Mode Conversion in Lens Guides with Imperfect Lenses

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A coherent Gaussian beam transmitted through many imperfect lenses suffers a distortion of its profile. Particularly smooth polishing errors generate parasitic modes which travel with a slightly different propagation constant and about the same low loss as the beam. While the two modes of lowest order essentially influence position and width of the beam, all higher-order modes deform the profile and may hamper position control and detection if they build up to sufficient power. The calculations show that this effect can be reduced to a negligible amount if the beam cross-section is of the order or smaller than the dimensions of the irregularities. This is in agreement with experiments. The perturbation of the beam in the air path between the lenses is also investigated and it is shown from experimental data that this effect is negligible in a properly shielded underground lens guide.

I. INTRODUCTION

There has been much uncertainty about the optical quality required for the components in an optical transmission link. Particularly for a lens guide with thousands of lenses, this is a major cost factor. It has been shown that systematic lens aberrations may lead to a severe degeneracy of a transmitted laser beam,[1] but hardly anything is known about random errors. Previous work in this field dealt with antenna or imaging problems,[2, 3, 4] but none of these theories can be applied to iterative structures.

The theory presented here was developed in parallel with experiments in a half-mile underground lens guide designed to gain data about the required component quality.[5] This guide employed antireflection-coated quartz lenses separated by about 140 m. A loss of roughly 1 percent per lens was measured, so that a transmission over 100 miles without amplification seems feasible. Systematic aberrations are negligible as compared with random surface irregularities.

These irregularities are of various nature and origin. There are minute scratches in the polished surface and tiny holes or craters in the antireflection coatings. Both cause a wide angle scattering and part of the measured overall loss without considerably changing the intensity profile or the phasefront of the transmitted light beam.

On the other hand, the polishing process achieves a spherical surface only to a certain degree, so there are always small smooth protuberances and recesses called "polishing errors." They show up in an interferometer check and their magnitude is usually given in fringes or wavelength of the light used in the interferometer. This magnitude defines the quality of the lens.

It is this imperfection which will be of interest here, for, without introducing immediate loss, it distorts the light beam in a way that may lead to complete deformation of the intensity profile when occurring repetitively. The consequence may eventually be an additional loss. Furthermore, it influences the choice of the receiving technique used at the end of a long lens guide because the efficiency of a heterodyne system will depend on how well the signal and local oscillator beams can be matched. Thirdly, it affects the applicability and design of beam position control systems which probably will have to be employed in some sections of the lens guide to provide for occasional realignment.[6, 7]

Refractive index variations in the atmosphere between the lenses are of course an additional source of beam distortion. Though weak in a shielded underground lens guide the influence might be comparable to that of imperfect optical components. The calculations in the last part of this work consider these index variations using the model of an imperfect waveguide.[8] Though not as general or accurate as previous work[9] this approach has the advantage that it yields simple formulae for the case of weak coupling. By inserting some experimental data the influence of the air paths and the optical components will be compared.

II. THE STATISTICAL FEATURES OF IMPERFECT LENSES

Restricting the following calculations to smooth irregularities has two consequences. First, in the proximity of the lens surface the approximations of geometrical optics may be applied, which means that the wavefront emerging from the surface exhibits a phase deviation but no amplitude change.

If there is, for example, a protuberance of magnitude δ at a certain point of a lens surface, the phase retardation of a light ray passing this point will be

$$\varphi = k \, \Delta n \, \delta, \tag{1}$$

where k is the propagation constant of the light outside the lens and Δn is the refractive index change at the surface.

Second, the surface irregularities and consequently also the phase deviations may be described by a random function which is both well-behaved (with at least the first derivative being finite) and homogeneous over the whole surface, since the irregularities were generated everywhere by the same process.

To proceed in the mathematical description, some assumptions must be made which seem to be reasonable for the random function under consideration, but will not be proved as valid here. One may conceptually construct an ensemble of identical optical surfaces which exhibit different point-by-point deviations, but are statistically equivalent. It is assumed that averages over the surface are replaceable by ensemble averages, and that δ and therefore φ are Gaussianly distributed, have zero mean, and variance Δ^2 and Φ^2, respectively. Obviously, the correct lens surface can always be defined in such a way that the mean value of δ is zero.

