Phase Progression in Conical Waveguides

By ELLIOTT R. NAGELBERG

(Manuscript received August 10, 1967)

We studied the phase progression properties of normal modes in a conical waveguide in order to develop techniques for analysis of multimode microwave antennas. We found that the large-order asymptotic expansions of Bessel functions developed by F. W. J. Olver are most appropriate for such calculations by virtue of their simplicity and uniformity with respect to argument. These expansions are applied to analysis of the conical TE_{11} and conical TM_{11} modes and, in addition, to an examination of the "quasi-cylindrical approximation" in which the conical waveguide is regarded as a cylindrical waveguide with gradually changing cross section.

I. INTRODUCTION

For most applications to microwave communication systems, waveguides are designed in such a way that only the dominant mode can propagate. This has been the case principally for practical reasons, as evidenced for example by problems encountered in the development of millimeter-wave systems using the higher-order TE_{01}° mode.[1] Since the waveguide in this case must be oversized, small geometrical asymmetries due to errors in fabrication, bends, and other structural perturbations cause coupling to unwanted modes, which can result in a significant degradation in performance.

On the other hand, there has been considerable interest during the past several years in techniques which require the controlled excitation of higher-order modes combined with the dominant mode in, for example, a conical waveguide. Two such applications have been the $TE_{11}^{<} - TM_{01}^{<}$* precision autotrack system for the *Telstar*® satellite,[2] and the $TE_{11}^{<} - TM_{11}^{<}$ dual mode conical horn[3] which has been suggested as a primary feed configuration for low-noise satellite communication antennas.

---

* The notation $TE_{mn}^{<}$ or $TM_{mn}^{<}$ will be used to designate conical waveguide modes.

A common feature of these techniques is the necessity for maintaining a high degree of phase coherence among the various modes of propagation. It is therefore required, in order to design such systems and predict the effects of frequency, temperature and structural variations, to accurately determine the phase progression properties of the guided wave fields.

The paper's contents may be summarized as follows: We first describe the conical waveguide modes, which are vector eigenfunctions of Maxwell's equations in what is essentially spherical geometry. We note that although these solutions are well known in principle, the actual computation of their phase progression properties is not straightforward. It is, therefore, necessary to consider the problem of numerically evaluating both the eigenvalues and vector eigenfunctions so that we can apply these results to actual antenna problems. In order to do this we utilize certain uniform asymptotic expansions due to F. W. J. Olver[4] which are found to be well suited to such calculations. We thereby observe that a very common method of determining phase progression, which might be termed the quasi-cylindrical approximation, is not particularly accurate, and the errors associated with this method are evaluated.

Rationalized MKS units and the (suppressed) harmonic time dependence $e^{-i\omega t}$ will be used throughout.

II. MODES IN A CONICAL WAVEGUIDE

The normal modes characteristic of a conical waveguide are derived in the usual manner by finding separable solutions to Maxwell's equations in spherical coordinates, subject to the boundary condition that the components of electric field tangent to the lateral surface must vanish. The solutions thus derived may be partitioned into two types, $\text{TE}^{<}$ modes for which the electric field is transverse to the direction of propagation (the r-direction), and $\text{TM}^{<}$ modes for which the magnetic field is transverse to the direction of propagation. In terms of the coordinate system shown in Fig. 1, the components of electric field, for example, are given by,[5]

$\text{TM}^{<}$

$$E_r^{\text{TM}} = A\,\frac{\mu(\mu+1)}{(kr)^{\frac{3}{2}}}\,H^{(1)}_{\mu+\frac{1}{2}}(kr)P^m_\mu(\cos\theta)e^{im\varphi}$$

$$E_\theta^{\text{TM}} = \frac{A}{kr}\,\frac{d}{d(kr)}\left[\sqrt{kr}\,H^{(1)}_{\mu+\frac{1}{2}}(kr)\right]\frac{d}{d\theta}\,P^m_\mu(\cos\theta)e^{im\varphi} \tag{1}$$

