Contraction Maps and Equivalent Linearization*

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This study is primarily concerned with the question: If the method of equivalent linearization indicates the existence of a periodic solution, is there actually a periodic solution near the approximation of equivalent linearization? To answer this question, we use a modification of the contraction mapping fixed point theorem. We discuss applications to differential equations and difference-differential equations (with forcing functions). Also, we show that our use of contraction maps is not applicable (without modification) to autonomous systems because the mapping evaluated in the neighborhood of a periodic solution to an autonomous system is not a contraction in a space of periodic functions.

I. INTRODUCTION

The method of equivalent linearization is a most valuable technique to investigate nonlinear phenomena, particularly nonlinear oscillations. It has its roots in the method of Krylov and Bogoliubov and is related to (or equivalent to, depending on the specific definitions) the method of harmonic balance, Galerkin's method, and the describing function method used by control engineers. The purpose of the present study is to develop a new technique for investigating the method of equivalent linearization.

We shall be primarily concerned with the following question: If the method of equivalent linearization indicates the existence of a periodic solution x_0, is there actually a periodic solution near x_0? To answer this question we first introduce a convenient modification of the contraction mapping fixed point theorem which is actually more general than just applicable to the question posed above.† We apply

* Taken from a dissertation submitted to the Faculty of the Polytechnic Institute of Brooklyn in partial fulfillment of the requirements for the degree of Doctor of Philosophy (System Science), 1967.

† Appendix A contains some reading suggestions for engineers interested in this work but who are not familiar with the mathematics used.

our approach to systems described by nonautonomous differential equations. Then we show that there is no essential difficulty in also handling difference-differential equations.

We shall try to clearly indicate what our method can and cannot do. The discussion of autonomous systems is particularly important in this regard. The relation of the present study to previous work is discussed in Section VIII.

II. THE METHOD OF EQUIVALENT LINEARIZATION*

Consider the following vector differential equation

$$\dot{x}(t) = f(x(t), t) = A(t)x(t) + n(t, x(t)), \tag{1}$$

where

$$f(x, t) = f(x, t + T) \tag{2}$$

for all (x, t) of interest. This, of course, includes the case of $f(x, t)$ independent of t, i.e.,

$$f(x, t) = f(x). \tag{3}$$

We shall be concerned with the situation that permits an equivalent representation of (1):

$$x = LN(x), \tag{4}$$

where x now represents a vector function, L is a linear operator, and N is a nonlinear operator (these terms will be made more precise later). If it is assumed that $LN(x)$ has the following Fourier series,

$$LN(x)(t) \sim \frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos k \frac{2\pi}{T} t + b_k \sin k \frac{2\pi}{T} t \right). \tag{5}$$

Then we define $\tilde{L}N(x)$ as follows:

$$\tilde{L}N(x)(t) = a_1 \cos \frac{2\pi}{T} t + b_1 \sin \frac{2\pi}{T} t. \tag{6}$$

That is, $\tilde{L}$ extracts the fundamental component of the Fourier series.

The method of equivalent linearization seeks a solution of the equation

$$x = \tilde{L}N(x). \tag{7}$$

This study will be primarily concerned with the following problem:

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* See Minorsky,[1] p. 350, for a discussion of the relationship of the method of equivalent linearization to the method of Krylov and Bogoliubov.

Given an x_0 satisfying (7), is there an x^* satisfying (4) and if there is, how are x_0, and x^* related?

Note that (4) is a functional relation more general than (1) and our method will be correspondingly applicable to a more general problem.

The above discussion is now related to the method of describing functions* as commonly used by control engineers. They are concerned with the feedback loop shown in Fig. 1. The linear operator L

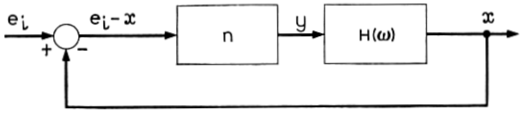

Fig. 1 — Feedback Loop.

is represented in this case by a transfer function $H(\omega)$ (see Kaplan[4] for a definition and discussion of transfer functions) and the nonlinear operator N is represented by a nonlinear function,

$$y = n(e_i - x),$$

where e_i is an input function. The engineer replaces the nonlinear function n by its describing function which is defined loosely as the complex ratio of the fundamental component of the output to a sinusoidal input. That is, if†

$$n(A \sin \omega t) = \sum_{k=1}^{\infty} a_k \sin k\omega t + b_k \cos k\omega t \left(\omega = \frac{2\pi}{T}\right)$$

then the describing function of n is

$$\frac{\sqrt{a_1^2 + b_1^2} \,\Big/ \tan^{-1} \dfrac{b_1}{a_1}}{A}.$$

Note that while the describing function may be dependent on both A and ω, it is still a relatively simple matter to replace n by its describ-

* For further discussion of the use of describing functions by engineers see e.g., Truxal[2] or Graham and McRuer.[3] They give further references and historical background. The describing function method is associated with the names of Tustin, Goldfarb, Oppelt, Kochenberger, Dutilh, and Nichols and Kreezer. Also see Minorsky,[1] Chap. 17 for a discussion of the work of Theodorchik and Blaquière. The work of E. C. Johnson is discussed in Ref. 2.

† The constant term is assumed zero.

ing function, then to consider it as a "linear" (or quasi-linear) operator and then use standard techniques for linear systems. Of course, such a procedure should be mathematically justified and, in fact, that is the purpose of this study.

Before we embark on our investigation, it is well to review the arguments used by engineers in their justification of the method. These arguments seem to be plausible and they are suggestive of what may be expected of a more rigorous investigation. If it is assumed that the combination of n and $H(\omega)$ operating on a sinusoidal function is primarily fundamental (i.e., the harmonics are "small" compared to the fundamental) then it would be expected that the describing function method might not be too inaccurate. The harmonics will be small if one or both of the following are satisfied:

(*i*) the nonlinearity n is "not too nonlinear"

(*ii*) the transfer function $H(\omega)$ is low-pass, i.e., it attenuates harmonics much more than the fundamental. (It is assumed that no subharmonics arise).

We shall use Duffing's equation,

$$\ddot{y} + ay + by^3 = f \cos \omega t,$$

as a running example to illustrate the methods discussed. We show here how this differential equation corresponds to a feedback control problem and then make no further explicit reference to feedback systems. The appropriate feedback system is shown in Fig. 2.

The next section contains an approach to a problem much more general than the problem of equivalent linearization posed in this section. The remainder of the study will be primarily devoted to adapting the more general approach to the specific problem of equivalent linearization.

It may be noted that we are not getting more abstract in the next section just for the sake of abstraction. It should be clear to the reader

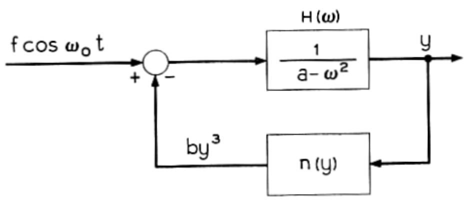

Fig. 2 — Feedback equivalent of Duffing's equation.

that the method of equivalent linearization leads to an integro-differential equation rather than an ordinary differential equation because Fourier coefficients are determined by integration (this is also pointed out in Bass[5], p. 898). We then cannot expect the theory of ordinary differential equations to answer our questions and we are led quite naturally to considering more general equations. In particular, the theory of operator equations in a Banach space is shown to provide the tools appropriate to the task. As an added bonus for the abstraction, we develop an approach which is applicable to problems unrelated to equivalent linearization.

