Deformation of Gas Lenses by Gravity

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Gravity forces cause distortions in tubular gas lenses. A theory is derived here which yields excellent quantitative agreement with measured distortions for various tube lengths, diameters, and gases. It is shown that in a gas lens of optimum design the displacement of the optical center has a maximum at the end of the lens. The amount of displacement increases with the fourth power of the tube diameter and with the square of the gas pressure.

I. INTRODUCTION

If a cool gas is blown into a hot tube (Fig. 1), the gas heats up first at the wall of the tube and remains cool longer at its center. The density therefore, is higher in the center of the tube and decreases toward the wall. The increase in density is accompanied by an increase in dielectric constant. In this way the gas acts as a positive lens.[1,2]

At the same time, however, the cooler gas tends to sink down because of gravity, thus causing an asymmetric density profile in a horizontal tube.[3] Though a simple approach already gives an estimate of this effect,[4] a more rigorous theory is derived here using a perturbation calculation which determines the transverse convection currents from the unperturbed temperature profile and then uses the currents to correct the temperature profile.

II. TRANSVERSE CONVECTION CURRENTS

The tube walls are at a temperature T_W and ΔT degrees warmer than the entering gas. Heat diffuses toward the axis and determines the temperature field. Using the coordinate system shown in Fig. 1, the temperature field may be approximated by[2]

$$T = T_W - \Delta T\left[1 - 2\,\frac{x^2 + y^2}{a^2} + \left(\frac{x^2 + y^2}{a^2}\right)^2\right]e^{-z/s}, \tag{1}$$

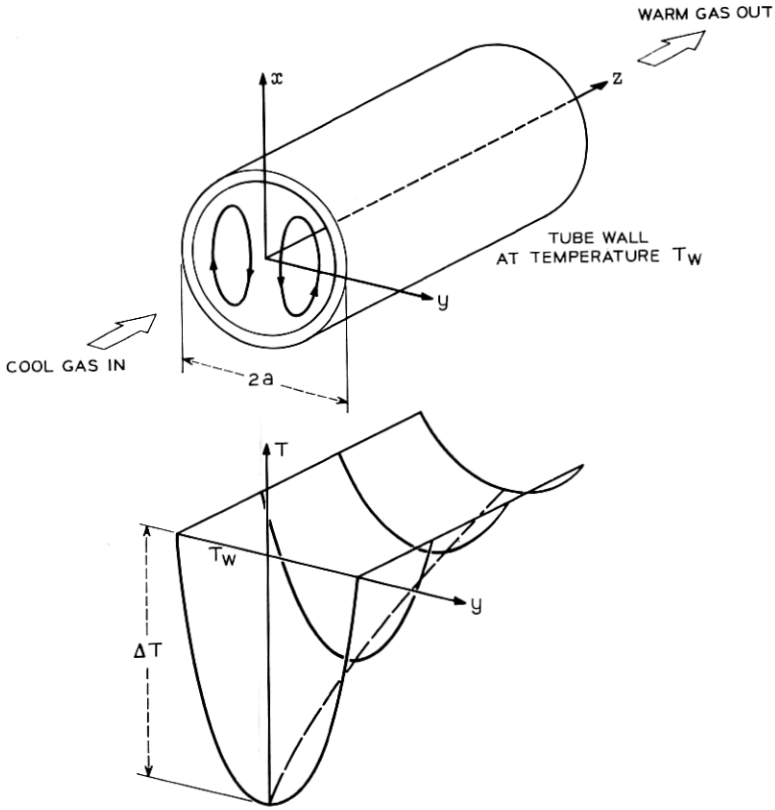

Fig. 1 — Convection currents and temperature distribution in a gas lens.

where a is the tube radius and s a decay length given by the formula

$$s = \frac{a^2 v_{zo}}{7.3\alpha}. \tag{2}$$

v_{zo} is the gas velocity along the axis and α the thermal diffusivity defined as the ratio of heat conductivity κ to heat capacity:

$$\alpha = \frac{\kappa}{\rho c_p}. \tag{3}$$

The heat capacity is written here as the product of density ρ and specific heat at constant pressure.