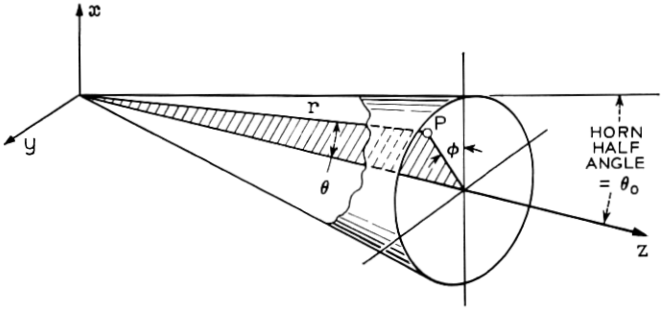

Fig. 1—Conical horn geometry.

$$E_\varphi^{TM} = \frac{imA}{kr \sin \theta} \frac{d}{d(kr)} [\sqrt{kr}\, H_{\mu+\frac{1}{2}}^{(1)}(kr)] P_\mu^m(\cos \theta) e^{im\varphi},$$

TE<

$$E_r^{TE} = 0$$

$$E_\theta^{TE} = \frac{mB}{\sin \theta \sqrt{kr}} H_{\nu+\frac{1}{2}}^{(1)}(kr) P_\nu^m(\cos \theta) e^{im\varphi} \tag{2}$$

$$E_\varphi^{TE} = \frac{iB}{\sqrt{kr}} H_{\nu+\frac{1}{2}}^{(1)}(kr) \frac{d}{d\theta} [P_\nu^m(\cos \theta)] e^{im\varphi}.$$

In (1) and (2), P_ν^m (cos θ) denotes the associated Legendre function (m = integer) and $H_\nu^{(1)}(x)$ represents the Hankel function of the first kind, corresponding to outgoing waves under the assumed time dependence $e^{-i\omega t}$. The constant k is the free-space wave number.

The eigenvalues ν and μ are found as solutions of the respective characteristic equations

$$\left[\frac{d}{d\theta} P_\nu^m(\cos \theta)\right]_{\theta=\theta_0} = 0 \tag{3}$$

$$P_\mu^m(\cos \theta_0) = 0 \tag{4}$$

for a specified horn half angle θ_0.

These eigenvalues can be computed by a variety of numerical methods. For example, one can represent the associated Legendre function in terms of the hypergeometric function[6] as

$$P_\nu^m(\cos \theta) = C \sin^{m/2} \theta\, F\left(1 + m + \nu,\, m - \nu;\, m + 1;\, \frac{1 - \cos \theta}{2}\right), \tag{5}$$

where C is a constant. The ν- or μ-zeros for a given θ_0 can then be found by a variety of root finding techniques. It is, however, worth noting that a first-order approximation may be determined from the formula

$$P_\nu^m(\cos\theta) \approx J_m(\sqrt{\nu(\nu+1)}\,\theta) + \mathcal{O}(\theta^2), \tag{6}$$

where J_m denotes a Bessel function. Since the roots of Bessel functions are well tabulated,[7] (6) can be conveniently used to provide either an estimate of the eigenvalue or a starting value for an iterative algorithm. The phase errors associated with this approximation will be discussed in a later section.

To indicate the behavior of the zeros and to provide helpful information for design of dual mode conical horns, a partial list of ν and μ values, computed using (5), has been prepared and is given in the Appendix.

III. BEHAVIOR OF THE RADIAL FUNCTIONS. PRECISE CALCULATION OF PHASE PROGRESSION.

Having obtained the appropriate eigenvalues as defined by (3) and (4), we may then proceed to the more interesting calculation of the radial dependence. In principle, the phase shift between the two spherical surfaces $r = r_1$ and $r = r_2$ is given by

$$\delta\alpha^{\text{TE}} = \arg\,[H^{(1)}_{\nu+\frac{1}{2}}(kr_2)] - \arg\,[H^{(1)}_{\nu+\frac{1}{2}}(kr_1)] \tag{7}$$

$$\delta\alpha^{\text{TM}} = \arg\left\{\frac{d}{dx}\,[\sqrt{x}\,H^{(1)}_{\mu+\frac{1}{2}}(x)]\right\}_{x=kr_2} - \arg\left\{\frac{d}{dx}\,[\sqrt{x}\,H^{(1)}_{\mu+\frac{1}{2}}(x)]\right\}_{x=kr_1} \tag{8}$$

for the TE and TM modes, respectively, where arg ( ) denotes the phase angle associated with a complex number.

