Restoration of Photographs Blurred by Image Motion

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The blurring of photographs by image motion during exposure is studied by means of a simple model. Conditions under which it is possible to recover the unblurred image are determined and some methods of restoration are described.

I. INTRODUCTION AND SUMMARY

This paper is concerned with the feasibility of restoring photographs that have been blurred during exposure by relative motion between the camera and the entire scene being photographed.* It is assumed that all objects of the scene are at rest relative to each other. Several simple mathematical models of this situation are investigated.

Section II treats the case of uniform translation between film and image. During exposure an area, A, of the image crosses over the margins onto the film. It is shown that unique restoration of the scene from the blurred photograph is, in general, impossible without *a priori* knowledge of certain portions of the undistorted image of area A. An algorithm is given for the restoration when this *a priori* knowledge is available, and a filtering technique is described that covers a case of frequent interest,—the photographing of a small object viewed against a uniform background.

The restoration techniques require knowledge of the translation undergone. Section III describes a method of estimating this displacement from the blurred photograph.

In Section IV more general image motions are considered. The case of pure rotation has many features in common with that of pure translation. Estimation of the parameters of the motion, however, appears to be more difficult in this case.

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II. IMAGE TRANSLATION

We are concerned here with photographs blurred because of a uniform relative motion during exposure between the camera and the object being photographed. For mathematical simplicity, in this section we treat the problem as one-dimensional; the modifications necessary to describe the more accurate two-dimensional model are evident.

Let $g(x)$ denote the illuminance from a scene or object being photographed that would result along a line in the image plane of the camera if there were no relative motion between the camera and the object. We suppose $g(x)$ defined for all values of x. The film occupies the interval $|x| \leqq L$. Imagine now that during the exposure time T the image moves with constant velocity v along the image plane in the x-direction. The total light energy $\epsilon(x)$ incident on a point x in this plane is

$$\begin{aligned} \epsilon(x) &= c_1 \int_0^T g(x - vt)\, dt \\ &= c_2 \int_{x-a}^{x} g(y)\, dy, \end{aligned} \tag{1}$$

where $a = vT$. In appropriate units, the density of the photograph is then

$$f(x) = r[\epsilon(x)], \qquad |x| \leqq L, \tag{2}$$

where $r(\epsilon)$ is the response curve of the film. Our aim is to recover g, or a portion of g, from a knowledge of $f(x)$, $|x| \leqq L$. If we assume the film response is monotone and known, knowledge of $f(x)$ is equivalent by (2) to knowledge of $\epsilon(x) = r^{-1}[f(x)]$, $|x| \leqq L$. For our purposes, then, it suffices to assume $\epsilon(x)$ known, or equivalently, to assume that the film response is linear. Accordingly, we henceforth consider recovering the undistorted scene $g(x)$ from the blurred photograph

$$f(x) = \int_{x-a}^{x} g(y)\, dy, \qquad |x| \leqq L, \tag{3}$$

where it is assumed that a is known. (The problem of estimating a is treated in Section III.)

From (3) we obtain at once

$$f'(x) = g(x) - g(x - a)$$

or

$$g(x) = f'(x) + g(x - a), \qquad |x| \leqq L, \tag{4}$$

the basic recovery equation. If $g(x)$ were known for $-L - a \leqq x \leqq -L$, say

$$g(x) = \varphi(x), \qquad -L - a \leqq x \leqq -L, \tag{5}$$

then g could be determined at once from (4) across the entire film interval. One has

$$g(x) = f'(x) + \varphi(x - a) \qquad -L \leqq x \leqq -L + a$$

$$\begin{aligned} g(x) &= f'(x) + g(x - a) \\ &= f'(x) + f'(x - a) + \varphi(x - 2a), \end{aligned}$$
$$-L + a \leqq x \leqq -L + 2a$$

