Extensions to the Analysis of Regenerative Repeaters with Quantized Feedback

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The functional iterative approach given by Zador for calculating the average bit error probability in a regenerative repeater with quantized feedback is extended to the vector case. For a channel with a rational fraction transfer function, the vector extension permits us at least formally to deal with the following practical conditions:

(i) The pulse transmission plan is described by an m-ary alphabet with independent digits.
(ii) Perfect and imperfect low-frequency tail cancellation cases are considered.
(iii) High-frequency signal shaping and its interaction with the predominantly low-frequency tail are taken into account.

Expressions for error probability on the kth digit are derived in terms of the kth vector iterate of a known function. The restriction to independent noise samples is also removed. The resulting expression for kth bit error probability is then derived from an operational iteration procedure which acts on the k + 1 dimensional joint distribution of the noise samples.

I. INTRODUCTION

In the design of digital communication links, various reasons exist for the removal of low-frequency components during or prior to transmission of a pulse train. In the case of vestigial sideband (VSB) modulation[1] of data over voice-frequency channels, the dc and low-frequency signal components are removed at the transmitter before modulation and carrier reinsertion. This is required to insure satisfactory carrier recovery at the receiver for relatively low transmitted carrier power. In the T-1 Carrier System,[2] the loss of low-frequency information results from transformer coupling of an unbalanced repeater to the balanced line. In either event, the effect of low-frequency suppression is to cause the positive impulse response of the overall equalized medium to exhibit an undershoot which gives intersymbol interference.

One means of reducing the effect of low-frequency suppression in a regenerative repeater is to feed back a signal in an attempt to cancel the long transient tail. This method of compensation has been called quantized feedback and its use dates back to the 1920's (as noted by Bennett[3]). We assume that the reader is familiar with Bennett's excellent expository paper. Until recently, analysis of the effects of quantized feedback on average bit error probability in a noisy environment has received essentially no attention. The first to examine this problem were Anderson, Gerrish, and Salz[4] who considered the polar binary case, neglecting signal shaping and assuming perfect matching of the feedback cancellation signal to the input signal tail. They have obtained results, with the aid of the computer, that have provided insight into the problem. In addition, they have exposed computational difficulties involved in grinding out numerical results for any given set of system parameters.

A more analytical approach to the basic problem is found in Zador[5] who used the theory of generalized random jump processes[6] to obtain an iterative procedure for computing error probability. Unfortunately, the class of physical systems that can be handled by Zador's approach as originally stated is quite restrictive in the following sense (see Fig. 1):

(*i*) The transmitted message sequence is composed of independent binary digits.

(*ii*) The low-frequency behavior of the channel as represented by $G(s)$ is dominated by a single pole.

(*iii*) $G(s)$ and $H(s)$ are exact complements of each other so that perfect feedback tail cancellation is achieved.

(*iv*) The time dispersion of the transmitted pulses caused by the medium, $C(s)$, with or without equalization $E(s)$ is strictly limited to two pulse intervals.

(*v*) The noise samples at the input to the threshold detectors are assumed independent.

It is our intention here to remove some of the above restrictions. In particular, we extend Zador's approach along the following lines:

(*i*) By allowing a multilevel threshold device as a regenerator, the

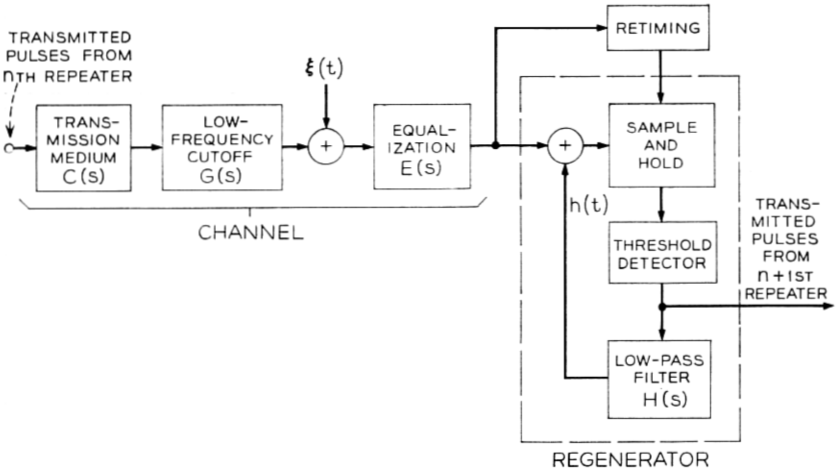

Fig. 1 — Block diagram of reconstructive repeater with quantized feedback.

allowable pulse transmission plan is extended to include m-ary alphabets with independent digits. (The ternary case is treated in detail.)

(*ii*) The high- and low-frequency behaviors of the channel may be individually characterized by rational functions. The implication of this is twofold. First, the predominantly low-frequency tail is now described by several exponentials. Secondly, the impulse response of the overall equalized medium $C(s)\ E(s)$ is not restricted to be time-limited.

(*iii*) The restriction to perfect tail cancellation is removed to allow for imperfections in the forward and/or feedback paths.

(*iv*) The more realistic case of correlated noise samples is examined.

Extensions (*i*), (*ii*), and (*iii*) are possible only through a vector approach based on Zador's original iteration scheme. The assumption of a nonflat noise spectrum as in (*iv*) leads to an *operational* iteration procedure for calculating bit error probability. It is to be emphasized that the question of computational procedures, which even in the simple binary case was a formidable task, grows considerably in complexity with the degree of generality assumed.

The generalizations listed above will be treated one at a time so as to demonstrate individually the necessary changes in Zador's original formulation. A review of his model is given in Section II.

Section III assumes an unrestricted ternary message sequence together with the remaining restrictions as imposed by Zador. An application of the results is given for a particular high-frequency behavior of the system. The response of the high-frequency portion of the channel, $C(s)\,E(s)$, to a transmitted rectangular pulse is assumed triangular in shape and time-limited to two pulse intervals.

Section IV derives the general expression for error probability when the overall channel, $Y(s) = C(s)\,E(s)\,G(s)$ is assumed to be characterized by a rational function. The feedback network, $H(s)$, is designed to cancel only the low-frequency poles, i.e., those of $G(s)$. The special case of a binary input format is treated in detail.