The difficulty which arises when one attempts to utilize these expressions is essentially one of computation, due to the particular regime of order and argument frequently encountered in analysis of conical horn waveguides. We are particularly concerned here with the so-called transition region where the argument and order of the Hankel functions are large and comparable. For example, when $\theta_0 = 5°$, $\nu = 20.6155$ and $\mu = 43.4109$ (see Appendix), which means that we must allow for a range of arguments increasing from these values.

Asymptotic formulas for Bessel functions have, of course, been studied in great detail. Asymptotic *forms* in the transition region $kr \gtrsim \nu$ have been given, for example, by Watson and Langer.[8] Although of mathematical interest, these typically give only the limiting behavior of the function, with the remainder specified within some order. However, engineering design generally requires more precise results, which can be obtained only with the aid of complete asymptotic expansions.

Expansions particularly appropriate for our problem have been given by Olver.[4] These formulas represent the Bessel functions as asymptotic series in terms of reciprocal powers of the order, and are valid asymptotic expansions for all values of argument. Although their derivation is very complicated, and will not be discussed here, we shall state the general form of the result and indicate several simplifications which are valid for most problems involving phase progression in conical horns.

The complete asymptotic expansion for the Hankel function of the first kind, following Olver's notation, is given by

$$H_\nu^{(1)}(\nu x) \approx \left(\frac{4\zeta}{1-x^2}\right)^{\frac{1}{4}}\left\{\frac{A_i(\nu^{\frac{2}{3}}\zeta) - iBi(\nu^{\frac{2}{3}}\zeta)}{\nu^{\frac{1}{3}}} \cdot \sum_{n=0}^{\infty}\frac{A_n(\zeta)}{\nu^{2n}} + \frac{Ai'(\nu^{\frac{2}{3}}\zeta) - iBi'(\nu^{\frac{2}{3}}\zeta)}{\nu^{5/3}}\sum_{n=0}^{\infty}\frac{B_n(\zeta)}{\nu^{2n}}\right\}. \tag{9}$$

In this expression, Ai, Ai', Bi, and Bi' represent Airy functions and their derivatives,[9] and ζ is a constant related to x by the formula

$$\zeta = -\{\tfrac{3}{2}(x^2-1)^{\frac{1}{2}} - \tfrac{3}{2}\sec^{-1} x\}^{\frac{2}{3}}. \tag{10}$$

The coefficients $A_n(\zeta)$ and $B_n(\zeta)$ are determined through an auxiliary sequence $\{U_n(t)\}$ defined by the recursion formula

$$\begin{aligned} U_0(t) &= 1 \\ U_n(t) &= \frac{t^2(1-t^2)}{2}U'_{n-1}(t) + \frac{1}{8}\int_0^t (1-5t^2)U_{n-1}(t)\,dt, \end{aligned} \tag{11}$$

where the prime denotes differentation. The A_n and B_n can then be found using the relations

$$A_n(\zeta) = \sum_{m=0}^{2n}\frac{b_m U_{2n-m}(\tau)}{\zeta^{3m/2}} \tag{12}$$

$$B_n(\zeta) = -\sum_{m=0}^{2n+1}\frac{a_m U_{2n-m+1}(\tau)}{\zeta^{(3m+1)/2}}, \tag{13}$$

where $\tau = (1-x^2)^{-\frac{1}{2}}$ and $\{a_m\}$, $\{b_m\}$ are given by

$$a_0 = b_0 = 1 \tag{14}$$

$$a_m(m > 0) = \frac{(2m+1)(2m+3)\cdots(6m-1)}{m!\,(144)^m} \tag{15}$$

$$b_m(m > 0) = -\frac{6m+1}{6m-1}\,a_m\,. \tag{16}$$

Since for the range $x > 1$, which is of interest here, ζ is negative and τ is imaginary, it is also necessary to define the proper branches, which are as follows:

$$\tau = i(x^2 - 1)^{-\frac{1}{2}} \tag{17}$$

$$\frac{1}{\zeta^{\frac{1}{2}}} = \frac{i}{(-\zeta)^{\frac{1}{2}}} \tag{18}$$

$$\frac{1}{\zeta^{3m/2}} = \frac{i^{3m}}{(-\zeta)^{3m/2}}. \tag{19}$$

Using a table of Airy Integrals,[9] one can proceed to evaluate the required Hankel functions to whatever accuracy is needed. As an indication of the number of terms required in a typical calculation, it has been observed by J. A. Cochran and C. M. Nagel* that for $\nu \gtrsim 10$, four decimal place accuracy can be obtained simply by using terms including B_0 and A_1. The coefficients required for most horn calculations are thus given by