$$g(x) = \sum_{j=0}^{k-1} f'(x - ja) + \varphi(x - ka), \tag{6}$$
$$-L + (k - 1)a \leqq x \leqq -L + ka$$
$$k = 1, 2, \cdots, K$$

$$g(x) = \sum_{j=0}^{K} f'(x - ja) + \varphi[x - (K + 1)a],$$
$$-L + Ka \leqq x \leqq L,$$

where $K = [2L/a]$ is the largest integer not greater than $2L/a$. Similarly, if $g(x)$ is known on any interval H of length a contained in the interval $I \equiv (-L - a, L)$, (4) can be used to determine g first in the intervals of length a adjacent to H and then successively to determine g throughout I. More generally, if g is known on a set S of intervals in I whose translates by various multiples of a form a set containing an interval of length a in I, then g can be determined everywhere in I by repeated application of (4). We call such a set S an admissible *a priori* set.

Two quite different cases of restoration are now evident: (*i*) g known beforehand on an admissible *a priori* set; (*ii*) g not so known. In the former case, exact restoration is possible in principle. In the latter case, unique restoration is *not* possible. Indeed, a given blurred photograph f could arise from infinitely many different scenes. For example, if no *a priori* knowledge of g is available, choose $g(x) = \varphi(x)$ for $-L - a \leqq x < -L$ with φ arbitrary. Use (6) then to determine g for $-L \leqq x \leqq L$. This scene g will give rise to a blurred photograph differing from f by at most a constant. (By judicious choice of background, and by moving the camera, it is possible to make the devil appear as only a slightly-blurred saint!) Similar considerations show that if

g is not known beforehand on some admissible *a priori* set, its values can be assigned arbitrarily on some set of points in I and determined elsewhere to give a scene that could produce a given blurred photograph. There seems to be little useful that can be said, in general, about restoration of blurred photographs when g is not known on some admissible *a priori* set.

A case of importance in practice where something of value can be said concerns the restoration of a photograph of a small object moving across a uniform background. We suppose the background corresponds to photographic density zero and that the blurred object image is smaller than the photograph. Specifically, assume that it is known *a priori* that the unblurred object image $g(x)$ would be nonzero only in the interval $x_0 \leqq x \leqq x_0 + (p - 1)a$, where $x_0 \geqq -L$, $x_0 + pa \leqq L$. The blurred photograph then would have a density different from zero only for $-L \leqq x_0 \leqq x \leqq x_0 + pa \leqq L$. We define f everywhere by taking

$$f(x) \equiv 0, \qquad x < x_0\,, \qquad x > x_0 + pa. \tag{7}$$

We define $g = 0$ for $x < x_0$ and $x > x_0 + (p - 1)a$. In this case, the solution of form (6) becomes simply

$$g(x) = \sum_{j=0}^{p-1} f'(x - ja), \qquad x \leqq x_0 + pa \tag{8}$$

$$g(x) = \sum_{j=1}^{p} f'(x - ja), \qquad x_0 + pa < x \leqq x_0 + (p + 1)a$$

$$\vdots$$

$$g(x) = \sum_{j=n}^{p+n-1} f'(x - ja), \qquad x_0 + (p + n - 1)a < x \leqq x_0 + (p + n)a$$

$$n = 1, 2, \cdots . \tag{9}$$

Because of our assumptions, the sums in (9) must give zero for $n = 1, 2, \cdots$ and x in the indicated ranges. They are in this sense nugatory. Equation (8) gives $g = 0$ for $x < x_0$ because of (7). In the range of interest $x_0 \leqq x \leqq x_0 + pa$, it gives a simple algorithm for obtaining a true picture of the object.

Equation (8) can be instrumented in many ways. The derivative f' of the blurred photo extended by (7) can be obtained as a transparency by optical filtering techniques. The sum (8) then can be found by p-tuple exposure of a film with the image of f' being translated by an amount a by a mirror between each exposure.