For simplicity, the two-dimensional model shown in Fig. 1 is used at the beginning. δ and φ are now functions of the surface coordinate x only. The covariance

$$F(x_1 - x_2) = \langle \varphi(x_1)\varphi(x_2) \rangle \tag{2}$$

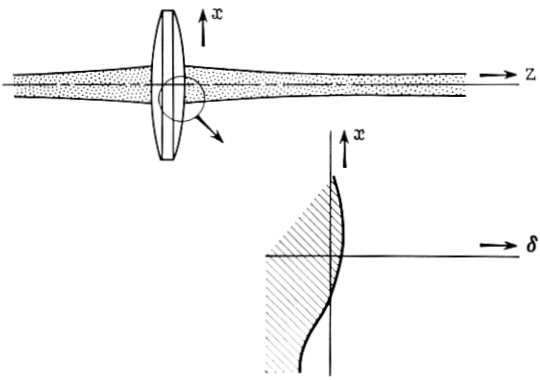

Fig. 1 — Two-dimensional model of an imperfect lens.

can be shown to exist and be a function of the distance $x_1 - x_2$ only because of the assumed features of $\varphi(x)$. For later calculations the identity

$$\langle \exp i[\varphi(x_1) - \varphi(x_2)] \rangle = \exp (F - \Phi^2) \tag{3}$$

is needed, which can be derived from those assumptions also.[3]

The fact that φ is smooth and stationary suggests a Gaussian covariance

$$F = \Phi^2 \exp [-(x_1 - x_2)^2/v^2], \tag{4}$$

where v is a correlation length determined by the dimension of those protuberances and recesses on the optical surface.

III. COUPLING TO PARASITIC MODES

A coherent light beam with Gaussian field profile conserves itself from lens to lens in a lens guide if it enters with the right phasefront curvature and the right half-width w of the field profile.[11] This "Gaussian beam" is the lowest order of an infinite set of modes which can propagate in such a lens guide. All these modes have the same phase fronts, slightly different propagation constants and a field profile that can be described by the orthogonal set of hyperbolic cylinder functions

$$D_n\left(2 \frac{x}{w}\right) = e^{-x^2/w^2} He_n\left(2 \frac{x}{w}\right), \tag{5}$$

where He_n are the hermite polymonials.[10, 11] Note that $He_0 = 1$ and, therefore, D_0 describes the Gaussian beam profile.

The higher the mode number, the further the profile extends about the lens area. As will be shown, the smooth irregularities under consideration here generate mainly low order modes and those to an amount that the comparatively small losses at the lens apertures are negligible. It seems justified, therefore, to consider the lenses as unbounded.

Assume that a perfect Gaussian beam traverses the optical surface in Fig. 1. Then the emerging wave function is

$$u(x) = D_0 \exp [i\varphi(x)] \tag{6}$$

which, on the other hand, can be expanded into the infinite series

$$u(x) = \sum_{n=0}^{\infty} c_n D_n . \tag{7}$$

The expansion coefficient c_n describes the coupling or scattering from the Gaussian beam to the n^{th} mode. Using (6), (7), and the orthogonality relation given in Ref. 10, one finds

$$c_n = \frac{\sqrt{2/\pi}}{n!\,w} \int_{-\infty}^{+\infty} D_0 D_n \exp(i\varphi)\, dx. \tag{8}$$

Multiplying this by its conjugate complex and averaging over the ensemble yields the average power coupled to the n^{th} mode from a Gaussian beam of unit power passing one distorted surface. The calculation is shown in the Appendix. The result is

$$p_n = \frac{\sqrt{2/\pi}}{n!\,2^n w} \int_{-\infty}^{+\infty} D_0 D_{2n} \exp[F(\sqrt{2}\,\xi) - \Phi^2]\, d\xi. \tag{9}$$

For a Gaussian correlation function F as defined in (4), the most useful representation is an expansion in powers of the variance Φ^2

$$p_n = \frac{(2n)!}{2^{2n}(n!)^2} \exp(-\Phi^2) \sum_{q=0}^{\infty} \frac{\Phi^{2q}}{q!} \frac{(qw^2/v^2)^n}{(1 + qw^2/v^2)^{n+\frac{1}{2}}}. \tag{10}$$

This formula is valid for any value of Φ and v. In practice, the converted power is only a small part of the total beam power and therefore,

$$\Phi w/v \ll 1. \tag{11}$$

In the case of a lens guide, this is a necessary condition for reconversion from parasitic modes into the beam to be negligible.

