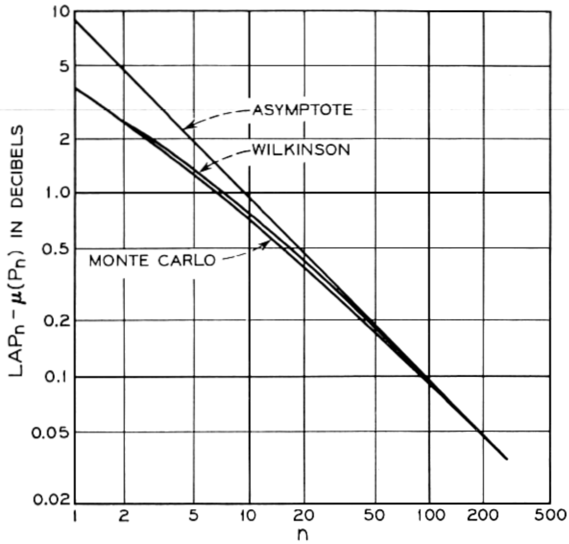

Fig. 6—Comparison between three estimates for LAP$_n$-$\mu(P_n)$. The components are truncated normal; $\mu = 0$, $\sigma = 6$, truncation at $\pm 3.5\sigma$.

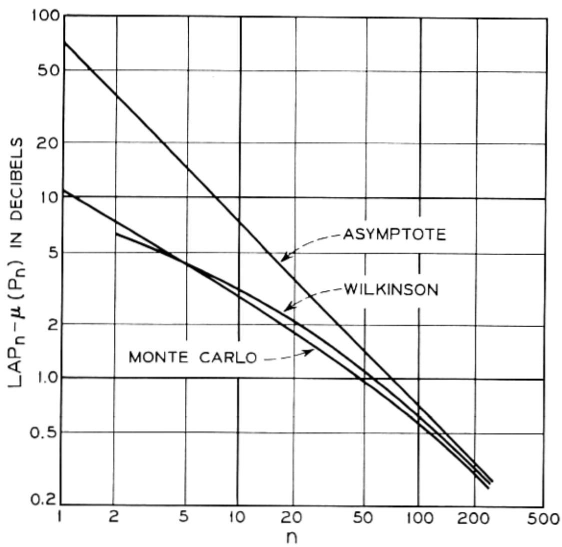

Fig. 7—Comparison between three estimates for LAP$_n$-$\mu(P_n)$. The components are truncated normal; $\mu = 0$, $\sigma = 10$, truncation at $\pm 3.5\sigma$.

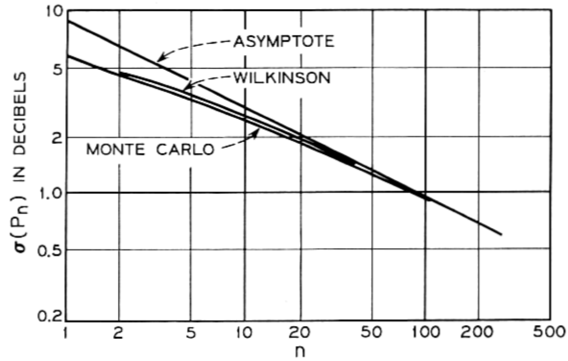

Fig. 8—Comparison between three estimates for $\sigma(P_n)$. The components are truncated normal; $\mu = 0$, $\sigma = 6$, truncation at $\pm 3.5\sigma$.

of $\sigma(P_n)$ for $\sigma = 6$ and 10, respectively. The figures serve as a basis for conjecturing that the asymptote provides an upper bound for $\sigma(P_n)$. Furthermore, the figures indicate as above the degree of agreement between the analytic approximations and the Monte Carlo results, and they show that the analytical approximations are better for low than for high values of σ.