Section V modifies the results of Section III by including the case of imperfect match of the $G(s)$ and $H(s)$ characteristics.

Section VI begins with Zador's original assumptions on the signaling format, channel, and feedback network characteristics, but removes the restriction of independent noise samples. An expression for kth bit error probability is derived from an *operational* iteration procedure which acts on the $k + 1$ dimensional joint distribution of the noise samples. The analogy between this scheme and the functional iteration proposed by Zador for the uncorrelated noise case is demonstrated.

II. REVIEW OF ZADOR'S MODEL

We begin with a brief review of Zador's mathematical assumptions and emphasize their physical significance. Consider once again the repeater-to-repeater transmission link illustrated in Fig. 1. The output of the nth repeater at time rT is a binary rectangular pulse* $d_r p(t-rT)$ where

$$p(t) = p_0 \qquad |t| \leqq t_0$$

$$= 0, \qquad |t| > t_0$$

$d_r = \pm 1$, and $1/T$ is the pulse rate of the system. Zador does not explicitly describe the high-frequency behavior of the system. The class of channels that satisfies his underlying assumptions is discussed below. Let the response of $C(s)E(s)$ to the pulse $p(t)$, denoted by $z(t)$, be time limited to $2T$, and zero at its end points. It is understood that in practice these conditions are usually met only approximately. Then, by passing $z(t)$ through a single pole high-pass filter, $G(s)$, the part of the resulting

* Zador assumes ± 1 impulses as repeater output. As we shall see, in the sampled systems we consider, this modification has no effect on the ensuing analysis.

response, $g(t)$, for $t \geqq 2T$ is dominated by a single exponential. If $s(t)$ is sampled at time $t = T$ and held until $t = T + t_0$, then by employing an ideal slicer element as a threshold detector a unit rectangular output pulse, $b(t)$ is regenerated. Furthermore, by passing this pulse through $H(s)$, the response tail of $s(t)$ for $t \geqq 2T$ may be exactly cancelled in the absence of noise and circuit imperfections. These observations are illustrated in Fig. 2 for a triangular pulse shape $z(t)$. The response of $H(s)$ to the regenerator output pulse is denoted by $h(t)$. Turning now to a sample notation, let g_k, h_k, and b_k represent the values of $g(t)$, $h(t)$, and $b(t)$, respectively, at time $(k + 1)T$, $k = 0, 1, 2$. Then, from Fig. 2, it is obvious that the following conditions must hold, in general, independent of the waveshape of $z(t)$ within the $2T$ interval:

$$(i) \qquad g_0 > 0, \qquad h_0 = 0$$

$$(ii) \qquad h_i + g_i = 0 \qquad i = 1, 2, \cdots .$$

$$(iii) \qquad g_i = r g_{i-1} \qquad i \geqq 2$$

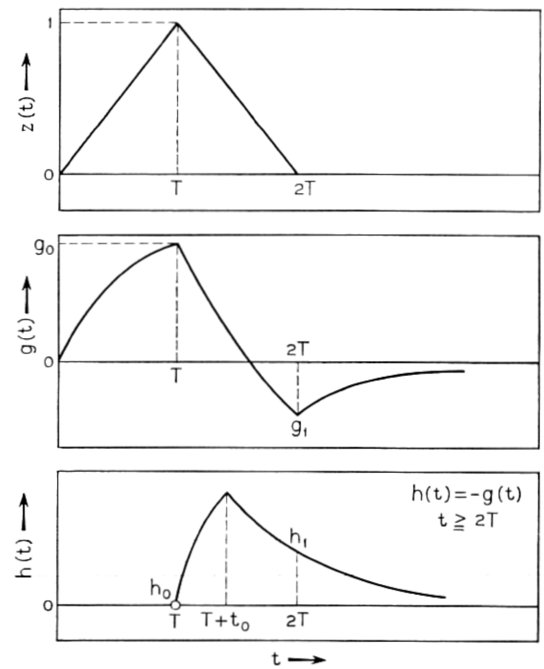

Fig. 2 — System pulse responses.

where r is related to the single pole, α, of $G(s)$ by $r = e^{-\alpha T}$. Condition (*iii*) is clear upon noting that the response of a single pole high-pass filter to a time limited signal of width $2T$ has a single exponential response for values of $t \geqq 2T$. Statements (*i*) to (*iii*) as above are identical with Zador's restrictions on the system as reported in Ref. 5.* The shape of $z(t)$ is solely used in determining the two dependent quantities g_0 and g_1. For a triangular $z(t)$ waveshape of unity height (Fig. 2) and $G(s) = s/s + \alpha$,

$$g_0 = \frac{1}{\alpha T} [1 - e^{-\alpha T}]$$

$$g_1 = -\frac{1}{\alpha T} [1 - e^{-\alpha T}]^2 .$$

III. TERNARY PULSE TRANSMISSION

When considering a ternary system, the only essential modification of the model suggested by Zador is an ideal slicer with positive and negative pulse detection thresholds set at $+a_0$ and $-a_1$, respectively.

Letting s_k denote the total reshaped input at the $k + 1$th timing instant, and c_k the feedback voltage at the same instant in time as before, the slicing operation is described by

$$\begin{aligned} b_k &= 1 \quad \text{if} \quad s_k + n_k + c_k \geqq a_0 \\ &= 0 \quad \text{if} \quad -a_1 < s_k + n_k + c_k < a_0 \\ &= -1 \quad \text{if} \quad s_k + n_k + c_k \leqq -a_1 , \end{aligned}$$

where

$$s_k = \sum_{i=0}^{k} g_{k-i} d_i \qquad k = 0, 1, \cdots$$

$$c_k = \sum_{i=0}^{k} h_{k-i} b_i \qquad k = 0, 1, \cdots$$

and n_k is a sample from a stationary noise process $n(t)$ having a fixed but arbitrary distribution function $N(x)$, and independent samples. The process $n(t)$ is actually the result of passing the additive white noise process in the system, $\xi(t)$, through $E(s)$. We assume, however, that the correlation between noise samples introduced by the above is small and can be ignored as a first approximation. When this as-

* Note that Zador also requires $g_i <$ for $i \geqq 1$. This restriction is not necessary although it is often true.

sumption is invalid, the method discussed in Section VI must be used.