Two cases are of interest: Φ is large, say, of the order of 1 rad or larger, but (11) is satisfied since v is large at the same time. A series expansion in powers of Φ, as in (10), is not very useful in this case. However, expanding F in powers of w/v and truncating after the quadratic term yields for (9)

$$p_n = \frac{(2n)!}{2^{2n}(n!)^2} \frac{(\Phi^2 w^2/v^2)^n}{(1 + \Phi^2 w^2/v^2)^{n+\frac{1}{2}}}. \tag{12}$$

Probably of more importance are optical surfaces which cause a small rms phase distortion Φ but have a correlation length v of the order or even smaller than the beam width w. Then (10) may be used and terms with $q > 2$ in (10) may be neglected. Note that for both (10) and (12) a summation over all n yields unity. No power is lost in the conversion process. p_o is the power that is left in the Gaussian beam and $1 - p_o$, consequently, the conversion loss.

IV. THE THREE-DIMENSIONAL REPETITIVE STRUCTURE

To investigate the three-dimensional model, the additional assumption is made that the irregularities are isotropic over the optical area. The correlation function (4) may then be extended to two dimensions by

$$F = \Phi^2 \exp\left[-\frac{(x_1 - x_2)^2 + (y_1 - y_2)^2}{v^2}\right]. \tag{13}$$

The modes of the new model are defined by two numbers n and m. It can be shown that groups of modes with the same

$$r = n + m \tag{14}$$

are degenerate, that is, they travel with the same propagation constant.[11]

The coupling coefficients for the three-dimensional model must be evaluated from double integrals which are separable if higher orders of Φ^2 may be neglected. One finds for the average power coupled from a unit power beam

$$p_{00} = 1 - \frac{\Phi^2}{1 + v^2/w^2}, \tag{15}$$

and

$$p_{nm} = \frac{(2n)!}{2^{2n}(n!)^2} \frac{(2m)!}{2^{2m}(m!)^2} \frac{\Phi^2 v^2/w^2}{(1 + v^2/w^2)^{n+m+1}} \quad \text{for} \quad n, m = 1, 2, \cdots .$$

Physically more meaningful is the computation of the average power that is coupled to a complete group of degenerate modes:

$$P_0 = 1 - \frac{\Phi^2}{1 + v^2/w^2},$$

and

$$P_r = \frac{\Phi^2 v^2/w^2}{(1 + v^2/w^2)^{r+1}} \qquad r = 1, 2, \cdots . \tag{16}$$

It has been shown in Ref. 1 that an optical surface can be adjusted in such a way that no power is coupled to the first group of parasitic modes. If this is done, the power loss is a minimum and the power kept in the beam may be found from (16) to be

$$\bar{P}_0 = 1 - \frac{\Phi^2}{(1 + v^2/w^2)^2}. \tag{17}$$

If, furthermore, one is free to adjust the width of the transmitted beam at the receiving end, say by a telescope arrangement or by adapting the local oscillator beam to the width of the signal beam, one can minimize the losses even further. In this case no power is coupled to the second mode group either.[1] The power kept in the beam may be found from (16) to be

$$\bar{P}_0 = 1 - \frac{\Phi^2}{(1 + v^2/w^2)^3}. \tag{18}$$

It seems reasonable to assume that the irregularities on both sides of a lens surface are uncorrelated, in which case the powers generated in both conversion processes simply add. Fig. 2 shows the conversion loss $1 - 2P_0$ and the powers in the three parasitic modes of lowest order versus w/v for a lens quality of $\lambda/10$. For the first approximation, it is assumed that such a lens has an rms deviation of $\Delta = \lambda/10$ though actually the rms value should be somewhat smaller. Δn in (1) is 0.5. The loss increases rapidly with decreasing correlation length. For $v < w$, the loss approaches the value Φ^2. For a correlation length larger than the beam width, almost all the losses are found in the first parasitic mode.

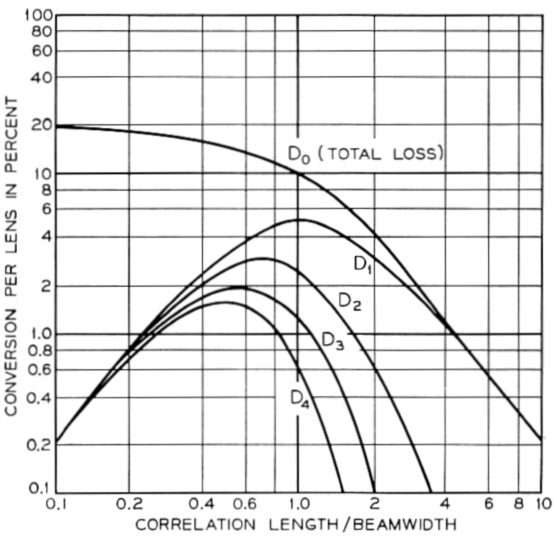

Fig. 2 — Average loss and power coupled to higher mode groups by a lens of quality $\lambda/10$.

