Optimal Routing in Connecting Networks Over Finite Time Intervals

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A telephone connecting network is given, and with full information at all times about its state, routing policies are sought which minimize the expected number of attempted calls denied service in some finite interval. In this paper, the search is pursued as a mathematical problem in the context of a standard traffic model in terms of optimal control theory and dynamic programming. Certain combinatorial properties of the network, earlier found to be the key to minimizing the loss, also turn out to be relevant here: they lead to policies which differ from optimal policies only in accepting all unblocked call attempts, and provide a "practical" solution of the problem posed. In many cases, the policies found vindicate heuristic policies earlier conjectured to be optimal.

I. INTRODUCTION AND SUMMARY

We study the problem of optimally routing calls in a telephone connecting network during a finite time interval $[0, t]$ over which the traffic intensity need not be constant. The present work reports on extensions of earlier results[1] on routing in telephone networks with constant traffic intensity; the principal novelty lies in the fact that whereas previously we minimized the probability of blocking* here we seek to minimize simply the expected total number of call attempts denied service in a given fixed time interval $[0, t]$ on which the traffic intensity may vary.

A traffic model, the same as that used in Ref. 1, is described (Sections II to IV), and the problem is formulated mathematically in the manner of optimal control theory (Section V). The associated Hamilton–Jacobi equation is written and it is noted that this equation has a solution constructible in terms of functions satisfying nonlinear integral

* Defined asymptotically as the stochastic limit, as t becomes large, of the *fraction* of attempted calls blocked or rejected in $[0, t]$.

equations derived from the principle of optimality, (Section VI). An isotony theorem, based on the same combinatorial properties as were used in Ref. 1 to minimize the loss fraction, then exposes the optimal policies to within rejection of unblocked attempted calls. That is, policies are arrived at which differ from optimal policies only in that the latter might reject some unblocked calls at some times (Sections VII and VIII); these policies are the same as those that were arrived at in minimizing the loss.

II. STATES, EVENTS, AND ASSIGNMENTS

The mathematical model of Ref. 1 will be used. The elements of this model separate naturally into combinatorial ones and probabilistic. The former arise from the structure of the connecting network and from the ways in which calls can be put up in it; the latter represent assumptions about the random traffic the network is to carry. The combinatorial and structural aspects are discussed in this section; terminology and notation for them are introduced. The probabilistic aspects are considered in a later section.

A connecting network ν is a quadruple $\nu = (G, I, \Omega, S)$, where G is a graph depicting network structure, I is the set of nodes of G which are inlets, Ω is the set of nodes of G that are outlets, and S is the set of permitted states. Variables x, y, z at the end of the alphabet denote states, while u and v (respectively) denote a typical inlet and a typical outlet. A state x can be thought of as a set of disjoint chains on G, each chain joining I to Ω. Not every such set of chains represents a state: sets with wastefully circuitous chains may be excluded from S. It is possible that $I = \Omega$, that $I \cap \Omega = \theta =$ null set, or that some intermediate condition obtain, depending on the "community of interest" aspects of the network ν.

The set S of states is partially ordered by inclusion $\leqq$, where $x \leqq y$ means that state x can be obtained from state y by removing zero or more calls. If x and y satisfy the same assignment of inlets to outlets, i.e., are such that all and only those inlets $u \in I$ are connected in x to outlets $v \in \Omega$ which are connected to the same v in y (though possibly by different routes), then we say that x and y are equivalent, written $x \sim y$.

The set S of states determines another set $\mathcal{E}$ of events, either hangups (terminations of calls), successes (successful call attempts), or blocked or rejected calls (unsuccessful call attempts). The occurrence of an event in a state may lead to a new state obtained by adding or removing a call in progress, or it may, if it is a blocked call or one that is rejected, lead to no change of state. Not every event can occur in every state: naturally, only those calls can hang up in a state which are in progress in that state, and only those inlet-outlet pairs can ask for a connection between them in a state that are idle in that state. The notation e is used for a (general) event, h for a hangup, and c for an attempted call. If e can occur in x we write $e \varepsilon x$. A call $c \varepsilon x$ is blocked in a state x if there is no $y \varepsilon S$ which covers x in the sense of the partial ordering $\leqq$ and in which c is in progress. For $h \varepsilon x$, $x - h$ is the state obtained from x by performing the hangup h.