$$A_0 = 1 \tag{20}$$

$$A_1 = -\frac{81\tau_1^2 + 462\tau_1^4 + 385\tau_1^6}{1152} - \frac{7(3\tau_1 + 5\tau_1^3)}{1152\zeta_1^{\frac{3}{2}}} + \frac{455}{4608\zeta_1^3} \tag{21}$$

$$B_0 = \frac{3\tau_1 + 5\tau_1^3}{24\zeta_1^{\frac{1}{2}}} - \frac{5}{48\zeta_1^2}\,, \tag{22}$$

where $\tau_1 = (x^2 - 1)^{-\frac{1}{2}}$ and $\zeta_1 = -\zeta$.

IV. APPLICATIONS

In this section we shall discuss several applications of the preceding results. After presenting examples of phase progression for different modes we consider the phase errors introduced by approximating the eigenvalues μ and ν. Finally, we examine what might be called the

* Private communication.

"quasi-cylindrical approximation," in which phase progression is calculated by considering the horn to be a cylindrical waveguide with slowly varying cross section.

4.1 *Phase progression of the* $TE_{11}^{<}$ *and* $TM_{11}^{<}$ *modes. Effect of errors in* ν *and* μ.

A qualitative understanding of the phase progression properties of conical waveguide modes can be achieved by regarding a horn as a cylindrical waveguide with gradually increasing cross section. Although such a model has limitations, which will be discussed later in Section 4.2, it correctly predicts the fact that the phase progression rates for both classes of modes begin at relatively low values and increase monotonically toward that corresponding to the far field of a spherical wave in free space. This limiting behavior is reached when the conditions $kr \gg \nu$ or $kr \gg \mu$ are satisfied, corresponding to conical TE or conical TM modes, respectively.

Fig. 2 shows, for example, a direct computation of the phase shift as a function of $kr_2 - kr_1$ for the conical TE_{11} mode for half angles $\theta_0 =$ 3°, 10°. The value kr_1 is in each case taken to be that corresponding to the cut-off cross section of a cylindrical waveguide, i.e.,

$$kr_1 = \frac{\zeta}{\sin \theta_0}, \tag{23}$$

where $\zeta \approx 1.84118$. Fig. 3 shows analogous results for the $TM_{11}^{<}$ mode,

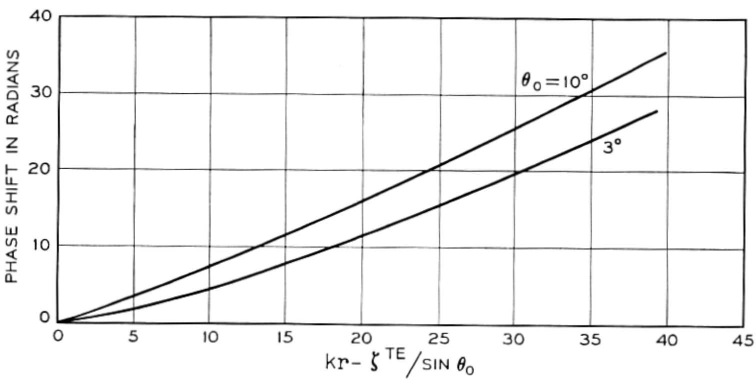

Fig. 2—Phase shift for the $TE_{11}^{<}$ mode relative to the cross section at which cut off would occur for a cylindrical waveguide. $\zeta_{TE} \approx 1.84118$.

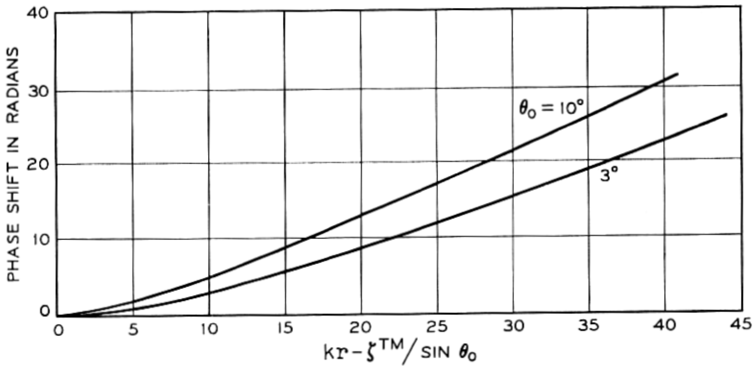

Fig. 3—Phase shift for the $TM^{<}_{11}$ mode relative to the cross section at which cut off would occur for a cylindrical waveguide. $\zeta_{TM} \approx 3.83171$.

the principal difference being in the more gradual increase in phase near kr_1.