The temperature is related to the density ρ and the pressure p by the gas equation

$$p = R\rho T. \tag{4}$$

The density determines the gravitational forces $\mathbf{g}\rho$ which drive the gas particles in the transverse direction. The transverse components of the velocity field $\mathbf{v}(x, y, z)$ can be found from Newton's Law

$$\mathbf{g}\rho = \text{grad}\, p - \nu\rho\nabla^2\mathbf{v} + \rho\frac{d\mathbf{v}}{dt}, \tag{5}$$

where ν is the kinematic viscosity determining the frictional forces. The acceleration is described by the total differential $d\mathbf{v}/dt$ and for the steady state takes the form

$$\frac{d\mathbf{v}}{dt} = (\mathbf{v}\ \text{grad})\mathbf{v}. \tag{6}$$

In the problem under consideration the gas may be treated as a quasi-incompressible (Boussinesq) fluid. That means that variations of density may be neglected, except insofar as they modify the action of gravity. Forming the curl of (5) therefore, yields

$$\text{curl}\,(\mathbf{g}\rho) = -\nu\rho\ \text{curl}\ \nabla^2\mathbf{v} + \rho\ \text{curl}\frac{d\mathbf{v}}{dt}. \tag{7}$$

Using (4) and rearranging (7) one finds

$$\mathbf{g} \times \left(\frac{\text{grad}\, T}{T} - \frac{\text{grad}\, p}{p}\right) = \nu\ \text{curl curl curl}\ \mathbf{v} + \text{curl}\frac{d\mathbf{v}}{dt}. \tag{8}$$

Here grad p/p can be neglected compared with grad T/T, and T in the denominator will be replaced by the mean (absolute) temperature T_o. Finally, by inserting (6) one finds

$$\frac{1}{T_o}(\mathbf{g} \times \text{grad}\, T) = \nu\ \text{curl curl curl}\ \mathbf{v} + \text{curl}\,(\mathbf{v}\ \text{grad})\mathbf{v}. \tag{9}$$

To solve this equation, a tentative velocity distribution is introduced which represents the flow lines shown in Fig. 1. The unknown coefficients are chosen in such a way that the equation

$$\text{div}\ \mathbf{v} = 0 \tag{10}$$

is fulfilled, which assumes that the gas is incompressible. Then the velocity components

$$
\begin{aligned}
v_x &= -\frac{v_o}{a^4}(a^2 - x^2 - y^2)(a^2 - x^2 - 5y^2) \\
v_y &= -4\frac{v_o}{a^4}xy(a^2 - x^2 - y^2) \\
v_z &= \frac{v_{zo}}{a^2}(a^2 - x^2 - y^2)
\end{aligned} \tag{11}
$$

result which leaves only the coefficient v_o unknown, since the velocity v_{zo} is determined by the forced laminar flow in the tube. v_o is the vertical gas velocity at the tube center caused by the gravitational forces. It may be assumed to be much smaller than the longitudinal velocity v_{zo}. Though v_o is a function of z the variation of $\mathbf{v}$ in the z-direction is negligible compared to its variation in the cross-sectional plane and has to be considered only in the acceleration term where $\partial v_o/\partial z$ occurs multiplied with the velocity v_{oz}.

With these approximations, v_o can be determined by inserting (1) and (11) into (9) which yields

$$4g\frac{\Delta T}{T_o}e^{-z/s}y = 192\frac{v_o}{a^2}y + 18v_{zo}\frac{\partial v_o}{\partial z}. \tag{12}$$

Third- and higher-order products of x and y are neglected in this equation since they are only important at the wall of the tube and contribute little at the tube center.

Equation (12) is a linear inhomogeneous differential equation in z with the solution

$$v_o = \frac{g}{\nu}\frac{\Delta T}{T_o}\frac{a^2}{48}\frac{s}{s-q}(e^{-z/s} - e^{-z/q}), \tag{13}$$

where

$$q = \frac{3}{32}\frac{a^2}{\nu}v_{zo}\,. \tag{14}$$

A discussion of (13) is postponed in order to proceed with the calculation of the lens disturbance by using the derived convection currents to correct the temperature profile which, in turn, gives the density distribution and the lens profile.