We denote by A_x the set of states that are immediately above x in the partial ordering $\leqq$, and by B_x the set of those that are immediately below. Thus,

$$A_x = \{\text{states accessible from } x \text{ by adding a call}\}$$

$$B_x = \{\text{states accessible from } x \text{ by a hangup}\}.$$

For an event $e \varepsilon x$, the set A_{ex} is to consist of those states $y \neq x$ to which the network might pass upon the occurrence of e in x. Thus, if e is a blocked call, $A_{ex} = \theta$; also

$$\bigcup_{h \varepsilon x} A_{hx} = B_x$$

$$\bigcup_{\substack{c \varepsilon x \\ c \text{ not blocked in } x}} A_{cx} = A_x \, .$$

The number of calls in progress in state x is denoted by $|x|$. The number of call attempts $c \varepsilon x$ which are not blocked in x is denoted by $s(x)$, for "successes in x." The functions $|\cdot|$ and $s(\cdot)$ defined on S play important roles in the stochastic process to be used for studying routing. In addition, we use

β_x = number of idle inlet-outlet pairs blocked in state x

α_x = number of idle inlet-outlet pairs in state x,

and note that $\alpha = \beta + s$.

It can be seen, further, that the set S of states is not merely partially ordered by $\leqq$, but also forms a semilattice, or a partially ordered system with intersections, with $x \cap y$ defined to be the state consisting of those calls and their respective routes which are common to both x and y.

An assignment is a specification of what inlets should be connected to what outlets. The set A of assignments can be represented as the set of all fixed-point-free correspondences from I to Ω. The set A is partially ordered by inclusion, and there is a natural map $\gamma(\cdot): S \to A$ which takes each state $x \in S$ into the assignment it realizes; the map $\gamma(\cdot)$ is a semilattice homomorphism of S into A, since

$$x \geqq y \quad \text{implies} \quad \gamma(x) \geqq \gamma(y),$$

$$\gamma(x \cap y) \leqq \gamma(x) \cap \gamma(y).$$

We denote by F_x the set of calls that are *free* or idle in x, i.e.,

$$F_x = \{c: c \text{ is idle in } x\} = \{\gamma(y - x): y \in A_x\},$$

where $y - x$ is the state obtained from y by removing all the calls of $x \leqq y$.

III. PROBABILISTIC ASSUMPTIONS

A Markov stochastic process x_t taking values on S is used as a mathematical description of an operating connecting network subject to random traffic. Specifically, the Markov process of Ref. 1 will be used, with the modification that the calling-rate per idle inlet-outlet pair can depend on time. This model can be paraphrased in the informal terminology of "rates" by two simple assumptions:

(*i*) The hang-up rate per call in progress is unity.

(*ii*) The calling-rate between an inlet and a distinct outlet, both idle at time u, is $\lambda(u) \geqq 0$.

The transition probabilities of x_t will be described after a discussion of system operation and routing.

IV. ROUTING POLICIES

It will be assumed here, as in Ref. 1, that attempted calls to busy terminals are rejected, and have no effect on the state of the network; similarly, blocked attempts to call an idle terminal are refused, with no change of state. Attempts to place a call are completed instantly with some choice of route, or are rejected, in accordance with some *routing policy*.