III. THE USE OF THE CONTRACTION MAPPING FIXED POINT THEOREM WITH DERIVATIVES IN A BANACH SPACE

Let X be a complete metric space (with metric d) containing the the closed set Ω and let P map Ω into itself. P is a contraction mapping if

$$d(P(x), P(x')) \leqq \alpha d(x, x') \qquad (x, x' \varepsilon \Omega) \tag{8}$$

with $\alpha < 1$. The contraction mapping theorem† states that if P is a contraction mapping then there is a unique $x^* \varepsilon \Omega$ such that $x^* = P(x^*)$, i.e., x^* is a fixed point of the operation P. x^* is the limit of a sequence $\{x_n\}$ where

$$x_{n+1} = P(x_n) \tag{9}$$

and x_0 is any element of Ω. Furthermore,

$$d(x_n, x_0) \leqq \frac{d(x_1, x_0)}{1 - \alpha} = \frac{d(P(x_0), x_0)}{1 - \alpha} \qquad n = 1, 2, \cdots . \tag{10}$$

In order to use the contraction mapping fixed point theorem it has to be shown that some neighborhood of x_0 is mapped into itself and that in this neighborhood the operation is contracting. Our approach will be simultaneously to determine a set containing x_0 which is mapped into itself along with the contraction constant α for the operation on that set. This is possible because of relationship (10). The use of operator derivatives will be seen to be convenient. The method will result in a relation in α for which it is desired to find solutions with $\alpha \varepsilon [0, 1)$.

The following is proven in Kantorovich and Akilov[6] (p. 661).

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† See Kantorovich and Akilov,[6] p. 627.

If X is a Banach space and P maps a convex closed subset Ω of X into itself and if P has a derivative* at every point of Ω, then

$$\sup_{x \in \Omega} \| P'(x) \| = \alpha < 1 \tag{11}$$

implies that P is a contraction on Ω (and thus, there is a unique fixed point of P in Ω).

The object then is to find a neighborhood of x_0 mapped into itself and in which the norm of the derivative is less than one. The following simple theorem is a help in this direction.

Theorem: Let B be a Banach space. F maps B into itself and $x_0 \in B$. It is assumed that

(i) F has a derivative at all $x \in B$

(ii) There is a nondecreasing function g such that if $x \in B$, then

$$\| F'(x) \| \leqq g(\| x - x_0 \|)$$

(iii) There is an $\alpha \in [0, 1)$ such that

$$g\left(\frac{k}{1 - \alpha}\right) \leqq \alpha,$$

where

$$k \geqq \| F(x_0) - x_0 \|$$

Then there is a unique $x^* \in \Omega$ such that

$$x^* = F(x^*),$$

where

$$\Omega = \left\{ x: x \in B, \| x - x_0 \| \leqq \frac{k}{1 - \alpha} \right\}$$

Proof: We will show that $\| F'(x) \| \leqq \alpha$ for all $x \in \Omega$ and that F maps Ω into itself and thus, there is a unique fixed point in Ω. If $x \in \Omega$, we have from *(ii)*

$$\begin{aligned} \| F'(x) \| &\leqq g(\| x - x_0 \|) \\ &\leqq g\left(\frac{k}{1 - \alpha}\right) \\ &\leqq \alpha. \end{aligned}$$

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* See Kantorovich and Akilov,[6] chap. XVII, for a general discussion of differentiation in Banach spaces. For convenience, Appendix B of this study repeats the definitions.

F maps Ω into itself because if $x \in \Omega$, then

$$\begin{aligned} \| F(x) - x_0 \| &\leqq \| F(x) - F(x_0) \| + \| F(x_0) - x_0 \| \\ &\leqq \alpha \| x - x_0 \| + k \\ &\leqq \alpha \frac{k}{1-\alpha} + k \\ &= \frac{k}{1-\alpha}. \end{aligned}$$

We further discuss the use of the contraction mapping theorem in Ref. 7. This modification of the contraction map theorem is less general than some other modifications but is simpler to apply when applicable.

IV. THE CHOICE OF BANACH SPACE

In order to use the result of the previous section, an appropriate Banach space must be chosen. For most of our investigation it will be found convenient to use the space of continuous periodic functions. Another space worth considering is the space of periodic functions square-integrable over a period. Before discussing the desirable characteristics of this space, a restrictive factor will be mentioned. The nonlinear operator of interest $y = N(x)$ is often defined by the ordinary function (satisfying the Carathéodory condition)

$$y(t) = n(t, x(t)). \tag{12}$$

A necessary condition that this operation map $L_2(0, T)$ into $L_2(0, T)$ is (see Krasnosel'skii,[8] p. 27) that for some $b > 0$ and some $a(t) \in L_2(0, T)$

$$| n(t, u) | \leqq a(t) + b \, | u | \qquad t \in [0, T]. \tag{13}$$

It is thus seen that the allowable nonlinearities are quite restricted. This is, in fact, the reason the present investigation will be carried out in a space of continuous functions where the requirement that a function map a continuous function into a continuous function is much more convenient. It should be noted, however, that in some cases one may focus attention on some subset of the Banach space and less restrictive requirements on the nonlinearities might be imposed. Also, for many control engineering problems the nonlinearities are Lipschitzian and the problems can be attacked in L_2.

The attractive feature of L_2 is that Fourier series results can be fully utilized (in particular, Parseval's relation). More generally, L_2

is a separable* Hilbert space with many useful properties and the trigonometric functions are a complete orthogonal system in L_2. The norms can often be conveniently evaluated in terms of quantities associated with the "transfer function" or "frequency response". For example, L may be defined by the set of complex numbers $\{\ldots, L_2, L_{-1}, L_0, L_1, L_2, \ldots\}$ (i.e., the transfer function evaluated at the fundamental and harmonic frequencies).† A simple sufficient condition for L to map L_2 into itself is that

$$\sup_n |L_n| < \infty .$$

The evaluation of $\| LN(x_0) - x_0 \|$ may be done as follows:

$$\begin{aligned} \| LN(x_0) - x_0 \| &= \| LN(x_0) - \tilde{L}N(x_0) \| \\ &\leqq \| L - \tilde{L} \| \cdot \| N(x_0) \| \\ \| L \| &= \sup_n | L_n |. \end{aligned}$$

The last relationship is proven in Appendix B of Sandberg.[10]

Despite the above mentioned attractive features of L_2, we chose to work in the space of continuous functions primarily because of the first-mentioned restriction placed on the nonlinearities in L_2. Also, the sup norm (uniform norm) in the space of continuous functions seems more appropriate in error analysis (the error between an approximation and an exact solution) than does the L_2 norm. The sup norm provides a bound on the magnitude of the error while the L_2 norm gives the integral of the square of the error.

Section V will give the details of working in the space of continuous periodic functions. First (in Section 5.1) an integral equation equivalent to the differential equation of interest will be derived. Then in Section 5.2, the derivatives will be determined and finally in Section 5.3, the quantity $\| F(x_0) - x_0 \|$ will be evaluated. Application of the results will then be seen to be rather straightforward.