An alternate restoration method suggested by (8) sheds some light on a filtering technique previously reported in the literature.[1] Let us define for all x

$$\hat{g}(x) \equiv \sum_{j=0}^{p-1} f'(x - ja) \tag{10}$$

with f defined everywhere by (7).

For $x \leqq x_0 + pa$, $\hat{g}$ will coincide with g, but for $x > x_0 + pa$ it gives values different from g. From the Fourier representation

$$f(x) = \int_{-\infty}^{\infty} e^{i\lambda x} F(\lambda)\, d\lambda$$

if follows that

$$f'(x - ja) = \int_{-\infty}^{\infty} e^{i\lambda x} i\lambda F(\lambda) e^{-ij\lambda a}\, d\lambda$$

so that (10) can be written

$$\begin{aligned}\hat{g}(x) &= \int_{-\infty}^{\infty} d\lambda\, e^{i\lambda x} i\lambda F(\lambda) \sum_{j=0}^{p-1} e^{-ij\lambda a} \\ &= \int_{-\infty}^{\infty} d\lambda\, e^{i\lambda x} \lambda F(\lambda) \frac{\sin(\lambda pa/2)}{\sin(\lambda a/2)}\, ie^{-i(p-1)(\lambda a/2)}\end{aligned}$$

which shows that $\hat{g}[x + (p - 1)(a/2)]$ can be obtained from the extended blurred photograph f by processing with a filter having transfer function

$$Y(\lambda) = \frac{i\lambda \sin(\lambda pa/2)}{\sin(\lambda a/2)}. \tag{11}$$

A different filter for restoration in the present case can be derived as follows. Recall our assumption that

$$g = 0 \quad \text{for} \quad x \leqq -L \quad \text{and} \quad x > L - a. \tag{12}$$

Then

$$\begin{aligned}f(x) &= \int_{x-a}^{x} g(y)\, dy \\ &= \int_{-\infty}^{\infty} h(x - y) g(y)\, dy\end{aligned} \tag{13}$$

holds true for all x. Here

$$h(x) = \begin{cases} 1, & -a \leqq x \leqq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Taking the Fourier transform of (13) yields

$$G(\lambda) = \pi e^{-(ia\lambda/2)} \frac{\lambda}{\sin(\lambda a/2)} F(\lambda) \tag{14}$$

which shows that $g[x + (a/2)]$ can be obtained from the extended blurred photograph by processing with a filter having transfer function

$$Y_\infty = \pi \frac{\lambda}{\sin(\lambda a/2)} \tag{15}$$

as has been reported previously.[1]

The filter (15) has poles at the points $\lambda = 2n\pi/a$, $|n| = 1, 2, \cdots$ and hence cannot be realized in practice. Some ad hoc scheme for assigning a finite value at these pole positions must be made. Just what these modified filters do to picture quality is not easy to analyze. The filter (15), could it be instrumented, would yield g, that is, a picture with infinite white skirts. The filter (11), on the other hand, has no poles and hence can be realized.* It restores g correctly in the interval $x_0 \leqq x \leqq x_0 + pa$ where this quantity is different from zero. It gives uninterpretable values for $x > x_0 + pa$ and the value zero for $x < x_0$. It would appear that the infinities in (15) with their attendant difficulties are due to insisting that the processed picture yield the value zero over an infinite region where from a priori knowledge one would accept no other value anyhow.

It is worth noting that if (12) is violated, then (13) does not hold for all x and one cannot write (14). These edge effects have been overlooked in past treatments of the problem based on (14).[1]

III. ESTIMATION OF MOTION PARAMETERS

The restoration technique of the preceding section presupposed knowledge of the direction and amount of the image displacement during exposure. We now consider how these quantities might be determined from the blurred photograph itself.