A routing policy over $[0, t]$ will be described by a measurable matrix-valued function of time, denoted by $R(u) = (r_{xy}(u))$, $x, y \in S$, $0 \leqq u \leqq t$, having the following properties and interpretation: for each $x \in S$, let Π_x be the partition of A_x induced by the relation $\sim$ of "having the same calls up," or satisfying the same assignment of inlets to outlets;

it can be seen that Π_x consists of exactly the sets A_{cx} for $c \in x$, c not blocked in x; for each $u \in [0, t]$, $Y \in \Pi_x$, $r_{xy}(u)$ for $y \in Y$ is a possibly improper probability distribution over Y, (that is, it may not sum to unity over Y),

$$r_{xx}(u) = s(x) - \sum_{y \in A_x} r_{xy}(u),$$

and $r_{xy}(u) = 0$ in all other cases.

The interpretation of the routing matrix $R(u)$ is to be this: any $Y \in \Pi_x$ represents all the ways in which a particular call c (free and not blocked in x) *could* be completed when the network is in state x; for $y \in Y$, $r_{xy}(u)$ is the chance that if call c is attempted in state x at time u, it will be completed by being routed through the network so as to take the system to state y. That is, we assume that if c is attempted in x at u, then with probability

$$1 - \sum_{y \in A_{cx}} r_{xy}(u) \qquad (1)$$

it is rejected (even though it is not blocked), and with probability $r_{xy}(u)$ it is assigned the route which would change the state x to y, for $y \in A_{cx}$. The possibly improper distribution of probability

$$\{r_{xy}(u),\ y \in Y\}$$

indicates how the calling-rate $\lambda(u)$ due to c at time u is to be spread over the possible ways of putting up the call c, while the improper part (1) is just the chance that it is rejected outright.

It is to be noted that, as in Ref. 1, routing is carried out with perfect information about the current state of the network. The problem of optimal routing with only partial information is much more difficult (than the problem to be considered here), and it is not taken up.

Alternatively, we may define the convex set C of all (routing) matrices $R = (r_{xy})$ such that $r_{xy} \geqq 0$, $r_{xy} = 0$ unless $y \in A_x$, and

$$\sum_{y \in A_{cx}} r_{xy} \leqq 1,$$

for $c \in x$ not blocked in x,

$$r_{xx} = s(x) - \sum_{y \in A_x} r_{xy},$$

and describe the routing policies as measurable functions on $[0, t]$ taking values in C.

A routing policy $R(\cdot)$ with $r_{xy}(u) \equiv 0$ or 1 is called a *fixed* policy.

V. FORMULATION OF THE PROBLEM

For the purpose of defining a Markov stochastic process it is convenient and customary to collect the probabilistic and operational assumptions made above in a time-dependent matrix $Q(\cdot)$ of *transition rates*. Indeed, each routing policy $R(\cdot)$ determines such a matrix function, and so a process, according to the relationship $Q = Q(R)$ given in detail by

$$q_{xy}(u) = \begin{cases} 1, & y \varepsilon B_x \\ \lambda(u) r_{xy}(u), & y \varepsilon A_x \\ -\mid x \mid - \lambda(u)[s(x) - r_{xx}], & y = x \\ 0, & \text{otherwise.} \end{cases}$$

If the routing policy $R(\cdot)$ is used, the transition probability matrix $P(u, t) = (p_{xy}(u, t))$, with

$$p_{xy}(u, t) = \Pr \{x_t = y \mid x_u = x\},$$

will develop according to the backward Kolmogorov equation

$$P(t, t) \equiv I, \qquad Q = Q(R)$$

$$\frac{\partial}{\partial u} P(u, t) = -Q(u)P(u, t), \qquad 0 \leqq u \leqq t.$$

In particular, if the system starts at 0 with an initial probability distribution given by the column vector $p(0)$, then its distribution $p(u)$ at time u is $[p(0)'P(0, u)]'$, which satisfies the equation $\dot{p}(u) = Q(u)'p(u)$. If the network is in state x at time u, the rate at which blocked or rejected calls are being generated is $\lambda(u)[r_{xx}(u) + \beta_x]$. Thus, with $r(u) = r(R(u))$ the vector function $\{r_{xx}(u), x \varepsilon S\}$, the total expected number of calls denied service during [0, t] is just

$$D = D(p(0), t) = \int_0^t p(u)'[r(u) + \beta]\lambda(u)\, du. \tag{2}$$

We may, therefore, state our routing problem thus: Minimize D subject to the conditions $p(0)$ given, $\dot{p}(u) = Q(u)'p(u)$, $Q = Q(R)$, $R(u) \varepsilon C$, for $u \varepsilon [0, t]$.

