# Equations Governing the Electrical Behavior of an Arbitrary Piezoelectric Resonator Having $N$ Electrodes*

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*In a paper by J. A. Lewis (B.S.T.J., 40, 1961, pp. 1259-1280) general formulas for the electrical admittance of a piezoelectric resonator, having essentially one pair of electrodes, were derived in terms of motional parameters associated with the normal modes of vibration of the device. The logical extension of this work to a resonator with N electrodes is presented here. Expressions are given for both the admittance and impedance matrices of the resonator. These matrices are expressed in terms of motional parameters associated with, respectively, (i) the normal modes of vibration with all electrodes connected together, and (ii) all electrodes left open circuited. The electrical equivalent circuit for the 2-port characteristics of the N electrode resonator is given for two particular examples.*

## I. INTRODUCTION

General formulas for the electrical admittance of a piezoelectric resonator having essentially one pair of electrodes were derived by Lewis.[1] These formulas are consistent with those derived earlier for special cases such as long bars and large plates (see, for example, Mason[2]). In Lewis' work the admittance function is expanded about its poles in an infinite series. The residue at one of these poles determines the strength of the contribution of the normal mode, associated with the pole, to the overall vibrational behavior of the resonator when it is driven at a frequency close to the natural frequency of the mode. Surprisingly, the work of Lewis seems to have seen little application, as far as can be judged, except for that of Lloyd and Redwood,[3] and Byrne, et al.[4]

* Most of the work described here is based on part of the author's doctoral thesis (University of London, 1966).

With the current interest in multi-electroded resonators, such as the monolithic crystal filter,[5] it is pertinent to consider the logical extension of the work of Lewis to the case of an arbitrary resonator having $N$ electrodes. A discussion of this problem has previously been presented by the author,[6] and also by EerNisse and Holland.[7]

Included in Section II of this paper are the basic equations governing the piezoelectric resonator, presented here for completeness.

In Section III various integral relations are derived for use in Section IV where the properties of the admittance and impedance matrices are investigated. The electrical equivalent circuits for two particular 2-port configurations of the $N$ electrode resonator are also derived in Section IV, in order to illustrate the application of the admittance and impedance matrices.

A brief list of the principal symbols used in the text is given below.

### 1.1 *List of Symbols*

| | |
|---|---|
| $\rho$ | Mass per unit volume. |
| $\rho_s$ | Mass per unit area of an electrode. |
| $u_i$ | Particle displacement vector. |
| $S_{kl}$ | Strain tensor. |
| $T_{kl}$ | Stress tensor. |
| $\tau_i$ | Traction (stress vector). |
| $\phi$ | Electric scalar potential. |
| $E_i$ | Electric field vector. |
| $D_i$ | Electric displacement vector. |
| $c^E_{ijkl}$ | Elastic stiffness tensor (measured at constant electric field). |
| $e_{nij}$ | Piezoelectric constant tensor. |
| $\epsilon^S_{mn}$ | Dielectric constant tensor (measured at constant strain). |
| $n_i$ | Unit vector normal to, and outwards from surface of body. |
| $\Phi_p$ | Electric potential on the $p$th electrode. |
| $Q_p$ | Total charge on the $p$th electrode. |
| $B$ | Volume of the body. |
| $A$ | Unelectroded area of the body. |
| $A_p$ | Area of the $p$th electrode. |
| $\omega$ | Angular frequency. |
| $\lambda \equiv \omega^2$. | |

The tensor components above are referred to orthogonal Cartesian coordinate axes $x_i$. The comma notation is used to indicate differentiation, e.g. $D_{i,j} = \partial D_i/\partial x_j$, and the repeated index summation convention is used, e.g., $D_{i,i} = D_{1,1} + D_{2,2} + D_{3,3}$.

## II. BASIC EQUATIONS OF A PIEZOELECTRIC RESONATOR

The equations describing the steady vibrations of a piezoelectric body are listed below.

The equations of motion:

$$\rho\lambda u_i + T_{ij,j} = 0. \tag{1}$$

The divergence equation of electrostatics (for an insulator):

$$D_{i,i} = 0. \tag{2}$$

The piezoelectric constitutive relations:

$$T_{ij} = c^E_{ijkl} S_{kl} - e_{nij} E_n \,, \tag{3}$$

$$D_m = e_{mkl} S_{kl} + \epsilon^S_{mn} E_n \,, \tag{4}$$

where

$$S_{kl} = \tfrac{1}{2}(u_{k,l} + u_{l,k}), \tag{5}$$

and

$$E_n = -\phi_{2_n} \,. \tag{6}$$

The symmetry relations

$$c^E_{ijkl} = c^E_{ijlk} = c^E_{jikl} = c^E_{klij} \,, \tag{7}$$

$$e_{nij} = e_{nji} \,, \tag{8}$$

$$\epsilon^S_{mn} = \epsilon^S_{nm} \,. \tag{9}$$

### 2.1 *Boundary Conditions*

The boundary conditions for the resonator shown in Fig. 1 will now be discussed.