It is of prime interest to examine the conditions under which the system will operate error-free in the absence of noise. For $0 < a_0$, $a_1 \leqq g_0$,

$$s_0 + c_0 = g_0 d_0$$

thus if,

$$\begin{aligned} a_0 &= 1, & s_0 + c_0 &= g_0, & b_0 &= 1 \\ &= 0, & s_0 + c_0 &= 0, & b_0 &= 0 \\ &= -1, & s_0 + c_0 &= -g_0, & b_0 &= -1, \end{aligned}$$
$$\text{or} \quad b_0 = d_0 .$$

Continuing, in this way $k = 1, 2, \cdots, k - 1$,

$$\begin{aligned} s_k + c_k &= g_0 d_k + \sum_{i=0}^{k-1} (g_{k-i} + h_{k-i}) d_i \\ &= g_0 d_k . \end{aligned}$$

Thus, if $b_m = d_m$ for $m = 0, 1, \cdots, k - 1$, then $b_k = d_k$ and the system operates error-free in the absence of noise.

For the more general case when noise is present,

$$s_k + c_k = \sum_{\substack{i=0 \\ b_i \neq d_i}}^{k-1} g_{k-1}(d_i - b_i) + g_0 d_k = x_k + g_0 d_k ,$$

where x_k represents the cumulative effect of any and all errors prior to time k.

Letting p and q denote the *à priori* probabilities of a plus one and minus one, respectively, the probability of error on the kth digit $p(k)$ can be written as

$$\begin{aligned} p(k) = p \operatorname{Prob} \{n_k + x_k \leqq a_0 - g_0\} + q \operatorname{Prob} \{n_k + x_k > -a_1 + g_0\} \\ + (1 - p - q) \operatorname{Prob} \{n_k + x_k \geqq a_0 ; n_k + x_k \leqq -a_1\}. \end{aligned}$$

The independence of n_k and x_k allows $p(k)$ to be expressed in terms of the noise distribution function $N(x)$ and the distribution function of x_k, $F_k(x)$ as follows:

$$p(k) = p \int_{-\infty}^{\infty} N(a_0 - g_0 - x) \, dF_k(x)$$

$$+ q \int_{-\infty}^{\infty} [1 - N(g_0 - a_1 - x)] \, dF_k(x)$$

$$+ (1 - p - q) \int_{-\infty}^{\infty} [N(-a_1 - x) + 1 - N(a_0 - x)] \, dF_k(x).$$

For the case of a zero mean symmetrical noise distribution and equal *à priori* probabilities for all input symbols (i.e., $p = q = (1 - p - q)$ it is easy to show that the optimum threshold settings are $\pm g_0/2$ with

$$p(k) = (1 - p) \int_{-\infty}^{\infty} [1 - N(g_0/2 - x) + N(-g_0/2 - x)] \, dF_k(x).$$

It now remains to show that the sequence of random variables $x_0, x_1, \cdots$ are representative of a random jump process studied in Ref. 6 and thus $p(k)$ can be expressed as the kth iterate of a known function evaluated at x_0 with a finite limit as $k \to \infty$.

Consider,

$$x_{k+1} = \sum_{i=0}^{k} g_{k+1-i}(d_i - b_i)$$

$$= g_1(d_k - b_k) + \sum_{i=0}^{k-1} r g_{k-i}(d_i - b_i)$$

$$x_{k+1} = g_1(d_k - b_k) + r x_k .$$

There are five possible transition states each of which takes place with probability depending on the value of x_k.

If $d_k = 1, \quad b_k = -1,$ then $x_{k+1} = r x_k + 2g_1$ with probability $p_1(x_k)$.

If $d_k = 1, \quad b_k = 0$

or , then $x_{k+1} = r x_k + g_1$ with probability $p_2(x_k)$.

$d_k = 0, \quad b_k = -1$

If $d_k = b_k$, then $x_{k+1} = r x_k$ with probability $p_3(x_k)$.

If $d_k = -1, b_k = 0$

or , then $x_{k+1} = r x_k - g_1$ with probability $p_4(x_k)$.

$d_k = 0, \quad b_k = 1$

If $d_k = -1, b_k = 1$, then $x_{k+1} = r x_k - 2g_1$ with probability $p_5(x_k)$.

The transition probabilities $p_n(x_k)$, $n = 1, 2, \cdots, 5$ are defined by

$$p_1(x_k) = pN(-a_1 - g_0 - x_k)$$

$$p_2(x_k) = p[N(a_0 - g_0 - x_k) - N(-a_1 - g_0 - x_k)] + (1 - p - q)[N(-a_1 - x_k)]$$

$$p_3(x_k) = 1 - p_1(x_k) - p_2(x_k) - p_4(x_k) - p_5(x_k)$$

$$p_4(x_k) = q[N(a_0 + g_0 - x_k) - N(-a_1 + g_0 - x_k)] + (1 - p - q)[1 - N(a_0 - x_k)]$$

$$p_5(x_k) = q[1 - N(a_0 + g_0 - x_k)].$$

Note,

$$p(k) = \int_{-\infty}^{\infty} [p_1(x) + p_2(x) + p_4(x) + p_5(x)]\, dF_k(x).$$

Defining $U^1[f(x)] = p_1(x)f(rx + 2g_1) + p_2(x)f(rx + g_1) + p_3(x)f(rx) + p_4(x)f(rx - g_1) + p_5(x)f(rx - 2g_1)$

and denoting the kth iterate of $U^1[f(x)]$ by $U^k[f(x)]$,

$$p(k) = U^k[p_1(x) + p_2(x) + p_4(x) + p_5(x)] \,|_{x=x_0=0} \,.$$

If $A(x)$ is the limiting distribution of $F_k(x)$, then

$$\lim_{k\to\infty} p(k) = \int_{-\infty}^{\infty} [p_1(x) + p_2(x) + p_4(x) + p_5(x)]\, dA(x)$$

$$= \lim_{k\to\infty} U^k[p_1(x) + p_2(x) + p_4(x) + p_5(x)] \,|_{x=x_0=0} \,.$$

A few remarks are now presented to indicate the obvious extension to the m-level (m-ary) pulse transmission scheme. A random jump process with $2m - 1$ transition states will result requiring an iteration function $U^1[f(x)]$ having $2m - 1$ terms. It should be indicated that computationally the amount of computer storage or operations required to evaluate $p(k)$ is of the order $(2m - 1)^k$.