III. DISPLACEMENT OF THE OPTICAL CENTER

The gravitational forces cause a continuous flow of cool gas toward the bottom of the pipe, which distorts the temperature profile more and more in the way shown in Fig. 2. The growing temperature gradient at the bottom, however, will increase the heat diffusion toward the center and counteract the convection effect. The equality of both effects is expressed by the equation

$$\alpha\nabla^2 T = \mathbf{v} \operatorname{grad} T \tag{15}$$

which determines the actual temperature profile under the boundary condition that $T = T_W$ at the tube wall.

Considering that the temperature function for axial direction is much less curved than the radial one, $\partial^2 T/\partial z^2$ may be neglected and (15) separated with respect to z.[6] This yields

$$\alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) = v_x \frac{\partial T}{\partial x} + v_z \frac{\partial T}{\partial y} + v_z \frac{T - T_W}{s}, \tag{16}$$

where $T - T_W$ is an exponential function of z as already introduced by (1) for the undisturbed temperature profile.

No straightforward solution of (16) is known. Assuming, however, that the gravity effect, to first order, tilts the temperature profile in the x-direction as shown in Fig. 2, the amount of this disturbance can be calculated. The assumption implies that by transforming $T(x, y, z)$ into new coordinates

$$\xi = x - \delta(T_W - T); \qquad \eta = y; \qquad \zeta = z \tag{17}$$

the undisturbed profile can be regained, which in the following is denoted by $\theta(\xi, \eta, \zeta)$. Since this is symmetric with respect to ξ, the corresponding transformation in (16) must generate a differential equation for θ which contains only even terms in ξ. The requirement that the odd terms

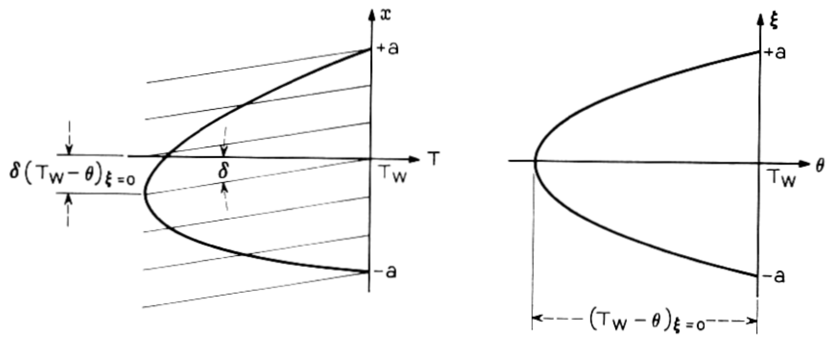

Fig. 2 — The temperature function $T(x)$ and its transformation into a symmetric function $\theta(x - \delta T)$.

cancel yields the following equation for δ:

$$2\alpha\delta\left(\frac{\partial\theta}{\partial\xi}\right)^2\frac{1}{\xi} = v_x\frac{\partial\theta}{\partial\xi} + 2\delta v_z\frac{\theta - T_W}{s}\frac{\partial\theta}{\partial\xi}. \tag{18}$$

The locus of the minimum of the temperature $T(x)$ is of particular interest, for this is the optical center of the distorted lens profile. Fig. 2 shows that this center occurs at a distance

$$d = \delta(T_W - \theta)_{\xi=0} \tag{19}$$

below the tube axis. Using for θ the undisturbed temperature profile given in (1) and solving (18) for $\delta(T_W - \theta)$ at $\xi = 0$ yields

$$d = \frac{v_o}{2\frac{v_{zo}}{s} + 8\frac{\alpha}{a^2}}. \tag{20}$$

By inserting (2) and (3) into (20) one finally finds

$$d = \frac{1}{750}\frac{ga^4}{\alpha^2}\frac{\Delta T}{T_o}\frac{q}{s - q}(e^{-z/s} - e^{-z/q}). \tag{21}$$