In view of the difficulty in computing the ν- and μ-zeroes of the Legendre functions, as required by (3) and (4), it is of practical interest to determine how an error in the eigenvalue will effect the calculation of phase shift. If, for example, we denote by Δ^{TE} the *error in phase shift* due to a small error $\delta\nu$ in the eigenvalue, then from (7) we

Table I—Maximum Phase Error Due to 0.10 Percent Miscalculation of ν or μ

$k(r_2 - r_1) = 100$

		TE_{11} mode	
θ_0	$\delta\nu$	$\Delta^{TE} = (\pi/2)\,\delta\nu$	Δ^{TE} actual
3°	0.035	3.15°	2.04°
5°	0.021	1.89°	1.26°
10°	0.010	0.90°	0.61°

TM_{11} mode

θ_0	$\delta\mu$	$\Delta^{TM} = (\pi/2)\,\delta\mu$	Δ^{TM} actual
3°	0.073	6.87°	4.68°
5°	0.043	3.87°	3.14°
10°	0.021	1.89°	1.16°

have that

$$\Delta^{\mathrm{TE}} = \left\{ \frac{\partial}{\partial \nu} \arg \left[H^{(1)}_{\nu+\frac{1}{2}}(kr_2) \right] - \frac{\partial}{\partial \nu} \arg \left[H^{(1)}_{\nu+\frac{1}{2}}(kr_1) \right] \right\}_{\nu=\nu_0} \cdot \delta\nu, \tag{24}$$

where ν_0 denotes the correct eigenvalue. Although the calculation required by (24) is, in general, very difficult, it is relatively simple to obtain an upper bound to Δ^{TE}. First, it can be shown (see Ref. 7, p. 368), that at the cutoff radius r_1, the argument kr_1 is approximately equal to the eigenvalue ν (or μ) and furthermore that the partial derivative at that value tends toward zero as ν or μ becomes very large. It follows that an upper bound on the error Δ^{TE} can be obtained by neglecting the second term on the right side of (24) and letting $kr_2 \to \infty$. In this way we find, from the asymptotic behavior of the Hankel functions (see Ref. 6, p. 85), that

$$\max \mid \Delta^{\mathrm{TE}} \mid = \mid \delta\nu \mid \frac{\pi}{2} \tag{25}$$

and, in a similar way for Δ^{TM},

$$\max \mid \Delta^{\mathrm{TM}} \mid = \mid \delta\mu \mid \frac{\pi}{2}. \tag{26}$$

In Table I we present a comparison between the actual computed error in differential phase shift, for an assumed relative 0.1 percent error in the eigenvalue, and the upper bound as determined by (25) and (26). The results indicate that the predicted estimates are quite reasonable. Note that the larger phase errors for smaller angles are due simply to the fact that the eigenvalue and hence the absolute error is greater.

The principal purpose of the previous exercise was to determine what error might be expected from using (6), which expresses the Legendre function in terms of a Bessel function. Results for the TM_{11} mode show that (6) is sufficiently accurate in predicting the μ-zeroes that the maximum differential phase error for horns up to 30° in half angle should be less than 1°. As might be expected, however the same approximation applied to the ν-zeroes of the derivative of the Legendre function is not as accurate. Nevertheless, as shown in Fig. 4, for a horn with half angle equal to 30° the maximum phase error is approximately 6°, which would ordinarily be acceptable.