IV. RATIONAL FUNCTION APPROXIMATIONS OF THE CHANNEL AND FEEDBACK NETWORKS

As the subtitle indicates, we are interested here in studying the repeater error performance under the assumption of a rational function approximation to the channel and feedback networks. This generalizes the assumptions of Section III in that (*i*) the tail of the pulse response, $g(t)$, is no longer described by a single exponential, and (*ii*) the high-frequency behavior of the channel allows its time response to exceed two pulse intervals. To isolate these effects, however, perfect feedback tail cancellation is still assumed and we return to a binary message format.

It is convenient to represent the output rectangular pulses of the nth repeater as the impulse response of a filter $F(s) = (1/s)[1 - e^{-st_0}]$ where t_0 is the pulse width. Including this filter in the forward path of Fig. 1, the overall channel link between repeaters, $T(s) = F(s)C(s)E(s)G(s)$, is assumed to be characterized by a rational function as follows:

$$T(s) = G_0 \frac{s^M}{\prod_{i=1}^{M} (s + \alpha_i)} \times \frac{P(s)}{\prod_{i=1}^{N} (s + \beta_i)}$$

with its associated impulse response

$$g(t) = \sum_{i=1}^{M} A_i e^{-\alpha_i t} + \sum_{i=1}^{N} B_i e^{-\beta_i t}.$$

Note, the impulse response of $T(s)$ is the same as the rectangular pulse response of $Y(s) = C(s)\ E(s)\ G(s)$ and is thus denoted as before by $g(t)$. All poles are assumed to be simple, but in general may be complex. The terminology used henceforth will refer to the set $\{\alpha_i\}$, $i = 1, 2, \cdots, M$ as *low-frequency poles* and the set $\{\beta_i\}$, $i = 1, 2, \cdots, N$ as *high-frequency poles.* The inference here is that the β_i's are predominantly responsible for signal shaping and the α_i's determine the low-frequency cutoff of the channel.

A low-pass quantized feedback path $H(s)$ is proposed which in the absence of noise would provide perfect low-frequency tail cancellation at all sampling instants beyond the input pulse peak (the effect of imperfect low-frequency compensation will be discussed in Section V).* Thus, if

$$H(s) = H_0 \frac{N(s)}{\prod_{i=1}^{M} (s + \alpha_i)} e^{-s\tau_0},$$

where τ_0 represents the physical delay in the feedback path beyond the occurrence of the input pulse peak at $t = t_{\max}$, then, the response to a

* It is to be emphasized at this point that all of the following is easily generalized in terms of MacColl's conception of quantized feedback[7] wherein restoration of both low- and high-frequency signal components is attempted.

positive regenerator output pulse at $t = t_{\max}$ would be

$$h(t) = \sum_{i=1}^{M} D_i e^{-\alpha_i(t-t_{\max}-\tau_0-t_0)} = \sum_{i=1}^{M} E_i e^{-\alpha_i t} \quad \text{for} \quad t \geqq t_{\max} + \tau_0 + t_0 \,.$$

Ideally, for perfect low-frequency tail compensation, we desire

$$\sum_{i=1}^{M} E_i e^{-\alpha_i t} + \sum_{i=1}^{M} A_i e^{-\alpha_i t} = 0$$

at all instants $t_{\max} + nT$, $n = 1, 2, \cdots$, where T is the uniform sampling period.

Letting h_k and g_k represent the values of the pulse responses $h(t)$ and $g(t)$, respectively, at the kth sample point the above statements may be expressed in brief as follows:

$$(i) \qquad h_0 = 0 \qquad g_0 > 0$$

$$(ii) \qquad h_i + g_i = \sum_{n=1}^{N} e_{i,n} \qquad i \neq 0$$

$$e_{i,n} = z_n e_{i-1,n} \begin{cases} n = 1, 2, \cdots, N \\ i \geqq 2 \end{cases}$$

$$e_{0,n} = 0$$

$$z_n = e^{-\beta_n T}$$

$$(iii) \qquad h_i = \sum_{n=1}^{M} h_{i,n}$$

$$h_{i,n} = r_n h_{i-1,n} \begin{cases} n = 1, 2, \cdots, M \\ i \geqq 2 \end{cases}$$

$$r_n = e^{-\alpha_n T}.$$

The term $e_{i,n}$ represents the residual intersymbol interference at the ith timing instant due to the nth high-frequency pole. To simplify what is to follow and at the same time allow a better comparison with the previous work of Zador, we introduce the following vector notation:

$$R = \begin{bmatrix} r_1 & 0 & 0 \cdots\cdots 0 \\ 0 & r_2 \cdots\cdots & 0 \\ & & \vdots \\ 0 & & \vdots \\ \vdots & & \vdots \\ 0 \cdots\cdots & & r_M \end{bmatrix}; \qquad Z = \begin{bmatrix} z_1 & 0 & 0 \cdots\cdots 0 \\ 0 & z_2 \cdots\cdots & 0 \\ & & \vdots \\ 0 & & \vdots \\ \vdots & & \vdots \\ 0 \cdots\cdots & & z_N \end{bmatrix}$$

$$H = \begin{bmatrix} h_{0,1} & h_{1,1} & h_{2,1} & h_{3,1} & \cdots & h_{k,1} \\ h_{0,2} & h_{1,2} & h_{2,2} & h_{3,2} & \cdots & h_{k,2} \\ \vdots & \vdots & \vdots & \vdots & & \\ h_{0,M} & h_{1,M} & h_{2,M} & h_{3,M} & \cdots & h_{k,M} \end{bmatrix}$$

$$G = \begin{bmatrix} g_{0,1} & g_{1,1} & g_{2,1} & g_{3,1} & \cdots & g_{k,1} \\ g_{0,2} & g_{1,2} & g_{2,2} & g_{3,2} & \cdots & g_{k,2} \\ \vdots & \vdots & \vdots & \vdots & & \\ g_{0,M+N} & g_{1,M+N} & g_{2,M+N} & g_{3,M+N} & \cdots & g_{k,M+N} \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & e_{1,1} & e_{2,1} & e_{3,1} & \cdots & e_{k,1} \\ 0 & e_{1,2} & e_{2,2} & e_{3,2} & \cdots & e_{k,2} \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & e_{1,N} & e_{2,N} & e_{3,N} & \cdots & e_{k,N} \end{bmatrix}.$$

Using H as an example, the ith row written as a column vector is denoted by $\mathbf{h}^i$ and the ith column by the vector $\mathbf{h}_i$. Also any vector written not in bold face is by definition the scalar representing the sum of its elements (e.g., $h^i = \sum_{p=1}^{k} h_{p,i}$). Finally, we denote the column vector obtained by summing all rows of H (i.e., whose ith component is h_i) by $\mathbf{h}$. All of the above statements are equally applied to the matrices R, Z, G, and E.