V. APPLICATION TO DIFFERENTIAL EQUATIONS

5.1 *The Equivalent Integral Equation*

Halanay[11] shows how to convert the quasi-linear differential equation

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* It is, of course, assumed that the measure is Lebesgue. Then L_2 is separable; see Kolmogorov and Fomin,[9] Vol. II, p. 88.

† No confusion should arise because of the double use here of the symbol L_2.

$$\frac{dx(t)}{dt} = A(t)x(t) + n(t, x(t)) \tag{14}$$

into an integral equation which is convenient for examination of periodic solutions. First, consider

$$\frac{dx(t)}{dt} = A(t)x(t) + f(t), \tag{15}$$

where $A(t)$ and $f(t)$ are both continuous and periodic of period T. The following theorem is proven in Halanay,[11] p. 223.

Theorem: A necessary and sufficient condition in order that, for any periodic function $f(t)$ of period T, system (15) admits periodic solutions of period T is that the corresponding homogeneous system

$$\frac{dy(t)}{dt} = A(t)y(t) \tag{16}$$

does not admit a non-trivial periodic solution of period T.

It is to be noted that if $Y(t)$ is the principal fundamental solution matrix for (16) then the existence of the inverse of $[I - Y(T)]$* is equivalent to the non-existence of a non-trivial periodic solution (of period T) to (16). Then the following proposition is proved in Halanay,[11] p. 225.

Proposition: If $[I - Y(T)]^{-1}$ exists, the unique periodic solution of the system (15) can be put in the form

$$x(t) = \int_0^T G(t, s)f(s)\, ds, \tag{17}$$

where

$$G(t, s) = \begin{cases} Y(t)[I - Y(T)]^{-1}Y^{-1}(s), & 0 \leqq s \leqq t \leqq T \\ Y(t + T)[I - Y(T)]^{-1}Y^{-1}(s), & 0 \leqq t < s \leqq T. \end{cases} \tag{18}$$

Since this reformulation into an integral equation is quite important, a sketch of the proof given in Halanay[11] will be given here. Solution of (15) is

$$x(t) = Y(t)x(0) + \int_0^t Y(t)Y^{-1}(s)f(s)\, ds. \tag{19}$$

* I is the identity matrix.

For periodicity,

$$x(T) = x(0) = Y(T)x(0) + \int_0^T Y(T)Y^{-1}(s)f(s)\,ds \tag{20}$$

or

$$[I - Y(T)]x(0) = \int_0^T Y(T)Y^{-1}(s)f(s)\,ds. \tag{21}$$

Since $[I - Y(T)]$ is assumed invertible, we can solve for $x(0)$ and get

$$x(t) = Y(t)[I - Y(T)]^{-1} \int_0^T Y(T)Y^{-1}(s)f(s)\,ds + \int_0^t Y(t)Y^{-1}(s)f(s)\,ds. \tag{22}$$

The form of $G(t, s)$ given in the statement of the proposition results from algebraic manipulation of (22).

Now suppose that

$$\frac{dx(t)}{dt} = A(t)x(t) + n(t, x(t)) \tag{23}$$

with $A(t)$ and $n(t, u)$ both being periodic of period T and $[I - Y(T)]$ invertible. It is assumed that $n(t, x(t))$ is continuous if $x(t)$ is continuous. Then, from the previous discussion, if we can find a continuous periodic x of period T satisfying

$$x(t) = \int_0^T G(t, s)n(s, x(s))\,ds, \tag{24*}$$

we have a periodic solution of (23). (It is easily shown that such an x satisfies (23); see Halanay,[11] p. 237).

The problem is thus reduced to finding a continuous solution of a nonlinear integral equation. We need only consider the interval [0, T] because $G(t, s)$ was constructed so that $x(0) = x(T)$.

5.2 *Computation of Derivatives*

It is shown here how to evaluate the Fréchet derivative† of the

* The nonlinear integral operation represented by the right hand of (24) is of the form sometimes referred to as a Hammerstein operator which is a special case of Uryson's operator defined by $\int_0^T K(t, s, x(s))\,ds$ (see Krasnosel'skii,[8] pp. 32, 46).

† The Fréchet derivative is actually more than what is required. The Gateaux derivative (which does not require uniform convergence) would suffice for much of what follows. However, since the convergence is indeed uniform in most cases of interest and since the uniformity is easy to demonstrate, we shall derive the Fréchet derivative. Furthermore, Fréchet derivatives are needed in Section VI.

mapping $y = F(x)$ defined by

$$y(t) = \int_0^T G(t, s)n(s, x(s)) \, ds \qquad t \in [0, T]. \tag{25}$$

This operation is assumed to map into itself the Banach space of real-valued n-vectors continuous on $[0, T]$ with norm

$$|| x || = \max_{i=1,\cdots,n} \max_{t \in [0,T]} | x_i(t) |. \tag{26}$$

To determine the derivative of the operation $y = F(x)$ it is convenient to express it as $y = LN(x)$, where $N(x)$ is defined by the nonlinear function $n(t, x(t))$ and L is the linear integral operator. Then $F'(x_0) = LN'(x_0)$ (see Kantorovich and Akilov,[6] p. 659). Thus, consider the mapping $y = N(x)$ defined by

$$\begin{aligned} y(t) &= n(t, x(t)) \\ &= \begin{bmatrix} n_1(t, x(t)) \\ \vdots \\ n_n(t, x(t)) \end{bmatrix}. \end{aligned} \tag{27}$$

It is assumed that $n(t, x(t))$ is continuous whenever $x(t)$ is continuous on $[0, T]$. For simplicity, the derivative will be determined for the case of

$$\begin{aligned} n_i(t, x(t)) &= 0 \qquad i = 1, 2, \cdots, n-1 \\ n_n(t, x(t)) &= p(t)h(x_1(t)) + r(t), \end{aligned} \tag{28}$$

where $p(t)$ and $r(t)$ are continuous functions of t with period T and $h(u)$ is a twice continuously differentiable function of u. This special case which covers our examples may arise, for example, when the matrix differential equation is actually derived from a scalar differential equation. The more general case offers no other difficulties than much more complicated notation (e.g., one must deal with matrices of partial derivatives).

The derivative operation $z = N'(x_0)x$ is defined by

$$z(t) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ p(t)h'(x_{01}(t))x_1(t) \end{bmatrix}, \tag{29}$$

where

$$x_0(t) = (x_{01}(t), \cdots, x_{0n}(t))^T, \qquad x(t) = (x_1(t), \cdots, x_n(t))^T,$$