Let us now view the $\mid S \mid$-dimensional probability vector $p = p(u)$ as a "state-variable" whose "motion" is governed by the linear differential equation $\dot{p}(u) = Q(u)'p(u)$. The criterion D is linear in $p(\cdot)$ and the matrix entries of the control $R(\cdot)$ appear as coefficients in the equation and in the criterion. The problem of minimizing D can be approached and solved by the now classical methods of the theory of optimal control.

VI. THE HAMILTON-JACOBI EQUATION

Let p, q be $|S|$-dimensional vector variables, and introduce the Hamiltonian function

$$H(p, u, q, R) = \lambda(u)p'(\beta + r) + \sum_x p_x\{\sum_{y \in B_x} q_y + \lambda(u) \sum_{y \in A_x} r_{xy}q_y - (\lambda(u)r_x + |x|)q_x\}.$$

Let H^* be the minimum of H for $R \in C$, i.e.,

$$H^*(p, u, q) = \min_{R \in C} \{\lambda(u)p'Hq + \lambda(u)p'Rq - \sum_x p_x[\lambda(u)r_x + |x|]q_x\},$$

where $H = (h_{xy})$ is the "hangup matrix" such that $h_{xy} = 1$ or 0 according as $y \in B_x$ or not. The Hamilton–Jacobi equation associated with the minimization of D above is just

$$\frac{\partial V}{\partial u} + H^*\left(p, u, \frac{\partial V}{\partial p}\right) = 0, \qquad 0 \leqq u \leqq t, \qquad p \geqq 0. \tag{3}$$
$$V(p, t) \equiv 0.$$

It follows from a known theorem[2] of the theory of optimal control that if we can find a continuously differentiable solution $V(p, u)$ of the Hamilton–Jacobi equation (3), then a control policy $R(\cdot) = (r_{xy}(\cdot))$ such that by components

$$R(u) \frac{\partial V}{\partial p}(p, u) = \min_{R \in C} R \frac{\partial V}{\partial p}(p, u), \qquad 0 \leqq u \leqq t$$

is optimal.

To find a solution of the Hamilton-Jacobi equation (3) let us consider the problem of starting the connecting system at a time $u < t$, and operating it until t so as to minimize the expected number of blocked calls over (u, t). We define, with t fixed, and $u < t$,

$E_x(u)$ = expected number of blocked calls in (u, t) using an optimal policy, starting in state x.

To solve the problem we note that two possibilities arise: Either an event occurs in (u, t), or else none does. In the latter case, the system stays in its initial state x throughout (u, t), and no calls are blocked during the interval. In the former case the first event e to occur does so at some time epoch $\tau \varepsilon (u, t)$ and can lead to one of the states in $A_{ex} \cup \{x\}$. The minimum expected blocking to be suffered in the remaining interval (τ, t) is just

$$\begin{cases} \min \{1 + E_x(\tau), \min_{y \varepsilon A_{cx}} E_y(\tau)\} & \text{if} \quad e = c \\ E_{x-h}(\tau) & \text{if} \quad e = h. \end{cases}$$

With

$$c_x(u) = |\, x \,| + \alpha_x \lambda(u),$$

$$C_x(u) = \int_0^u c_x(v)\, dv,$$

the probability density that the first event to occur does so at τ, and is e, equals

$$c_x(\tau) \exp \{-C_x(\tau) + C_x(u)\} \cdot \begin{cases} \dfrac{1}{c_x(\tau)}, & e = h \\[2ex] \dfrac{\lambda(\tau)}{c_x(\tau)}, & e = c. \end{cases}$$