Proper adjustment reduces the loss by this amount. This is shown in Fig. 3. For $v = 2w$, a factor of 4 is gained by adjustment. The loss decreases by more than an order of magnitude if the beam spread is neglected also. The lens quality in this example is $\lambda/10$. Fig. 4 compares the losses for lenses of various qualities. Fig. 5 gives the same quantities when proper adjustment of the lenses is taken into account. Fig. 6 in addition neglects spreading of the beam.

For hand-polished lenses, the correlation length can be expected to be of the order of cm. The beam width in a lens guide depends on the lens spacing and the wavelength of the transmitted light.[11] For lenses separated by 140 m and red light of 0.63 μ, the beam width is $2w = 1$ cm. Fig. 6 shows that in this case the conversion loss is less than 0.1 percent and therefore, a negligible amount of the total loss. Nevertheless, poorly attenuated parasitic modes may build up and distort the beam profile.

Certainly there is no correlation from lens to lens. So the average mode power simply increases proportionally to the number of lenses. The modes under consideration have about the same overall attenuation as the Gaussian beam. Therefore, after N lenses, the average

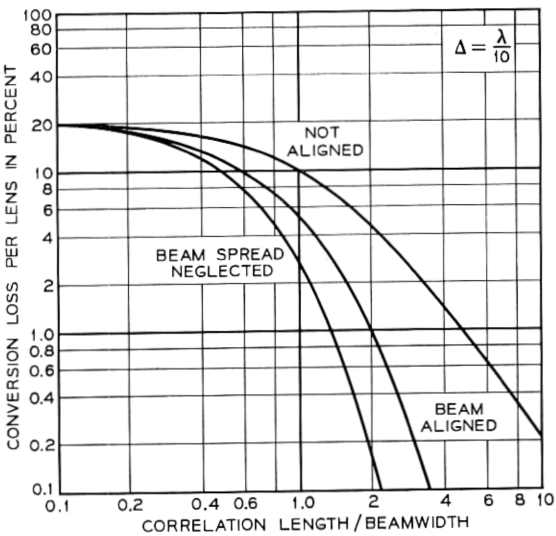

Fig. 3 — The conversion loss is reduced if the lens is adjusted and a beam spread tolerated (quality $\lambda/10$).

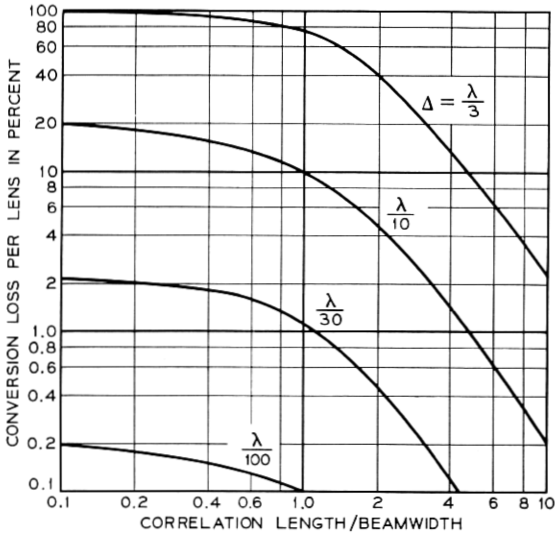

Fig. 4 — Average conversion loss for lenses of various qualities.

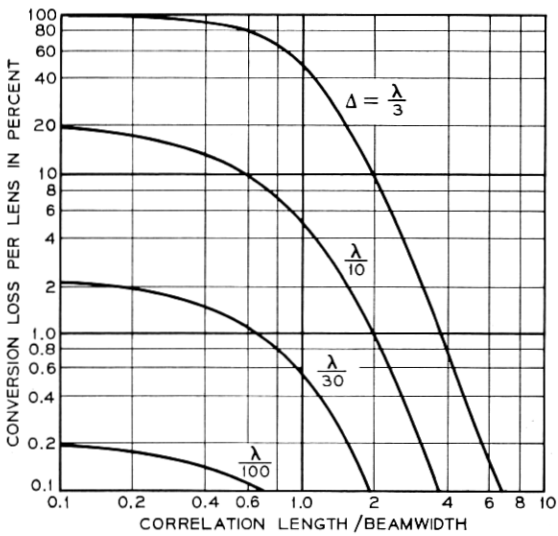

Fig. 5 — Average conversion loss for lenses of various qualities (beam aligned on axis).