On the unelectroded portion of the surface $A$

$$\tau_i = 0, \qquad \text{on } A, \tag{10}$$

$$D_i n_i = \epsilon_0 (E_i) \text{ ext } n_i = 0, \qquad \text{on } A, \tag{11}$$

that is, no surface tractions and zero external electric field exist normal to the surface. Note that (11) is an approximation which in practice is usually valid for materials with large values of $\epsilon^S_{nm}/\epsilon_0$. The driving electrodes are assumed to be very thin metallic conductors with infinite conductivity. Potential $\Phi_p$ and charge $Q_p$ exist on the electrode area $A_p$. External electrical connections to the electrodes will not be specified at present. We pause to note, however, that $\phi = 0$ at some point ex-

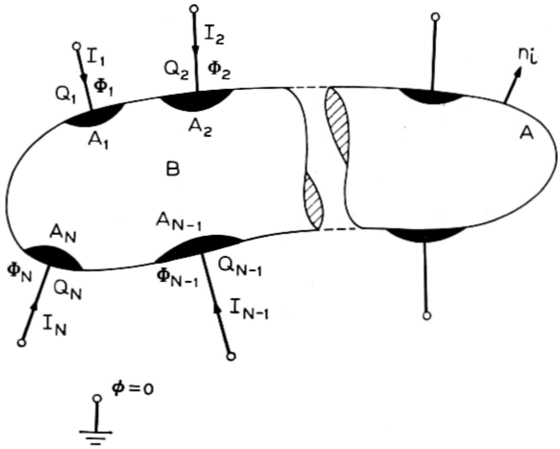


Fig. 1 — Arbitrary piezoelectric resonator with $N$ electrodes.

ternal to the resonator. Since we have neglected the effects of the external potential distribution, this "earth" point only has significance in connection with the topography of the external electrical circuit. The latter is assumed to interact only with the currents $I_p$ and potentials $\phi_p$ on the electrodes of the resonator. The mechanical properties of the electrode are assumed here to be nonexistent except for a surface mass density $\rho_s$. The surface of the resonator beneath an electrode is, therefore, assumed responsible only for exerting a force consistent with maintaining the acceleration of the electrode. The boundary conditions at the electrode can therefore be written as

$$\tau_i = \rho_s \lambda u_i , \qquad \text{on } A_p , \tag{12}$$

$$\phi = \text{constant}, \qquad \text{on } A_p \tag{13a}$$

and either

$$\phi = \Phi_p \qquad \text{or} \tag{13b}$$

$$\int_{A_p} D_i n_i \, dA_p = -Q_p . \tag{13c}$$

Note that the choice between (13b) or (13c) as a primary condition is unimportant.

## III. PROPERTIES OF SOLUTIONS

As is well known, the solution of the equations reviewed in Section II for a practical case is a formidable problem, and it is often necessary to resort to some approximate method of solution. In this paper, we will not discuss the methods for solving the equations, but rather the nature of the solution assuming that it has been found.

Equations (1) through (13) can be expressed in terms of $u_i$ and $\phi$, from which it follows that

$$\rho\lambda u_i + \overset{E}{c}_{ijkl} u_{k,jl} + e_{nij}\phi_{,nj} = 0, \tag{14}$$

$$e_{nkj} u_{k,jn} - \overset{S}{\epsilon}_{nm}\phi_{,nm} = 0, \tag{15}$$

subject to the boundary conditions

$$\overset{E}{c}_{ijkl} u_{k,l} n_j + e_{kij}\phi_{,k} n_j = 0, \qquad \text{on } A, \tag{16}$$

$$e_{jkl} u_{k,l} n_j - \overset{S}{\epsilon}_{jk}\phi_{,k} n_j = 0, \qquad \text{on } A, \tag{17}$$

with

$$\phi = \text{constant on } A_p \tag{18a}$$

and either

$$\phi = \Phi_p \qquad \text{on } A_p \tag{18b}$$

or

$$\int_{A_p} D_i n_i \, dA_p = -Q_p \,, \qquad \text{on } A_p \tag{18c}$$

and

$$\overset{E}{c}_{ijkl} u_{k,l} n_j + e_{kij}\phi_{,k} n_j = \rho_s \lambda u_i \,, \qquad \text{on } A_p \,. \tag{19}$$

We now note that (14) through (19) become homogeneous when $\Phi_p = 0$ for all $p$. This latter condition represents one of the eigenvalue problems associated with Fig. 1, namely, that of mechanical vibrations possible when all electrodes are connected directly to the reference point. Other eigenvalue problems associated with Fig. 1 include those where some electrodes are open-circuited ($Q_p = 0$) and the remainder are short-circuited ($\phi_r = 0$) (i.e., connected to the reference point).

### 3.1 *Reciprocal Theorem*

Consider two solutions of (1) through (13) denoted, respectively, by ($\lambda'$, $u_i'$, $\phi'$) and ($\lambda''$, $u_i''$, $\phi''$). The two solutions could be, for example, those associated with two different sets of forcing parameters at different frequencies.

From (1) we have

$$\int_B \rho\lambda'' u_i' u_i'' \, dB + \int_B u_i' T_{ij,j}'' \, dB = 0, \tag{20}$$

and from (2)

$$\int_B \phi' D''_{i,i} \, dB = 0, \tag{21}$$

where $B$ is volume of the body exclusive of the electrodes. Using the divergence theorem, (20) may be written

$$\int_B \rho\lambda'' u'_i u''_i \, dB - \int_B S'_{ij} T''_{ij} \, dB + \int_A u'_i T''_{ij} n_j \, dA = 0, \tag{22}$$

and (21)

$$- \int_B \phi'_{,i} D''_i \, dB + \int_A \phi' D''_i n_i \, dA = 0. \tag{23}$$

Subtracting (23) from (22), and substituting for the surface conditions given by equations (10) through (13) we have

$$\int_B \rho\lambda'' u'_i u''_i \, dB + \sum_{p=1}^{N} \int_{A_p} \rho_s \lambda'' u'_i u''_i \, dA - \sum_{p=1}^{N} \Phi'_p Q''_p$$
$$= \int_B (T''_{ij} S'_{ij} - E'_i D''_i) \, dB. \tag{24}$$