V. INFLUENCE OF TAILS

The results discussed thus far are all based on a truncation of the component distributions at $\pm 3.5\sigma$. Truncations at other points can easily be studied with the tools used. Thus, Table II summarizes results

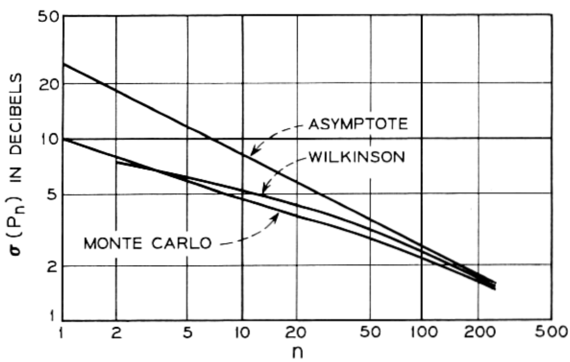

Fig. 9—Comparison between three estimates for $\sigma(P_n)$. The components are truncated normal; $\mu = 0$, $\sigma = 10$, truncation at $\pm 3.5\sigma$.

Mean $\mu = 0$, Standard Deviation before Truncation σ, Truncation at $\pm c\sigma$.

	c: / σ:	2.0 / 1	2.5 / 1	3.0 / 1	2.0 / 6	2.5 / 6	3.0 / 6	2.0 / 10	2.5 / 10	3.0 / 10
$n = 2$	m	3.06	3.06	3.07	4.3	4.4	4.6	5.9	6.3	6.4
	s	0.62	0.68	0.70	4.0	4.4	4.5	6.9	7.6	7.8
	g_1	−0.1	0.0	−0.1	−0.1	0.0	0.0	−0.2	−0.1	0.1
	g_2	−0.4	−0.3	−0.2	−0.5	−0.3	−0.1	−0.5	−0.3	−0.1
$n = 4$	m	6.09	6.10	6.10	8.1	8.4	8.6	10.9	11.5	11.8
	s	0.44	0.48	0.50	2.9	3.3	3.4	5.2	5.9	6.2
	g_1	−0.1	0.0	0.0	−0.3	−0.1	0.1	−0.3	−0.1	0.1
	g_2	−0.2	−0.1	0.0	−0.3	−0.3	−0.1	−0.4	−0.3	−0.1
$n = 8$	m	9.11	9.12	9.13	11.6	12.0	12.2	15.1	16.0	16.5
	s	0.31	0.34	0.35	2.06	2.39	2.60	3.7	4.5	4.9
	g_1	0.0	0.0	0.0	−0.3	−0.2	0.1	−0.3	−0.1	0.2
	g_2	−0.1	0.0	0.0	−0.1	−0.2	−0.1	−0.3	−0.4	−0.2
$n = 16$	m	12.12	12.14	12.15	14.80	15.30	15.53	18.9	20.0	20.6
	s	0.222	0.241	0.248	1.44	1.70	1.92	2.61	3.3	3.8
	g_1	0.0	0.0	0.0	−0.3	−0.1	0.1	−0.5	−0.1	0.2
	g_2	−0.1	0.0	0.0	0.1	−0.3	−0.2	0.0	−0.4	−0.3
$n = 32$	m	15.137	15.153	15.160	17.93	18.47	18.75	22.22	23.6	24.4
	s	0.157	0.170	0.174	1.00	1.21	1.40	1.81	2.39	2.96
	g_1	0.0	0.0	0.0	−0.3	−0.1	0.1	−0.4	−0.2	0.1
	g_2	−0.1	0.1	−0.1	0.2	−0.2	−0.2	0.1	−0.2	−0.4
$n = 64$	m	18.149	18.165	18.172	20.99	21.56	21.87	25.41	26.93	27.8
	s	0.110	0.121	0.125	0.70	0.86	1.00	1.25	1.70	2.20
	g_1	0.0	0.0	0.0	−0.2	−0.1	0.0	−0.3	−0.3	0.0
	g_2	0.0	0.0	−0.1	0.1	−0.1	−0.2	0.1	−0.1	−0.4
$n = 128$	m	21.160	21.176	21.183	24.03	24.61	24.93	28.51	30.10	31.10
	s	0.078	0.086	0.088	0.50	0.61	0.71	0.87	1.19	1.58
	g_1	0.0	0.0	0.0	−0.1	−0.1	0.0	−0.2	−0.2	0.0
	g_2	0.0	0.0	0.0	−0.1	−0.2	−0.1	0.1	−0.1	−0.2
$n = 256$	m	24.170	24.186	24.194	27.06	27.65	27.97	31.56	33.19	34.25
	s	0.055	0.060	0.062	0.35	0.43	0.50	0.61	0.84	1.13
	g_1	0.0	0.0	0.0	−0.1	−0.1	0.0	−0.2	−0.2	−0.1
	g_2	0.0	0.0	0.0	0.0	−0.1	0.0	0.1	0.0	−0.1