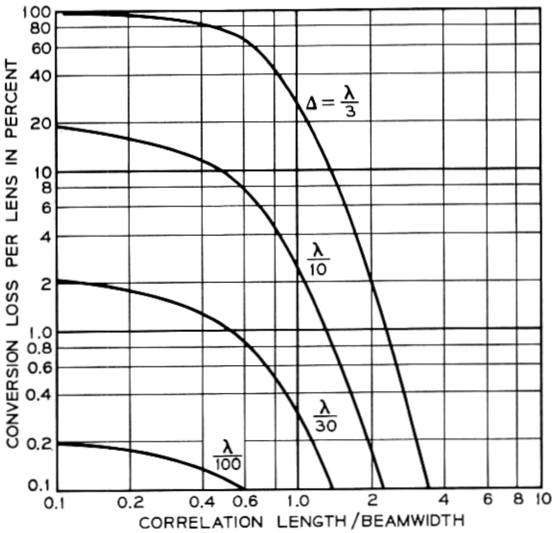

Fig. 6 — Average conversion loss for lenses of various qualities (beam spread tolerated).

power in the rth parasitic group of degenerate modes is

$$\frac{2NP_r}{1 - 2NP_0} \tag{19}$$

times the power in the Gaussian beam, provided that the conversion even after N lenses is small enough to permit the negligence of reconversion and higher-order loss terms. The average amplitude ratio is

$$\frac{\sqrt{2NP_r}}{1 - NP_r}. \tag{20}$$

The respective phases even of modes in the same group are undetermined.

To gain a conception of the distortion a situation is assumed in Fig. 7 to 9 where all modes are in phase. Fig. 7 shows a possible intensity profile after passing a lens of quality $\lambda/10$ and correlation length $v = 2w$. The result is mainly a displacement. In Fig. 8 the beam passed 10 lenses but these now are adjusted so that the beam stays on the guide axis. The main effect is a spreading. Fig. 9 is a sketch of the profile after 100 lenses, all adjusted, and the profile is

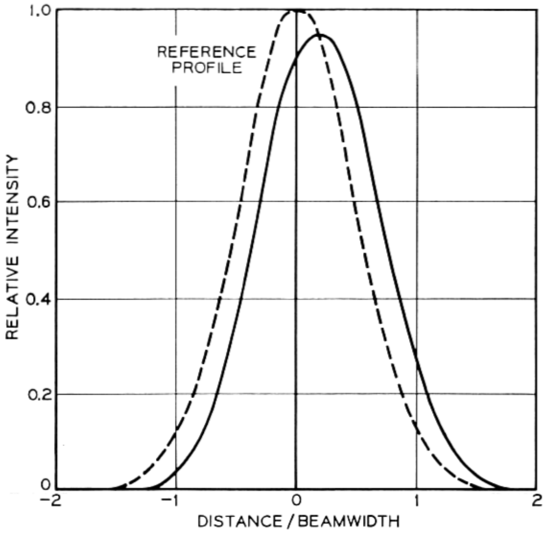

Fig. 7 — Possible profile distortion caused by a lens of quality λ/10.

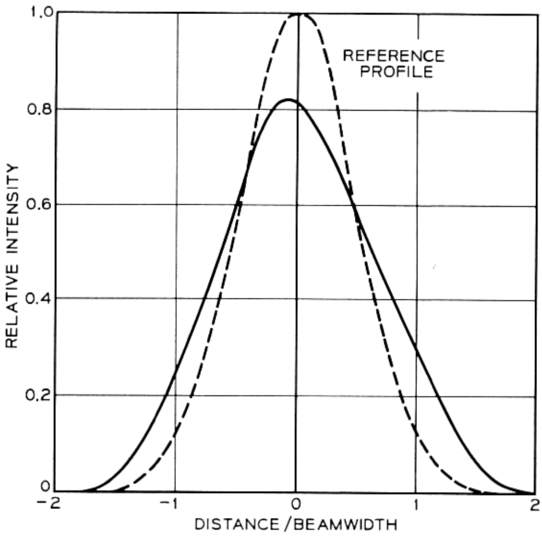

Fig. 8 — Possible profile distortion after 30 lenses of quality λ/10 (beam aligned on axis).