4.2 *Evaluation of the Quasi-Cylindrical Approximation*

The difficulty of making precise calculations of phase progression in conical horns has led to the use of an approximate formula derived

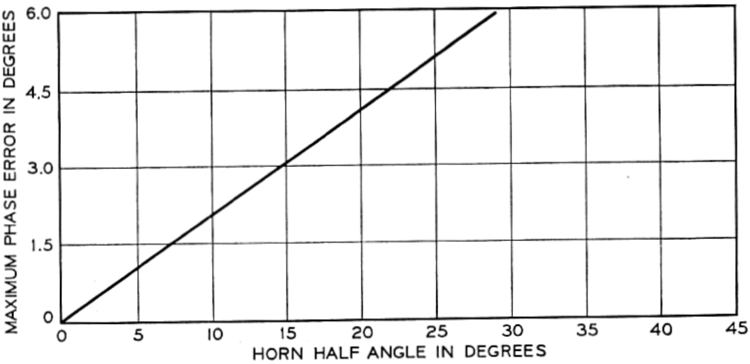

Fig. 4—Maximum error in phase shift using Bessel approximation to the Legendre function.

by assuming that the horn behaves as a cylindrical waveguide with gradually increasing cross sectional radius. The phase shift is then determined simply by integrating the local waveguide propagation constant, with the result that

$$\delta\beta \approx \frac{1}{\tan\theta_0}\left[\sqrt{(kr\sin\theta_0)^2 - \zeta^2} - \zeta\cos^{-1}\frac{\zeta}{kr\sin\theta_0}\right]_{r_1}^{r_2}, \tag{26}$$

where ζ is the characteristic value for the particular mode (e.g., for the TE_{11}° mode $\zeta \approx 1.841$ and for the TM_{11}° mode $\zeta \approx 3.832$). This formula is, in fact, asymptotic to the true phase shift in certain limits. For example, let $(kr \sin \theta_0)$ be fixed and let $\theta_0 \rightarrow 0$. Then it can be shown that (26) becomes essentially equivalent to the simple Debye approximation. (See Ref. 7, p. 366.) However, this formula is known to be invalid in the range where order and argument are comparable. Nevertheless, it is useful to investigate the properties of (26) in order to determine what errors accompany its use.

Fig. 5 shows the resulting error in differential phase shift when the quasi-cylindrical approximation is applied to the $\mathrm{TE}_{11}^{<}$ mode, with $kr_1 = \zeta/\sin\theta_0$ (corresponding to the cut-off diameter) and $\theta_0 = 5°$. The error is seen to grow very rapidly at first, showing that the quasi-uniform approximation predicts too slow an increase in propagation constant with increasing cross section. Eventually, the error curve approaches a linear variation. This asymptote can actually be predicted fairly well by using the large argument behavior of the Hankel function, combined with the fact that for large ν (see Ref. 7, p. 368),

$$\arg H_\nu^{(1)}(\nu) \approx -\pi/3. \tag{27}$$

By letting $kr \sin \theta_0 \to \infty$ in (26) and using (6), which relates ζ to ν, we find that in the limit of large kr

$$\delta\alpha^{\mathrm{TE}} - \delta\beta^{\mathrm{TE}} \approx \frac{\pi}{12} - \zeta_{\mathrm{TE}}\left(\frac{1}{\theta_0} - \frac{1}{\tan \theta_0}\right)\frac{\pi}{2} + \left(1 - \frac{\sin \theta_0}{\tan \theta_0}\right)kr \tag{28}$$

which, for small θ_0 reduces to

$$\delta\alpha^{\mathrm{TE}} - \delta\beta^{\mathrm{TE}} \approx \frac{\pi}{12} - \frac{\zeta_{\mathrm{TE}}\pi}{6}\,\theta_0 + \frac{\theta_0^2}{2}\,kr. \tag{29}$$

This result, shown in Fig. 5 as the dotted line, is seen to predict very accurately the asymptotic behavior of the error.

Fig. 6 shows the corresponding error for the $\mathrm{TM}_{11}^{<}$ mode. In contradistinction to the previous example, the quasi-cylindrical approximation at first predicts too high a phase progression rate, but eventually also conforms to the linear error predicted by the last term of (28). The formula analogous to (29) is given by

$$\delta\alpha^{\mathrm{TM}} - \delta\beta^{\mathrm{TM}} = -\frac{\pi}{12} - \frac{\zeta_{\mathrm{TM}}\pi}{6}\,\theta_0 + \frac{\theta_0^2}{2}\,kr. \tag{30}$$

This result, shown in Fig. 6 as the dotted line, is also seen to correctly describe the asymptote.