of Monte Carlo evaluations for symmetric truncations at $\pm 2\sigma$, $\pm 2.5\sigma$, and $\pm 3\sigma$ for $\sigma = 1$, 6, and 10 dB, respectively, and with the same range of n-values as considered previously. A study of the table reveals that the truncation point can have a considerable influence on the distribution of the resulting power sum. To exemplify this, Fig. 10 shows plots of the standard deviation of the power sum of 256 components as a function of the truncation point c. The plots cover a wider range of c-values and σ-values than found in Tables I and II. The extensions are based on the Wilkinson approximation.

The plots in Fig. 10 exhibit the important trend that the influence of the truncation point increases with an increase of the component standard deviation σ. The same conclusion can be drawn from a study of the c-dependence of the mean $\mu(P_n)$ or of the quantity $LAP_n - \mu(P_n)$.

Table II contains several cases of negative skewness of P_n. Hence, the earlier observation that P_n is in general positively skewed does not apply for c-values below 3.5.

VI. CONCLUDING REMARKS

The extension of the results given here to an even larger number of components ($n > 256$) is straightforward, but the computer time needed can easily become excessive. The agreement betweent asymptotic expressions and Monte Carlo results for large enough n does, however, indicate that the Monte Carlo technique is not necessary for

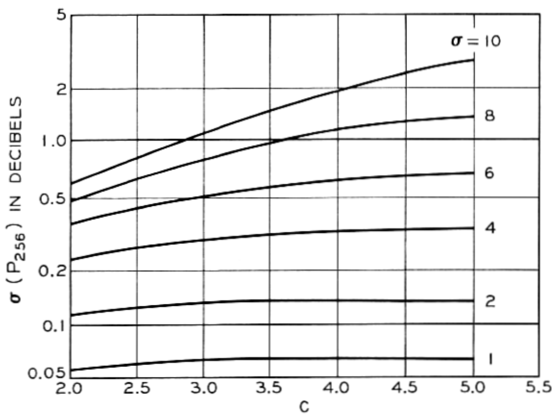

Fig. 10—Estimates of $\sigma(P_{256})$. The components are truncated normal; $\mu = 0$, truncation at $\pm c\sigma$.

power sum evaluations beyond a certain n-value, namely, the one where the asymptotic expressions become sufficiently accurate.

Finally, we note that the problem of evaluating the distribution of the power sum of nontruncated normal components has not been brought closer to its solution by the results presented here. This problem is certainly of mathematical interest even though it represents a physically unrealistic situation. Some Monte Carlo studies with larger values for the truncation points have indicated that the convergence of the power sum to normality is much less rapid in this case, and that considerably larger values of the measures of skewness and excess can occur than those contained in Table I.

VII. ACKNOWLEDGMENTS

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APPENDIX A

Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random variables. Put

$$S_n = 10^{X_1/10} + \cdots + 10^{X_n/10}$$

and let $\theta = E[10^{X_1/10}]$. In order to prove the asymptotic results in the main body of the paper, we need the following.

Lemma: *Suppose*

$$Q(x) = x^l\left[\int_0^1 \frac{t^m\, dt}{1+xt}\right]^j, \qquad x > -1$$

where l, j, m *are nonnegative integers. If* $E10^{2lX_1/10}$ *and* $E10^{-jX_1/10}$ *are bounded, then*

$$E\left[Q\left(\frac{S_n - n\theta}{n\theta}\right)\right] = O(n^{-l/2}) \quad \text{as} \quad n \to \infty.$$

