Off-Axis Wave-Optics Transmission in a Lens-Like Medium with Aberration

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Normal modes and their propagation constants have been found for a two-dimensional lens-like medium in which the transverse refractive index varies essentially with quadratic law but has perturbing terms of higher order. In such a realistic guiding medium with aberrations, those modes are used to find the field configuration of a Gaussian beam of half width W entering off-axis.

Close to the input the beam oscillates periodically with amplitude x_i as if the medium were aberration-free, but slowly the beam cross-section changes shape, breaking up in several maxima, and increases size, reaching a maximum approximately equal to $2(W + x_i)$. Afterwards the beam slowly shrinks back to the starting field configuration and the process repeats again.

These results are applicable to a sequence of lenses with aberrations and become important when the lenses are closely spaced. If redirectors are to be used to compensate for lens misalignments, the corrections must be made before a large break-up of the beam occurs.

I. INTRODUCTION

Transmission through a lens-like medium consisting of a dielectric rod in which the dielectric constant decreases radially with quadratic law has been studied because, first, it is closely related to light transmission in a periodic sequence of gaseous lenses,[1,2,3] second, it helps to understand the filamentary nature of the oscillations in ruby lasers,[4] and third, if the dielectric constant is complex, it describes the gain medium in a gaseous laser.[5]

Whenever the radial dependence of the dielectric constant is quadratic, any paraxial beam propagates undulating periodically about the axis, and the field reproduces itself after each period. This is true only within some approximations.* Part of the object of this paper is to find the distance over which the approximations hold and how it depends upon the beam shape and displacement from the axis. The mathematical description of Gaussian beams in such an aberration-free lens-like quadratic medium has been achieved via two techniques. The first consists of expanding the input beam in terms of on-axis normal modes[1] and summing them up everywhere. The second technique consists of solving differential equations that elegantly relate the location of the beam axis, the spot size, and the curvature of the phase front at any place along the beam.[7] Nevertheless, we show in this paper that if the lens-like medium has aberrations, no matter how small, the electromagnetic field of a paraxial beam changes shape radically as it travels along the guide. The distorted beam cannot be described any more by the beam axis location, the spot size, and the phase front curvature, and consequently, the off-axis beam must be calculated following the first technique.

The on-axis normal modes and their propagation constants are found in Section II using a first-order perturbation technique; the off-axis beam is calculated in Section III; the limits of validity of the previous results are given in Section IV and conclusions are reached in Section V.

The modes in lens-like media with arbitrarily large perturbations have been studied by S. E. Miller[8] and J. P. Gordon.[9] However, they have not given a quantitative description of an off-axis beam.

Beam deformations similar to those analyzed in this paper have been obtained by D. Marcuse[10] who calculated, via a computer, the transmission of an off-axis Gaussian beam through a sequence of curved and distorted lenses.

II. MODES IN A LENS-LIKE MEDIUM WITH SMALL ABERRATION

Consider a lens-like medium in which the refractive index

$$n = n_o\left[1 - \left(\frac{\pi x}{L}\right)^2 - \sum_\alpha a_\alpha\left(\frac{\pi x}{L}\right)^\alpha\right]^{\frac{1}{2}} \tag{1}$$

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* A. H. Carter has found[6] that if the dielectric constant of the lens-like medium varies radially, not with quadratic law, but with the square of the hyperbolic secant law, the periodicity is not restricted to paraxial beams and since there are no approximations involved, the results hold for any length. Unfortunately, the functions describing the modes are hypergeometric and they cannot be easily handled as the parabolic cylinder functions of the quadratic medium. Because of this and also because our main interest is the study of transmission through non-ideal media anyhow, we will keep on basing our calculations on the quadratic medium and we will even call it ideal.

varies only in the transverse direction x; n_o is the refractive index at $x = 0$, and L and a_α are constants whose physical significance will be shown later. The quadratic law of the dielectric constant is slightly perturbed by the summation in the integers $\alpha = 1, 2, 3 \cdots$.

We study here modes propagating along z in a medium with variation only in the x direction; nevertheless, the extension to the case in which the dielectric constant varies also along the other transverse direction y, can be easily achieved as shown in Ref. 1.

