Communications and Radar Receiver Gains for Minimum Average Cost of Excluding Randomly Fluctuating Signals in Random Noise

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The problem of automatic gain control is approached from a statistical point of view. A simple generic equation is found whose solution yields the required receiver gain or attenuation for minimum average cost of excluding (from the receiver's limited dynamic range) randomly fluctuating signals in random noise. A canonical phase-incoherent link is considered and the resulting transcendental equation is solved using an iterative technique. The analysis and the results obtained apply to both linear and nonlinear incoherent receivers including those of the logarithmic or lin-log type and to a range of fluctuation models including Rician, Rayleigh, and nonfluctuating cases. It is shown that the optimum receiver gain is relatively insensitive to the ratio of costs of saturation at the upper and lower dynamic range bounds, differing at most by about 3 dB from the optimum for the equal cost (minimum exclusion probability) case for typical parameters. The effect of noise introduced by the gain adjustment cascade itself is discussed.

The results, presented in concise normalized form, are applicable to a wide range of signal, noise, and channel conditions and have important implications for communications through fading channels as well as for radar observation of fluctuating targets.

I. INTRODUCTION

Since the ability of both communications and radar receivers to perform satisfactorily can be seriously degraded when the signal amplitude does not lie within the dynamic range of the receiver, the setting of receiver gain to minimize or prevent saturation at the upper and lower dynamic range bounds is an important problem. The problem arises in various forms. In simple receivers, the gain might be fixed to optimize performance for nominal signal and noise parameters. More complicated receivers can adjust the gain automatically by any of several methods. The most common AGC circuits, for example, use a time averaged baseband signal as an indication of signal strength. Another possibility is to have the gain adjusted on command from a digital computer. This latter configuration has important implications for communications terminals which can use sophisticated techniques for estimation of signal and noise parameters as well as for certain radars which must observe from look-to-look radar targets of different cross-section which have been illuminated by various transmitted waveforms. The analysis presented here does not depend on the particular configuration and is applicable to both linear and nonlinear receivers including those of the logarithmic and lin-log type. The application to a nonlinear receiver can be accomplished by referring the overall dynamic range of the signal processing chain to a point before the nonlinearity.

In a recent correspondence, Ward[1] determined the placement of dynamic range bounds to minimize the probability of excluding a Rayleigh distributed signal. This was extended by Rappaport[2] who determined dynamic range bounds for minimum probability of excluding a signal from the dynamic range of incoherent radar or communications receivers. The viewpoint taken there[2] considered randomness due *either* to background noise *or* target fluctuations (channel fading). This paper considers several further generalizations of the problem. The case in which the randomness is due to both causes together is treated. In addition, the criterion for optimization is taken as the minimum average cost of exclusion. The required receiver gains as well as the optimum dynamic range bounds are determined.

The present paper proceeds from the specific to the general. That is, first the determination of optimum dynamic range bounds for minimum exclusion probability with non-fluctuating target (no channel fading) is presented.

The criterion is then generalized from minimum exclusion probability to minimum average exclusion cost; the former being a special case of the latter. Finally, dynamic range bounds and receiver gains for minimum average exclusion cost in an environment of fluctuating targets or channel fading is determined. It is assumed throughout that the signal, noise, and fluctuation parameters are known to the receiver. By letting the parameters involved assume certain values the relations for the fluctuating case reduce to the nonfluctuating case. Hence, the general treatment presented here includes either criteria and either the case of fluctuating or nonfluctuating SNR. Rician and Rayleigh SNR fluctuations are considered.

Fig. 1 shows a model for an incoherent radar or communications link. The blocks labeled K_1, K_2, and K_3 represent variable gain devices (perhaps, variable attenuation pads) whose total gain $K = K_1K_2K_3$ is to be adjusted so that the random signal appearing at (E) is in some sense confined to a specified range. The model used for the propagation medium and/or target is described in Section IV. Extension of the explicit results obtained here to an important class of nonlinear receivers by conceptually including a zero-memory nonlinear device between points (D) and (E) will be described subsequently. The other blocks in the figure require no further explanation. The figure is presented so that the reader can obtain a clear understanding of where in the signal processing chain various quantities arising in the following analysis are being determined. However, the analysis applies to incoherent signal processing links in general and is not constrained, for example, by the number of components or IF frequencies that may be used.

The receiver structure shown in Fig. 1 may be used for recovering the envelope of a transmitted sinusoidal signal or it may represent an optimum incoherent receiver for the detection of finite duration signals of known form in a background of Gaussian noise. The probability density function (pdf) of interest in the former case is that of the voltage appearing at the input to the video processing

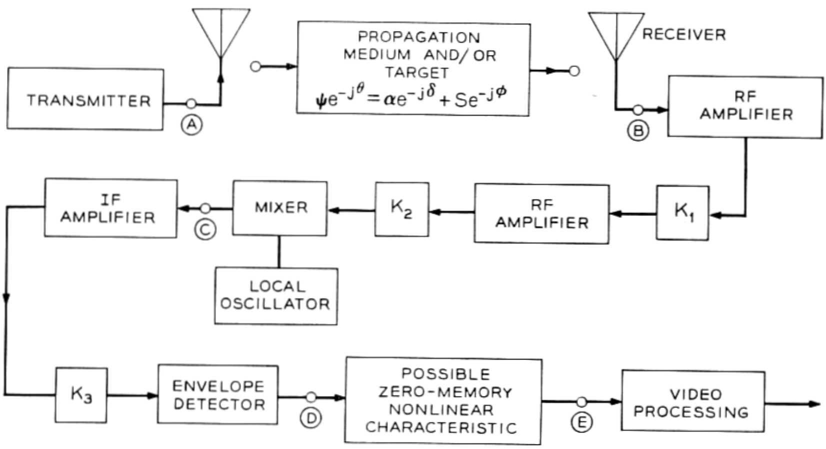

Fig. 1 — Model for an incoherent radar or communications link.