A salient feature of these results is that the linear portion of these curves is quite independent of the type and order of the mode being

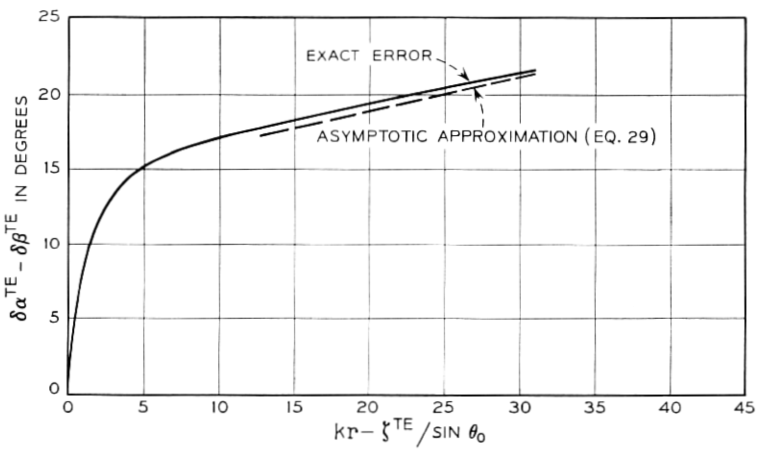

Fig. 5—Error in phase shift due to quasi-cylindrical approximation for the $\mathrm{TE}_{11}^{<}$ mode. $\zeta_{\mathrm{TE}} \approx 1.84118$.

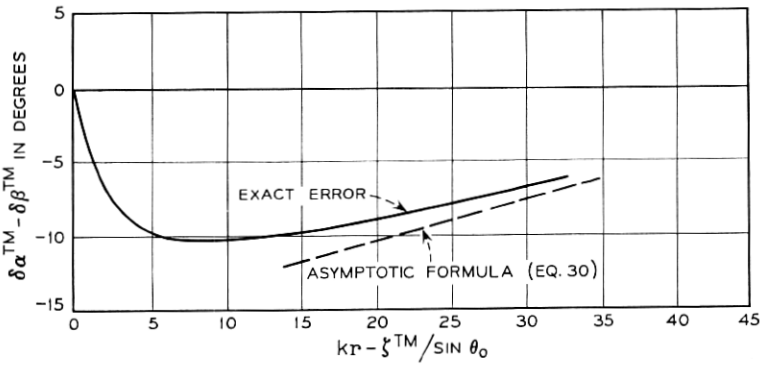

Fig. 6 — Error in phase shift due to quasi-cylindrical approximation for $TM_{11}^{<}$ mode. $\zeta_{TM} \approx 3.83171$.

considered. This immediately implies that the error in *differential phase shift between modes* resulting from the quasi-cylindrical approximation is *always bounded* at a value easily predicted by (29) and (30). This latter result is considered to be one of the more significant conclusions of this study.

V. SUMMARY AND CONCLUSIONS

In this paper, we have considered the phase progression properties of conical waveguide modes. The principal difficulties have been in computing Bessel functions over their so-called transition region. It is suggested that, in view of the typically large orders involved, the asymptotic expansions due to Olver are the most applicable. An examination of the quasi-cylindrical approximation has shown that this latter formula, although not necessarily accurate for evaluating phase progression of a particular mode, can be used to determine differential phase shift between modes with an error which is bounded over the conical region.

VI. ACNKOWLEDGMENTS

The author would like to thank Dr. J. Alan Cochran for helpful comments on the properties of Olver's expansions, and Miss Eileen Foley and Mr. C. M. Nagel, who were responsible for the computational aspects of this study.

APPENDIX

Roots of P_μ^1 *(cos* θ*) = 0 and* $d/d\theta\ P_\nu^1$ *(cos* θ*) = 0 for a Given Value of* θ

In order to assist in design of dual-mode conical horns, we have prepared an accurate table of the roots of the associated Legendre Function and its derivative for a given value of the horn half angle θ. These values of ν and μ may then be used to evaluate the vector wave functions which characterize the propagation of the $TE_{11}^{<}$ and $TM_{11}^{<}$ modes in the horn section.

The computational program consisted of using the hypergeometric series representation for the Legendre function, and determining the zeroes by a standard root finding method. The program was terminated when the value of the function was less than 10^{-6} in amplitude.