Assuming that the only component of electric field $\bar{E}$ is parallel to the y axis and independent of y the wave equation is

$$\frac{d^2\bar{E}}{dx^2} + \frac{d^2\bar{E}}{dz^2} + \left(\frac{2\pi}{\lambda}\right)^2\left[1 - \left(\frac{\pi x}{L}\right)^2 - \sum_\alpha a_\alpha\left(\frac{\pi x}{L}\right)^\alpha\right]\bar{E} = 0, \tag{2}$$

where λ is the plane wave wavelength in a medium of refractive index n_o.

Let us call

$$\xi = \frac{2x}{W} \tag{3}$$

a new transverse variable normalized to the beam half-width

$$W = \frac{\sqrt{\lambda L}}{\pi} \tag{4}$$

in the unperturbed medium, and also

$$\bar{E} = E_p \exp(-i\gamma_p z). \tag{5}$$

E_p, the transverse field distribution of the pth mode, is a function of ξ exclusively and

$$\gamma_p = \frac{2\pi}{\lambda}\sqrt{1 - \left(p + \tfrac{1}{2} + \sum_\alpha a_\alpha f_\alpha\left(\frac{\lambda}{L}\right)^{\alpha/2-1}\right)\frac{\lambda}{L}} \tag{6}$$

is the propagation constant of the pth mode along the z axis. The summation $\sum_\alpha a_\alpha f_\alpha(\lambda/L)^{\alpha/2-1}$, is the perturbation due to the aberrations and the unknown f_α will be determined later.

Now we substitute (3), (4), and (5) in (2) and obtain an equation in ξ exclusively for the transverse field E_p,

$$\frac{d^2E_p}{d\xi^2} + \left\{p + \frac{1}{2} - \left(\frac{\xi}{2}\right)^2 + \sum_\alpha a_\alpha\left(\frac{\lambda}{L}\right)^{\alpha/2-1}\left[f_\alpha - \left(\frac{\xi}{2}\right)^\alpha\right]\right\}E_p = 0. \tag{7}$$

For $a_\alpha = 0$, the solution[11] is the parabolic cylinder function $D_p(\xi) = \exp(-\xi^2/4)H_p(\xi)$ which is a product of the Gaussian function $\exp(-\xi^2/4)$ and the Hermite polynomial of integer and positive order p.

For $a_\alpha \neq 0$ it is shown in Appendix A that if

$$\sum_\alpha a_\alpha \left(\frac{\lambda}{L} p\right)^{\alpha/2-1} \ll 1 \tag{8}$$

the differential equation (7) can be solved by using first-order stationary perturbation theory.[12] Then, the transverse field distribution of the pth mode in the perturbed medium expressed in terms of the normal modes of the unperturbed medium is

$$E_p = D_p(\xi) + \sum_{n \neq p} c_n D_n(\xi); \tag{9}$$

the sum extends from 0 to ∞ excluding $n = p$ and the values of c_n are given in (54). We also find in (55) that

$$f_\alpha = \begin{cases} 2^{-3\alpha/2}\alpha!\,p! \displaystyle\sum_{m=0}^{\alpha/2} \frac{2^m}{(m!)^2(p-m)!\left(\frac{\alpha}{2}-m\right)!} & \text{if } \alpha \text{ is even}, \\ 0 & \text{if } \alpha \text{ is odd}. \end{cases} \tag{10}$$

Further on, specific values of f_α will be needed, so some of them are given in Table I. From the table and (6) we conclude that if the medium has only antisymmetric perturbation (α odd), the propagation constants of its modes are identical to those of the modes in the unperturbed medium. Physically, this means that to a first order, the change in phase velocity introduced by the perturbation on one side of the guide is canceled by the other.

III. BEAM IN THE PERTURBED MEDIUM

We want to calculate the field $F(z)$ everywhere in the perturbed medium such that at the origin it coincides with a prescribed function

TABLE I—SOME SPECIFIC VALUES OF f_α

α	f_α
1	0
2	$\frac{1}{4}(1 + 2p)$
3	0
4	$\frac{3}{16}(1 + 2p + 2p^2)$
5	0
6	$\frac{15}{64}(1 + \frac{8}{3}p + 2p^2 + \frac{4}{3}p^3)$

$F(0)$. In terms of the normal modes E_p, the field is

$$F(z) = \sum_{p=0}^{\infty} A_p E_p \exp(-i\gamma_p z), \tag{11}$$

and at the origin

$$F(0) = \sum_{p=0}^{\infty} A_p E_p . \tag{12}$$

We simplify next $A_p E_p$ and the propagation constant γ_p.