[point (E) in Fig. 1] while in the latter case the pdf of concern is that of the voltage at the same point in the receiver at the decision time only. Either of these cases gives rise to a Rician pdf. Differences between the two are reflected only in the definition of a suitable SNR.[3, 4]

The analysis presented here can be easily extended to determine optimum gain settings for an important class of nonlinear incoherent receivers; namely, those in which the nonlinearity can be represented by a memoryless nonlinear device acting on the envelope of the received signal. For example, logarithmic or lin-log receivers can be represented by a logarithmic or lin-log characteristic inserted between points (D) and (E) regardless of whether the actual nonlinear device is a video or IF amplifier. One needs only to first refer the dynamic range and equivalent limiting voltages from the output of the nonlinear characteristic [point (E)] to the corresponding values at the input to the characteristic [point (D)] and then to determine the gain setting by considering only the linear portion of the receiver. In what follows it will be assumed that this first step has been taken if necessary and only the linear incoherent receiver will be treated explicitly.

II. DYNAMIC RANGE BOUNDS FOR MINIMUM EXCLUSION PROBABILITY WITH NONFLUCTUATING SNR

For nonfluctuating SNR the voltage gain of the radar or communications link is fixed. It is convenient to assume (without loss of generality) that the voltage gain ψ, of the propagation medium and/or target [i.e., the portion of the link from (A) to (B) in Fig. 1] is unity. In the more general case of fluctuating SNR, the voltage gain of this portion of the link will be treated as a random quantity.

The optimum placement of dynamic range bounds for incoherent receivers is determined by the pdf of the envelope detected signal, v, which appears at point (D) in Fig. 1. Let σ be a normalization parameter and define

$$R = \text{normalized envelope of received signal} = v/\sigma$$
$$a = \text{lower normalized bound of dynamic range}$$
$$ad = \text{upper normalized bound of dynamic range}$$
$$D = 20 \log_{10} d = \text{dynamic range in dB},$$

where these quantities are referred to point (D) in Fig. 1. If $p_\gamma(R)$ denotes the pdf of the normalized envelope, the corresponding exclusion probability is

$$P_e(a, d) = 1 - \int_a^{ad} p_\gamma(R)\, dR. \tag{1}$$

To minimize $P_e(a, d)$ with respect to a for fixed dynamic range d, (1) is differentiated with respect to a and this derivative is set to zero. This yields the necessary optimization condition

$$p_\gamma(a) = d p_\gamma(ad) \tag{2}$$

which must be solved for a. Consider an optimum incoherent receiver for detection of signals of known form in Gaussian noise. Let $2\sigma^2$ be the mean square value of the signal voltage envelope, v, when only noise is present. In this case, the pdf of the normalized signal envelope is

$$p_o(R) = R \exp[-R^2/2]. \tag{3}$$

The optimization condition (2) for this case leads to

$$A^2 = \frac{2 \ln(d)}{d^2 - 1} \triangleq A_o^2 \tag{4}$$

in which $A \triangleq a/\sqrt{2}$ is the optimum normalized lower dynamic range bound. When signal-plus-noise is present the probability density of the normalized envelope has a Rician distribution[3,4]

$$p_\gamma(R) = R \exp[-(R^2 + \gamma^2)/2] I_o(\gamma R) \tag{5}$$

in which

$I_o(x)$ = modified Bessel function of first kind and order zero
γ = voltage signal-to-noise ratio for $\psi = 1$.

In this case condition (2) gives the following transcendental equation which must be solved for the optimum normalized lower dynamic range bound $A = a/\sqrt{2}$:[2]

$$A^2 = A_o^2 + \frac{1}{(d^2 - 1)} [\ln I_o(A d\gamma\sqrt{2}) - \ln I_o(A\gamma\sqrt{2})]. \tag{6}$$

The minimum exclusion probability becomes

$$P_e(a, d) = 1 - Q(\gamma, A\sqrt{2}) + Q(\gamma, Ad\sqrt{2}) \tag{7}$$

in which $Q(\alpha, \beta)$ is the tabulated[5] Q-function defined by

$$Q(\alpha, \beta) = \int_\beta^\infty \xi \exp[-(\xi^2 + \alpha^2)/2] I_o(\alpha\xi)\, d\xi. \tag{8}$$

Solutions to (6) for various γ and d are presented in Ref. 2 along with minimum exclusion probabilities for this case. The solutions can be obtained from Fig. 2 with γ used in place of $\tilde{\gamma}$ and A in place of $\tilde{A}$.

III. DYNAMIC RANGE BOUNDS FOR MINIMUM AVERAGE EXCLUSION COST WITH NONFLUCTUATING SNR

In certain situations it may not be desirable to use dynamic range bounds which minimize the exclusion probability. It may be reasonable to favor saturation at one dynamic range bound to the other. In the case of a radar, for example, the signal is invisible if it falls beneath the lower dynamic range bound, while if the receiver saturates at the upper dynamic range bound the presence of the signal would be detected although its information content would be corrupted by the limiting. In such cases, a more reasonable criterion might be to minimize the average cost of excluding the signal from the receiver's dynamic range.

Suppose that when the signal falls below the lower bound a loss, c_1, is incurred, while saturation at the upper bound causes a loss, c_2

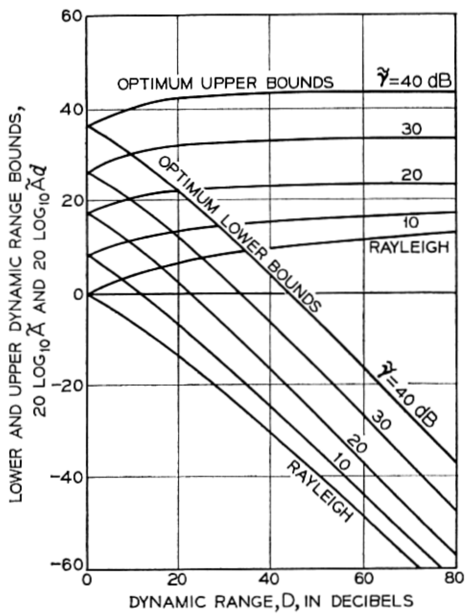

Fig. 2 — Optimum dynamic range centering for $\nu_{\text{dB}} = 0$.