In terms of the above, (*i*), (*ii*), and (*iii*) may now be rewritten as:

(*i*) $\qquad h_0 = 0, \qquad g_0 > 0$

(*ii*) $\qquad \mathbf{h} + \mathbf{g} = \mathbf{e}$

$\qquad \mathbf{e}_i = Z\mathbf{e}_{i-1}, \qquad i \geqq 2$

(*iii*) $\qquad \mathbf{h}_i = R\mathbf{h}_{i-1}, \qquad i \geqq 2.$

Some further interpretation of the above statements in terms of the actual system operation might prove helpful at this point. (*i*) indicates a positive input pulse peak ($g_0 > 0$) and a delay in the feedback path ($h_0 = 0$). Statements (*ii*) indicate that perfect feedback tail cancellation is achieved at the sample points starting with the second except for the effects of the high-frequency poles ($\beta_1, \beta_2, \cdots, \beta_N$). In contrast to the previous sections, we do not assume that the high-frequency components of the response have died out before the occurrence of the next input pulse peak. Feedback cancellation of only the channel low-frequency poles is described by statements (*iii*).

For the binary message case, the threshold detector box of Fig. 1 reduces to a simple ideal slicer element operating between +1 and −1 levels. The input sequence $\{d_i\}$ is a random train of +1 and −1 impulses represented by the vector $\mathbf{d}$ with elements d_i.

The total reshaped input at the $k + 1$th timing instant, s_k, and the feedback voltage at the same instant, c_k, are described by,

$$\begin{aligned} s_k &= (\mathbf{d} * \mathbf{g})_k \qquad k = 0, 1, 2, \cdots, \\ c_k &= (\mathbf{b} * \mathbf{h})_k \end{aligned}$$

where the kth element of $\mathbf{b}$, $b_k = \operatorname{sgn}\{s_k + c_k + n_k\}$ is the kth regenerator output digit. The notation $(\mathbf{a} * \mathbf{b})_k$ represents the convolution of two $k + 1$ dimensional vectors $\mathbf{a}$ and $\mathbf{b}$ (i.e., $\sum_{i=0}^{k} a_i b_{k-i}$).

Considering first operation in the absence of noise, we see by inspection $b_0 = d_0$. (This tacitly assumes that no intersymbol interference due to precursors is present.) Proceeding as in Zador,[5] if $b_m = d_m$ for $m = 0, 1, \cdots, k - 1$, then

$$s_k + c_k = (\mathbf{d} * \mathbf{e})_k = g_0 d_k + (\mathbf{d} * \mathbf{e})_{k-1}.$$

From this, one concludes that if

$$g_0 > \sum_{i=1}^{k-1} | e_i |,$$

then $b_k = d_k$ and the eye is open. The system will therefore operate error-free in the absence of noise for *any* length input sequence if

$$g_0 > \sum_{i=1}^{\infty} | e_i |.$$

If all N high-frequency poles $(\beta_1, \beta_2, \cdots, \beta_N)$ have positive residues, the above criterion reduces to

$$g_0 > \sum_{n=1}^{N} \frac{e_{1,n}}{1 - z_n}.$$

The above implies that the eye is open if the total high-frequency contribution at all sample points beyond the first is smaller than the pulse peak.

More specifically, the values of $e_{1,n}$ and z_n may be related to the allowable amount of degradation of the eye. That is, for any eye which is X percent closed.

$$\sum_{n=1}^{N} \frac{e_{1,n}}{1 - z_n} = \frac{X}{100} g_0 .$$

Turning now to the more realistic situation in the presence of noise

$$s_k + c_k = g_0 d_k + (\mathbf{d} * \mathbf{g})_{k-1} + (\mathbf{b} * \mathbf{h})_{k-1} .$$

Consider subdividing the vector $\mathbf{d}$ into two parts $\mathbf{d}'$ and $\mathbf{d}''$ in such a way as to separate the input digits into two classes corresponding to $b_i = d_i$ and $b_i \neq d_i$ respectively. That is,

$$\begin{aligned} d_i' &= d_i \,; \qquad d_i'' = 0 \qquad \{i;\, b_i = d_i\} \\ &= 0 \qquad\qquad\; = d_i \qquad \{i;\, b_i \neq d_i\}. \end{aligned}$$

(Obviously $\mathbf{d} = \mathbf{d}' + \mathbf{d}''$.)
Then, using (*ii*),

$$\mathbf{u}_k = (\mathbf{d} * E^T)_{k-1} \,;\; \mathbf{v}_k = -2(\mathbf{d} * H^T)_{k-1}$$

$$\begin{aligned} s_k + c_k &= g_0 d_k + u_k + v_k \\ &= g_0 d_k + x_k \,, \end{aligned}$$

where $(\mathbf{d} * G)_{k-1}$ is a vector whose ith component is the convolution of $\mathbf{d}$ with the ith column of G. Again omission of the bold face notation indicates summation over all the components and T is the transpose operator.

The first term in x_k denoted by u_k represents intersymbol interference due to residual high-frequency tail components irrespective of previous decisions. The second term v_k again represents the cumulative effect of any and all errors prior to time k.

The expression for error probability on the kth digit is identical to that given by Zador, namely,

$$p(k) = p \int_{-\infty}^{\infty} N(-g_0 - x)\, dF_k(x) + q \int_{-\infty}^{\infty} [1 - N(g_0 - x)]\, dF_k(x).$$

The only difference being the nature of the distribution function $F_k(x)$.