Equation (24) is still valid when the primed and double-primed quantities are interchanged. Using this fact, we have

$$(\lambda'' - \lambda')\left[\int_B \rho u'_i u''_i \, dB + \sum_{p=1}^{N} \int_{A_p} \rho_s u'_i u''_i \, dA\right] - \sum_{p=1}^{N} (\Phi'_p Q''_p - \Phi''_p Q'_p)$$
$$= \int_B [(T''_{ij} S'_{ij} - E'_i D''_i) - (T'_{ij} S''_{ij} - E''_i D'_i)] \, dB. \tag{25}$$

The quantity on the right-hand side is zero by virtue of the constitutive equations (3) and (4).

Equation (25) then becomes

$$(\lambda'' - \lambda') V(u'_i u''_i) = \sum_{p=1}^{N} (\Phi'_p Q''_p - \Phi''_p Q'_p), \tag{26}$$

where

$$V(u'_i u''_i) = \int_B \rho u'_i u''_i \, dB + \sum_{p=1}^{N} \int_{A_p} \rho_s u'_i u''_i \, dA. \tag{27}$$

Equation (26) is a special case of the reciprocal theorem given by Lewis[1] and discussed by Love[8] for the purely elastic case.

### 3.2 *Orthogonality of the Eigensolutions*

Consider two eigensolutions, $(\lambda^{(n)}, u_i^{(n)}, \phi^{(n)})$ and $(\lambda^{(m)}, u_i^{(m)}, \phi^{(m)})$ of the same homogeneous boundary problem. That is, $\Phi_p^{(m)} = 0$ if $\Phi_p^{(n)} = 0$ and $Q_r^{(m)} = 0$ if $Q_r^{(n)} = 0$. Thus, for two solutions of the same eigenset

$$\sum_{p=1}^{N} \left( \Phi_p^{(m)} Q_p^{(n)} - \Phi_p^{(n)} Q_p^{(m)} \right) = 0, \tag{28}$$

and from (26)

$$V(u_i^{(n)} u_i^{(m)}) = 0, \qquad \lambda^{(n)} \neq \lambda^{(m)}. \tag{29}$$

Thus, two solutions of the same eigenset satisfy the orthogonality condition (29). Also we have from (24), the Rayleigh quotient for the eigenvalue $\lambda^{(n)}$

$$\lambda^{(n)} = \omega_n^2 = \frac{2 \int_B H(u_i^{(n)}, \phi^{(n)})\, dB}{V(u_i^{(n)} u_i^{(n)})}, \tag{30}$$

where

$$H(u_i, \phi) = \tfrac{1}{2}(T_{ij} S_{ij} - E_i D_i). \tag{31}$$

### 3.3 *Expansion in Terms of Eigensolutions*

The solution to the inhomogeneous boundary value problem indicated by Fig. 1 can be expanded in terms of any of the sets of eigensolutions. These expansions are very important when electrical behavior is of prime interest. We will show here how the forced vibrational solution $(\lambda, u_i, \phi)$ may be expressed in terms of two of the possible expansions, namely: (*i*) the eigensolutions $(\lambda^{S(n)}, u_i^{S(n)}, \phi^{S(n)})$ which correspond to the normal modes of vibration of the resonator with all its electrodes connected to the reference point, and (*ii*) the eigensolutions $(\lambda^{O(n)}, u_i^{O(n)}, \phi^{O(n)})$ for the normal modes with all the electrodes open circuited.

For expansion (*i*) we set

$$u_i = u_i^{(o)} + \sum_{n=1}^{\infty} a^{(n)} u_i^{S(n)} \tag{32}$$

and

$$\phi = \phi^{(o)} + \sum_{n=1}^{\infty} a^{(n)} \phi^{S(n)}; \tag{33}$$

and for expansion (*ii*)

$$u_i = u_i^{(o)} + \sum_{n=1}^{\infty} b^{(n)} u_i^{O(n)} \tag{34}$$

and

$$\phi = \phi^{(o)} + \sum_{n=1}^{\infty} b^{(n)} \phi^{O(n)}, \tag{35}$$

where $(u_i^{(o)}, \phi^{(o)})$ is the solution to the boundary value problem of (1) through (13), as $\lambda \to 0$.

Since we have not specified the means by which the electrodes are connected to the external electric circuit we will allow the parameters $\Phi_p$ and $Q_p$ to be of the general form

$$\Phi_p = \Phi_p^{(o)} \exp(j\omega t), \qquad Q_p = Q_p^{(o)} \exp(j\omega t), \tag{36}$$

$$\Phi_p^{(o)} = | \Phi_p^{(o)} | \exp(j\theta_p), \qquad Q_p^{(o)} = | Q_p^{(o)} | \exp(j\psi_p). \tag{37}$$

Although it is immaterial how the charges $Q_p$ and potentials $\Phi_p$ are set up in relation to the external circuit, $Q_p$ and $\Phi_p$ are of course not independent.