(c_1, $c_2 > 0$). The expected or average cost of excluding the signal from the dynamic range is

$$L = c_1 \int_0^a p_\gamma(R)\,dR + c_2 \int_{ad}^\infty p_\gamma(R)\,dR. \tag{9}$$

It is convenient to divide (9) by c_1 to obtain the normalized average exclusion cost

$$l = \int_0^a p_\gamma(R)\,dR + \nu \int_{ad}^\infty p_\gamma(R)\,dR \tag{10}$$

in which ν is the cost ratio, $\nu \triangleq c_2/c_1$. It is seen that for $\nu = 1$ the normalized average exclusion cost, l, reduces to the exclusion probability. In order to minimize the average exclusion cost, the derivative of (10) with respect to a is set to zero. One then finds that the optimum lower normalized dynamic range bound a must satisfy

$$p_\gamma(a) = \nu\, d p_\gamma(ad), \tag{11}$$

which reduces to (2) for $\nu = 1$ as it should. For the incoherent receiver substitution of (5) in (11) leads to the following transcendental equation for the optimum normalized lower dynamic range bound*

$$A^2 = A_c^2 + A_o^2 + \frac{1}{(d^2 - 1)} [\ln I_o(A\,d\gamma\sqrt{2}) - \ln I_o(A\gamma\sqrt{2})] \tag{12}$$

in which by definition

$$A_c^2 = \frac{\ln \nu}{d^2 - 1}. \tag{13}$$

It is noted that it is entirely possible for c_2 to be less than c_1 making A_c^2 negative. However, the sum $A_c^2 + A_o^2$ is positive if νd^2 is greater than unity. Using (12) it is seen that for $\gamma = 0$, i.e., Rayleigh distributed envelope, the optimum normalized lower bound can be determined explicitly from

$$A_R^2 = A_c^2 + A_o^2. \tag{14}$$

When the cost ratio, ν, is unity (14) reduces to (4) as expected. It is convenient to measure the cost ratio in dB using

$$\nu_{\mathrm{dB}} = 20 \log_{10} \nu. \tag{15}$$

* The desired lower dynamic range bound is the positive real root of (12).

Thus, positive values of ν_{dB} imply $c_2 > c_1$, while negative values imply $c_2 < c_1$. $\nu_{dB} = 0$ is the case of minimum exclusion probability. If $\nu_{dB} = -2D$, then A_R^2 is 0 and $A^2 = 0$ solves (12) for any γ. For the foregoing formulation to be meaningful, A^2 must be positive. Hence, (4), (12), (13), and (14) require that $\nu_{dB} > -2D$. If this constraint is not satisfied, then the average exclusion cost, l, is not stationary with respect to A. For given γ, d, and ν it is generally necessary to solve (12) for A. This equation is of the general form $x = f(x)$. A proposed scheme to find the solution is to iterate $x_{i+1} = f(x_i)$ beginning with an approximate solution x_o. It can be shown that this scheme will converge if the magnitude of the derivative of the RHS, $|f'(x)|$, is less than unity in the neighborhood of the solution, x_s. Moreover the convergence is faster as $|f'(x_s)|$ is closer to zero. In order to speed convergence an *extrapolated* iteration scheme can be used by introducing another parameter, β. Consider the equation

$$x = f(x) - \beta[x - f(x)]. \tag{16}$$

Provided $\beta \neq -1$ the solution to this equation is the same as that of $x = f(x)$. If one could choose

$$\beta = \frac{f'(x_s)}{1 - f'(x_s)} \qquad f'(x_s) \neq 1, \tag{17}$$

the derivative of the RHS of (16) would be zero at x_s. However, since x_s is not known at the outset the approximate solutions are substituted for x_s in (17) to speed convergence.

Using this approach (12) can be solved to any desired accuracy with the aid of the iteration formulas

$$A_{i+1}^2 = F_i - \beta_i(A_i^2 - F_i) \qquad \beta_i \neq -1 \tag{18}$$

$$F_i = A_c^2 + A_o^2 + \frac{1}{(d^2 - 1)} [\ln I_o(A_i d\gamma\sqrt{2}) - \ln I_o(A_i\gamma\sqrt{2})] \tag{19}$$

$$G_i = \frac{\gamma\sqrt{2}}{2(d^2 - 1)A_i}\left[d\,\frac{I_1(A_i d\gamma\sqrt{2})}{I_o(A_i d\gamma\sqrt{2})} - \frac{I_1(A_i\gamma\sqrt{2})}{I_o(A_i\gamma\sqrt{2})}\right] \tag{20}$$

$$\beta_i = G_i/(1 - G_i) \qquad G_i \neq 1 \tag{21}$$

in which $I_n(x)$ denotes the modified Bessel function of the first kind and nth order. One may begin with small values of γ, $i = 0$, A_o^2 given by (4) and A_c^2 given by (13). The iteration is stopped when $|A_{i+1} - A_i|$ is less than the allowable error. Equation (12) was solved by this method for various values of ν_{dB}, D(dB), and γ(dB).

For $d \gg 1$ an approximate solution to (12) can be found explicitly.

Neglecting the second term in brackets in comparison with the first and taking $I_o(x) \approx e^x$ reduces (12) to a quadratic equation in Ad which has the solution $Ad \approx (\gamma/\sqrt{2}) + \sqrt{(\gamma^2/2) + \ln \nu\, d^2}$. Thus, the optimum normalized upper bounds (as shown in Fig. 2 for $\nu = 1$) continue to increase slowly as D increases.