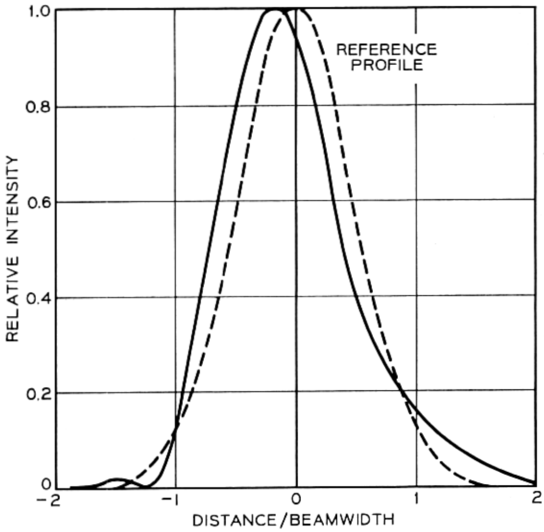

Fig. 9 — Possible profile distortion after 900 lenses of quality λ/10 (beam spread tolerated).

reduced to the nominal beam width. There is a slight tilt of the profile and a side lobe, but no basic destruction of the beam.

V. CONVERSION IN THE ATMOSPHERE BETWEEN LENSES

Similar to the mode coupling at certain cross-sections of the lens guide there can be mode coupling all along the guide if there is a source of distortion. In the case of a gas between the lenses the source may be the random fluctuation of the refractive index of this gas.

For simplicity let us return to the two-dimensional model of Fig. 1. Here Δn, the deviation from the mean index n_0, is a function of x and z. Consider slabs of thickness Δz cut perpendicular to the guide axis. A light beam traversing a slab at z suffers a distortion of its phase front

$$\varphi(x, z) = k \, \Delta z \, \Delta n(x, z). \tag{21}$$

This causes a conversion into parasitic modes which can be calculated from (8). The validity of this model has been investigated in Ref. 8. Its usefulness lies in its physical simplicity which allows controlled approximations.

Contrary to the lens irregularities δ, the index variations Δn are only "locally homogeneous," which makes it necessary to use structure functions instead of covariance functions for the statistical description. In two dimensions the structure function of φ is

$$S(x_1 - x_2, z_1 - z_2) = \langle [\varphi(x_1, z_1) - \varphi(x_2, z_2)]^2 \rangle. \tag{22}$$

Instead of (3) the identity

$$\langle \exp i[\varphi(x_1, z_1) - \varphi(x_2, z_2)] \rangle = \exp [-\tfrac{1}{2} S]. \tag{23}$$

will be used later, which may be derived in the same way as (3).[3] From Kolmogoroff's theory for a locally isotropic turbulent flow one finds[12]

$$S = k^2 (\Delta z)^2 \sigma [(x_1 - x_2)^2 + (z_1 - z_2)^2]^{\frac{1}{3}}. \tag{24}$$

σ is called the refractive index structure constant and measures the strength of the index fluctuations.

If Δz, the thickness of the slabs, is made very small the coupling per slab will be proportional to Δz, say $c_n \Delta z$ for the nth mode. Assume that a Gaussian beam of unit amplitude traverses the air path from one lens to the next. Then in every slab an amplitude $c_n \Delta z$ is generated in the nth mode. Assume that the coupling to all parasitic modes is so small that reconversion can be neglected. Then, at the end of the air path of length L, the amplitude in the nth mode is

$$a_n = \int_0^L c_n(z) \exp(-in\theta z/L)\, dz. \tag{25}$$

$n\theta/L$ describes the phase lag between the fundamental and the nth mode as they travel along the path.[1] For a confocal system $\theta = \pi/2$, i.e., the phase retardation of the first parasitic mode with respect to the fundamental is $\pi/2$ from one lens to the next. Considering that the patches of correlated index variations are much smaller than the lens distance, it can be expected that the phase between low-order modes changes only a negligible amount within the area of correlated coupling. In the following, therefore, the phase lag will be neglected.

For the evaluation of (25) it has to be considered also that c_n is a function of the beam width w which varies slowly along the transmission path. In a confocal lens guide the modes are $\sqrt{2}$ times wider at a lens than in the center between two lenses.[11] It is by a factor of this order that the results will deviate from the true values if for the following w is kept constant and equal to the width at a lens.