Since the approximately calculated normal modes of the medium with aberration are orthogonal to the first order of the perturbations $a_\alpha(\lambda/L)^{\alpha/2-1}$, we calculate, with the help of (9), each term of the summation to be

$$A_p E_p = B_p D_p(\xi) + O[a_\alpha(\lambda/L)^{\alpha/2-1}], \tag{13}$$

where

$$B_p = \frac{\int_{-\infty}^{\infty} F(0) D_p(\xi)\, d\xi}{\int_{-\infty}^{\infty} D_p^2(\xi)\, d\xi} \tag{14}$$

is the amplitude of the pth mode of an expansion of the input field $F(0)$ in terms of modes of the unperturbed guide and the term of the order of $a_\alpha(\lambda/L)^{\alpha/2-1}$, given only for completeness, is

$$O\left[a_\alpha\left(\frac{\lambda}{L}\right)^{\alpha/2-1}\right] = \sum_{\substack{n=0 \\ n\neq p}}^{\infty} c_n \frac{D_n(\xi)\int_{-\infty}^{\infty} F(0) D_p(\xi)\, d\xi + D_p(\xi)\int_{-\infty}^{\infty} F(0) D_n(\xi)\, d\xi}{\int_{-\infty}^{\infty} D_p^2(\xi)\, d\xi}.$$

Substituting (13) in (11) one obtains

$$F(z) = \sum_{p=0}^{\infty} B_p D_p(\xi) \exp(-i\gamma_p z) + \sum_{p=0}^{\infty} O\left[a_\alpha\left(\frac{\lambda}{L}\right)^{\alpha/2-1}\right] \exp(-i\gamma_p z). \tag{15}$$

We will find that for large z the first summation yields configurations which depart grossly from the field in the unperturbed medium, therefore, according to (8), the second summation whose amplitude is of the order of $a_\alpha(\lambda/L)^{\alpha/2-1}$ will be neglected.

Furthermore, the propagation constant γ_p, (6), which is exact for the ideal quadratic medium and good to first order of $\sum_\alpha a_\alpha(\lambda/L)^{\alpha/2-1}$

can be expanded in series

$$\gamma_p = \frac{2\pi}{\lambda}\left\{1 - \frac{1}{2}\left[p + \frac{1}{2} + \sum_\alpha a_\alpha f_\alpha\left(\frac{\lambda}{L}\right)^{\alpha/2-1}\right]\frac{\lambda}{L} - \frac{1}{8}\left[p + \frac{1}{2} + \sum_\alpha a_\alpha f_\alpha\left(\frac{\lambda}{L}\right)^{\alpha/2-1}\right]^2\frac{\lambda^2}{L^2} - \cdots\right\}. \tag{16}$$

The third and higher-order terms can be neglected only if their contribution to the phaseshift of the highest-order mode P of significant amplitude is small, that is, if

$$\frac{\pi\lambda z}{4L^2}\left[P + \frac{1}{2} + \sum_\alpha a_\alpha f_\alpha\left(\frac{\lambda}{L}\right)^{\alpha/2-1}\right]^2 \ll 1.$$

According to (47) in the Appendix, once the inequality (8) is satisfied, the summation is negligible compared to P and consequently the inequality above reduces to

$$\frac{\pi\lambda z}{4L^2}P^2 \ll 1. \tag{17}$$

This expression establishes the range of validity z, of the description of a beam composed essentially of P modes at wavelength λ, traveling in a lens-like medium characterized by L. Since the inequality is independent of a_α we conclude that the finite range of validity z is determined not by the perturbation of the medium, but by the approximation involved in the expansion (16) of the propagation constants thus affecting the description of the beam even in the ideal quadratic medium.