The solutions obtained using (18) to (21) show that the optimum lower bounds for a wide range of cost ratios do not differ appreciably from those for $\nu_{\rm dB} = 0$. The difference (in dB) in optimum lower bounds for values of $\nu_{\rm dB} = \pm 25$ and $\nu_{\rm dB} = 0$, and for $\nu_{\rm dB} = \pm 50$ and $\nu_{\rm dB} = 0$ for various values of $\tilde{\gamma}$ and d are shown in Fig. 3(a) and (b). (For non-fluctuating SNR take $\tilde{\gamma}$ in the figures as γ and $\tilde{A}$ as A.) It can be seen that for any given values of ν and d, the largest difference is for $\tilde{\gamma} = 0$. This maximum deviation can be determined explicitly. From (14)

$$A_R^2 = A_o^2(1 + A_c^2/A_o^2). \tag{22}$$

Equations (4), (13), and (15) then yield

$$20 \log_{10} A_R - 20 \log_{10} A_o = 10 \log_{10} \left(1 + \frac{\nu_{\rm dB}}{2D}\right). \tag{23}$$

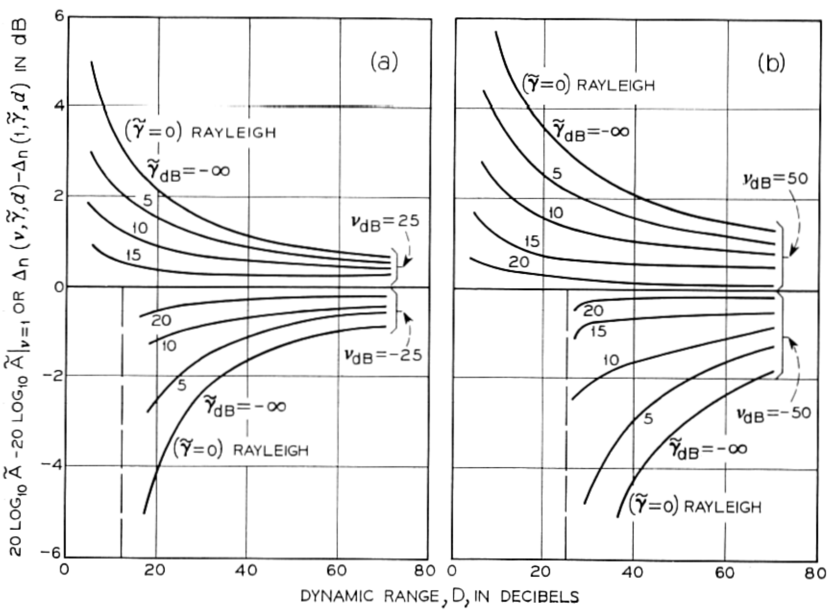

Fig. 3.—Increase in optimum normalized dynamic range bounds or in required receiver attenuation due to nonunity cost ratio. (a) $\nu_{\rm dB} = -25{,}25$; (b) $\nu_{\rm dB} = -50{,}50$.

Thus, the maximum difference in optimum dynamic range bounds is determined by the ratio of the cost ratio in dB to the dynamic range in dB. A plot of (23) is shown in Fig. 4.

The fact that the optimum bound is relatively insensitive to cost ratio at least for large D and large $\tilde{\gamma}$ is an important one since exact assessment of the costs c_1 and c_2 is difficult or impossible. However, this analysis shows that for large $\tilde{\gamma}$ and D an optimum solution for $\nu_{dB} = 0$ is nearly optimum for $-D \leqq \nu_{dB} \leqq 2D$. For $\tilde{\gamma} = 0$ the optimum dynamic range bounds are most sensitive to cost ratio but in this region differ only by about 3 dB from the optimum for $\nu = 1$.

When the optimum normalized lower bound, A, is determined, the normalized minimum average cost of excluding the signal is given in this case by

$$l = 1 - Q(\gamma, A\sqrt{2}) + \nu Q(\gamma, Ad\sqrt{2}). \tag{24}$$

These minimum average exclusion costs are shown in Fig. 5, in which the parameter $\tilde{\gamma}$ is to be taken as γ. For $\nu_{dB} = 0$, (24) becomes the minimum exclusion probability (7).

IV. A MODEL FOR TARGET FLUCTUATION AND FADING CHANNELS

Thus far this paper has considered the case where the SNR at the receiver is fixed. However, in the case of radar the target cross section

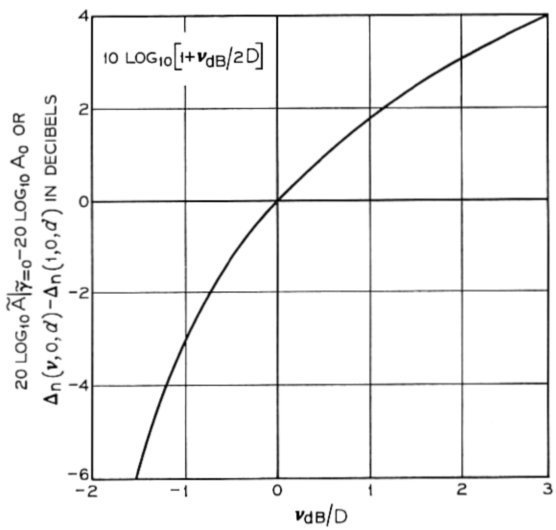

Fig. 4 — Maximum change in optimum normalized dynamic range bounds or in required receiver attenuation due to nonunity cost ratio.

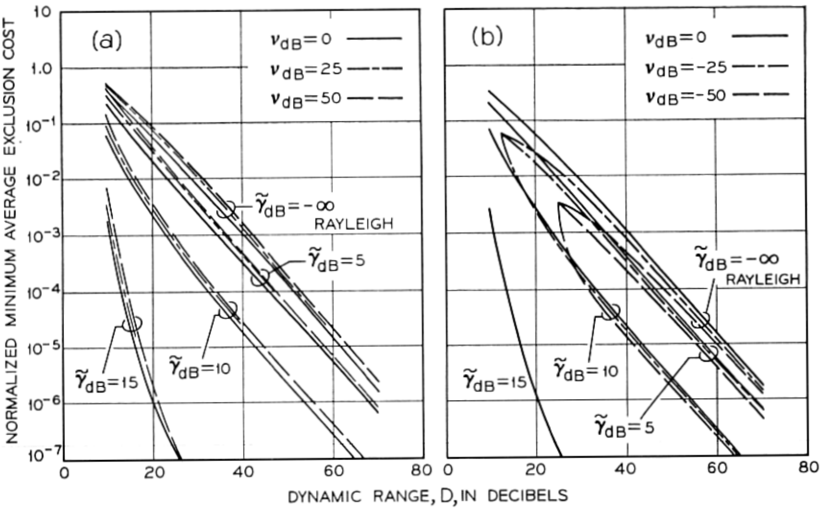

Fig. 5 — Minimum average exclusion costs.

presented to the receiver generally varies, while in communications links the phenomenon of fading generally causes fluctuation in the received SNR. Consider the case in which the SNR fluctuation is slow so that it is essentially fixed for the duration of a given signal but will fluctuate over longer time intervals.

