A Normal Limit Theorem for Power Sums of Independent Random Variables

By N. A. MARLOW

(Manuscript received June 8, 1967)

Suppose that

$$P_n = 10 \log_{10} [10^{X_1/10} + \cdots + 10^{X_n/10}],$$

where $\{X_n\}$ is a sequence of independent random variables. The main result of this paper shows that under very general conditions on the sequence $\{X_n\}$, the power sums P_n will be asymptotically normally distributed. This result supports a commonly used normal approximation, and shows why many physical quantities obtained by power addition of random variables tend to be normally distributed in dB.

I. INTRODUCTION

In many areas of transmission engineering, logarithms of sums of powers are considered in the form

$$P_n = 10 \log_{10} [10^{X_1/10} + \cdots + 10^{X_n/10}],$$

where $X_1, \ldots, X_n$ are random variables. Specifically, if $X_1, \ldots, X_n$ are power levels in dB such that

$$X_j = 10 \log_{10} (w_j/w_0) \qquad j = 1, 2, \cdots, n,$$

where $w_0, w_1, \cdots, w_n$ are powers (e.g., expressed in watts), then the power level in dB of the sum $w \equiv w_1 + \cdots + w_n$ is given by the so-called "power sum,"

$$P_n = 10 \log_{10} (w/w_0) = 10 \log_{10} [10^{X_1/10} + \cdots + 10^{X_n/10}].$$

Quite often $X_1, \cdots, X_n$ are taken to be mutually independent random variables with specified distributions, and it is of interest to determine properties of their power sum P_n.

A major difficulty encountered in working with power sums is that the distribution and moments of such a sum usually cannot be expressed in simple closed form. This includes, for example, the important case when $X_1, \cdots, X_n$ are mutually independent and each has a truncated normal distribution. Even in the simpler case when $X_1, \cdots, X_n$ are mutually independent, identically distributed, and X_1 is normal, the problem is intractable. The difficulty and importance of the general problem, in turn, has led to a number of methods for approximating the distribution of a power sum.[1, 2, 3, 4, 5, 6, 7, 8, 9]

In the present paper, the asymptotic distribution of a power sum is studied. The main result is a limit theorem which shows that under very general conditions on the components $X_1, X_2, \cdots$, the corresponding power sums P_n will be asymptotically normal as $n \to \infty$. The particular *form* of the result is as follows: Given a sequence $\{X_n\}$ of mutually independent random variables satisfying certain conditions, there exist sequences of constants $\{c_n\}$ and $\{d_n\}$ such that

$$\lim_{n\to\infty} P\{[(P_n - c_n)/d_n] \leqq x\} = [1/\sqrt{2\pi}] \int_{-\infty}^{x} \exp[-t^2/2]\, dt. \tag{1}$$

The conditions for (1) to hold are the central concern of this paper, but the implications of the results are equally important. In particular, one of the oldest and most useful approximations to the distribution of a power sum is a normal approximation. This approximation was first used at Bell Telephone Laboratories in 1934 by R. I. Wilkinson,[2] and is based on the fact that many observed power sum distributions are "nearly normal." This includes power sum distributions obtained by numerical convolution, and empirical distributions of physical quantities such as noise levels on trunks and connections where the resultant noise (on a dB scale) can be viewed as an approximate power sum.[10, 11] The limit theorem proved in this paper thus provides mathematical support for a normal approximation, and substantially explains why many physical quantities obtained by power addition of random variables tend to be normally distributed in dB.

II. A NORMAL LIMIT THEOREM FOR POWER SUMS

2.1 *Discussion*

Before stating the main results, it is instructive to show informally why one would expect power sums to be asymptotically normal. To take a simple case, suppose that $\{X_n\}$ is a sequence of mutually independent, identically distributed random variables such that

$$\tau^2 \equiv \mathrm{Var}\,[10^{X_1/10}]$$

is finite. Let $\theta = E10^{X_1/10}$ and put

$$S_n = 10^{X_1/10} + \cdots + 10^{X_n/10}.$$

Then by the law of large numbers, one expects that for large n,

$$\frac{S_n}{n\theta} \approx 1.$$

Next, note that if $x \approx 1$, then $\log_e x \approx x - 1$ so for large n

$$\log_e \frac{S_n}{n\theta} \approx \frac{S_n - n\theta}{n\theta}.$$

Multiplication by $(\theta \sqrt{n})/\tau$ then gives

$$\frac{\theta\sqrt{n}}{\tau} \log_e \frac{S_n}{n\theta} \approx \frac{S_n - n\theta}{\tau\sqrt{n}}. \tag{2}$$

But, by the central limit theorem, the right-hand side of (2) is asymptotically normal with mean 0 and variance 1. Thus, it is strongly suggested that

$$\lim_{n\to\infty} P\left\{\frac{\theta\sqrt{n}}{\tau}\left[\log_e S_n - \log_e (n\theta)\right] \leqq x\right\} = [1/\sqrt{2\pi}] \int_{-\infty}^{x} \exp\left[-t^2/2\right] dt.$$

This, and more, is indeed true as will be shown.