Hence, applying the "principle of optimality," we conclude that the vector function $E(u)$, $0 \leqq u \leqq t$, satisfies the equation

$$E_x(u) = \int_u^t \exp \{-C_x(\tau) + C_x(u)\} \cdot [\sum_{y \varepsilon B_x} E_y(\tau) + \lambda(\tau) \sum_{c \varepsilon x} \min \{1 + E_x(\tau), \min_{y \varepsilon A_{cx}} E_y(\tau)\}]\, d\tau. \tag{4}$$

We now observe that if $E(\cdot)$ satisfies (4), then the scalar function $V(p, u) = p'E(u)$ satisfies the Hamilton-Jacobi equation. This is of course not surprising since the equation for $E(\cdot)$ was obtained from the optimality principle. To see it we differentiate (4) with respect to u, obtaining

$$\begin{aligned} \frac{\partial}{\partial u} E_x(u) &= (|\, x \,| + \lambda(u)\alpha_x)E_x(u) - \sum_{y \varepsilon B_x} E_y(u) - \lambda(u)\beta_x \\ &\qquad - \lambda(u) \sum_{c \varepsilon x} \min \{1 + E_x(u), \min_{y \varepsilon A_{cx}} E_y(u)\} \\ &= [|\, x \,| + \lambda(u)s(x)]E_x(u) - \sum_{y \varepsilon B_x} E_y(u) - \lambda(u)\beta_x \\ &\qquad - \lambda(u) \sum_{\substack{c \varepsilon x \\ c \text{ not blocked in } x}} \min \{1 + E_x(u), \min_{y \varepsilon A_{cx}} E_y(u)\}. \end{aligned}$$

Now note that

$$\sum_{\substack{c \varepsilon x \\ c \text{ not blocked in } x}} \min \{1 + E_x(u), \min_{y \varepsilon A_{cx}} E_y(u)\}$$

$$= \sum_{\substack{c \varepsilon x \\ c \text{ not blocked in } x}} \min_{R \varepsilon C} \{(1 - \sum_{y \varepsilon A_{cx}} r_{xy})[E_x(u) + 1] + \sum_{y \varepsilon A_{cx}} r_{xy} E_y(u)\}$$

$$= \min_{R \varepsilon C} \{ \sum_{\substack{c \varepsilon x \\ c \text{ not blocked in } x}} (1 - \sum_{y \varepsilon A_{cx}} r_{xy})[E_x(u) + 1] + \sum_{y \varepsilon A_{cx}} r_{xy} E_y(u)\}$$

$$= \min_{R \varepsilon C} \{[s(x) - \sum_{y \varepsilon A_x} r_{xy}][E_x(u) + 1] + \sum_{y \varepsilon A_x} r_{xy} E_y(u)\}$$

$$= \min_{R \varepsilon C} \{r_{xx}[E_x(u) + 1] + \sum_{y \varepsilon A_x} r_{xy} E_y(u)\}.$$

Therefore,

$$\frac{\partial}{\partial u} E_x(u) + \min_{R \varepsilon C} \{-(| x | + \lambda(u)[s(x) - r_{xx}])E_x(u)$$
$$+ \sum_{y \varepsilon B_x} E_y(u) + \lambda(u)[\beta_x + r_{xx}] + \sum_{y \varepsilon A_x} r_{xy} E_y(u)\} = 0.$$

Now with $V = p'E$ and $\partial V/\partial p = E$, $r = r(R) = \{r_{xx}, x \varepsilon S\}$, $p \geqq 0$,

$$\frac{\partial V}{\partial u} + \min_{R \varepsilon C} \{\lambda(u) p'(\beta + r) - \sum_x p_x(| x | + \lambda(u)[s(x) - r_{xx}])E_x(u)$$
$$+ \sum_x p_x(\sum_{y \varepsilon B_x} + \sum_{y \varepsilon A_x} r_{xy})E_y(u)\} = 0.$$

This is the Hamilton–Jacobi equation. It follows that the minimum of D is achieved by a fixed policy, as could be expected on intuitive grounds.