Proof: Let

$$I_m(x) = \int_0^1 \frac{t^m}{1+xt}\, dt, \qquad x > -1.$$

Then $Q(x) = x^l[I_m(x)]^j$, and it follows from the Cauchy-Schwarz inequality that

$$\left| E\left[Q\left(\frac{S_n - n\theta}{n\theta}\right)\right]\right|^2 \leqq E\left(\frac{S_n - n\theta}{n\theta}\right)^{2l} E\left[I_m\left(\frac{S_n - n\theta}{n\theta}\right)\right]^{2j}.$$

The asymptotic behavior of the central moment of S_n of order $2l$ is found from Cramér.[10] Hence,

$$\left| E\left[Q\left(\frac{S_n - n\theta}{n\theta}\right)\right]\right|^2 = O(n^{-l})E\left[I_m\left(\frac{S_n - n\theta}{n\theta}\right)\right]^{2j} \quad \text{as} \quad n \to \infty.$$

To complete the proof, it suffices to show that

$$E\left[I_m\left(\frac{S_n - n\theta}{n\theta}\right)\right]^{2j} = O(1) \quad \text{as} \quad n \to \infty.$$

To show this, we again apply the Cauchy-Schwarz inequality. Thus,

$$I_m^2(x) \leqq \int_0^1 t^{2m}\, dt \int_0^1 \frac{dt}{(1 + xt)^2} = \frac{1}{2m + 1}\frac{1}{x + 1}.$$

Hence,

$$E\left[I_m\left(\frac{S_n - n\theta}{n\theta}\right)\right]^{2j} \leqq \left(\frac{\theta}{2m + 1}\right)^j E\left(\frac{n}{S_n}\right)^j.$$

Consider now the function $u(x) = 1/x^j$, which is convex on $(0, \infty)$ for $j \geqq 0$. By Jensen's inequality it follows that if $\alpha_1, \cdots, \alpha_n, y_1, \cdots, y_n$ are non-negative real numbers such that $\alpha_1 + \cdots + \alpha_n = 1$, then

$$u(\alpha_1 y_1 + \cdots + \alpha_n y_n) \leqq \alpha_1 u(y_1) + \cdots + \alpha_n u(y_n).$$

In particular,

$$\begin{aligned}(n/S_n)^j &= u(S_n/n) \leqq (1/n)[u(10^{X_1/10}) + \cdots + u(10^{X_n/10})] \\ &= (1/n)[10^{-jX_1/10} + \cdots + 10^{-jX_n/10}].\end{aligned}$$

Hence,

$$E(n/S_n)^j \leqq E[10^{-jX_1/10}].$$

The right-hand side of this inequality is finite by assumption, so the proof is complete.

APPENDIX B

Derivation of the Wilkinson Results for Truncated Normal Components

As in Appendix A, let $X_1, X_2, \ldots, X_n$ be independent, identically distributed random variables, and assume further that they all have a truncated normal distribution. The density function of X_1 is then

$$g(x) = \begin{cases} 0, & x < \mu - c\sigma, \quad x > \mu + c\sigma \\ \dfrac{1}{\Phi(c) - \Phi(-c)} \dfrac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right), & \\ & \mu - c\sigma \leqq x \leqq \mu + c\sigma \end{cases}$$

where Φ stands for the standardized normal distribution function.

Now let W_i be the nonnegative random variable that expresses the power corresponding to X_i, i.e.,

$$W_i = 10^{X_i/10}.$$

The density function of W_1 is

$$f(w) = \begin{cases} 0, \quad w < 10^{(\mu - c\sigma)/10}, \quad w > 10^{(\mu + c\sigma)/10} & \\ \dfrac{1}{\Phi(c) - \Phi(-c)} \dfrac{1}{\sqrt{2\pi}\,\sigma\lambda w} \exp\left(-\dfrac{(\log w - \lambda\mu)^2}{2\lambda^2\sigma^2}\right), & \\ & 10^{(\mu - c\sigma)/10} \leqq w \leqq 10^{(\mu + c\sigma)/10} \end{cases}$$

The moments of W_1 are therefore,

$$EW_1^k = \int_0^\infty w^k f(w)\, dw = \exp\left(k\lambda\mu + \tfrac{1}{2}k^2\lambda^2\sigma^2\right)T_c(k\sigma),$$

where

$$T_c(\sigma) = \frac{\Phi(c - \lambda\sigma) - \Phi(-c - \lambda\sigma)}{\Phi(c) - \Phi(-c)}$$

accounts for the effect of the truncation. We note that $T_c(\sigma) \to 1$ as $c \to \infty$.