(superscript T denotes transpose) and

$$h'(x_{01}(t)) = \frac{dh(u)}{du}\bigg|_{u=x_{01}(t)}. \tag{30}$$

To prove this, it must be shown that

$$\lim_{\mu \to 0} \frac{N(x_0 + \mu x) - N(x_0)}{\mu} = P'(x_0)x, \tag{31}$$

that is,

$$\lim_{\mu \to 0} \max_{t \in [0,T]} \left| \frac{p(t)h(x_{01}(t) + \mu x_1(t)) - p(t)h(x_{01}(t))}{\mu} - p(t)h'(x_{01}(t))x_1(t) \right| = 0, \tag{32}$$

with that convergence being uniform with respect to all x with $|| x || = 1$. Since

$$\left| \frac{p(t)h(x_{01}(t) + \mu x_1(t)) - p(t)h(x_{01}(t))}{\mu} - p(t)h'(x_{01}(t))x_1(t) \right| = | p(t) |$$

$$\cdot \left| \frac{h(x_{01}(t)) + \mu x_1(t)h'(x_{01}(t)) + \frac{(\mu x_1(t))^2}{2} h''(x_{01}(t) + \theta(t)\mu x_1(t)) - h(x_{01}(t))}{\mu} - h'(x_{01}(t))x_1(t) \right| \qquad 0 < \theta(t) < 1$$

$$= | p(t) | \cdot \left| \frac{\mu}{2} \right| \cdot | x_1(t) |^2 \, | h''(x_{01}(t) + \theta(t)\mu x_1(t)) |$$

$$\leqq \left| \frac{\mu}{2} \right| \max_{t \in [0,T]} | p(t) | \max_{z \in Z} | h''(z) | \qquad (\text{for } | \mu | \leqq 1 \text{ and } || x || = 1), \tag{33}$$

where

$$Z = \{z : z = w + v; w = x_{01}(t), t \in [0, T]; | v | \leqq 1\} \tag{34}$$

the uniform convergence relationship is seen to be satisfied.

To summarize the result, the first derivative operation $y = F'(x_0)x$

is defined by

$$y(t) = \int_0^T G(t, s) \begin{bmatrix} 0 \\ \vdots \\ p(s)h'(x_{01}(s))x_1(s) \end{bmatrix} ds. \tag{35}$$

Loosely speaking, the derivatives of the integral operators are obtained by differentiating under the integral.

5.3 *Evaluation of* $\| LN(x_0) - x_0 \|$

A bound on $\| LN(x_0) - x_0 \|$ is given here. The relationship

$$\| LN(x_0) - x_0 \| \leqq \| LN(x_0) \| + \| x_0 \| \tag{36}$$

is too gross an estimate. The evaluation is simplified if the following relation is used:

$$LN(x_0) - x_0 = (L - \tilde{L})N(x_0). \tag{37}^*$$

Recall that the operation $\tilde{L}$ suppresses all frequency terms except the fundamental.

Consider the same system as in Section 5.2 and assume that

$$p(t)h(x_1(t)) + r(t) = \sum_{k=1}^{\infty} a_K \cos k \frac{2\pi}{T} t + b_K \sin k \frac{2\pi}{T} t. \tag{38}$$

and also, for simplicity, that A is a constant matrix. Then,

$$\begin{aligned} &\| LN(x_0) - x_0 \| \\ &= \max_{i=1,\cdots,n} \max_{t\epsilon[0,T]} \left| \int_0^T G_{in}(t, s) \sum_{k=2}^{\infty} \left(a_K \cos k \frac{2\pi}{T} s + b_K \sin k \frac{2\pi}{T} s \right) ds \right| \\ &\leqq \sum_{k=2}^{\infty} (| a_K | + | b_K |) \max_{i=1,\cdots,n} \max_{t\epsilon[0,T]} \int_0^T | G_{in}(t, s) | \, ds. \end{aligned} \tag{39}$$

5.4 *Example*

Consider Duffing's equation†

$$\ddot{y} + ay + by^3 = f \cos \omega t \ (a > 0). \tag{40}$$

Equivalent linearization indicates that

$$y = A \cos \omega t \tag{41}$$

* From this expression, it is seen that $\| LN(x_0) - x_0 \|$ may be regarded as a quantitative measure of characteristics (*i*) and (*ii*) mentioned in section II.

† Duffing's equation is discussed in great detail in Stoker.[12] Also see Graham and McRuer[3] for a treatment of Duffing's equation as a feedback control problem.

with

$$\tfrac{3}{4}bA^3 + (a - \omega^2)A - f = 0 \tag{42}$$

is an approximate solution of the equation.

Letting

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{43}$$

$$x_1 = y \tag{44}$$

$$x_2 = \dot{x}_1 \tag{45}$$

the corresponding vector differential equation is

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ f \cos \omega t - bx_1^3 \end{bmatrix}. \tag{46}$$

The fundamental solution matrix for

$$\dot{y} = \begin{bmatrix} 0 & 1 \\ -a & 0 \end{bmatrix} y \tag{47}$$

is

$$Y(t) = \begin{bmatrix} \cos \sqrt{a}\, t & \dfrac{\sin \sqrt{a}\, t}{\sqrt{a}} \\ -\sqrt{a} \sin \sqrt{a}\, t & \cos \sqrt{a}\, t \end{bmatrix}. \tag{48}$$

$G(t, s)$ is given by

$$G(t, s) = \begin{cases} \dfrac{1}{2 \sin \left(\dfrac{\sqrt{a}\, T}{2}\right)} \begin{bmatrix} \sin \sqrt{a} \left(\dfrac{T}{2} - t + s\right) & \dfrac{1}{\sqrt{a}} \cos \sqrt{a} \left(\dfrac{T}{2} - t + s\right) \\ -\sqrt{a} \cos \sqrt{a} \left(\dfrac{T}{2} - t + s\right) & \sin \sqrt{a} \left(\dfrac{T}{2} - t + s\right) \end{bmatrix}, & 0 \leqq s \leqq t \leqq T \\[2ex] \dfrac{1}{2 \sin \left(\dfrac{\sqrt{a}\, T}{2}\right)} \begin{bmatrix} -\sin \sqrt{a} \left(\dfrac{T}{2} + t - s\right) & \dfrac{1}{\sqrt{a}} \cos \sqrt{a} \left(\dfrac{T}{2} + t - s\right) \\ -\sqrt{a} \cos \sqrt{a} \left(\dfrac{T}{2} + t - s\right) & -\sin \sqrt{a} \left(\dfrac{T}{2} + t - s\right) \end{bmatrix}, & 0 \leqq t < s \leqq T. \end{cases} \tag{49}$$

The integral operation of interest, $y = F(x)$, is defined by

$$y(t) = \int_0^T G(t, s) \begin{bmatrix} 0 \\ f \cos \omega s - b x_1^3(s) \end{bmatrix} ds. \tag{50}$$

The approximate solution (obtained from harmonic balance) is x_0, i.e.,

$$x_0(t) = \begin{bmatrix} x_{01}(t) \\ x_{02}(t) \end{bmatrix} = \begin{bmatrix} A \cos \omega t \\ -\omega A \sin \omega t \end{bmatrix} \tag{51}$$

with ω and A being related by (42).