The coefficients $a^{(n)}$ in the first expansion can be found by noting that the equations of motion (1) and boundary conditions (12) require

$$\rho\lambda u_i^{(o)} = \rho \sum_{n=1}^{\infty} (\lambda^{S(n)} - \lambda) a^{(n)} u_i^{S(n)}, \qquad \text{in } B \tag{38}$$

and

$$\rho_s\lambda u_i^{(o)} = \rho_s \sum_{n=1}^{\infty} (\lambda^{S(n)} - \lambda) a^{(n)} u_i^{S(n)}, \qquad \text{on } A_p . \tag{39}$$

On multiplying (38) and (39) by $u_i^{S(m)}$ and carrying out the indicated integrations and adding we have:

$$\int_B \rho\lambda u_i^{(o)} u_i^{S(m)} \, dB + \sum_{p=1}^{N} \int_{A_p} \rho_s \lambda u_i^{(o)} u_i^{S(m)} \, dA_p$$
$$= \sum_{n=1}^{\infty} a^{(n)} (\lambda^{S(n)} - \lambda) \left[ \int_B \rho u_i^{S(n)} u_i^{S(m)} \, dB + \sum_{p=1}^{N} \int_{A_p} \rho_s u_i^{S(n)} u_i^{S(m)} \, dA_p \right]. \tag{40}$$

We note from (27) that (40) may be written

$$\lambda V(u_i^{(o)} u_i^{S(m)}) = \sum_{n=1}^{\infty} a^{(n)} (\lambda^{S(n)} - \lambda) V(u_i^{S(n)} u_i^{S(m)}). \tag{41}$$

From the orthogonality condition (29), all terms on the right are zero except the term in $a^{(m)}$, giving

$$a^{(m)} = \frac{\lambda}{\lambda^{S(m)} - \lambda} \frac{V(u_i^{(o)} u_i^{S(m)})}{V(u_i^{S(m)} u_i^{S(m)})}. \tag{42}$$

By a similar argument we have for the coefficients in the second expansion

$$b^{(m)} = \frac{\lambda}{(\lambda^{O(m)} - \lambda)} \frac{V(u_i^{(o)} u_i^{O(m)})}{V(u_i^{O(m)} u_i^{O(m)})}. \tag{43}$$

Remembering the definition of $u_i^{(o)}$, $u_i^{S(m)}$ and $u_i^{O(m)}$ we have the following as a consequence of the reciprocal theorem (26):

$$\lambda^{S(m)} V(u_i^{(o)} u_i^{S(m)}) = \sum_{p=1}^{N} \Phi_p^{(o)} Q_p^{S(m)}, \tag{44}$$

and

$$\lambda^{O(m)} V(u_i^{(o)} u_i^{O(m)}) = -\sum_{p=1}^{N} Q_p^{(o)} \Phi_p^{O(m)}, \tag{45}$$

since

$$\Phi_p^{S(m)} = 0 \quad \text{and} \quad Q_p^{O(m)} = 0.$$

Using (44) and (45) with (42) and (43)

$$a^{(m)} = \frac{\lambda \sum_{p=1}^{N} \Phi_p^{(o)} Q_p^{S(m)}}{(\lambda^{S(m)} - \lambda)\lambda^{S(m)} V(u_i^{S(m)} u_i^{S(m)})} \tag{46}$$

$$b^{(m)} = \frac{-\lambda \sum_{p=1}^{N} Q_p^{(o)} \Phi_p^{O(m)}}{(\lambda^{O(m)} - \lambda)\lambda^{O(m)} V(u_i^{O(m)} u_i^{O(m)})}. \tag{47}$$

From (46), we see immediately that the contribution of the $S(m)$th mode (eigensolution) in the expansion (32) and (33) is dominant when $\lambda \rightarrow \lambda^{S(m)}$, if $\Phi_p^{(o)}$ is held constant with frequency. We also note that the amplitude of $a^{(m)}$ depends on the charge on the electrodes when the resonator is executing free vibrations corresponding to the $S(m)$th mode (i.e., with all electrodes short-circuited).

Equation (47) shows similarly that the contribution of the $O(m)$th mode is dominant in the expansion of (34) and (35) when $\lambda \rightarrow \lambda^{O(m)}$ if $Q_p^{(o)}$ is held constant. Also the amplitude $b^{(m)}$ depends on the potentials on the electrodes when they are open circuit with the resonator executing free vibrations corresponding to the $O(m)$th mode. When applying the expansions (*i*) and (*ii*) it should be realized that assumptions have been made concerning the completeness of the eigensets.

## IV. THE ELECTRICAL ADMITTANCE AND IMPEDANCE MATRICES

### 4.1 *General Considerations*

The admittance matrix $y_{pq}$ for the $N$-electrode piezoelectric resonator is defined by

$$I_p = \sum_{q=1}^{N} y_{pq}\Phi_q \,. \tag{48}$$

Similarly $z_{pq}$, the impedance matrix, is here defined by

$$\Phi_p = \sum_{q=1}^{N} z_{pq}I_q + R, \tag{49}$$

where

$$I_p = j\omega Q_p \tag{50}$$

and $R$ is a constant depending on the external circuit configuration. The relationships (48) and (49) are postulated on the basis that the equations of the resonator are linear and that their use is restricted to steady vibrations.