2.2 *The Main Result*

The normal limit theorem for power sums is a consequence of the following result which will first be proved:

Lemma 1: Let $\{S_n\}$ be a sequence of positive random variables. Suppose there exist sequences of positive real numbers $\{a_n\}$ and $\{b_n\}$, and a distribution F such that

(i) At each point of continuity of F,

$$\lim_{n\to\infty} P\left\{\frac{S_n - a_n}{b_n} \leqq x\right\} = F(x)$$

(ii) $$\lim_{n\to\infty} (b_n/a_n) = 0.$$

Then at each point of continuity of F,

$$\lim_{n\to\infty} P\{(a_n/b_n) \log_e (S_n/a_n) \leqq x\} = F(x).$$

Proof: Let x be a continuity point of F, and let $\epsilon > 0$ be given. Because F has at most a countable number of discontinuities, there is a $\delta > 0$ such that F is continuous at $x + \delta$ and

$$F(x + \delta) - F(x) < \epsilon. \tag{3}$$

Next, define

$$U_n = (S_n - a_n)/b_n \quad \text{and} \quad V_n = (a_n/b_n) \log_e (S_n/a_n).$$

Then

$$| P\{V_n \leqq x\} - F(x) | \leqq | P\{V_n \leqq x\} - P\{U_n \leqq x\} | + | P\{U_n \leqq x\} - F(x) |.$$

By assumption (i) therefore,

$$\overline{\lim_{n\to\infty}} | P\{V_n \leqq x\} - F(x) | \leqq \overline{\lim_{n\to\infty}} | P\{V_n \leqq x\} - P\{U_n \leqq x\} |.$$

Let

$$\Delta_n(x) = | P\{V_n \leqq x\} - P\{U_n \leqq x\} |.$$

To complete the proof it suffices to show that

$$\overline{\lim_{n\to\infty}} \Delta_n(x) = 0.$$

To prove this note first from the inequality $\log_e x \leqq x - 1$, $x > 0$, that $V_n \leqq U_n$ for all n. Thus,

$$\begin{aligned} \Delta_n(x) &= P[\{V_n \leqq x\} \cap \{U_n > x\}] \\ &= P\{x < U_n \leqq (a_n/b_n)[\exp (b_n x/a_n) - 1]\}. \end{aligned}$$

Using the inequality $e^y - 1 \leqq ye^y$, $-\infty < y < \infty$, it follows that

$$0 \leqq \Delta_n(x) \leqq P\{x < U_n \leqq x \exp (b_n x/a_n)\}.$$

By assumption, $(b_n/a_n) > 0$ for all n and $\lim_{n\to\infty} (b_n/a_n) = 0$. Thus, there exists a natural number N such that $n \geqq N$ implies

$$x < x \exp (b_n x/a_n) \leqq x + \delta.$$

So if $n \geqq N$,

$$0 \leqq \Delta_n(x) \leqq P\{x < U_n \leqq x + \delta\}.$$

Because x and $x + \delta$ are continuity points of F, it follows by assumption (i) and inequality (3) that

$$0 \leqq \overline{\lim_{n\to\infty}}\, \Delta_n(x) \leqq F(x+\delta) - F(x) < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the proof is complete.

The importance of Lemma 1 is that it gives a sufficient condition to go from limit theorems for sums of random variables to limit theorems for logarithms of sums. In the important case of power sums of independent random variables, general conditions for asymptotic normality can thus be obtained from classical central limit theory as shown in the next result.

Theorem 1: Let $\{X_n\}$ be a sequence of mutually independent random variables and suppose that

$$\tau_j^2 \equiv \text{Var}\,[10^{X_j/10}]$$

is finite for every j. Let $\theta_j = E10^{X_j/10}$ and put .

$$M_n = \sum_{j=1}^{n} \theta_j\,, \qquad s_n^2 = \sum_{j=1}^{n} \tau_j^2\,.$$

Denote the distribution of $10^{X_j/10}$ by $H_j(x)$, and let

$$P_n = 10\,\log_{10}\,[10^{X_1/10} + \cdots + 10^{X_n/10}].$$

If the following conditions are satisfied:

(*i*) *The Lindeberg Condition: For every $\epsilon > 0$,*

$$\lim_{n\to\infty} \frac{1}{s_n^2} \sum_{j=1}^{n} \int_{A_{jn}} (x - \theta_j)^2\, dH_j(x) = 0,$$

where

$$A_{jn} = \{x\colon |\, x - \theta_j \,| \geqq \epsilon s_n\}$$

(*ii*)
$$\lim_{n\to\infty} (s_n/M_n) = 0$$

it will follow that

$$\lim_{n\to\infty} P\{(\lambda M_n/s_n)[P_n - 10\,\log_{10} M_n] \leqq x\} = \Phi(x) \tag{4}$$

where $\lambda = (\log_e 10)/10$ and

$$\Phi(x) = [1/\sqrt{2\pi}] \int_{-\infty}^{x} \exp\,[-t^2/2]\, dt.$$

Proof: Let

$$S_n = 10^{X_1/10} + \cdots + 10^{X_n/10}.$$

Then condition (*i*) implies that

$$\lim_{n\to\infty} P\left\{\frac{S_n - M_n}{s_n} \leqq x\right\} = \Phi(x)$$

(cf. Feller,[12] p. 256). With the identifications $a_n = M_n$ and $b_n = s_n$ it follows from condition (*ii*) and Lemma 1 that

$$\lim_{n\to\infty} P\{(M_n/s_n) \log_e (S_n/M_n) \leqq x\} = \Phi(x).$$

The assertion of the theorem then follows by changing to logarithms with base 10.

An interesting thing to note is that if the conditions of Theorem 1 are satisfied then the sum of powers

$$S_n = 10^{X_1/10} + \cdots + 10^{X_n/10}$$

and the power sum in dB, $P_n = 10 \log_{10} S_n$, will both be asymptotically normal. Thus, not only will normality be observed on a "power scale" but on a "dB scale" as well.

2.3 *Identically Distributed Components*

The preceding result implies the asymptotic normality of P_n when the components are identically distributed. To show this, suppose that $\{X_n\}$ is a sequence of mutually independent, identically distributed random variables with $H(x) = P\{10^{X_1/10} \leqq x\}$. Let

$$\tau^2 = \text{Var}\,[10^{X_1/10}]$$

and $\theta = E10^{X_1/10}$. If τ^2 is finite, condition (*ii*) of Theorem 1 is clearly satisfied since

$$\frac{s_n}{M_n} = \frac{\tau}{\theta\sqrt{n}}.$$

Condition (*i*) is also satisfied because if $\epsilon > 0$,

$$\frac{1}{s_n^2} \sum_{j=1}^{n} \int_{A_{jn}} (x - \theta_j)^2 \, dH_j(x) = \frac{1}{\tau^2} \int_{A_n} (x - \theta)^2 \, dH(x) \to 0 \text{ as } n \to \infty,$$

where $A_n = \{x : | x - \theta | \geqq \epsilon\tau \sqrt{n} \}$. It thus follows that

$$\lim_{n\to\infty} P\left\{\lambda \frac{\theta\sqrt{n}}{\tau} [P_n - 10 \log_{10} (n\theta)] \leqq x\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\,[-t^2/2]\, dt,$$

hence, P_n is asymptotically normal with mean $10 \log_{10}(n\theta)$ and variance $\tau^2/(n\lambda^2\theta^2)$.

2.4 *Bounded Components*

Suppose next that $\{X_n\}$ is a sequence of mutually independent random variables and that the following conditions are satisfied:

(*i*) There exist constants b and B such that

$$0 < b \leqq 10^{X_j/10} \leqq B \text{ for all } j$$

(*ii*) $$s_n^2 \to \infty \quad \text{as} \quad n \to \infty .$$

The conditions of Theorem 1 are easily shown to be satisfied in this case, and it follows that P_n will be asymptotically normal. Note that condition (*i*) will be satisfied whenever $10^{X_j/10}$ represents power from a physical source. Condition (*ii*), on the other hand, will be satisfied if $\tau_j^2 \geqq c > 0$ for some fixed c and an infinite number of indices j.

III. THE NORMAL LIMIT THEOREM AND WILKINSON'S NORMAL APPROXIMATION

One of the most useful approximations to the distribution and moments of a power sum is based on a normal approximation as mentioned in the introduction. The method consists of approximating the distribution of P_n by a normal distribution so that

$$P\{P_n \leqq x\} \approx P\{\alpha\xi + \beta \leqq x\},$$

where ξ is normal with mean 0 and variance 1. Writing as before,

$$M_n = E10^{P_n/10} \quad \text{and} \quad s_n^2 = \text{Var}\,[10^{P_n/10}],$$

the parameters α and β are chosen so that

$$M_n = E[10^{(\alpha\xi+\beta)/10}]$$

and

$$s_n^2 = \text{Var}\,[10^{(\alpha\xi+\beta)/10}]$$

which is equivalent to equating means and variances on a "power scale." If ξ is normal with mean 0 and variance 1 then