From Section 5.3 we have that

$$\begin{aligned} \| LN(x_0) - x_0 \| &\leqq T \max_{i=1,2} \max_{t,s\epsilon[0,T]} | G_{i2}(t, s) | \frac{| bA^3 |}{4} \\ &\leqq TC \frac{| bA^3 |}{4}, \end{aligned} \tag{52}$$

where

$$C = \frac{1}{2 \left| \sin \left(\frac{\sqrt{a}\, T}{2} \right) \right|} \max \{1, 1/\sqrt{a}\}. \tag{53}$$

The derivative operation $z = F'(x_0)x$ is given by (see Section 5.2)

$$\begin{aligned} z(t) &= \int_0^T G(t, s) \begin{bmatrix} 0 \\ -3bx_{01}^2(s) x_1(s) \end{bmatrix} ds \\ &= \int_0^T \begin{bmatrix} -G_{12}(t, s) 3bx_{01}^2(s) & 0 \\ -G_{22}(t, s) 3bx_{01}^2(s) & 0 \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} ds. \end{aligned} \tag{54}$$

The norm of the derivative operation at an arbitrary point x (not necessarily at x_0 as above) is evaluated as follows:

$$\begin{aligned} \| F'(x) \| &\leqq \max_{i=1,2} \max_{t\epsilon[0,T]} \int_0^T | G_{i2}(t, s) 3bx_1^2(s) | \, ds \\ &\leqq 3 | b | CT(\max_{t\epsilon[0,T]} | x_1(t) |)^2 \\ &\leqq 3 | b | CT(\max_{t\epsilon[0,T]} | x_{01}(t) | + \max_{t\epsilon[0,T]} | x_1(t) - x_{01}(t) |)^2 \\ &\leqq 3 | b | CT(| A | + \| x - x_0 \|)^2. \end{aligned} \tag{55}$$

The above relation defines the nondecreasing function g such that

$$\| F'(x) \| \leqq g(\| x - x_0 \|). \tag{56}$$

to use the theorem of Section II, let

$$\Omega = \left\{ x: \| x - x_0 \| \leqq \frac{k}{1 - \alpha} \right\} \tag{57}$$

where

$$k = CT \frac{|bA^3|}{4} \tag{58}$$

(see (52)).

If an $\alpha \varepsilon [0, 1)$ can be found satisfying

$$3 \mid b \mid CT \left[\mid A \mid + \frac{CT \dfrac{\mid bA^3 \mid}{4}}{1 - \alpha} \right]^2 \leqq \alpha \tag{59}$$

then there is an $x^* \varepsilon \Omega$ such that $x^* = F(x^*)$, i.e., Duffing's equation has a periodic solution in the neighborhood of the approximation obtained by harmonic balance.

Rather than just solve the cubic relation (59) for particular numerical values of a, b, f, and ω (which is, of course, the thing to do if one is given a particular equation of interest) we shall obtain some general results. Consider a, f, and ω fixed and $f \neq 0$. Since† for any $\alpha \varepsilon [0, 1)$

$$\lim_{b \to 0} 3 \mid b \mid CT \left[\mid A \mid + \frac{CT \dfrac{\mid bA^3 \mid}{4}}{1 - \alpha} \right]^2 = 0 \tag{60}$$

it is seen that for b sufficiently small, there will be an $\alpha \varepsilon [0, 1)$ satisfying (59) (and thus a periodic solution neighboring the approximation). Note that while the result has been stated as an asymptotic result, it is possible to determine quantitatively what is meant by "sufficiently small". This is in contrast to most asymptotic analyses based on "small nonlinearities".

5.5 *Special Cases*

In many cases, it is not necessary to convert the differential equation into a vector integral equation. For example, let the system be

† See Appendix C for details.

described by the equation

$$a_0x^{(m)}(t) + a_1x^{(m-1)}(t) + \cdots + a_mx(t) = n(t, x(t)), \tag{61}$$

where the $a_0, \ldots, a_m$ are constants, $n(t, x)$ is continuous in t when x is continuous in t, and

$$n(t + T, u) = n(t, u). \tag{62}$$

A periodic solution to the differential equation will satisfy the following integral equation

$$x(t) = \int_0^T W_T(t - u)n(u, x(u))\, du, \tag{63}$$

where $W_T(t-u)$ is the appropriate convolution kernel (see Kaplan,[4] chap. 4 for details). All the manipulations of previous sections will be somewhat simplified as a result of not having to deal with matrices. In particular, the example of Section 5.4 could be repeated with some simplification. We omit the details because they exactly parallel the previous case. We felt it would be more useful to work out the details of the more complicated case. As the order of the differential equation increases it clearly becomes more advantageous to avoid the use of matrices.

It may be noted that there is a finite Fourier transform $Y(in\omega)$ (again see Kaplan,[4] chap. 4) associated with the differential equation (61)

$$Y(in\omega) = \frac{1}{a_0(in\omega)^m + \cdots + a_m}. \tag{64}$$

This $Y(p)$, considered as a function of a complex variable p, evidently can only have poles and cannot have finite zeros. In many electrical engineering applications (e.g., control systems, networks) the relevant transfer function has both poles and zeroes. In these cases, we would start with the transfer function, rather than a differential equation of the form (61), find the corresponding convolution integral and then apply our method. For other applications it must, of course, be verified that the appropriate conditions are satisfied.

5.6 *Autonomous Systems*†

The describing function method has been used by control engineers primarily for the prediction of self-oscillations (i.e., with no forcing

† Since the actual oscillation of an autonomous system may have a different period than that of the approximation, it is usually convenient to normalize the time variable and have the period be a parameter.

functions). It would seem at first glance that our approach should be appropriate for analysis of this problem. Suppose the describing function method indicates that there exists a non-trivial periodic solution, x_0, to the operator equation

$$x_0 = \tilde{L}N(x_0). \tag{65}$$

Usually, $N(0) = 0$, so that it is of interest to investigate whether there is any non-trivial solution to the exact equation near x_0. If our method is successful, then we can guarantee that Ω does not contain the trivial solution $(x = 0)$ if

$$\frac{\| F(x_0) - x_0 \|}{1 - \alpha} < \| x_0 \| \tag{66}$$

since the fixed point x* satisfies

$$\| x^* - x_0 \| \leqq \frac{\| F(x_0) - x_0 \|}{1 - \alpha}. \tag{67}$$

Unfortunately, an attempt to use the approach in the autonomous case will be unsuccessful. The reason for the failure of our approach is due to the nature of the fixed point: the mapping is not a contraction in a neighborhood of the fixed point.† The discussion below will clarify this point.

Assume that the differential equation of interest is

$$\dot{x}(t) = Ax(t) + n(x(t)) \tag{68}$$

with A a constant real valued matrix and $n(x)$ is a real-valued function having continuous partial derivatives with respect to all of the elements of the vector x. Suppose that there is a continuous periodic x^* of period T satisfying (68). Then x^* satisfies the equivalent integral equation:

$$x^*(t) = \int_0^T G(t, s)n(x^*(s))\, ds. \tag{69}$$

The equation of first variation corresponding to (68) is

$$\dot{y}(t) = Ay(t) + \frac{\partial n}{\partial x^*} y(t), \tag{70}$$

where $\partial n/\partial x^*$ is a matrix with entries $\partial n_i/\partial x_j$ evaluated along the

† Actually, this should not be surprising since if $x(t)$ is a periodic solution, then so is $x(t + \epsilon)$.

trajectory defined by $x^*(t)$. As is easily shown (by differentiation; see, e.g., Hochstadt,[13] p. 251) the time derivative of $x^*(t)$ satisfies the equation of first variation and thus the equivalent integral equation

$$\dot{x}^*(t) = \int_0^T G(t, s) \frac{\partial n}{\partial x^*} \dot{x}^*(s)\, ds. \tag{71}$$

Now consider the derivative operation $y = LN'(x^*)x$ associated with the integral operation of (69). The derivative operation is defined by

$$y(t) = \int_0^T G(t, s) \frac{\partial n}{\partial x^*} x(s)\, ds. \tag{72}$$

We have shown above that $\dot{x}^*$ satisfies this last equation or

$$\dot{x}^* = LN'(x^*)\dot{x}^*. \tag{73}$$

Then

$$\| \dot{x}^* \| \leqq \| LN'(x^*) \| \cdot \| \dot{x}^* \|. \tag{74}$$

Since $\| \dot{x}^* \| > 0$, we must have that

$$\| LN'(x^*) \| \geqq 1 \tag{75}$$

or LN cannot be a contraction in a neighborhood of x^*. We leave open, however, the possibility that the contraction mapping theorem might be applicable in a subspace.