The diffusivity α and the viscosity ν for perfect gases are related by Eukens formula[7]

$$\frac{\alpha}{\nu} = \frac{1}{4}\left(9 - 5\frac{c_v}{c_p}\right), \tag{22}$$

c_v being the specific heat at constant volume. As Table I shows, the decay lengths s and q given by (2) and (14) differ very little. Since (21) is not defined for $s = q$ it is more convenient to use the following approximation for (21):

$$d = \frac{1}{750}\frac{ga^4}{\alpha^2}\frac{\Delta T}{T_o}\frac{z}{s}e^{-z/s}, \tag{23}$$

which is valid for $z < 2sq\,|q - s|$.

In Fig. 3 the displacement of the center of the lens profile is plotted

TABLE I

	c_p/c_v	α/ν	q/s
He	1.66	1.55	1.03
N_2	1.41	1.35	0.93
CO_2	1.31	1.30	0.89
CH_4	1.31	1.30	0.89

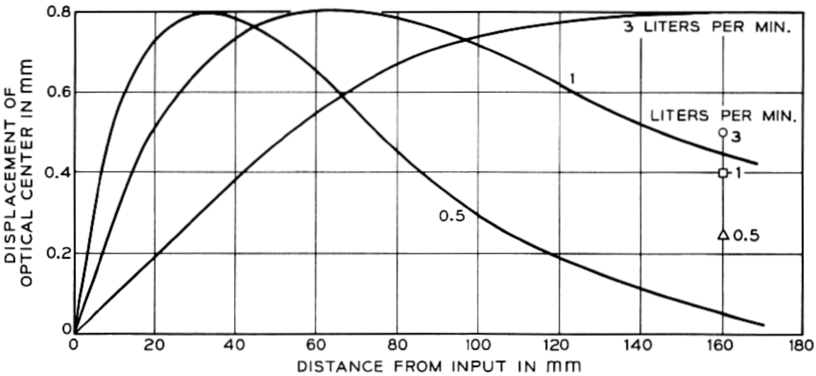

Fig. 3 — Displacement of the profile center in a tubular gas lens of ⅜-inch i. d. for flow rates of 0.5, 1, and 3 liters per minute using air. ($\alpha = 0.22$ cm²/s). Measured data by DeGano.[8]

versus the distance from the tube input for flow rates of 0.5, 1, and 3 liters per minute. A tube of $\frac{3}{8}$-inch diameter and 100°C wall temperature is assumed. The gas enters at room temperature. The mean temperature during the process is assumed to be $T_o = 50$°C. The gas is air with a diffusivity $\alpha = 0.27$ cm$^2/s$.

All curves show a linear increase of the displacement at the tube input, determined by the transverse acceleration of the gas. Further from the input the displacement follows the exponential decay of the temperature profile. The maximum displacement occurs at $z = s$. Measurements at the end of a 16-cm gas lens using the mentioned parameters are in fair agreement with the theory.[8]

In Fig. 4 the displacement is shown for a tube of $\frac{1}{4}$-inch diameter and two different gases: CO_2 with $\alpha = 0.125$ cm$^2/s$ and N_2 with $\alpha = 0.25$ cm$^2/s$. The temperatures are the same as in Fig. 3. The flow rate is 1 liter per minute. In this case, data are available for various tube lengths.[3] They show an excellent agreement with the predicted behavior of d versus z.

The focal length of the tubular gas lens has a minimum if the flow rate is chosen in such a way that s equals the tube length. The maximum displacement occurs at the end of such a lens and has the value

$$d_{\max} = \frac{1}{2040} \frac{ga^4}{\alpha^2} \frac{\Delta T}{T}. \tag{24}$$

A more useful measure for the gravity effect is the distance D by which a light beam has to be displaced off the tube axis to pass the lens without deflection. Integrating at $x = D$ over the tube length L one finds D from the requirement that the total deflection cancels:

$$\int_0^L \frac{\partial T}{\partial x}\bigg|_{x=D} dz = 0. \tag{25}$$

The development of the (disturbed) temperature field T about the axis yields for small distortion

$$\int_0^L [D - d(z)]e^{-z/s}\, dz = 0; \tag{26}$$

and finally, by using (23) one has

$$D = \frac{1}{3000}\frac{ga^4}{\alpha^2}\,\frac{1 - \left(\frac{2L}{s} + 1\right)e^{-2L/s}}{1 - e^{-L/s}}. \tag{27}$$