We suppose the blurred photograph density to be given by

$$f(x, y) = \int_0^T g(x - ut, y - vt)\, dt, \quad |x| \leqq L_1, \quad |y| \leqq L_2, \tag{16}$$

where $g(x, y)$ is the image that would result if there were no motion. Again to avoid edge effects we suppose $g(x, y)$ defined everywhere

* Because of the growing factor λ, both (15) and (11) must ultimately be cut off at some point beyond the largest spatial frequency of interest in the photographs.

and different from zero only in the rectangle $-L_1 \leqq x \leqq L_1 - uT$, $-L_2 \leqq y \leqq L_2 - vT$, so that defining $f = 0$ for $|x| > L_1$, $|y| > L_2$ we can write

$$f(x, y) = \int_0^T g(x - ut, y - vt)\, dt, \qquad -\infty \leqq x, y \leqq \infty. \tag{17}$$

Into (17) now introduce the Fourier representation

$$g(x, y) = \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\, G(\xi, \eta) e^{i[\xi x + \eta y]}.$$

There results

$$\begin{aligned} f(x, y) &= \int_0^T dt \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\, G(\xi, \eta) e^{i[\xi(x-ut)+\eta(y-vt)]} \\ &= \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} d\eta\, e^{i(\xi x + \eta y)} G(\xi, \eta) e^{-i[(\xi a + \eta b)/2]} \frac{\sin(\xi a + \eta b)/2}{(\xi a + \eta b)/2} \end{aligned} \tag{18}$$

on performing the t integration. Here

$$a = uT, \qquad b = vT.$$

Since (18) holds for all x and y, we see that the Fourier transform of f is given by

$$F(\xi, \eta) = G(\xi, \eta) e^{-i[(\xi a + \eta b)/2]} \frac{\sin(\xi a + \eta b)/2}{(\xi a + \eta b)/2}. \tag{19}$$

As seen from (19) the transform of the blurred photograph is zero on the family of parallel lines

$$\xi a + \eta b = 2n\pi \qquad n = \pm 1, \pm 2, \cdots .$$

These lines of zero density in F should provide a reasonable means of estimating the parameters a and b. Due to noise, the curves of zeros of F will not appear as straight lines. The job of fitting straight lines to these curves of zeros should be greatly simplified however by the knowledge that the lines are parallel and uniformly separated. Once the fitted lines are drawn, value of a and b are readily found.

IV. MORE GENERAL MOTION

In the present model, the blurred photograph that results from the general nondistance-distorting motion of a small object is

$$f(x, y) = \int_0^T dt\, g[(x - u) \cos \varphi - (y - v) \sin \varphi,$$
$$\cdot (x - u) \sin \varphi + (y - v) \cos \varphi]. \qquad (20)$$

Here $g(x, y)$ is the illuminance that would result if the object were at rest with respect to the film during the exposure, u and v are functions of t giving, respectively, the x and y coordinates of the origin of a coordinate system fixed with respect to the body, and $\varphi = \varphi(t)$ is the angle that this second coordinate frame makes with respect to the x–y frame. In the case of most immediate interest

$$\begin{aligned} u &= x_0 + \bar{u}t \\ v &= y_0 + \bar{v}t \\ \varphi &= \omega t. \end{aligned} \qquad (21)$$

One finds without difficulty that the Fourier transform of f and g are related by

$$F(\xi, \eta) = \int_0^T dt\, e^{-i[u\xi + v\eta]} G[\xi \cos \varphi - \eta \sin \varphi, \xi \sin \varphi + \eta \cos \varphi]. \quad (22)$$

This equation appears somewhat simpler in polar coordinates. We write

$$\xi = \rho \cos \theta \qquad \eta = \rho \sin \theta$$
$$u = V \cos \alpha \qquad v = V \sin \alpha$$

and set

$$F(\xi, \eta) = \hat{F}(\rho, \theta) \qquad G(\xi, \eta) = \hat{G}(\rho, \theta).$$

Then (22) becomes

$$\hat{F}(\rho, \theta) = \int_0^T dt\, e^{-i\rho V \cos(\theta - \alpha)} \hat{G}(\rho, \theta + \varphi). \qquad (23)$$

Here V, α and φ are functions of t.