The above reasoning also shows that there would be difficulty associated with using Newton's method for the problem. To seek a zero of $P(x)$, Newton's method uses the following iteration:

$$x_{n+1} = x_n - [P'(x_n)]^{-1}(P(x_n)) \qquad n = 0, 1, \cdots . \tag{76}$$

Letting

$$P(x) = x - F(x) \qquad (F(x) = LN(x)), \tag{77}$$

we are led to investigating the inveritibility of $I - F'(x)$ where I is the identity operator. Consider the operation

$$y = P'(x^*)x = (I - F'(x^*))x. \tag{78}$$

If $x = 0$, then $y = 0$. But because $F'(x^*)$ is associated with the equation of first variation there is also a nonzero x (the time derivative of $x^*(t)$) which results in $y = 0$. There is thus not a unique x satisfying $y = 0$ and $P'(x^*)$ is not invertible (see Kantorovich and Akilov,[6] p. 168).

5.7 *Properties of the Fixed Point—Dependence on Parameters; Stability*

The method of this investigation is to obtain a relation for α in terms of parameters of the differential equations. If certain conditions are satisfied and $0 \leqq \alpha < 1$, then α represents a contraction constant. If α is a contraction constant and, also, α depends continuously on a parameter then for sufficiently small changes of that parameter α will still be a contraction constant and a periodic solution is still guaranteed. Once again note that "sufficiently small" can be quantitatively determined if one wishes to do that. As an illustration, in the Duffing equation example, α depends continuously on b.

If our method indicates the existence of a periodic solution of period T then there is no neighboring solution of period T (for a sufficiently small neighborhood). This condition does not imply stability of the periodic solution. With a perturbation of the initial conditions, stability is concerned with closeness to (asymptotic stability is concerned with the eventual approaching of) the original periodic solution. A perturbation of the initial conditions of the period solution of period T may result in a solution not of period T and thus, not even considered in the Banach space used.

A sufficient condition for the asymptotic stability of a periodic solution (of period T) to a nonautonomous system is the asymptotic stability of the null solution of the corresponding equation of first variation (Hochstadt,[13] p. 251). We are only able to show that the equation of first variation may not have a (nontrivial) periodic solution of period T if the fixed point is a contraction. The nonexistence of a periodic solution to the equation of first variation is (along with a continuous differentiability requirement) a sufficient condition for the continuous dependence of the periodic solution on a parameter. This result is not identical to but is compatible with our initial comments on the continuity of the contraction constant with respect to a parameter.

5.8 *Perturbation Analysis*

A very common approach to nonlinear problems is to solve a linear problem ignoring the nonlinearity and then to use a series expansion or a perturbation about the linear solution (see, e.g., Hochstadt,[13] Sections 6.5, 7.4). As useful as these procedures are, they usually suffer from the defect of not providing adequate quantitative information about the nonlinear solution, i.e., it may not be possible to determine quantitatively what is meant by "sufficient small". Our use of the contraction mapping theorem may prove useful in this regard.

As an illustration, again consider Duffing's equation and assume that x_0 was obtained by ignoring the nonlinear term (by^3). In this case, $x_0 = F(x_0)$ is defined by

$$x_{01}(t) = \int_0^T G_{12}(t, s) f \cos \omega s \, ds. \tag{79}$$

Then arguments similar to (but simpler than) those of Section 5.4 show that if b is "small enough" there is a continuous periodic solution to (40) which is "close" to the linear approximation

$$x_{01}(t) = \frac{f \cos \omega t}{a - \omega^2}. \tag{80}$$

Note that "small enough" and "close" may be quantitatively evaluated.

VI. DIFFERENCE-DIFFERENTIAL EQUATIONS*

Our use of the contraction map fixed point therom is not limited to ordinary differential equations or integral equations. As a further example, consider the difference-differential equation represented by

$$x = LN(D_h x), \tag{81}$$

where $y = D_h x$ is defined by

$$y(t) = x(t - h). \tag{82}$$

If the Banach space B of interest is the space of continuous periodic functions, then

$$\| D_h \| = 1. \tag{83}$$

This follows easily from

$$\begin{aligned} \| D_h \| &= \sup \{ \| D_h x \| : x \in B, \| x \| = 1 \} \\ &= \sup \{ \max_t | x(t - h) | : x \in B, \max_t | x(t) | = 1 \}. \end{aligned} \tag{84}$$

Assume that N maps B into itself. If N is differentiable (i.e., has a Fréchet derivative) at x_0 then $N(D_h x)$ has a derivative at x_0 (Kantorovich and Akilov,[6] p. 658) given by $N'(D_h x_0) D_h$. Then $LN(D_h)$ has a derivative at x_0 given by $LN'(D_h x_0) D_h$. The norm is easily

* An interesting discussion of the problems of oscillations in difference-differential equations is given in Chap. 21 of Minorsky.[1] Halanay[11] contains much information (and references) on difference-differential equations.

evaluated:

$$\| LN'(D_h x_0)D_h \| \leqq \| L \| \cdot \| N'(D_h x_0) \|. \tag{85}$$

Conceptually, the introduction of the time delay offers no great difficulty as compared to the case without the time delay. However, it will generally complicate the arithmetic involved in examples, in particular, in obtaining the solution to $x_0 = \tilde{L}N(x_0)$. This relative lack of complication going from differential equations to difference-differential equations is not typical. In existence, uniqueness, and stability considerations one must consider initial function conditions in difference-differential equations while the initial conditions for differential equations are merely at one time (or perhaps boundary conditions at several times).

To illustrate the above remarks, consider the following difference-differential equation:

$$\ddot{y} + ay + by_h^3 = f \cos \omega t \tag{86}$$

$$y_h(t) = y(t - h) \tag{87}$$

This is Duffing's equation but with the argument of the cubic term retarded. The corresponding operator equation is

$$x = LN(x) = L[N_1(D_h x) + F], \tag{88}$$

where $y = N_1(x)$ is defined by the cubic nonlinearity, L is the same linear operator as in Section V, and

$$F(t) = \begin{bmatrix} 0 \\ f \cos \omega t \end{bmatrix}. \tag{89}$$

It is clear that

$$\| LN'(x_0) \| \leqq \| L \| \, \| N_1'(x_0) \| \tag{90}$$

and that the analysis will be completely analogous to that of Section V, except that the approximate solution, x_0, will be different. Note that the Banach space is the space of continuous periodic functions, not the space of functions continuous on one period.