Following Turin[6] it is assumed that the medium from the transmitter to the receiver can be characterized as propagating two components, a fixed or specular component and a completely random or scatter component. Thus corresponding to a transmitted signal Re $\{s(t) \exp (j\omega_c t)\}$ the reciever's IF signal [point (*C*) in Fig. 1] with $K \triangleq K_1 K_2 K_3 = 1$ is given by Re $\{x(t)\}$ where

$$x(t) = s(t - \tau)[\alpha \exp(-j\delta) + S \exp(-j\varphi)] \cdot \exp(j\omega_o t) + n(t) \exp(j\omega_o t). \qquad (25)$$

In (25) α and δ are fixed while S and φ are independent variates; S having a Rayleigh pdf with mean square $2\mu^2$ and φ a uniformly distributed phase over an interval of 2π. ω_c and ω_o denote the angular frequencies of the transmitted carrier and the receiver intermediate frequency, respectively. $n(t)$ is a narrowband Gaussian noise process. It can be shown[3, 6] that the joint distribution of the resultant amplitude, ψ, and phase, θ, of the sum of the fixed vector (α, δ) and the random vector (S, φ) is

$$p(\psi, \theta) = \frac{\psi}{2\pi\mu^2} \exp\left[-\frac{\psi^2 + \alpha^2 - 2\alpha\psi\cos(\theta - \delta)}{2\mu^2}\right]$$

$$\text{for} \quad 0 \leqq \psi \qquad 0 \leqq \theta - \delta \leqq 2\pi. \tag{26}$$

In (26) α^2 can be regarded as the strength of the fixed component while μ^2 is proportional to the strength of the scattered component. The quantity $\beta^2 \triangleq \alpha^2/\mu^2$ is twice the ratio of the energy received via the specular component to that received via the scatter component. The variates ψ and θ are, respectively, the instantaneous voltage gain and phase shift of the path from the transmitter to receiver and δ is the average phase shift of the path. Note that (26) is just the two dimensional Gaussian distribution in polar form. The pdf of ψ is found by integrating (26) over the range of θ giving[3,6]

$$p(\psi) = \frac{\psi}{\mu^2} \exp\left[-\frac{\psi^2 + \alpha^2}{2\mu^2}\right] I_o\left(\frac{\alpha\psi}{\mu^2}\right) \quad \text{for} \quad \psi \geqq 0 \tag{27}$$

$$= 0 \text{ elsewhere.}$$

Letting $r = \psi/\mu$, (27) becomes*

$$p_\beta(r) = r \exp\left[-(r^2 + \beta^2)/2\right] I_o(\beta r). \tag{28}$$

The voltage gain of the propagation medium and/or target [from (A) to (B) in Fig. 1] is $\psi = \mu r$. The model above is an adequate representation of propagation conditions which are encountered on ionospheric and tropospheric radio links.[6] The pdf (27) is sufficiently general since as β approaches zero (no specular component) (27) becomes the Rayleigh distribution with parameter μ while if β approaches infinity (presence of specular component only (27) may first be approximated by a Gaussian pdf of mean α and variance μ^2 and in the limit by a delta function, $\delta(\psi - \alpha)$ corresponding to the case of no SNR fluctuation. Radar target fluctuations have been described by Rayleigh statistics[7] a special case of the above model ($\beta = 0$). In this case, μ^2 is proportional to the average target cross-section. It is reasonable to expect that radar targets which can be modeled as a single large reflector plus a large number of independent scatterers will yield signal returns of the form (25). For this more general Rician fluctuating target $\mu^2(1+\beta^2/2)$ is proportional to the average target cross-section.

* Note that (28) is a pdf of the same form as (5) as it must be since either equation is the probability density of the magnitude of the sum of a constant vector and a Gaussian vector.

V. GENERAL CASE: DYNAMIC RANGE BOUNDS FOR MINIMUM AVERAGE EXCLUSION COST WITH FLUCTUATING SNR

The phenomena of radar target fluctuation and channel fading are evidenced by fluctuating SNR in the receiver. To account for these fluctuations, γ in (10) must be weighted by a random voltage gain μr, where μ is a parameter and r is a random variable whose pdf $\hat{p}_\beta(r)$ determines the form of the SNR fluctuation. The normalized average exclusion cost can then be obtained from (10) giving

$$\tilde{l} = \int_0^a \tilde{p}_{\mu\gamma}(R)\, dR + \nu \int_{ad}^\infty \tilde{p}_{\mu\gamma}(R)\, dR, \tag{29}$$

where

$$\tilde{p}_{\mu\gamma}(R) = \int_0^\infty \hat{p}_\beta(r) p_{r\mu\gamma}(R)\, dr. \tag{30}$$

In (30), γ is the voltage SNR at the receiver for unity channel gain, i.e., for $\psi = r\mu = 1$. The product $r\mu\gamma$ appearing in the integrand is the voltage SNR at the receiver for a particular realization of the random gain $\psi = r\mu$; that is, the "instantaneous" voltage SNR at the receiver.

The condition for minimization of the average exclusion cost (29) becomes

$$\tilde{p}_{\mu\gamma}(a) = \nu\, d\tilde{p}_{\mu\gamma}(ad). \tag{31}$$

Consider phase incoherent reception of signals in Gaussian noise with fading or target fluctuations described by the probability law (28), i.e., $\hat{p}_\beta(r) = p_\beta(r)$. In this case the integral appearing in (30) becomes

$$\tilde{p}_{\mu\gamma}(R) = \int_0^\infty Rt \exp\left[-(t^2 + \beta^2)/2\right] I_o(\beta t) \cdot \exp\left[-(\mu^2\gamma^2 t^2 + R^2)/2\right] I_o(\gamma R t)\, dt \tag{32}$$

which can be evaluated using an identity in Watson* giving

$$\tilde{p}_{\mu\gamma}(R) = \frac{R}{1+\mu^2\gamma^2} \exp\left[-\frac{(R^2 + \mu^2\gamma^2\beta^2)}{2(1+\mu^2\gamma^2)}\right] I_o\left(\frac{\mu\gamma\beta R}{1+\mu^2\gamma^2}\right). \tag{33}$$

To evaluate the average exclusion cost (29) one needs to integrate (32) or (33) with respect to R from some number η to ∞. This integral of (33) can be easily evaluated using the definition (8).