Once P is known, the inequality (8) determines the magnitude of the perturbation a_α for which the calculations are valid. Then, provided that inequalities (8) and (17) are satisfied the simplified version of the field (15) becomes

$$F(z) = \sum_{p=0}^{\infty} B_p D_p(\xi) \exp\left\{-i\left[\gamma_{pi} - \frac{\pi}{L}\sum_\alpha a_\alpha f_\alpha\left(\frac{\lambda}{L}\right)^{\alpha/2-1}\right]z\right\}, \tag{18}$$

in which

$$\gamma_{pi} = \frac{2\pi}{\lambda}\left[1 - \frac{\lambda}{2L}(p + \tfrac{1}{2})\right] \tag{19}$$

is the propagation constant of the pth mode in the unperturbed ideal quadratic medium.

Therefore, the field $F(z)$ is described by a summation of modes of the ideal quadratic law medium but with phase constants corresponding to modes in the perturbed medium. Actually, if $a_\alpha = 0$, we derive from (18) the field in the ideal quadratic medium

$$I(z) = \sum_{p=0}^{\infty} B_p D_p(\xi) \exp(-i\gamma_{pi} z). \tag{20}$$

In order to find out more about the field distribution along the perturbed guide, we will work out a typical example assuming that the only perturbation of the dielectric is of order 4. Dropping the subscript from a_4 and with the help of Table I we find that

$$\sum_\alpha a_\alpha f_\alpha \left(\frac{\lambda}{L}\right)^{\alpha/2-1} = af_4 \frac{\lambda}{L} = \frac{3}{16} a \frac{\lambda}{L} (1 + 2p + 2p^2).$$

The field (18) then results

$$F(z) = \exp\left(i \frac{\pi}{2} \frac{z}{D}\right) \sum_{p=0}^{\infty} B_p D_p(\xi) \exp\left\{-i\left[\gamma_{pi} - \frac{\pi}{D}(p + p^2)\right]z\right\}, \tag{21}$$

where the distance

$$D = \frac{8L^2}{3a\lambda} \tag{22}$$

has a physical significance to be described later.

I do not know how to add (21) in general but the summation can be performed for discrete and significant values of z. They are

$$z_{\mu,\nu} = (\mu + 2^{-\nu})D \tag{23}$$

where μ is an arbitrary integer and ν is a positive integer. The summation can also be performed for $z = (\mu - 2^{-\nu})D$ but the results are quite similar to those found for $z_{\mu\nu}$. For the particular values of z, given in (23), the exponential in (21) containing p^2 can be expressed as a sum of exponentials containing only p. As a matter of fact,

$$\exp[i\pi p^2(\mu + 2^{-\nu})]$$
$$= \exp\left\{-i\pi p\left[\mu + \frac{1}{(-\nu)!}\right]\right\} \sum_{q=0}^{2^\nu-1} G_q \exp(-i2^{1-\nu}\pi qp) \tag{24}$$

and

$$G_q = 2^{-\nu} \sum_{s=0}^{2^\nu-1} \exp[i\pi 2^{-\nu}s(s + 2q)]. \tag{25}$$

The correctness of this expansion can be verified by substituting (25) in (24) and performing the summation first in q and then in s.

The field (21) at $z = z_{\mu\nu}$ results with the help of (24)

$$F(z_{\mu\nu}) = \exp\left[i\frac{\pi}{2}(\mu + 2^{-\nu})\right]\sum_{q=0}^{2^\nu-1} G_q \exp(i\varphi_{\nu q}) I(\zeta_{\mu\nu q}), \tag{26}$$

where

$$\varphi_{\nu q} = -\frac{\pi}{2}\left(1 - \frac{4L}{\lambda}\right)\left[\frac{1}{2^\nu} - \frac{1}{(-\nu)!} - \frac{q}{2^{\nu-1}}\right] \tag{27}$$

and

$$\zeta_{\mu\nu q} = (\mu + 2^{-\nu})D + \left(\frac{1}{2^\nu} - \frac{1}{(-\nu)!} - \frac{q}{2^{\nu-1}}\right)L. \tag{28}$$

Ignoring the overall uninteresting phase $\pi/2(\mu + 2^{-\nu})$, equation (26) establishes that the field at $z_{\mu\nu}$ is made of 2^ν terms, each given by a factor $G_q \exp i\varphi_{\nu q}$ times the field distribution $I(\zeta_{\mu\nu q})$. As we saw before, this field (20) coincides with the field distribution that the input $F(0)$ would have in the aberration free medium at a distance $z = \zeta_{\mu\nu q}$ given in (28). Therefore, in the perturbed medium, at $z = z_{\mu\nu}$, the field $F(z_{\mu\nu})$ due to an arbitrary input $F(0)$ is described by the superposition of field distributions that the same input $F(0)$ would produce in the unperturbed medium at 2^ν cross-sections located at distances $\zeta_{\mu\nu q}$ from the origin, multiplied by certain phase shifts and amplitudes.