The recursive properties of the intersymbol interference x_k are now examined.

$$x_{k+1} = (\mathbf{d} * E^T)_k - 2(\mathbf{d}'' * H^T)_k .$$

If $b_k \neq d_k$, then

$$\begin{aligned} x_{k+1} &= (e_1 - 2h_1)d_k + \mathbf{z}^T(\mathbf{d} * E^T)_{k-1} - 2\mathbf{r}^T(\mathbf{d}'' * H^T)_{k-1} \\ &= (e_1 - 2h_1)d_k + \mathbf{z}^T\mathbf{u}_k + \mathbf{r}^T\mathbf{v}_k . \end{aligned}$$

If $b_k = d_k$, then

$$x_{k+1} = e_1 d_k + \mathbf{z}^T(\mathbf{d} * E^T)_{k-1} - 2\mathbf{r}^T(\mathbf{d}'' * H^T)_{k-1}$$
$$= e_1 d_k + \mathbf{z}^T\mathbf{u}_k + \mathbf{r}^T\mathbf{v}_k .$$

Letting $\boldsymbol{a} = -2\mathbf{h}_1$, the intersymbol interference sequence x_0, x_1, x_2, $\cdots$ may be expressed as a random jump process,[5,6] with the following transition states:

If $d_k = 1 \neq b_k$, then with probability $p_1(x_k)$

$$x_{k+1} = \mathbf{z}^T\mathbf{u}_k + e_1 + \mathbf{r}^T\mathbf{v}_k + a.$$

If $d_k = 1 = b_k$, then with probability $p_2(x_k)$

$$x_{k+1} = \mathbf{z}^T\mathbf{u}_k + e_1 + \mathbf{r}^T\mathbf{v}_k .$$

If $d_k = 1 \neq b_k$, then with probability $p_3(x_k)$

$$x_{k+1} = \mathbf{z}^T\mathbf{u}_k - e_1 + \mathbf{r}^T\mathbf{v}_k - a.$$

If $d_k = -1 = b_k$, then with probability $p_4(x_k)$

$$x_{k+1} = \mathbf{z}^T\mathbf{u}_k - e_1 + \mathbf{r}^T\mathbf{v}_k ,$$

where

$$p_1(x_k) = pN(-g_0 - x_k)$$
$$p_2(x_k) = p[1 - N(-g_0 - x_k)]$$
$$p_3(x_k) = q[1 - N(g_0 - x_k)]$$
$$p_4(x_k) = q[N(g_0 - x_k)].$$

In the above, $N(x)$ is the distribution function of the stationary noise process, and p and q are the *a priori* probabilities of a plus one and minus one, respectively. In terms of the above elementary probability density functions, the error probability on the kth digit may be expressed as:

$$p(k) = \int_{-\infty}^{\infty} [p_1(x) + p_3(x)]\, dF_k(x).$$

We propose a vector extension of Zador's procedure, namely; an $M + N$ dimensional iteration scheme in which *each* of $M + N$ variables is replaced by a linear transformation on itself during each iteration. To elucidate the meaning of $M + N$ dimensional iteration and at the same time recall some of our earlier vector notation, the first-order iteration function U^1f is written in summation notation as:

$$\begin{aligned} U^1 f(\boldsymbol{\theta}, \boldsymbol{\varphi}) &= p_1\left(\sum_{n=1}^{N} \theta_n + \sum_{m=1}^{M} \varphi_m\right) \cdot f\left[\sum_{n=1}^{N} (z_n\theta_n + e_{1,n}) + \sum_{m=1}^{M} (r_m\varphi_m + a_m)\right] \\ &+ p_2\left(\sum_{n=1}^{N} \theta_n + \sum_{m=1}^{M} \varphi_m\right) \cdot f\left[\sum_{n=1}^{N} (z_n\theta_n + e_{1,n}) + \sum_{m=1}^{M} (r_m\varphi_m)\right] \\ &+ p_3\left(\sum_{n=1}^{N} \theta_n + \sum_{m=1}^{M} \varphi_m\right) \cdot f\left[\sum_{n=1}^{N} (z_n\theta_n - e_{1,n}) + \sum_{m=1}^{M} (r_m\varphi_m - a_m)\right] \\ &+ p_4\left(\sum_{n=1}^{N} \theta_n + \sum_{m=1}^{M} \varphi_m\right) \cdot f\left[\sum_{n=1}^{N} (z_n\theta_n - e_{1,n}) + \sum_{m=1}^{M} (r_m\varphi_m)\right]. \end{aligned}$$

It follows that the probability of error on the kth digit is

$$p(k) = U^k[p_1 + p_3] \,|_{\boldsymbol{\theta}, \boldsymbol{\varphi}=0}$$

where U^k is the kth $M + N$ dimensional iterate of U^1. The convergence of $p(k)$ in the limit as $k \to \infty$ has not been examined for an $M + N$ dimensional branching process. From Zador's work on one-dimensional branching processes[6] we may conjecture that absolute system stability (i.e., all poles in left-half plane) implies convergence in the multidimensional case.

Although the notation in the foregoing analysis appears formidable (quite an understatement) the procedure and its usage are straightforward (at least analytically) for a particular example. At the expense of being redundant, we once again point out that even in simple cases, numerical results are hard to come by.

V. IMPERFECT LOW-FREQUENCY TAIL CANCELLATION

It is relatively simple at this point to include the effect of imperfect low-frequency cancellation in the results of Section IV. As an example, such a phenomenon might be caused by a delay of amount τ in the feedback path. Defining an L matrix by

$$L = \begin{bmatrix} 0 & l_{1,1} & l_{2,1} & l_{3,1} & \cdots & l_{k,1} \\ 0 & l_{1,2} & l_{2,2} & l_{3,2} & \cdots & l_{k,2} \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & l_{1,M} & l_{2,M} & l_{3,M} & \cdots & l_{k,M} \end{bmatrix},$$

where

$$l_{1,m} = h_{1,m}\left[1 - \exp\left(-\frac{\tau}{T}\log_e \frac{1}{r_m}\right)\right] \qquad m = 1, 2, \cdots, M,$$

statement (*ii*) of Section IV may be modified as follows:

$$(ii) \qquad \mathbf{h} + \mathbf{g} = \mathbf{e} + \mathbf{l}$$

$$\mathbf{l_i} = R\mathbf{l}_{\mathbf{i}-1} \qquad \mathbf{i} \geqq 2$$

$$\mathbf{e_i} = Z\mathbf{e}_{\mathbf{i}-1}\,.$$

The effect of this on the recursion relationship for x_k is as follows: If $b_k \neq d_k$, then

$$x_{k+1} = (e_1 + l_1 - 2h_1)d_k + \mathbf{z}^T\mathbf{u_k} + \mathbf{r}^T\mathbf{v_k} + \mathbf{r}^T\boldsymbol{\omega}_\mathbf{k}.$$

If $b_k = d_k$, then

$$x_{k+1} = (e_1 + l_1)d_k + \mathbf{z}^T\mathbf{u_k} + \mathbf{r}^T\mathbf{v_k} + \mathbf{r}^T\boldsymbol{\omega}_\mathbf{k}\,,$$

where $\boldsymbol{\omega}_\mathbf{k} = (\mathbf{d} * L^T)_{k-1}$.