* See Ref. 8, p. 395. Take Watson's $a = i\mu\gamma R$, $b = i\beta$, $\nu = 0$, $p^2 = (1 + \mu^2\gamma^2)/2$.

Performing the integration with respect to R in (32) then establishes the result†

$$\int_\eta^\infty p_{\mu\gamma}(R)\,dR = \int_0^\infty t \exp\left[-(t^2+\beta^2)/2\right] I_o(\beta t) Q(\mu\gamma t, \eta)\,dt$$
$$= Q\left(\frac{\mu\gamma\beta}{\sqrt{1+\mu^2\gamma^2}}, \frac{\eta}{\sqrt{1+\mu^2\gamma^2}}\right). \tag{34}$$

Using (34) in (29) yields

$$\bar{l} = 1 - Q\left(\frac{\mu\gamma\beta}{\sqrt{1+\mu^2\gamma^2}}, \frac{a}{\sqrt{1+\mu^2\gamma^2}}\right) + \nu Q\left(\frac{\mu\gamma\beta}{\sqrt{1+\mu^2\gamma^2}}, \frac{ad}{\sqrt{1+\mu^2\gamma^2}}\right) \tag{35}$$

in which a is the optimum normalized lower dynamic range bound obtained as a solution to (31). By substituting (33) in (31) and manipulating the result it can be shown that the optimum a must satisfy

$$\frac{a^2}{2(1+\mu^2\gamma^2)} = A_c^2 + A_o^2 + \frac{1}{(d^2-1)} \cdot \left[\ln I_o\left(\frac{\mu\gamma\beta a d}{1+\mu^2\gamma^2}\right) - \ln I_o\left(\frac{\mu\gamma\beta a}{1+\mu^2\gamma^2}\right)\right]. \tag{36}$$

Let

$$\tilde{A} = \frac{a}{\sqrt{2}\sqrt{1+\mu^2\gamma^2}} \tag{37}$$

and

$$\tilde{\gamma} = \frac{\mu\gamma\beta}{\sqrt{1+\mu^2\gamma^2}}. \tag{38}$$

From (33) it can be seen that if β is zero the mean square value of R is $2(1+\mu^2\gamma^2)$. Thus $\tilde{A}$ is the optimum lower dynamic range bound normalized to the rms voltage that would appear at the output of the envelope detector [point (D) in Fig. 1], if the specular component were zero. Since $\mu\beta = \alpha$ the quantity $\tilde{\gamma}$ in (38) is twice the ratio of the rms voltage that would appear at (D) when only the noiseless specular com-

† The integral to the right of the first equality in (34) would appear if the average cost for the nonfluctuating case is determined first as in (24) and then this cost is averaged over the random fluctuations of ψ.

ponent of (25) is present, to the rms voltage that would appear if only the scatter and noise components of (25) were present. The mean square voltage at the output of the envelope detector when specular, scatter, and noise components are present is $2\sigma^2[1 + \mu^2\gamma^2 + (\alpha^2\gamma^2)/2]$. Using (37) and (38) in (36) gives

$$\tilde{A}^2 = A_c^2 + A_o^2 + \frac{1}{(d^2 - 1)} [\ln I_o(\tilde{\gamma}\tilde{A}\, d\sqrt{2}) - \ln I_o(\tilde{\gamma}\tilde{A}\sqrt{2})] \tag{39}$$

and (35) becomes

$$\tilde{l} = 1 - Q(\tilde{\gamma}, \tilde{A}\sqrt{2}) + \nu Q(\tilde{\gamma}, \tilde{A}\, d\sqrt{2}). \tag{40}$$

Equations (39) and (40) are of the same form as (12) and (24), respectively, with γ replaced by $\tilde{\gamma}$ and A by $\tilde{A}$. These results show that the optimum dynamic range bounds and performance curves obtained previously for nonfluctuating SNR can be used directly for the more general case of fluctuating SNR by merely changing the variables via (37) and (38). Therefore, although there are two additional parameters in the fluctuating case it is not necessary to increase the number of curves to describe performance. For $\nu_{\text{dB}} = 0$ the criterion reduces to the minimum average exclusion probability as in the case of nonfluctuating SNR.

VI. RECEIVER GAIN REQUIRED FOR THE GENERAL CASE

The optimum gain or attenuation required for insertion in the signal processing chain at a point preceding the components which limit the dynamic range can now be calculated. Let c be the lowest voltage at which the signal processing chain can operate satisfactorily, referred to the output of the envelope detector.* Optimum dynamic range utilization requires that the signal be multiplied by a factor K such that the scaled lower normalized dynamic range bound is equal to the voltage c, when normalized to the same base. That is,

$$Ka = c/\sigma. \tag{41}$$

Substituting from (37) for a and transposing, (41) gives

$$K(\sigma\sqrt{2}/c)\sqrt{1 + \mu^2\gamma^2} = \tilde{A}^{-1} \tag{42}$$

in which $\tilde{A}$ is the solution to (39). Denote the LHS of (42) as K_n, the normalized voltage gain, and let Δ_n be the normalized required attenuation in dB. Then

$$\Delta_n = -20 \log_{10} K_n = 20 \log_{10} \tilde{A} \tag{43}$$

* Point (D) in Fig. 1.

which expresses the normalized required attenuation in dB as a function of the optimum normalized lower dynamic range bound. Since $\tilde{A}$ is the solution to (39) Δ_n depends only on ν, $\tilde{\gamma}$, and d. It is fortunate that the normalized results can be expressed in terms of only a few parameters since this permits a concise description of optimum performance for many signal, noise and channel conditions. Optimum normalized attenuation required for the case of minimum exclusion probability ($\nu_{dB} = 0$) is shown plotted in Fig. 6. From (43) the actual required attenuation in dB can be obtained. Let $\Delta = -20 \log_{10} K$ denote the actual required attenuation in dB. Then (42) and (43) yield

$$\Delta = \Delta_n(\nu, \tilde{\gamma}, d) + 20 \log_{10} (\sigma \sqrt{2}/c) + 10 \log_{10} (1 + \mu^2 \gamma^2) \qquad (44)$$

in which the functional dependence of Δ_n is shown explicitly.