VII. ISOTONY THEOREM

In Ref. 1 we introduced some combinatorial "monotone" properties of the partial ordering $(S, \leqq)$ of states which (when present) provide an intuitive and straightforward description of the routing choices for accepted calls which minimize the loss probability. These properties are also relevant to minimizing the criterion D of (2).

The properties in question can be paraphrased as follows: the relative merit of states vis à vis blocking is consistent or continuous, i.e., if a state x is "better" than another y, then the neighbors of x in $\leqq$ are in the same sense better than the corresponding neighbors of y. Specifically, we deal in detail only with the weakest property used in Ref. 1, and we say that a relation P on S has the *weak monotone property* if xPy implies

(*i*) $|x| = |y|$,
(*ii*) $\exists\, \mu: B_x \leftrightarrow B_y$ and $z \in B_x$ implies $zP\mu z$,
(*iii*) $\exists\, \nu: F_y \rightarrow F_x$ and (a) $c \in F_y$, $z \in A_{cy}$ imply $\exists\, w \in A_{(\nu c)x}$ with wPz,
(b) $c, c' \in F_y$, $\nu c = \nu c'$ imply $c = c'$.

We now prove the following isotony result:

Theorem: If P is a relation on S having the weak monotone property, then xPy implies

$$\frac{\partial V}{\partial p_x} \leqq \frac{\partial V}{\partial p_y}.$$

Proof: Define recursively $E_x(0, u) \equiv 0$,

$$\begin{aligned} E_x(1, u) &= \beta_x[C_x(t) - C_x(u)] \exp \{C_x(u) - C_x(t)\} \\ &= \Pr \{\text{first \& only event in } (u, t) \text{ is a blocked call} \mid x_u = x\}, \end{aligned}$$

$$\begin{aligned} E_x(n + 1, u) = \int_u^t & \exp \{C_x(u) - C_x(\tau)\} \\ & \cdot [\sum_{y \in B_x} E_y(n, \tau) + \beta_x \lambda(\tau)[E_x(n, \tau) + 1] \\ & + \lambda(\tau) \sum_{\substack{c \in x \\ c \text{ not blocked in } x}} \min \{1 + E_x(n, \tau), \min_{y \in A_{cx}} E_y(n, \tau)\}]\, d\tau. \end{aligned}$$

It follows easily that $E(1, u) \leqq E(u)$, and that $E(n + 1, u) \geqq E(n, u)$. Furthermore, standard methods[3] using the inequality

$$| \min_{1 \leqq i \leqq n} y_i - \min_{1 \leqq i \leqq n} (y_i + \epsilon_i) | \leqq \max_{1 \leqq i \leqq n} | \epsilon_i |$$

show that the functions $E_x(n, \cdot)$ converge monotonely as $n \rightarrow \infty$ to the unique solution of (3).

If now xPy, then $\beta_x \leqq \beta_y$, $c_x(\cdot) \equiv c_y(\cdot)$, and so $E_x(1, u) \leqq E_y(1, u)$. Assume as a hypothesis of induction that xPy implies $E_x(n, u) \leqq E_y(n, u)$, $0 \leqq u \leqq t$. Then with μ and ν as in the definition of the weak monotone property

$$E_z(n, u) \leqq E_{\mu z}(n, u) \quad \text{for} \quad z \in B_x$$

$$\begin{aligned} \min \{1 + E_x(n, u), \min_{z \in A_{(\nu c)x}} E_z(n, u)\} \\ \leqq \min \{1 + E_y(n, u), \min_{z \in A_{cy}} E_z(n, u)\} \end{aligned}$$