We will now derive various properties of $y_{pq}$ and $z_{pq}$. First we note from (13c) and the divergence theorem that

$$\sum_{p=1}^{N} I_p = j\omega \sum_{p=1}^{N} Q_p = -j\omega \sum_{p=1}^{N} \int_{A_p} D_i n_i \, dA_p = -j\omega \int_B D_{i,i} \, dB = 0. \tag{51}$$

Equation (51) is simply Kirchhoff's current law, for the conservation of charge. We note from (48) and (51) that

$$\sum_{p=1}^{N} \sum_{q=1}^{N} y_{pq}\Phi_q = 0, \tag{52}$$

and since $\Phi_q$ is arbitrary, the sum of each column of the $y_{pq}$ matrix is zero, i.e.,

$$\sum_{p=1}^{N} y_{pq} = 0. \tag{53}$$

We will now use the reciprocal theorem to show that both $y_{pq}$ and $z_{pq}$ are symmetric. Consider the solutions for two sets of potentials $\Phi_p'$

and $\Phi_p''$ having the same frequency. Then from (26)

$$\sum_{p=1}^{N} (\Phi_p' Q_p'' - \Phi_p'' Q_p') = 0. \tag{54}$$

Using (48) and (54)

$$\sum_{p=1}^{N} \sum_{q=1}^{N} (y_{pq} \Phi_p' \Phi_q'' - y_{pq} \Phi_p'' \Phi_q') = 0, \tag{55}$$

and (49) and (54)

$$j\omega \sum_{p=1}^{N} \sum_{q=1}^{N} (z_{pq} Q_q' Q_p'' - z_{pq} Q_q'' Q_p') + R \sum_{p=1}^{N} (Q_p' - Q_p'') = 0. \tag{56}$$

Since $\Phi_p'$ and $\Phi_p''$ are arbitrary in (55) we must have

$$y_{pq} = y_{qp} \ . \tag{57}$$

In (56) $Q_p'$ and $Q_p'$ are arbitrary and $\sum_{p=1}^{N} Q_p = 0$, so

$$z_{pq} = z_{qp} \ . \tag{58}$$

As a consequence of (57) and (53)

$$\sum_{q=1}^{N} y_{pq} = 0. \tag{59}$$

It has been shown in this section that the impedance and admittance matrices of an $N$-electrode piezoelectric resonator have properties similar in many ways to those of an $N$-terminal passive electrical network.[9]

### 4.2 *Expansions for the Electrical Parameters*

We may now make use of the eigensolution expansions of (32)–(33) and (34)–(35) to inquire into the admissible forms for $y_{pq}$ and $z_{pq}$ as functions of frequency.

For expansion (a)

$$Q_p = Q_p^{(o)} + \sum_{m=1}^{\infty} a^{(m)} Q_p^{S(m)} \tag{60}$$

$$\Phi_p = \Phi_p^{(o)} + \sum_{m=1}^{\infty} a^{(m)} \Phi_p^{S(m)} = \Phi_p^{(o)}, \quad \text{all } \Phi_p^{S(m)} \text{ being zero.} \tag{61}$$

Then using (46) and (60)

$$Q_p = Q_p^{(o)} + \sum_{q=1}^{N} \sum_{m=1}^{\infty} \frac{\lambda C_{pq}^{(m)} \Phi_q^{(o)}}{(\lambda^{S(m)} - \lambda)}, \tag{62}$$

where

$$C_{pq}^{(m)} = \frac{Q_p^{S(m)} Q_q^{S(m)}}{\lambda^{S(m)} V(u_i^{S(m)} u_i^{S(m)})}. \tag{63}$$

For expansion (b)

$$Q_p = Q_p^{(o)} + \sum_{m=1}^{\infty} b^{(m)} Q_p^{O(m)} = Q_p^{(o)}, \quad \text{all } Q_p^{O(m)} \text{ being zero,} \tag{64}$$

and

$$\Phi_p = \Phi_p^{(o)} + \sum_{m=1}^{\infty} b^{(m)} \Phi_p^{O(m)}. \tag{65}$$

So using (47) and (65)

$$\Phi_p = \Phi_p^{(o)} - \sum_{q=1}^{N} \sum_{m=1}^{\infty} \frac{\lambda F_{pq}^{(m)} Q_q^{(o)}}{(\lambda^{O(m)} - \lambda)}, \tag{66}$$

where

$$F_{pq}^{(m)} = \frac{\Phi_p^{O(m)} \Phi_q^{O(m)}}{\lambda^{O(m)} V(u_i^{O(m)} u_i^{O(m)})}. \tag{67}$$

We now define the charge-potential relations for the solution of the static boundary value problem ($\lambda = 0$) as follows:

$$Q_p^{(o)} = \sum_{q=1}^{N} C'_{pq} \Phi_q^{(o)} \tag{68}$$

and

$$\Phi_p^{(o)} = \sum_{q=1}^{N} F_{pq}^{(o)} Q_q^{(o)} + R. \tag{69}$$

It is assumed, from now on, that the static parameters such as $C'_{pq}$ are such that quadratic forms like $C'_{pq}\Phi_p\Phi_q$ are positive definite. Proof of this depends on energetic considerations.