From (43) and (44) it can be seen that the difference in optimum receiver attenuations is the same as the difference in optimum normalized dynamic range bounds. Hence, Fig. 3(a) and (b) also show the differences

$$\Delta_n(\nu, \tilde{\gamma}, d) - \Delta_n(1, \tilde{\gamma}, d) \qquad (45)$$

for values of $\nu_{dB} = \pm 25, \pm 50$, and various values of $\tilde{\gamma}$. For given ν, $\tilde{\gamma}$, and d one can, therefore, determine $\Delta_n(\nu, \tilde{\gamma}, d)$ by finding $\Delta_n(1, \tilde{\gamma}, d)$

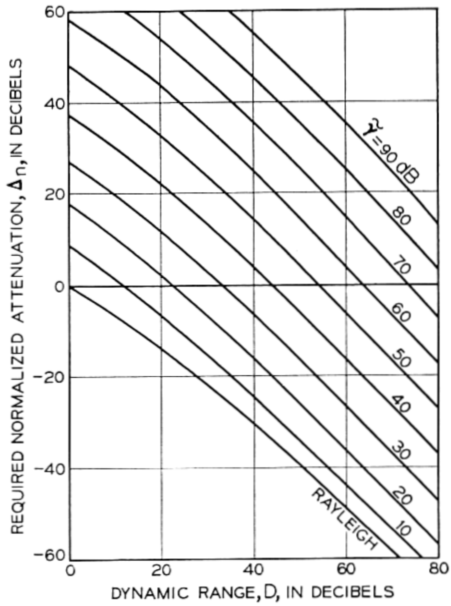

Fig. 6 — Required attenuation for minimum exclusion probability. $\nu_{dB} = 0$.

in Fig. 6 and adding the difference (45) found in Fig. 3. Fig. 4 is a plot of the differences

$$\Delta_n(\nu, 0, d) - \Delta_n(1, 0, d), \tag{46}$$

that is, a plot of (45) for zero SNR. For given ν and d these differences are of the same sign as (45) but are always larger in magnitude. Hence, Fig. 4 shows the maximum change in optimum receiver attenuation due to a nonunity cost ratio.

The definitions of the parameters appearing in (44) are summarized by the following list:

ν = cost ratio

σ^2 = noise power with no fluctuation

γ = voltage signal-to-noise ratio for unity propagation and/or target gain (i.e., $\psi = 1$)

μ^2 = strength of scatter component of the propagation medium

β^2 = twice the ratio of strength of specular component of the medium to that of the scatter component

$\mu^2(1 + \beta^2/2)$ = for the case of Rician fluctuating radar targets this quantity is proportional to the average target cross-section over all target fluctuations. $\beta = 0$ corresponds to the case of Rayleigh fluctuating targets

d = dynamic range of receiver

$$\tilde{\gamma} = \mu\gamma\beta/\sqrt{1 + \mu^2\gamma^2} = 2 \times \frac{\text{rms voltage at } (D) \text{ for noiseless specular component only}}{\text{rms voltage at } (D) \text{ for scatter and noise components only}}.$$

In the general analysis presented here, which includes fluctuating or nonfluctuating SNR and the criteria minimum average exclusion cost or minimum exclusion probability, special cases which may arise in various applications are represented when the parameters take on particular values. Some special cases are shown in Table I. The entries in the table are for either criterion.

TABLE I — CONSTRAINTS ON PARAMETERS FOR SPECIAL CASES

Constraints on parameters	Type of fluctuation or fading	Type of envelope detected signal
$\mu > 0, \beta > 0, \gamma > 0$	Rician	Rician
$\mu > 0, \beta = 0, \gamma > 0$	Rayleigh	Rayleigh
$\mu \to 0, \beta \to \infty, \mu\beta = \alpha, \gamma > 0$	none	Rician
$\mu\beta = \alpha, \gamma = 0$	none	Rayleigh

It is noted that for μ, β, and γ greater than zero one has the general case of Rician SNR fluctuation and Rician envelope detected signal.

When β is zero the medium from transmitter to receiver does not propagate any specular component and the envelope of the received signal has a Rayleigh pdf independent of the other parameters. Setting β to zero in (33) shows that the envelope of the received signal in this case has a mean square of $2\sigma^2(1 + \mu^2\gamma^2)$. When γ is zero only noise at the receiver is demodulated again giving rise to a Rayleigh distributed envelope but of mean square $2\sigma^2$ independent of the other parameters. In each of these two cases (i.e., $\beta = 0$ and $\gamma = 0$), the optimum *normalized* lower bound is found from (14), $\tilde{A}_R = \sqrt{A_c^2 + A_o^2}$. Since the minimum average exclusion cost (40) depends on the value of $\tilde{A}$ the minimum costs are equal for these two cases. However, it can be seen from (44) that the *actual* optimum lower bounds or *actual* required attenuations for these cases differ. This is because the quantity, $\sigma\sqrt{1 + \mu^2\gamma^2}$, to which the received signal voltage envelope is normalized is different in these instances. Note that in the former case ($\beta = 0$) the required receiver attenuation (44) is affected by the randomness of the scatter component while in the latter case ($\gamma = 0$) it is not. This can also be seen from (25). When β is zero there is no specular component and the received signal (25) depends upon the scatter component while if γ is zero the entire first term can be omitted and the received signal consists of only noise.

When μ goes to zero and β approaches infinity such that $\mu\beta = \alpha$ (a constant), the medium from transmitter to receiver propagates only a specular component with a voltage gain of α. In this case there is no SNR fluctuation (nonfluctuating case) and the envelope of the detected signal is Rician if $\gamma > 0$ and Rayleigh if $\gamma = 0$. Letting $\mu = 0$ and $\mu\beta = \alpha$ in (33) and (38) shows that for this case the SNR at the reciever is $\alpha\gamma$, a result which is clear from (25) if the scatter component of the medium is deleted. There is no essential loss in generality in this case if α is taken as unity. With $\mu = 0$ and $\alpha \triangleq \mu\beta = 1$ in (33) that equation reduces to the pdf considered in Sections II and III.