TABLE II—ROOTS OF P_μ^1 (cos θ) = 0

θ	μ	θ	μ
3.00°	72.6819	17.00°	12.4239
3.50	62.3379	17.50	12.0552
4.00	54.3874	18.00	11.7070
4.50	48.2893	18.50	11.3777
5.00	43.4109	19.00	11.0657
5.50	39.4196	19.50	10.7697
6.00	36.0935	20.00	10.4885
6.50	33.2792	20.50	10.2211
7.00	30.8669	21.00	9.9664
7.50	28.7764	21.50	9.7235
8.00	26.9471	22.00	9.4918
8.50	25.3332	22.50	9.2703
9.00	23.8985	23.00	9.0585
9.50	22.6110	23.50	8.8557
10.00	21.4597	24.00	8.6613
10.50	20.4146	24.50	8.4749
11.00	19.4645	25.00	8.2960
11.50	18.5970	25.50	8.1241
12.00	17.8019	26.00	7.9589
12.50	17.0704	26.50	7.7998
13.00	16.3952	27.00	7.6467
13.50	15.7700	27.50	7.4992
14.00	15.1894	28.00	7.3570
14.50	14.6490	28.50	7.2197
15.00	14.1446	29.00	7.0871
15.50	13.6728	29.50	6.9591
16.00	13.2304	30.00	6.8354
16.50	12.8149		

TABLE III—ROOTS OF $d/d\theta\, P_\nu^1(\cos\theta) = 0$

θ	ν	θ	ν
3.00	34.6743	17.00	5.7637
3.50	29.6526	17.50	5.5881
4.00	25.8867	18.00	5.4224
4.50	22.9581	18.50	5.2657
5.00	20.6155	19.00	5.1174
5.50	18.6992	19.50	4.9768
6.00	17.1026	20.00	4.8432
6.50	15.7518	20.50	4.7163
7.00	14.5943	21.00	4.5955
7.50	13.5913	21.50	4.4804
8.00	12.7139	22.00	4.3706
8.50	11.9400	22.50	4.2658
9.00	11.2522	23.00	4.1656
9.50	10.6370	23.50	4.0697
10.00	10.0835	24.00	3.9779
10.50	9.5828	24.50	3.8900
11.00	9.1279	25.00	3.8056
11.50	8.7126	25.50	3.7246
12.00	8.3321	26.00	3.6467
12.50	7.9822	26.50	3.5719
13.00	7.6593	27.00	3.4999
13.50	7.3605	27.50	3.4306
14.00	7.0831	28.00	3.3638
14.50	6.8250	28.50	3.2995
15.00	6.5842	29.00	3.2374
15.50	6.3591	29.50	3.1775
16.00	6.1481	30.00	3.1196
16.50	5.9500		

REFERENCES

1. Morgan, S. P., Mode Conversion Losses in Transmission of Circular Electric Waves through Slightly Non-Cylindrical Guides, J. Appl. Phys., *21*, 1950, p. 329.
2. Cook, J. S. and Lowell, R., The Autotrack System, B.S.T.J., *42*, July 1963, pp. 1283–1307.
3. Potter, P. D., A New Horn Antenna with Suppressed Sidelobes and Equal Beamwidths, Microwave J. *6*, 1963, p. 71.
4. Olver, F. W. J., The Asymptotic Expansion of Bessel Functions of Large Order, Phil. Trans., A, *247*, 1954, p. 328. Also see Olver's Tables of Bessel Functions of Moderate or Large Orders, National Physical Laboratory Mathematical Tables, *6*, 1962, Her Majesty's Stationery Office, London.
5. Borgnis, F. and Papas, C. H., Electromagnetic Waveguides and Resonators, *Encyclopedia of Physics,* Volume XVI, Springer, Berlin, 1958, p. 356.
6. Erdelyi, A., et al., *Higher Transcendental Functions, Volume I,* McGraw-Hill Book Company, Inc., New York, 1953, p. 148.
7. Abramowitz, M. and Stegun, I. A., *Handbook of Mathematical Functions,* U. S. Government Printing Office, Washington, D. C., 1964, p. 409.
8. Erdelyi, A., et al., *Higher Transcendental Functions, Volume II,* McGraw-Hill Book Company, Inc. New York, 1953, p. 89.
9. Miller J. C. P., *The Airy Integral,* Cambridge University Press, Cambridge (G. B.), 1946.