In Fig. 5 the displacement D is plotted versus the flow rate for CO_2 in a 7-inch tube assuming the same temperatures as in Figs. 3 and 4. Data measured by Steier[3] show good agreement with the theory. For $L > s$

$$D \approx \frac{1}{3000}\frac{ga^4}{\alpha^2} \tag{28}$$

is a good approximation. According to this formula, the optical center of a CO_2 lens of optimum design would occur outside the tube if the tube diameter is larger than 1 cm.

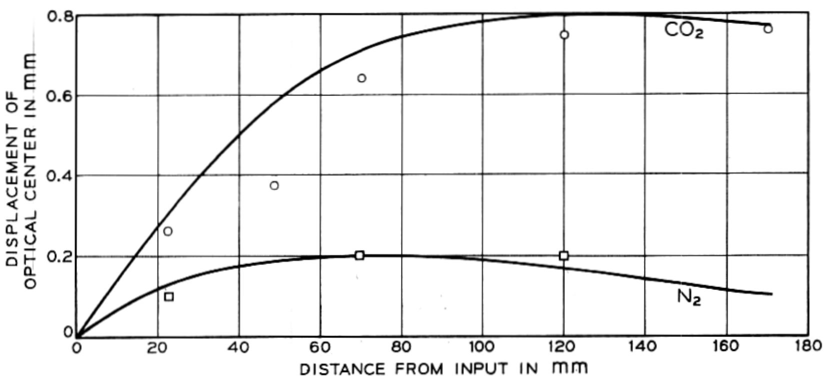

Fig. 4 — Displacement of the profile center in a tubular gas lens of ¼-inch i. d. for a flow rate of 1 liter per minute using CO_2 ($\alpha = 0.1$ cm^2/s) or N_2 ($\alpha = 0.2$ cm^2/s). Measured data by Steier.[3]

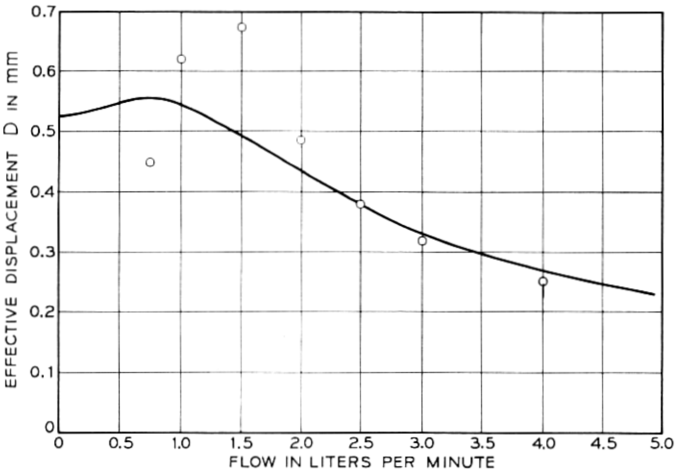

Fig. 5 — Displacement of the optical center of a tubular gas lens using CO_2 and a tube of 7 inches long and ¼-inch diameter. Measured data by Steier.[3]

IV. CONCLUSIONS

The calculations show that the temperature distribution in a gas-filled tube undergoes a distortion which increases with the fourth power of the tube radius. A square law dependence on pressure is predicted for the range of 0.05 to 50 atmospheres where the thermal conductivity is independent of the pressure and therefore, the diffusivity $\alpha \propto 1/p$.

As a measure of the distortion, the displacement of the effective optical center in a tubular gas lens is calculated. Using CO_2 at room temperature and a tube of 10-mm diameter at 100°C wall temperature the optical center occurs at the bottom of the tube.

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