$$\sum_{z \varepsilon B_y} E_z(n, u) + \beta_y \lambda(u)[E_y(n, u) + 1]$$
$$+ \lambda(u) \sum_{\substack{c \varepsilon y \\ c \text{ not blocked in } y}} \min \{1 + E_y(n, u), \min_{z \varepsilon A_{cy}} E_z(n, u)\}$$
$$\geqq \sum_{z \varepsilon B_x} E_z(n, u) + \beta_x \lambda(u)[E_x(n, u) + 1] + \lambda(u)(\beta_y - \beta_x)[E_y(n, u) + 1]$$
$$- \lambda(u)[E_y(n, u) + 1] \sum_{\substack{c \varepsilon x \\ c \notin \text{rng}\, \nu \\ c \text{ not blocked in } x}} 1$$
$$+ \lambda(u) \sum_{\substack{c \varepsilon x \\ c \text{ not blocked in } x}} \min \{1 + E_x(n, u), \min_{z \varepsilon A_{cx}} E_z(n, u)\}.$$

It can be seen that with $| X |$ the cardinality of a set X,

$$\beta_y - \beta_x \geqq | \{c \varepsilon x : c \notin \text{rng}\, \nu, c \text{ not blocked in } x\} |,$$

whence $E_x(n+1, u) \leqq E_y(n+1, u)$. Since $\partial V / \partial p = E$ and $E(n, u) \uparrow E(u)$, the theorem follows.

VIII. THE NATURE OF THE OPTIMAL POLICIES

Where it is applicable, the isotony theorem allows us to infer the optimal routes for accepted calls. Its relevance to the optimal policies for networks for which there is a relation P with the weak monotone property is this: Let $c \varepsilon x$ be a call that is not blocked in state x, so that $A_{cx} \neq \theta$, and suppose that there is at least one $y \varepsilon A_{cx}$ such that yPz for every $z \varepsilon A_{cx}$. It follows from the isotony theorem that at any time u, such a y is at least as good a way of routing c (if c is attempted at u) as any other state of A_{cx}. The only action which might conceivably be better in this situation than accepting c and routing it so as to take the system to y is rejecting c altogether. Such a rejection would be optimal if and only if

$$1 + \frac{\partial V}{\partial p_x} \leqq \frac{\partial V}{\partial p_y};$$

for u's close to t, clearly, this is false. In these circumstances a policy that routes c in x so as to take the system to y can differ (so far as x and c are concerned) from an optimal policy only in the respect that the latter might reject c in x.

In Ref. 1, the notation

$$\sup_P A_{cx}$$

was used for the set

$$\{y: z \in A_{cx} \text{ implies } yPz\} \cap A_{cx} ,$$

whenever this set was nonempty. The set $\sup_P A_{cx}$ consists precisely of the possible states to which an optimal policy takes the system from state x if it accepts the attempted call c.

The preceding observations are summarized in the *Corollary: If P on S has the weak monotone property then there exists an optimal policy $R(\cdot)$ such that $c \in x$, $y \in A_{cx} r_{xy}(u) > 0$, $0 < u < t$ imply*

$$y \in \sup_P A_{cx} .$$

The theory of routing for minimal D constructed here can be developed in greater detail in the fashion of the optimal routing theorems of Section XVIII of Ref. 1; however, the isotony theorem and corollary embody the basic idea, and we shall leave the topic at this stage.

REFERENCES

1. Beneš, V. E., Programming and Control Problems Arising from Optimal Routing in Telephone Networks, B.S.T.J., *45*, November, 1966, pp. 1373–1438.
2. Athans, M. and Falb, P. L., *Optimal Control*, McGraw-Hill Book Co., Inc., New York, 1966, p. 360, Theorem 5–13.
3. Bellman R., *Stability Theory of Differential Equations*, McGraw-Hill Book Co., Inc., New York, 1953, pp. 65–9.