We now put

$$\lambda^{O(m)} = \omega_{Am}^2 \quad \text{and} \quad \lambda^{S(m)} = \omega_{Rm}^2, \tag{70}$$

and define

$$C_{pq}^{(o)} = C'_{pq} - \sum_{m=1}^{\infty} C_{pq}^{(m)}. \tag{71}$$

We obtain from (62) and (71) the admittance matrix of (48) in the form

$$y_{pq} = j\omega\left\{C_{pq}^{(o)} + \sum_{m=1}^{\infty} \frac{\omega_{Rm}^2 C_{pq}^{(m)}}{(\omega_{Rm}^2 - \omega^2)}\right\}. \tag{72}$$

From (66) and (69) the impedance matrix of (49) is of the form

$$z_{pq} = \frac{1}{j\omega}\left\{F_{pq}^{(o)} - \sum_{m=1}^{\infty} \frac{\omega^2 F_{pq}^{(m)}}{(\omega_{Am}^2 - \omega^2)}\right\}. \tag{73}$$

Restricting our interest for the moment to an element $y_{pq}$ of the admittance matrix, we observe that the form of (72) is analogous in form to the admittance of the electrical network in Fig. 2. However, from (59), which is valid for all frequencies, we require

$$\sum_{q=1}^{N} C_{pq}^{(m)} = 0, \tag{74}$$

but from (63)

$$C_{pp}^{(m)} > 0, \tag{75}$$

so

$$\sum_{q=1,\, q\neq p}^{N} C_{pq}^{(m)} < 0. \tag{76}$$

Therefore, several elements of $C_{pq}^{(m)}$ may be negative.

The form of an element of the impedance matrix (72) suggests an analogy with the circuit of Fig. 3 but, in view of the proceeding discussion, it is again probable that several of the elements $F_{pq}^{(m)}$ are negative. It should be noted, however, that $F_{pp}^{(m)} > 0$.

### 4.3 *Driving Point Functions*

From (49) the driving point impedance $Z_{pq}^D$, at the two terminals $p$-$q$, when all other terminals are left open-circuit is given by

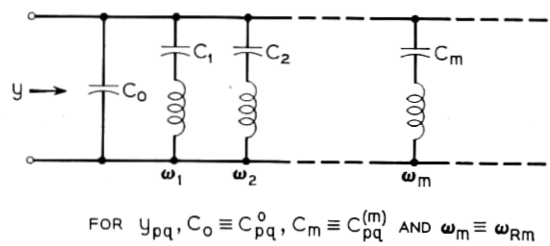


Fig. 2 — Electrical network for admittance representation.

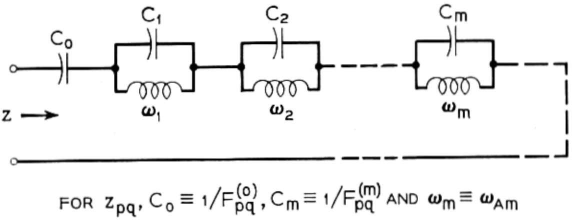


Fig. 3 — Electrical network for impedance representation.

$$\begin{aligned} Z^D_{pq} &= (\Phi_p - \Phi_q)/I_p \\ &= z_{pp} - 2z_{pq} + z_{qq} \, . \end{aligned} \tag{77}$$

In terms of the expansions for $z_{pq}$ given by (73), we have from (77)

$$Z^D_{pq} = \frac{1}{j\omega}\left[F^{(o)} - \sum_{m=1}^{\infty} \frac{\omega^2 F^{(m)}}{\omega^2_{Am} - \omega^2}\right], \tag{78}$$

where

$$F^{(m)} = \frac{(\Phi^{O(m)}_p - \Phi^{O(m)}_q)^2}{\lambda^{O(m)} V(u^{O(m)}_i u^{O(m)}_i)} \tag{79}$$

and

$$F^{(o)} = F^{(o)}_{pp} - 2F^{(o)}_{pq} + F^{(o)}_{qq} \, . \tag{80}$$

So, clearly $F^{(m)} > 0$ and therefore, the analogue circuit of Fig. 3 is "physical" for $Z^D_{pq}$.

At first sight, it would appear that one could easily derive a driving point admittance for the *p-q port* when all other terminals are shorted. In fact, it must be found by appropriate manipulation of either $y_{pq}$ or $z_{pq}$ and, in general, the expression includes many elements of either matrix. We can, however, define a driving point admittance for the *p-terminal*, when all other terminals are connected to the reference point, i.e.,

$$Y^D_p = y_{pp} \, . \tag{81}$$

The analogue electrical circuit for $y_{pp}$, namely Fig. 2, is again physical.

### 4.4 "*Black Box*" *Matrices for a Two-Port*

Before calculating any "black box" transfer matrices it is convenient to define a transformed admittance matrix valid for the resonator and its external circuit. In Fig. 4, the terminals of the resonator are all interconnected, there being a physical component with admittance $y_{pq}^E$ connected between terminals $p$ and $q$. The currents flowing into the $N$-terminal network and resonator as a whole are

$$I_p' = \sum_{q=1}^{N} y_{pq}'\Phi_q \,, \tag{82}$$

where

$$y_{pq}' = y_{pq} - y_{pq}^E \,, \tag{83}$$

and

$$y_{pp}^E = -\sum_{q=1,\, q\neq p}^{N} y_{pq}^E \,. \tag{84}$$

$y_{pq}$ is defined by (48) and $y_{pq}^E = y_{qp}^E$, as can be seen from Fig. 4. We may now form two-port networks. For the purposes of further discussion, any connections made externally to the two-port will be assumed to be consistent with

$$I_s' = -I_p' \,, \quad I_r' = -I_q' \,, \quad V_p = \Phi_p - \Phi_s \,, \quad V_q = \Phi_q - \Phi_r \,. \tag{85}$$

### 4.5 *Electrically Symmetric Two-Port Resonator*

Further discussion, with all $y_{pq}^E$ finite for the $N$-terminal resonator, will not be continued. The reduction of a $N$-terminal network to a 2-port is discussed by Weinberg.[7]

#### 4.5.1 *Two-Port With N-2 Terminals Shorted*

We will now consider a simple case where all terminals except $p$ and $q$ are connected directly to $s$, and an admittance $y_{pq}^E$ is connected