With this in mind the expected power coupled to the nth mode can be calculated by multiplying (25) by its conjugate complex and taking the average, which in this case will be an average over an infinite time. The evaluation agrees with the one outlined in the appendix. Since S is proportional to the square of the small increment Δz the exponential function in (23) may be expanded up to the linear term of the argument. The average power in the nth mode is finally

$$p_n = \frac{\sqrt{2/\pi}}{n!\,2^n w} \int_{-\infty}^{+\infty} D_0 D_{2n}[1 - \tfrac{1}{2} S_L(\sqrt{2}\,\xi)]\, d\xi \tag{26}$$

with

$$S_L(x_1 - x_2) = \int_0^L \int_0^L S(x_1 - x_2\,, z_1 - z_2)\, dz_1\, dz_2\,. \tag{27}$$

S_L has been calculated elsewhere in connection with the investigation of a plane wave propagating in a turbulent flow.[13] It is called the phase structure function of a plane wave and describes the statistics of the phases in a phase front that has traversed a turbulent air path. Equation (26) states that under the employed approximations the parasitic power arriving at the path end can be calculated from the intensity profile of an undisturbed beam multiplied by the phase structure function of a plane wave at the path end. From Ref. 13 one finds

$$S_L(x_1 - x_2) = 2.91 k^2 L \sigma (x_1 - x_2)^{5/3}. \tag{28}$$

As long as the function S_L is of the form ξ^α for $\alpha > -1$ the following general solution can be found for (26):

$$p_n = \delta_{0n} - S_L(\sqrt{2}\,w) \frac{\Gamma\left(\dfrac{1}{2} + \dfrac{\alpha}{2}\right)}{\Gamma(\frac{1}{2})} \binom{\alpha/2}{n} (-1)^n 2^{\alpha/2} \tag{29}$$

with

$$\delta_{0n} = \begin{cases} 1 & \text{for} \quad n = 0 \\ 0 & \text{for} \quad n = 1, 2, \cdots . \end{cases}$$

For the three-dimensional case again groups of modes with equal propagation constants are combined. The expected power in the rth group can be calculated from an expression similar to (26) but with double integrals for the x and y coordinates. It is

$$P_r = \delta_{0r} - S_L(\sqrt{2}\,w) \frac{\alpha}{2} \Gamma\left(\frac{\alpha}{2}\right) \binom{\alpha/2}{n} (-1)^n 2^{\alpha/2}. \tag{30}$$

In the case of the plane wave approximation $\alpha = 5/3$. $1 - P_r$ is the total average power loss for a Guassian beam. Note that the summation over all P_r yields unity. No power is dissipated.

The application of the waveguide model is no longer useful if the refractive index variations are so large that reconversion from parasitic modes into the fundamental must be considered. The mode conversion at the path end, however, may then still be calculated from the undisturbed profile multiplied by the appropriate phase structure function. Only, the phase fluctuations at the path end will then be so large that an expansion of the exponential function in (26) is no longer valid. For a configuration close to confocal the phase structure function S_L of the plane wave will be a good starting point to calculate the expected powers

$$p_n = \frac{\sqrt{2/\pi}}{n!\, 2^n w} \int_{-\infty}^{+\infty} D_0 D_{2n} \exp\left[-\tfrac{1}{2} S_L(\sqrt{2}\,\xi)\right] d\xi. \tag{31}$$

A better approximation would have to consider the amplitude variations at the path end in (34) as well. It has been shown elsewhere that its neglection results in an error of the order of two only.[14]

The results given in Fig. 10 for the three-dimensional case were calculated from an expression similar to (31). Fig. 10 shows the loss

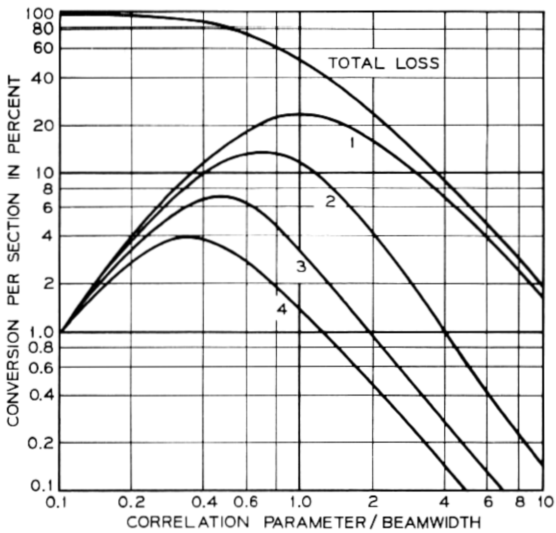

Fig. 10 — Average loss and power coupled to higher mode groups versus the phase correlation parameter $(2.91\ k^2 L\sigma)^{-3/5}$ for a turbulent medium.

in the beam and the powers in the first four parasitic mode groups versus the correlation parameter

$$v = (2.91k^2L\sigma)^{-3/5} \tag{32}$$

which is a measure for the correlation at the path end. This plot allows a comparison with Fig. 2.