If $d_k = 1 \neq b_k$, then with probability $p_1(x_k)$

$$x_{k+1} = \mathbf{z}^T\mathbf{u_k} + e_1 + \mathbf{r}^T\mathbf{v_k} + a + \mathbf{r}^T\boldsymbol{\omega}_\mathbf{k} + l_1.$$

If $d_k = 1 = b_k$, then with probability $p_2(x_k)$

$$x_{k+1} = \mathbf{z}^T\mathbf{u_k} + e_1 + \mathbf{r}^T\mathbf{v_k} + \mathbf{r}^T\boldsymbol{\omega}_\mathbf{k} + l_1.$$

If $d_k = 1 \neq b_k$, then with probability $p_3(x_k)$

$$x_{k+1} = \mathbf{z}^T\mathbf{u_k} - e_1 + \mathbf{r}^T\mathbf{v_k} - a + \mathbf{r}^T\boldsymbol{\omega}_\mathbf{k} - l_1.$$

If $d_k = -1 = b_k$, then with probability $p_4(x_k)$

$$x_{k+1} = \mathbf{z}^T\mathbf{u_k} - e_1 + \mathbf{r}^T\mathbf{v_k} + \mathbf{r}^T\boldsymbol{\omega}_\mathbf{k} - l_1\,,$$

where $p_1(x_k)$, $p_2(x_k)$, $p_3(x_k)$, and $p_4(x_k)$ are still defined as in Section IV.

The kth bit probability of error is now evaluated by a $2M + N$ dimensional iteration scheme where the first-order iteration function U^1f is written as

$$U^1f(\boldsymbol{\theta}, \boldsymbol{\varphi}, \boldsymbol{\gamma}) = p_1\left(\sum_{n=1}^{N}\theta_n + \sum_{m=1}^{M}\varphi_m + \sum_{m=1}^{M}\gamma_m\right)$$

$$\cdot f\left[\sum_{n=1}^{N}(z_n\theta_n + e_{1,n}) + \sum_{m=1}^{M}(r_m\varphi_m + a_m) + \sum_{m=1}^{M}(r_m\gamma_m + l_{1,m})\right]$$

$$+ p_2\left(\sum_{n=1}^{N} \theta_n + \sum_{m=1}^{M} \varphi_m + \sum_{m=1}^{M} \gamma_m\right)$$

$$\cdot f\left[\sum_{n=1}^{N} (z_n\theta_n + e_{1,n}) + \sum_{m=1}^{M} (r_m\varphi_m) + \sum_{m=1}^{M} (r_m\gamma_m + l_{1,m})\right]$$

$$+ p_3\left(\sum_{n=1}^{N} \theta_n + \sum_{m=1}^{M} \varphi_m + \sum_{m=1}^{M} \gamma_m\right)$$

$$\cdot f\left[\sum_{n=1}^{N} (z_n\theta_n - e_{1,n}) + \sum_{m=1}^{M} (r_m\varphi_m - a_m) + \sum_{m=1}^{M} (r_m\gamma_m - l_{1,m})\right]$$

$$+ p_4\left(\sum_{n=1}^{N} \theta_m + \sum_{m=1}^{M} \varphi_m + \sum_{m=1}^{M} \gamma_m\right)$$

$$\cdot f\left[\sum_{n=1}^{N} (z_n\theta_n - e_{1,n}) + \sum_{m=1}^{M} (r_m\varphi_m) + \sum_{m=1}^{M} (r_m\gamma_m - l_{1,m})\right].$$

It once again follows that the probability of error on the kth digit is

$$p(k) = U^k[p_1 + p_3] \,|_{\boldsymbol{\theta},\boldsymbol{\varphi},\boldsymbol{\gamma}=0} ,$$

where U^k is the kth $2M + N$ dimensional iterate of U^1.

To reward the reader for his patience up to this point, we will at least demonstrate that the general expression for $p(k)$ given above reduces to Zador's result for the single low-frequency pole, perfect cancellation case. The assumption of no high-frequency signal shaping and perfect cancellation imply that $\boldsymbol{\theta}$, $\mathbf{e}_1$, and $\boldsymbol{\gamma}$, $\mathbf{l}_1$ are, respectively, zero. Furthermore, a single low-frequency pole results in r_1, φ_1, and a_1 being the only nonzero components of $\mathbf{r}$, $\boldsymbol{\varphi}$, and $\mathbf{a}$, respectively. Under these conditions,

$$p(k) = U^k[p_1 + p_3] \,|_{\varphi_1=0} ,$$

where

$$U^1 f = p_1(\varphi) f(r_1\varphi_1 + a_1) + p_2(\varphi_1) f(r_1\varphi_1) + p_3(\varphi_1) f(r_1\varphi_1 - a_1) + p_4(\varphi_1) f(r_1\varphi_1)$$

which is identical to Zador's result upon combining $p_2(\varphi_1)$ and $p_4(\varphi_1)$.

VI. THE EFFECT OF NOISE CORRELATION

In this part, the emphasis is placed upon removing the restriction of uncorrelated noise while at the same time arranging the results in a form which allows easy comparison with the uncorrelated case. The approach to be followed is the reformulation of Zador's work into an *operational* iteration procedure which acts on the joint distribution of the noise samples. The details are presented for the simple binary case with perfect feedback cancellation considered by Zador. With sufficient patience, extension to the more general situations covered in the foregoing sections can be accomplished, but that is not done here.