Under these general conditions, I have been unable to find a practical method for obtaining the undistorted scene g from f, either in the space domain, or from the transform statements (22) and (23). Even in the case of combined uniform translation and uniform rotation given by (21) no method is as yet evident.

The case of pure uniform rotation, $\bar{u} = \bar{v} = 0$ can, however, be treated and complements the case of pure translation ($\omega = 0$) already discussed in Sections II and III. Working directly in the space domain,

(20) becomes

$$f(x, y) = \int_0^T dt\, g[(x - x_0) \cos \omega t - (y - y_0) \sin \omega t,$$
$$\cdot (x - x_0) \sin \omega t + (y - y_0) \cos \omega t].$$

Introduce polar coordinates located at the center of rotation

$$x - x_0 = \rho \cos \theta$$
$$y - y_0 = \rho \sin \theta$$
$$\hat{f}(\rho, \theta) = f(x, y), \qquad \hat{g}(\rho, \theta) = g(\rho \cos \theta, \rho \sin \theta). \tag{24}$$

We now have

$$\hat{f}(\rho, \theta) = \int_0^T dt\, \hat{g}(\rho, \theta + \omega t)$$
$$= \frac{1}{\omega} \int_\theta^{\theta + \omega T} d\theta'\, \hat{g}(\rho, \theta')$$

which is basically of the form (3) already treated. The basic restoration equation is

$$\hat{g}(\rho, \theta) = -\omega \frac{d}{d\theta} \hat{f}(\rho, \theta) + \hat{g}(\rho, \theta + \omega T), \tag{25}$$

where ρ is to be regarded as a parameter, $\hat{g}$ and $\hat{f}$ are periodic in θ with period 2π and the equation holds for all values of θ.

If $\hat{g}(\rho, \theta)$ is known *a priori* as a function of θ along an arc of angular extent ωT radians, (25) can be used successively to determine $\hat{g}$ for all θ. It is not hard to show that *if $\hat{g}$ is not known a priori on a θ set of angular measure ωT, unique restoration is impossible.* Indeed, there exists a scene with values assigned arbitrarily (except for an additive constant) in a wedge of angle ωT which, when rotated, will give rise to any preassigned blurred photograph.

In the case of a blurred photograph of a rotating unknown object, for example, if the center of rotation is within the body, unique restoration is impossible in the neighborhood of this center. If restoration is to be made, one must use some form of *a priori* knowledge to specify $\hat{g}$ or an estimate of $\hat{g}$ in some angular interval of amount ωT.

Restoration by means of the difference equation (25) presupposes knowledge of ω and [from (24)] the center of rotation x_0, y_0. We have not found a simple way of estimating these parameters. Unlike the case of pure translation, the Fourier transform of the blurred picture,

$$\hat{F}(\rho, \theta) = e^{-i\rho(x_0 \cos\theta + y_0 \sin\theta)} \int_0^T dt\, \hat{G}(\rho, \theta + \omega t),$$

does not seem to offer special clues. If the object has some straight line edges, their initial and final positions may show clearly enough in the blurred photograph to allow estimates of x_0, y_0 and ωT to be made. For example, if l_1 and l_1' are the lines along the initial and final positions of some edge of the body, the angle between l_1 and l_1' is clearly ωT. Let l_2 and l_2' be lines along the initial and final position of some other straight line feature of the object and let P be the intersection of l_1 with l_2 and P' be the intersection of l_1' with l_2'. The center of rotation O must lie on the perpendicular bisector of the segment $\overline{PP'}$, and its position is chosen so that $\angle POP' = \omega T$. It is likely that, in practice, restoration with several different trial values of the parameters will have to be made and the best result selected.

REFERENCES

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