For large v the curves in Fig. 10 turn into straight lines indicating a functional dependence $w^{5/3}$ as given by the approximate formula (30). Also for large v the ratio between P_1 and the total loss is a constant close to 1. By measuring P_1 the total power loss can be found.

This has been done in an experimental underground lens guide using a photoresistor bridge.[7] The first mode group consists of two modes of equal average power orientated in perpendicular planes. The photoresistor bridge described in Ref. 7 measures the instantaneous amplitude of one of these modes as compared to the amplitude in the fundamental. Actually if this ratio is of the order of some percent or smaller it is equal to the ratio of bridge signal V to bridge battery voltage V_0 (see Ref. 7). The variance of this signal is the ratio of expected first mode power to beam power and twice that is the loss. Neglecting a seasonal slow beam drift a variance of 3 10^{-7} was measured in a 400-foot section of the underground lens guide. The loss is consequently of the order of 10^{-6} of the total power.

It cannot be asserted here that this loss is indeed due to atmospheric effects. Microseisms may cause fluctuations of the lens positions that lead to disturbances of the same order. The measurement must be understood merely as an upper limit for the conversion caused by a well-shielded air path. The conversion expected from the lenses is several orders of magnitude larger, but, being independent of time, it can only be measured in a large number of sections to represent a reasonable average. An experiment of this kind is described in Ref. 5.

VI. CONCLUSIONS

In a lens guide with widely separated solid lenses, aberrations are negligible as compared to random surface irregularities. How much a Gaussian beam is distorted by the irregularities depends not only on the rms deviation Δ, but also very strongly on the dimension of the irregularities as compared to the beam cross-section.

For a beam of width $2w$, which loses no power into the two parasitic modes of lowest order, the conversion loss is proportional to $(w/v)^6$

where v is a correlation length defining the dimension of the irregularities. Consequently, w should be made small enough to assure that the conversion loss is negligible compared to all other losses and that the beam profile distortion caused by the generated parasitic modes is tolerable.

A beam with nominal width of 1 cm seems to satisfy these conditions if lenses with a quality $\lambda/10$ are used. These calculations are based on a conservative estimate of 1 cm for the correlation length. In this case, the conversion losses are smaller than 0.1 percent per lens and 1/10 of all other losses. The profile after 100 lenses may, at best, exhibit small side lobes with a peak intensity of the order of 1 percent of the beam peak intensity.

Refractive index variations in the air path between the lenses also lead to a conversion loss. It grows with $w^{5/3}$ for weak distortions. In a 400-foot section of the underground lens guide described in Ref. 7 an upper bound for this loss was measured to be 10^{-6} of the total power.

APPENDIX

A meaningful measure for the effect of surface imperfections is the power coupled from the Gaussian beam into parasitic modes, averaged over the ensemble of equivalent surfaces:

$$p_n = n!\,\langle c_n c_n^* \rangle. \tag{33}$$

The coupling coefficients c_n are given by (8). After changing the order of integration and averaging process and by replacing the ensemble average by an average over the surface, one gets

$$p_n = \frac{2}{n!\,\pi w^2} \iint D_0(x_1) D_n(x_1) D_0(x_2) D_n(x_2) \langle e^{i[\varphi(x_1) - \varphi(x_2)]} \rangle \, dx_1 \, dx_2 \,, \tag{34}$$

where all integrals here and in the following extend from $-\infty$ to $+\infty$. To separate the double integral in (34), it is appropriate to change to new coordinates

$$\xi = \frac{x_1 - x_2}{\sqrt{2}} \quad \text{and} \quad \eta = \frac{x_1 + x_2}{\sqrt{2}} \tag{35}$$

in the plane of integration. From the properties of the hyperbolic cylinder functions, the identity

$$D_n(x_1) D_n(x_2) = \sum_{p=0}^{n} (-2)^{-n} \binom{n}{p} D_{2p}(\xi) D_{2n-2p}(\eta) \tag{36}$$

can be derived. The use of (35), (36), and (3) turns (34) into

$$p^n = \frac{2}{n!\,\pi w^2} \sum_{p=0}^{n} (-2)^{-n} \binom{n}{p} \int D_0 D_{2n-2p}\, d\eta \int D_0 D_{2p} e^{F(\sqrt{2}\xi)-\Phi^2}\, d\xi \quad (37)$$

and because of the orthogonality of the D_n, this can finally be simplified to

$$p_n = \sqrt{2/\pi}\, \frac{(-2)^{-n}}{n!\, w} \int D_0 D_{2n} e^{F(\sqrt{2\xi})-\Phi^2}\, d\xi. \quad (38)$$

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