To review, the operation of the simplified system may be described by the equation

$$b_k = \operatorname{sgn}\{n_k + g_0 d_k + x_k\}, \tag{1}$$

where

$$x_k = 2 \sum_{\substack{i=0 \\ b_i = -d_i}}^{k-1} g_{k-i} d_i \tag{2}$$

represents the intersymbol interference accumulated at time t_k as a result of errors ($d_i \neq b_i$) prior to that time.

The system output b_k is in error when

$$\begin{aligned} n_k + x_k &< -g_0 \quad \text{and} \quad d_k = 1 \\ n_k + x_k &> g_0 \quad \text{and} \quad d_k = -1. \end{aligned} \tag{3}$$

Since the noise samples are not assumed to be independent, the random variables n_k and x_k are not independent. Hence, the distribution of the effective noise $n_k + x_k$ is not simply the convolution of the distributions of n_k and x_k. Instead, the expression for error probability $p(k) =$ prob $\{b_k \neq d_k\}$ must be written as

$$p(k) = p \int_{-\infty}^{\infty} \int_{-\infty}^{-g_0 - x_k} m_2(n_k, x_k)\, dn_k\, dx_k + q \int_{-\infty}^{\infty} \int_{g_0 - x_k}^{\infty} m_2(n_k, x_k)\, dn_k\, dx_k, \tag{4}$$

where $m_2(n_k, x_k)$ is the joint density function of n_k and x_k and p and q are the *a priori* probabilities of a $+1$ and -1, respectively.

A careful examination of the branching process described in Refs. 5 and 6 for the uncorrelated noise case shows that a similar process governs in the correlated noise case. Define the integral operators $p_1(x)$, $p_2(x)$, $p_3(x)$ by

$$\begin{aligned} p_1(x) &= p \int_{-\infty}^{-g_0 - x} \\ p_2(x) &= \int_{-\infty}^{\infty} - p \int_{-\infty}^{-g_0 - x} - q \int_{g_0 - x}^{\infty} \\ p_3(x) &= q \int_{g_0 - x}^{\infty}. \end{aligned} \tag{5}$$

Note that the action of each of these three operators on a single dimension Gaussian density function results in the three transition probabilities defined by (13) of Ref. 5. If $f(x)$ is defined analogously (but operationally) as

$$f(x) = p_1(x) + p_3(x) = p \int_{-\infty}^{-g_0-x} + q \int_{g_0-x}^{\infty} , \tag{6}$$

then the first-order iterative *operator* $Uf(x)$ is expressed as in Zador, namely:

$$Uf(x) = p_1(x)f(rx - a) + p_2(x)f(rx) + p_3(x)f(rx + a) \tag{7}$$

with $a = -2g_1$. We note that after separating $f(x)$ into its two components parts, each term of (7) represents a double integration and thus (7) has meaning only when applied to a second-order density function. Proceeding as in Zador, the error probability on the $k+1$th digit in a random input sequence is expressed as the kth iterate of the operator $Uf(x)$ acting on the $k+1$ dimensional joint density function of the noise process v_{k+1} $(\gamma_1, \gamma_2, \cdots, \gamma_{k+1})$ evaluated at $x = 0$, i.e.,

$$p(k) = U^k f(x)[v_k(\gamma_1 , \gamma_2 , \cdots , \gamma_{k+1})] \big|_{x=0} .^* \tag{8}$$

The meaning of iteration for the operators defined here is the same as in Zador's functional case. As an example, we write out $p(1)$ in detail:

$$\begin{aligned} p(1) = Uf(x)[v_2(\gamma_1 , \gamma_2)] \big|_{x=0} &= p^2 \int_{-\infty}^{-g_0} \int_{-\infty}^{-g_0+a} v_2(\gamma_1 , \gamma_2) \, d\gamma_1 \, d\gamma_2 \\ &+ pq \int_{-\infty}^{-g_0} \int_{g_0+a}^{\infty} v_2(\gamma_1 , \gamma_2) \, d\gamma_1 \, d\gamma_2 + p \int_{-\infty}^{\infty} \int_{-\infty}^{-g_0} v_2(\gamma_1 , \gamma_2) \, d\gamma_1 \, d\gamma_2 \\ &+ q \int_{-\infty}^{\infty} \int_{g_0}^{\infty} v_2(\gamma_1 , \gamma_2) \, d\gamma_1 \, d\gamma_2 - p^2 \int_{-\infty}^{-g_0} \int_{-\infty}^{-g_0} v_2(\gamma_1 , \gamma_2) \, d\gamma_1 \, d\gamma_2 \\ &- pq \int_{-\infty}^{-g_0} \int_{g_0}^{\infty} v_2(\gamma_1 , \gamma_2) \, d\gamma_1 \, d\gamma_2 - pq \int_{g_0}^{\infty} \int_{-\infty}^{-g_0} v_2(\gamma_1 , \gamma_2) \, d\gamma_1 \, d\gamma_2 \\ &- q^2 \int_{g_0}^{\infty} \int_{g_0}^{\infty} v_2(\gamma_1 , \gamma_2) \, d\gamma_1 \, d\gamma_2 + pq \int_{g_0}^{\infty} \int_{-\infty}^{-g_0-a} v_2(\gamma_1 , \gamma_2) \, d\gamma_1 \, d\gamma_2 \\ &+ q^2 \int_{g_0}^{\infty} \int_{g_0-a}^{\infty} v_2(\gamma_1 , \gamma_2) \, d\gamma_1 \, d\gamma_2 . \end{aligned} \tag{9}$$

* The convergence of the *operational* iteration procedure defined by (7) and (8) has not yet been proven. Nonetheless, we proceed with our results.

The above expression for $p(1)$ can be simplified for a symmetric density function v_2. It is further emphasized that the arguments of each term in the operator as defined by (7) determine the limits on the integrals in (9) (i.e., the region of integration).

VII. CONCLUSIONS

The analysis presented in this paper might in a broad sense be described as vector and operational extensions of the work of Zador. In addition to simply considering a vector of low-frequency poles, however, the vector approach has enabled us to remove certain other restrictions from the basic regenerator problem such as lack of high-frequency signal shaping and perfect tail cancellation. Although, the question of convergence of the operational iteration scheme for correlated noise samples remains as yet unanswered, the formulation itself, is of interest. Little has been suggested for solving the exact computational problem. A future paper will discuss some useful approximations to cases of relatively low dimensionality. This will generalize results given in Ref. 8.

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