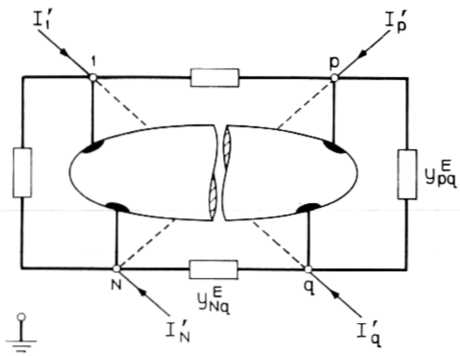


Fig. 4 — External electrical connections to the resonator.

between $p$ and $q$ as shown in Fig. 5. We will also assume that the construction of the resonator is such that it is electrically symmetric with respect to the ports. We have under these circumstances

$$I'_p = y'_{pp} V_p + y'_{pq} V_q \tag{86}$$

$$I'_q = y'_{pq} V_p + y'_{pp} V_q , \tag{87}$$

where

$$y'_{pp} = y_{pp} + y^E_{pq} \tag{88}$$

and

$$y'_{pq} = y_{pq} - y^E_{pq} . \tag{89}$$

We now consider the electrical lattice network of Fig. 6 as an analogue of transfer characteristics of (86) and (87). The analogue (Fig. 6) is physical if $Y_a$ and $Y_b$ are realizable with physical components.

Now

$$Y_a = y_{pp} - y_{pq} , \tag{90}$$

and

$$Y_b = y_{pp} + y_{pq} . \tag{91}$$

Using (72) and (63), $Y_a$ and $Y_b$ can be expressed in terms of the eigensolution expansion as follows:

$$Y_a = j\omega\left[(C^{(o)}_{pp} - C^{(o)}_{pq}) + \sum_{m=1}^{\infty} \frac{(Q_p^{S(m)} Q_p^{S(m)} - Q_p^{S(m)} Q_q^{S(m)})}{(\lambda^{S(m)} - \lambda) V(u_i^{S(m)} u_i^{S(m)})}\right], \tag{92}$$

$$Y_b = j\omega\left[(C^{(o)}_{pp} + C^{(o)}_{pq}) + \sum_{m=1}^{\infty} \frac{(Q_p^{S(m)} Q_p^{S(m)} + Q_p^{S(m)} Q_q^{S(m)})}{(\lambda^{S(m)} - \lambda) V(u_i^{S(m)} u_i^{S(m)})}\right]. \tag{93}$$

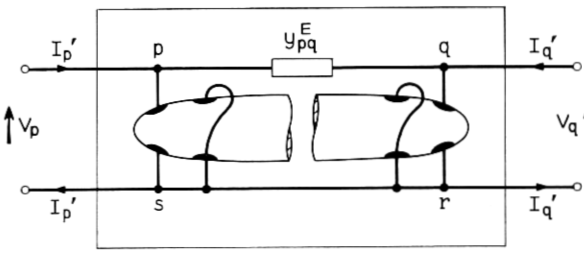


Fig. 5 — Two-port system for $N$-2 electrodes short-circuited.

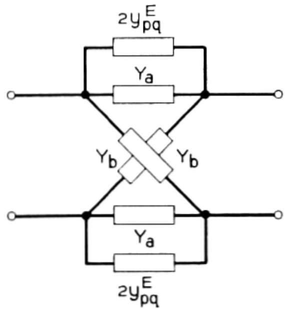


Fig. 6 — Electrical analogue of Fig. 5.

Also since we have taken $y_{pp} = y_{qq}$, then from (63), either

$$Q_p^{S(m)} = -Q_q^{S(m)} \quad \text{or} \quad Q_p^{S(m)} = Q_q^{S(m)}. \tag{94}$$

It therefore follows that the $S(m)$th eigensolution may only contribute to one of $Y_a$ and $Y_b$, depending on sign of $Q_p^{S(m)}/Q_q^{S(m)}$. We also see that the electric circuit of Fig. 2 is a physical analogue for both $Y_a$ and $Y_b$.

#### 4.5.2 *Two-Port With (N-4) Terminals Open Circuit*

The symmetrical resonator with ($N$-4) terminals open circuit is shown in Fig. 7. We now use the $z_{pq}$ matrix of (49), (67), and (73) to derive the impedance matrix of this two-port, again subject to the restrictions of (85). We find that

$$V_p = Z_{pp}I_p + Z_{pq}I_q \tag{95}$$

$$V_q = Z_{pq}I_p + Z_{pp}I_q , \tag{96}$$

where

$$Z_{pp} = z_{pp} + z_{ss} - 2z_{ps} \tag{97}$$

and

$$Z_{pq} = z_{pq} + z_{sr} - z_{pr} - z_{sq} . \tag{98}$$

From (82) and (67)

$$Z_{pq} = \frac{1}{j\omega}\left[G_{pq}^{(o)} - \sum_{m=1}^{\infty} \frac{\lambda G_{pq}^{(m)}}{(\lambda^{O(m)} - \lambda)}\right], \tag{99}$$

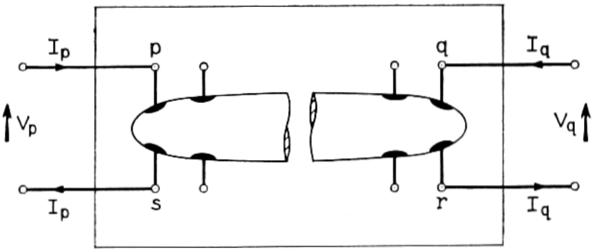


Fig. 7 — Two-port system for $N$-4 electrodes open-circuited.