To obtain the equivalent linearization approximation let

$$\begin{aligned} y(t) &= A \cos \omega t + B \sin \omega t \\ &= C \sin (\omega t + \theta), \end{aligned} \tag{91}$$

where

$$C = \sqrt{A^2 + B^2}$$
$$\theta = \tan^{-1}\left(\frac{A}{B}\right). \tag{92}$$

Substitution of this function into (86) yields

$$-\omega^2 C \sin(\omega t + \theta) + aC \sin(\omega t + \theta) + \tfrac{3}{4}C^3 b\,[\cos \omega h \sin(\omega t + \theta) - \sin \omega h \cos(\omega t + \theta) + \text{third harmonics}] = \text{f} \cos \omega t. \tag{93}$$

The approximation is obtained by neglecting the third harmonics and equating coefficients of $\cos \omega t$ and $\sin \omega t$. It is interesting to compare the equivalent linearization solution obtained for the difference-differential equation with that obtained for the following differential equation (Duffing's equation with a damping term):

$$\ddot{y} + k\dot{y} + ay + by^3 = f \cos \omega t. \tag{94}$$

Substituting (91) into (94) yields

$$-\omega^2 C \sin(\omega t + \theta) + k\omega C \cos(\omega t + \theta) + aC \sin(\omega t + \theta) + \tfrac{3}{4}bC^3 \sin(\omega t + \theta) + \text{third harmonics} = f \cos \omega t. \tag{95}$$

Comparing (93) and (95) it is seen that, as far as harmonic balance is concerned, the effect of the lag is to introduce a damping term with damping coefficient k,

$$k = -(\sin \omega h)\tfrac{3}{4}C^2 b/\omega \tag{96}$$

(Also, one other term is multiplied by $\cos \omega h$).

For some parameter values, the equivalent damping is negative. Because of the negative damping, it appears that the periodic solution is not asymptotically stable. We say "appears that" rather than making a more definite statement for the following reason. While it seems plausible that the stability properties of the solution of the equation of equivalent linearization should carry over to the actual solution, the mathematical proof does not seem so obvious.

VII. RELATION TO PREVIOUS WORK*

As mentioned previously, the method of equivalent linearization has its roots in the method of Krylov and Boguliubov. For an ac-

* The literature on equivalent linearization is vast. We shall thus discuss only those references which seem most pertinent. Even in those cases, we shall discuss only those aspects which are directly related to the present study. The reader should consult these references for many other interesting ideas.

count of the very important work of Krylov, Boguliubov, and Mitropolsky in this area, see Minorsky.[1] Their work is primarily of the asymptotic type, i.e., leading to statements of the form, "for sufficiently small μ, there exists . . . ." We may view our approach as using a fixed point theorem to be able to determine quantitatively what is meant by "sufficiently small" for a somewhat different but related problem.

Bass[5] considers the justification of the method of equivalent linearization in the autonomous case. In view of our comments on the inapplicability of the contraction map fixed point theorem, it is of interest to note that Bass uses a much more sophisticated fixed point theorem. Much of his analysis is interesting and important but his final results are unfortunately difficult to apply (as Bass himself points out).

Sandberg[10] considers the operator equation*,

$$x = LN(x + f) \tag{97}$$

and the equivalent linearization approximation

$$x_0 = \tilde{L}N(x_0 + f). \tag{98}$$

Sandberg's analysis is carried out in the space of periodic functions square integrable over a period. He presents conditions under which there exists a unique periodic response to an arbitrary periodic input with the same period as well as an upper bound on the mean square error in using equivalent linearization. He also gives conditions under which sub-harmonics and self-sustained oscillations cannot occur. Sandberg's method is to determine conditions that guarantee that LN† is a contraction mapping in the whole space. As mentioned previously, we do not try to obtain a contraction mapping in the whole space but only in a neighborhood of x_0. We thus free ourselves from Lipschitz type requirements. It may be noted that many nonlinearities encountered in engineering are non-differentiable and Lipschitzian (e.g., piecewise linear functions such as saturation-type nonlinearities). For these, Sandberg's analysis is applicable while ours is not because we have required differentiability. Thus, Sandberg's work and ours complement each other in this regard. Also, Sandberg very fruitfully uses Fourier transform results in his analysis of feedback systems.

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* This is the same notation as in Section II except that in Section II we did not explicitly show the dependence on a forcing function. That is, $y = N(x)$ could be defined by $y(t) = n(x(t)+f(t))$ or by $y(t) = n(x(t)) + f(t)$.

† Actually, an operator related to LN.

Cesari[14] considers the real differential system

$$\begin{aligned} \dot{x} &= g(x,\, t) \\ x &= (x_1\,,\, \cdots\,,\, x_n) \\ \dot{x}_j &= g_j(x_1\,,\, \cdots\,,\, x_n\,,\, t) \qquad j = 1,\, \cdots\,,\, n. \end{aligned} \tag{99}$$

(For the specific conditions imposed on the above functions see Ref. 14). If

$$x_j(t) \sim a_{j0} + \sum_{s=1}^{\infty} (a_{js} \cos s\omega t + b_{js} \sin s\omega t) \tag{100}$$

and m is a positive integer the vector function $Px = (P_1x_1 \ldots, P_nx_n)$ is defined by

$$P_jx_j(t) = a_{j0} + \sum_{s=1}^{m} (a_{js} \cos s\omega t + b_{js} \sin s\omega t) \qquad j = 1,\, \cdots\,,\, n. \tag{101}$$

The operation $H(x) = (X_1, \ldots, X_n)$ is defined by

$$X_j(t) = \sum_{s=m+1}^{\infty} \frac{1}{s\omega} (-b_j \cos s\omega t + a_{js} \sin s\omega t) \qquad j = 1,\, \cdots\,,\, n. \tag{102}$$

The operation $F(x)$ is defined by

$$F(x) = H(I - P)g(x), \tag{103}$$

where I is the identity map and

$$\begin{aligned} g(x) &= (g_1x,\, \cdots\,,\, g_nx) \\ g_jx &= g_j[x(t),\, t] \qquad (j = 1,\, \cdots\,,\, n). \end{aligned} \tag{104}$$

Letting $T = P + F$, Cesari determines conditions for the existence of fixed points of $x = Tx$. He uses both Banach's fixed point theorem (contraction mapping theorem) and Schauder's fixed point theorem (which does not give uniqueness but requires weaker conditions). He then shows that if y is a fixed point, it satisfies

$$\dot{y}_i = g_i(y(t),\, t) + P_i(\dot{y}_i - g_iy). \tag{105}$$

If

$$P_i(\dot{y}_i - g_iy) = 0, \qquad j = 1,\, \cdots\,,\, n \tag{106}$$

then

$$\dot{y}(t) = g(y(t),\, t) \tag{107}$$

Cesari discusses the solution of (106) which may be reduced to determining a Galerkin approximation. Cesari worked in the space of square integrable (periodic) functions and Knobloch[15] adapted his approach to the space of continuous functions.

While the above method has the use of truncations of Fourier series in common with our approach, there seems to be a closer relationship between Urabe's approach and ours.