$$E[10^{(\alpha\xi+\beta)/10}] = e^{\lambda\beta}e^{\frac{1}{2}(\lambda\alpha)^2}$$

and

$$\text{Var}\,[10^{(\alpha\xi+\beta)/10}] = e^{2\lambda\beta}[e^{2\lambda^2\alpha^2} - e^{\lambda^2\alpha^2}],$$

where

$$\lambda = (\log_e 10)/10.$$

Solving the above equations for α and β, the approximation then asserts that P_n is normal with

$$E(P_n) = \beta = 10 \log_{10} M_n - 5 \log_{10} [1 + (s_n/M_n)^2] \tag{5}$$

and

$$\text{Var}(P_n) = \alpha^2 = \frac{10}{\lambda} \log_{10} [1 + (s_n/M_n)^2]. \tag{6}$$

In light of the normal limit theorem, it is quite natural to assume that P_n is approximately normal, provided the conditions of the theorem are satisfied, and n is large. On the other hand, the estimates given by (5) and (6) are different from those based on (4):

$$E(P_n) \doteq 10 \log_{10} M_n \tag{7}$$

$$\text{Var}(P_n) \doteq s_n^2/(\lambda M_n)^2. \tag{8}$$

The difference, however, is easily resolved once it is realized that if condition (*ii*) of Theorem 1 is satisfied then (5) and (6) are asymptotically equivalent to (7) and (8). In fact, it is a simple matter to show (cf. Feller,[12] p. 246) that if the conditions of Theorem 1 are satisfied then

$$\lim_{n\to\infty} P\{[(P_n - u_n)/\sqrt{v_n}] \leqq x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp[-t^2/2]\, dt, \tag{9}$$

where

$$u_n = 10 \log_{10} M_n - 5 \log_{10} [1 + (s_n/M_n)^2]$$

and

$$v_n = \frac{10}{\lambda} \log_{10} [1 + (s_n/M_n)^2].$$

In numerical applications, the normal approximation based on (9) is to be favored over that based on (4). In the first place, when $X_1, \cdots, X_n$ are mutually independent, identically distributed, and X_1 has a truncated normal distribution, Monte Carlo studies by I. Nåsell[9] have shown that the mean and variance estimates given by (5) and (6) are better than those given by (7) and (8) (although for large n and small variance of X_1 there is hardly any difference). Secondly, the normalizing factors in (9) were obtained quite naturally by equating moments on a power scale. This is analogous to the situation in classical central limit theory when the sequence $(S_n - M_n)/s_n$ converges in distribution to the standard normal. The normalizing factors M_n and s_n are not the only ones that give this result, but they are chosen in a natural way to insure that for every n, the mean and variance of $(S_n - M_n)/s_n$ agrees with its asymptotic distribution.

IV. ACKNOWLEDGMENTS

I would like to thank D. A. Lewinski and I. Nåsell for their helpful comments and suggestions.

REFERENCES

1. Dixon, J. T., unpublished work, 1932.
2. Wilkinson, R. I., unpublished work, 1934.
3. Holbrook, B. D. and Dixon, J. T., Load Rating Theory for Multichannel Amplifiers, B.S.T.J., *18,* October, 1939, p. 624.
4. Curtis, H. E., Probability Distribution of Noise Due to Fading on Multisection FM Microwave Systems, IRE Trans. Commun. Syst., September, 1959, p. 161.
5. Fenton, L. F., The Sum of Log-Normal Probability Distributions in Scatter Transmission Systems, IRE Trans. Commun. Syst., March 1960, p. 57.
6. Roberts, J. H., Sums of Probability Distributions Expressed in Decibel Steps, Proc. IEE, *110,* No. 4, April, 1963, p. 692.
7. Cyr, M. H. and Thuswaldner, A., Multichannel Load Calculation Using the Monte Carlo Method, IEEE Trans. Commun. Tech., *COM-14,* No. 2, April, 1966, p. 177.
8. Derzai, M., Power Addition of Independent Random Variables Normally Distributed on a dB Scale, IEEE Int. Conv. Record, Part I, March, 1967, p. 40.
9. Nåsell, I., Some Properties of Power Sums of Truncated Normal Random Variables, B.S.T.J., this issue, p. 2091.
10. Nåsell, I., The 1962 Survey of Noise and Loss on Toll Connections, B.S.T.J., *43,* March, 1964, p. 697–718.
11. Lewinski, D. A., A New Objective for Message Circuit Noise, B.S.T.J., *43,* March, 1964, p. 719–740.
12. Feller, W., *An Introduction to Probability Theory and Its Applications,* Vol. II, John Wiley and Sons, Incorporated, New York, 1966.