where

$$G_{pp}^{(o)} = F_{pp}^{(o)} + F_{ss}^{(o)} - 2F_{ps}^{(o)}, \tag{100}$$

$$G_{pp}^{(m)} = \frac{(\Phi_p^{O(m)} - \Phi_s^{O(m)})^2}{\lambda^{O(m)} V(u_i^{O(m)} u_i^{O(m)})}, \tag{101}$$

$$G_{pq}^{(m)} = \frac{(\Phi_p^{O(m)} - \Phi_s^{O(m)})(\Phi_q^{O(m)} - \Phi_r^{O(m)})}{\lambda^{O(m)} V(u_i^{O(m)} u_i^{O(m)})}, \tag{102}$$

$$G_{pq}^{(o)} = F_{pq}^{(o)} + F_{sr}^{(o)} - F_{pr}^{(o)} - F_{sq}^{(o)}. \tag{103}$$

Also since the resonator has been taken to be symmetrical, i.e.,

$$Z_{pp} = Z_{qq}, \tag{104}$$

it follows, from 67, that

$$(\Phi_p^{O(m)} - \Phi_s^{O(m)}) = \pm(\Phi_q^{O(m)} - \Phi_r^{O(m)}). \tag{105}$$

If we represent the transfer equations (95) and (96) in terms of the lattice analogue of Fig. 8, subject to the restrictions of (85), we have

$$Z_a = \frac{1}{j\omega}\left[(G_{pp}^{(o)} - G_{pq}^{(o)}) - \sum_{m=0}^{\infty} \frac{\lambda}{\lambda^{O(m)}} \frac{(\Phi_p^{O(m)} - \Phi_s^{O(m)})(\Phi_p^{O(m)} - \Phi_s^{O(m)} - \Phi_q^{O(m)} + \Phi_r^{O(m)})}{(\lambda^{O(m)} - \lambda) V(u_i^{O(m)} u_i^{O(m)})}\right] \tag{106}$$

$$Z_b = \frac{1}{j\omega}\left[(G_{pp}^{(o)} + G_{pq}^{(o)}) - \sum_{m=0}^{\infty} \frac{\lambda}{\lambda^{O(m)}} \frac{(\Phi_p^{O(m)} - \Phi_s^{O(m)})(\Phi_p^{O(m)} - \Phi_s^{O(m)} + \Phi_q^{O(m)} - \Phi_r^{O(m)})}{(\lambda^{O(m)} - \lambda) V(u_i^{O(m)} u_i^{O(m)})}\right]. \tag{107}$$

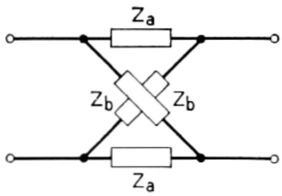


Fig. 8 — Electrical analogue of Fig. 7.

We note by virtue of (105) that the $O(m)$th eigensolution only contributes to one of $Z_a$ and $Z_b$ depending on the sign of $(\Phi_p^{O(m)} - \Phi_s^{O(m)})/(\Phi_q^{O(m)} - \Phi_r^{O(m)})$. It also follows that each of $Z_a$ and $Z_b$ have the circuit of Fig. 3 as a physical analogue.

## V. DISCUSSION AND CONCLUSIONS

It has been shown how the electrical behavior of a piezoelectric resonator with $N$ electrodes, represented by an admittance or impedance matrix, can be determined from the eigensolutions for free vibrations of the resonator. The results for the admittance obtained by Lewis[1] are contained here as the special case $N = 2$ in (72). The impedance of the two-electrode resonator is described by (73). This result was not given by Lewis[1] since he did not consider the alternative expansion of the open circuit eigensolutions. It could be argued that since impedance is simply the reciprocal of admittance, residues of one could be found from the other. This would, however, involve rather cumbersome calculations. Furthermore, if an approximate theory is used to generate the eigensolutions, the expansions may only be valid in a small frequency range, thus making the calculation of, say, the residues of the impedance from the admittance expansion impossible.

Returning to the general case of the $N$-electrode resonator, the electrical behavior can be predicted in terms of the admittance or impedance matrices of (72) and (82). If external electrical components are to be connected between the $N$ electrodes, and a two-port composite network is to be formed, its transfer characteristics can be deduced from either matrix. In the particular case of the symmetric resonator with $N$-2 of its electrodes connected together, the admittance matrix provides simple results, whereas the impedance matrix is useful for the case of $N$-4 open-circuited electrodes.

Before concluding, it might well be asked what reasons there are for preferring a motional parameter representation of the electrical characteristics over the direct method of determining the impedance or admittance matrices from the solution to the inhomogeneous boundary value problem. These are essentially twofold. Firstly, if a two-port $N$-electrode reasonator is to be designed to realize some transfer function or driving impedance, or so on, an appropriate synthesis procedure will usually automatically yield these motional parameters, leaving only the task of physically realizing a resonator with these parameters! Secondly, if the inhomogeneous boundary value problem is being solved, the inevitable numerical calculations are least likely to be accurate in just those ranges which are of prime interest, namely, the poles and zeros of the admittance or impedance matrices. Furthermore, considerable computing time would be lost, compared with the motional parameter method, if some transfer characteristic of the derived two-port were to be obtained for a large number of frequencies in a narrow band.

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