Urabe[16] considers the real nonlinear periodic system

$$\frac{dx}{dt} = X(x, t),$$

where $X(x, t)$ is periodic in t of period 2π. If

$$x_m(t) = a_0 + \sqrt{2} \sum_{n=1}^{m} a_n \cos nt + b_n \cos nt, \tag{108}$$

a Galerkin approximation* of order m is obtained if one can determine the $2m + 1$ coefficients $a_0, a_1, b_1, \ldots, a_m, b_m$ that satisfies the following equation:

$$\frac{dx_m}{dt} = \frac{1}{2\pi} \int_0^{2\pi} X[x_m(s), s]\, ds + \frac{1}{\pi} \sum_{n=1}^{m} \left\{ \cos nt \int_0^{2\pi} X[x_m(s), s] \cos ns\, ds + \sin nt \int_0^{2\pi} X[x_m(s), s] \sin ns\, ds \right\}. \tag{109}$$

Urabe considers the problem of determining whether there is an exact periodic solution near an approximate (Galerkin) solution, x_0. He determines conditions under which an iteration starting at x_0 converges to an exact periodic solution. His proof, while not explicitly mentioning a fixed point theorem, is closely related to the contraction mapping fixed point theorem and uses the fact that a contracting iteration sequence must stay within a certain sphere centered about the initial point.† Our approach is in the same spirit but we take a more general viewpoint at the beginning. The basic theorem is derived in an arbi-

* The method of equivalent linearization is essentially a first-order Galerkin approximation.

† It may be shown that Urabe's result (Proposition 3, p. 125 of Ref. 16) is essentially equivalent to requiring that the operator derivative have norm less than one (a contraction) in the appropriate sphere.

trary Banach space where derivatives of operators are fruitfully used. The more general viewpoint is very simple conceptually and also permitted the easy extension to difference-differential equations. Urabe in Ref. 16, and also in Ref. 17 and 18, considers many aspects of Galerkin's method for differential equations not touched on in our study. Also, see his comments on Cesari's method on p. 121 of Ref. 16.

VIII. CONCLUDING REMARKS

The development of analytical methods (other than asymptotic methods) for the equivalent linearization technique with autonomous systems remains a very important area for investigation.* Whether a modification of the contraction mapping theorem (perhaps using a subspace) might be applied to this problem remains to be seen. In connection with autonomous systems, a question perhaps more important than the one we have considered (if equivalent linearization indicates a periodic solution, does there actually exist one?) is the following: If a non-trivial periodic solution exists, will the method of equivalent linearization indicate it? A typical engineering use of the describing function is to determine conditions under which no self-sustained oscillations are predicted. The engineer would like these same conditions to also imply that there are no oscillations in the original (exact) system. Urabe[16] has shown that the existence of a periodic solution will (under certain conditions) imply the existence of a Galerkin approximation of sufficiently high order. The equivalent linearization technique is essentially a first-order Galerkin approximation and the first-order approximation may not be high enough to indicate the existence of a periodic solution according to a result of the type of Ref. 16. It would be very useful to determine conditions that would answer the question. This question is related to that rased by Aizerman's conjecture.

Leaving the problem of autonomous systems we find our adaptation of the contraction mapping theorem to be quite convenient in analyzing equivalent linearization in forced systems. The calculations are straightforward and require no difficult mathematical argument in the execution of the basic idea. It is hoped that the method may prove useful in justifying and refining approximations.

It should be clear that our approach is easily adapted to the dual-

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* A theory of autonomous systems, due to Urabe, is outlined in Chap. 3 of Halanay.[11]

input describing function approximation (see, e.g., Gibson,[19] p. 402).* The remarks in Section VI concerning difference-differential equations apply in that case also. That is, the only essential difficulty is in obtaining the dual-input describing function solution (which has nothing to do with our method of investigating the accuracy of such a solution). It should be noted that the dual-input describing function method has been used primarily for two sinusoids with commensurate frequencies (one an integral multiple of the other) and is actually equivalent to a Galerkin approximation. When the ratio of the two frequencies is irrational, we are in the realm of almost-periodic functions where analysis can get much more complicated. Boyer has presented an interesting approximate method of analysis (an account of which is given in Gibson,[19] p. 408ff.) for an input consisting of two sinusoids with incommensurate frequencies but with one much larger than the other. Analysis of this method would be of interest.

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APPENDIX A

Guide to Some Mathematical Background Reading for Engineers

The purpose of this appendix is to make reading suggestions to engineers interested in this work but who are not familiar with the mathematics used. The primary reference on functional analysis used for this work is Kantorovich and Akilov.[6] A more elementary introduction is given in Kolmogorov and Fomin.[9]† Lucid introductions to the theory of differential equations are given in Hochstadt[13] and Struble.[20] The theory of oscillations is extensively covered in Minor-

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* The dual-input describing function was apparently first used by J. C. West, J. L. Douce, and R. K. Livesly. In Ref. 7 there is an example of the existence of a subharmonic solution to Duffing's equation. This is actually an example of the dual-input describing function.

† The reader should be cautioned that some of the terminology is not standardized among American and Russian writers. For example, Kantorovich and Akilov do not require a compact set to be closed while most American authors do. Also, a linear operator is necessarily bounded according to Kantorovich and Akilov but not necessarily bounded according to most American writers (Kolmogorov and Fomin's definition agrees with American writers on this point).

sky[1] which also has a discussion on difference-differential equations. Kaplan treats Fourier series and finite Fourier transforms on an elementary level.[4] Further information on Fourier series (but still on an elementary level) can be found in Tolstov.[21]

APPENDIX B

Derivatives in a Banach Space

The following material is abstracted from Chap. XVII of Kantorovich and Akilov.[6]

Let P map an open subset Ω of a Banach space X into a subset Δ of another Banach space Y. Let $x_0 \in \Omega$ and suppose that there exists a linear* operation U mapping X into Y such that for every $x \in X$

$$\lim_{t\to 0} \frac{P(x_0 + tx) - P(x_0)}{t} = U(x). \tag{110}$$

The linear operation U is said to be the derivative of the operation P at the point x_0. We write this

$$U = P'(x_0). \tag{111}$$

The derivative thus defined is the Gateaux or weak derivative and $U(x)$ is the Gateaux differential.

If the convergence relationship of (110) is satisfied uniformly with respect to all $x \in X$ with $\| x \| = 1$, then the operation P is differentiable at the point x_0 and the derivative $P'(x_0)$ is called the Fréchet or strong derivative.

APPENDIX C

To discuss the satisfaction of (59), let

$$Z = 3 \mid b \mid CT\left[\mid A \mid + \frac{CT \dfrac{\mid bA^3 \mid}{4}}{1 - \alpha}\right]^2.$$

Consider a, f, and ω fixed with $f \neq 0$ and let $\alpha \in [0, 1)$. To show that

$$\lim_{b\to 0} Z = 0$$

we must show that

$$\lim_{b\to 0} \mid bA^2 \mid = 0$$

* Kantorovich and Akilov[6] include boundedness in their definition of linear.

since

$$Z = 3CT\left\{|\, bA^2 \,| + \frac{\frac{CT}{2} \,|\, bA^2 \,|^2}{1 - \alpha} + \left(\frac{CT}{1 - \alpha}\right)^2 \frac{|\, bA^2 \,|^3}{16}\right\}.$$

From (42) we have that

$$bA^2 = \frac{4}{3}\left(\frac{f}{A} + \omega^2 - a\right).$$

Also

$$\lim_{b \to 0} A = \frac{f}{a - \omega^2}$$

so that

$$\lim_{b \to 0} bA^2 = 0.$$

REFERENCES

Several of these works refer to still other works which contain helpful information. For example, see the first footnote on page 2407.

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