The mean and variance of W_1 are found to be

$$\theta = EW_1 = \exp\left(\lambda\mu + \tfrac{1}{2}\lambda^2\sigma^2\right)T_c(\sigma) \tag{21}$$

and

$$\tau_2 = \operatorname{Var}(W_1) = \exp\left(2\lambda\mu + \lambda^2\sigma^2\right)T_c^2(\sigma)[\exp(\lambda^2\sigma^2)U_c(\sigma) - 1], \tag{22}$$

where

$$U_c(\sigma) = \frac{T_c(2\sigma)}{T_c^2(\sigma)}.$$

Now let P_n be the power sum of $X_1, X_2, \cdots, X_n$ and take $\mu = 0$. Furthermore, let P_n be approximated by a normally distributed random variable P_{nw}. The independence of the X_i's then allows us to establish two equations by adding the means and variances of the W_i's to get the mean and variance, respectively, of

$$S_{nw} = 10^{P_{nw}/10} = W_1 + \cdots + W_n .$$

Relations (21) and (22) allow mean and variance of S_{nw} to be expressed in terms of mean and variance of P_{nw}. Hence, we get

$$n \exp (\tfrac{1}{2}\lambda^2\sigma^2)T_c(\sigma) = \exp [\lambda\mu(P_{nw}) + \tfrac{1}{2}\lambda^2\sigma^2(P_{nw})]$$

and

$$n \exp (\lambda^2\sigma^2)T_c^2(\sigma)[\exp (\lambda^2\sigma^2)U_c(\sigma) - 1] = \exp [2\lambda\mu(P_{nw}) + \lambda^2\sigma^2(P_{nw})][\exp \lambda^2\sigma^2(P_{nw}) - 1].$$

Solving these two equations for $\mu(P_{nw})$ and $\sigma^2(P_{nw})$ we find

$$\mu(P_{nw}) = LAP_n - \frac{1}{2\lambda} \log \left[1 + \frac{\exp (\lambda^2\sigma^2)U_c(\sigma) - 1}{n}\right]$$

and

$$\sigma^2(P_{nw}) = \frac{1}{\lambda^2} \log \left[1 + \frac{\exp (\lambda^2\sigma^2)U_c(\sigma) - 1}{n}\right],$$

where

$$LAP_n = \frac{1}{\lambda} \log n + \tfrac{1}{2}\lambda\sigma^2 + \frac{1}{\lambda} \log T_c(\sigma).$$

REFERENCES

1. Dixon, J. T., unpublished work, 1932.
2. Wilkinson, R. I., unpublished work, 1934.
3. Holbrook, B. D. and Dixon, J. T., Load Rating Theory for Multichannel Amplifiers, B.S.T.J., *18,* October, 1939, p. 624.
4. Curtis, H. E., Probability Distribution of Noise Due to Fading on Multisection FM Microwave Systems, IRE Trans. Commun. Syst., September, 1959, p. 161.
5. Roberts, J. H., Sums of Probability Distributions Expressed in Decibel Steps, Proc. IEE, *110,* No. 4, April, 1963, p. 692.
6. Derzai, M., Power Addition of Independent Random Variables Normally Distributed on a dB Scale, 1967 IEEE International Convention Record, I, p. 40. March, 1967.
7. Fenton, L. F., The Sum of Log-Normal Probability Distributions in Scatter Transmission Systems, IRE Trans. Commun. Syst., March, 1960, p. 57.
8. Marlow, N. A., A Normal Limit Theorem for Power Sums of Independent Random Variables, B.S.T.J., this issue, p. 2081.
9. Cyr, M. H. and Thuswaldner, A., Multichannel Load Calculation Using the Monte Carlo Method, IEEE Trans. Commun. Tech., *COM-14,* No. 2, April, 1966, p. 177.
10. Cramér, H., *Mathematical Methods of Statistics,* Princeton, 1945, p. 346.