For $\gamma > 0$, $\mu > 0$, β finite, the optimum gain settings for the fluctuating and nonfluctuating cases differ and the minimum average exclusion cost (probability) for the fluctuating case will be greater for the same values of γ, d, ν, and α.

VII. EFFECT OF NOISE INTRODUCED BY THE GAIN ADJUSTMENT CASCADE

In the foregoing discussion, the attenuation or gain required for optimum dynamic range utilization has been idealized as a multiplicative parameter. These results apply when the noise introduced by the gain adjustment cascade itself is virtually independent of its gain. This condition is often realized in practice. When this condition is not satisfied some modification is necessary. The phenomenon can be represented by using an equivalent noise source at the point in the signal processing chain where the dynamic range calculations are being made. The quantities σ and γ used previously must be replaced by equivalent σ_e and γ_e, respectively. Let $\mathfrak{F}_o(\mathcal{G})$ be the operating noise figure[9] of the cascade when it is set for an available gain of $\mathcal{G}$ (dB). Then the equivalent noise power when the gain is $\mathcal{G}$, is

$$\sigma_e^2(\mathcal{G}) = \frac{\sigma^2 \mathfrak{F}_o(\mathcal{G})}{\mathfrak{F}_o(\mathcal{G}_o)} \tag{47}$$

in which $\mathcal{G}_o$ is the gain for which the SNR is γ and the noise power is σ^2. The equivalent SNR is determined by

$$\gamma_e^2(\mathcal{G}) = \gamma^2 \frac{\mathfrak{F}_o(\mathcal{G}_o)}{\mathfrak{F}_o(\mathcal{G})}. \tag{48}$$

In the case where the noise depends upon the gain, K, ($\mathcal{G} = 20 \log_{10} K$), both the quantities γ and a in (29) or (35) depend upon gain. Hence, the optimization condition must be found by differentiating (29) or (35) with respect to K (or $\mathcal{G}$) rather than a and setting that derivative to zero. However, this condition is generally too complicated to be useful and it is usually better to evaluate (29) or (35) for various $\mathcal{G}$ to determine the optimizing value. For the general case of the incoherent receiver the normalization for R in (29) is with respect to σ_e rather than σ. In addition γ_e and σ_e must be used. Define

$$a_e = c/[K\sigma_e(\mathcal{G})] \tag{49}$$

and

$$\tilde{A}_e(\mathcal{G}) = \frac{a_e}{\sqrt{2}\sqrt{1 + \mu^2\gamma_e^2}}. \tag{50}$$

Then

$$\begin{aligned} \tilde{A}_e(\mathcal{G}) &= \tilde{A}\left[\frac{\mathfrak{F}_o(\mathcal{G}_o)}{\mathfrak{F}_o(\mathcal{G})}\right]^{\frac{1}{2}}\left[\frac{1 + \mu^2\gamma^2}{1 + \mu^2\gamma_e^2}\right]^{\frac{1}{2}} \\ &= \frac{c \cdot 10^{(-\mathcal{G}/20)}}{\sigma\sqrt{2}\sqrt{1 + \mu^2\gamma_e^2}}\left[\frac{\mathfrak{F}_o(\mathcal{G}_o)}{\mathfrak{F}_o(\mathcal{G})}\right]^{\frac{1}{2}}. \end{aligned} \tag{51}$$

From (38), (47), and (48) one can find

$$\tilde{\gamma}_e(\mathcal{G}) = \frac{\mu\beta\gamma_e(\mathcal{G})}{[1 + \mu^2\gamma_e^2(\mathcal{G})]^{\frac{1}{2}}}. \tag{52}$$

In order to find the optimum receiver gain (47), (48), (51), and (52) are used in conjunction with

$$\tilde{l} = 1 - Q[\tilde{\gamma}_e(\mathcal{G}), \sqrt{2}\, \tilde{A}_e(\mathcal{G})] + \nu Q[\tilde{\gamma}_e(\mathcal{G}), d\sqrt{2}\, \tilde{A}_e(\mathcal{G})] \tag{53}$$

which must be minimized with respect to $\mathcal{G}$. It is easiest to use a numerical method which requires only successive evaluation of (53) for various values of $\mathcal{G}$, as opposed to methods requiring analytical evaluation of the derivative of (53). In the important case where only a finite number of gain settings $\mathcal{G}_i$, $(i = 1, 2, \cdots N)$ are possible, minimization of (53) is easy requiring at most N evaluations for any given set of parameters. Likewise when $-\tilde{l}$ is a unimodal function of $\mathcal{G}$ any of various search methods can be used.[10]

VIII. SUMMARY AND COMMENTS

This paper considers the general problem of determining optimum receiver gains for radar and communications receivers. Dynamic range bounds and receiver gains are determined which yield minimum average cost of excluding fluctuating signals in noise. The analysis is general enough to include minimum exclusion probability as a special case as well as a range of fluctuation models including Rician, Rayleigh, and nonfluctuating cases. The analysis is applicable to both linear and nonlinear receivers and has important implications for certain radar processors and communications terminals which can use sophisticated techniques for signal and noise parameter estimation. The results are presented in a concise normalized form making them applicable to a wide range of signal, noise, and channel conditions. It is shown that the optimum receiver gain is relatively insensitive to cost ratio for $-D \leqq \nu_{\text{dB}} \leqq 2D$ differing at most by about 3 dB from the optimum gain for $\nu = 1$. The effect of noise introduced by the gain adjustment cascade is discussed.

The analysis presented assumes that certain signal, noise, and channel parameters are known to the receiver. In practice the receiver would be required to estimate these parameters. When these estimates are good, performance of the system will approach that described here. An extension of this work is to study both the optimization problem and the deterioration in performance when the parameters are not known to the receiver. Optimum dynamic range utilization for various coherent and partially coherent receivers can also be studied.

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