Let us extend the example assuming the input $F(0)$ to be an off-axis Gaussian of half-width $W = \sqrt{\lambda L/\pi}$. In the aberration-free medium the beam trajectory is a sinusoid of period $2L$, and at all cross-sections the field is Gaussian of half-width W (see Fig. 1(a)). Now we proceed to calculate the field $F(z_{\mu\nu})$ in the perturbed medium at the specific abscissas $z_{\mu\nu}$ (23) assuming the same off-axis Gaussian input $F(0)$.

For $\nu = 0$, that is at abscissas

$$z_{\mu o} = (\mu + 1)D \tag{29}$$

the field in the perturbed medium is derived from (26) and (25) to be

$$F(z_{\mu o}) = \exp\left[i\frac{\pi}{2}(\mu + 1)\right] I(\zeta_{\mu oo}). \tag{30}$$

Except for the phase shift $\pi/2(\mu + 1)$ the field in the perturbed medium, $F(z_{\mu o})$ coincides with the Gaussian field in the unperturbed medium, $I(\zeta_{\mu oo})$ at the same abscissa

$$z_{\mu o} = \zeta_{\mu oo} = (\mu + 1)D. \tag{31}$$

The distance $D = 8L^2/3a\lambda$ is then the distance between successive

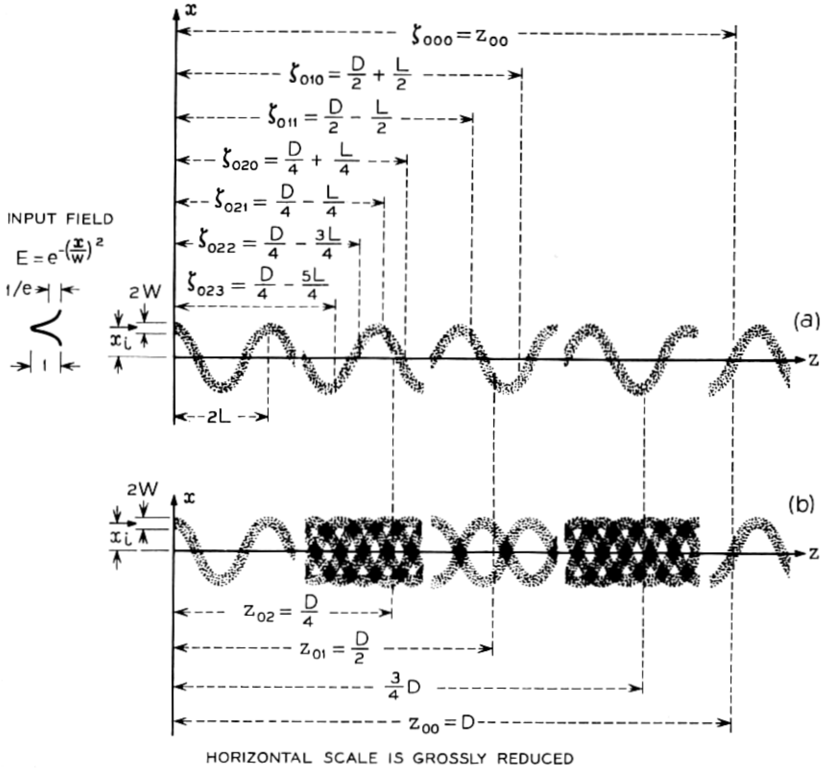

Fig. 1 — (a) Off-axis beam in ideal medium, $n = n_o[1 - (\pi x/L)^2]^{\frac{1}{2}}$. (b) Off-axis beam in perturbed medium, $n = n_o[1 - (\pi x/L)^2 - a(\pi x/L)^4]^{\frac{1}{2}}$. $D = 8L^2/3a\lambda$ and $w = \sqrt{\lambda L/\pi}$.

Gaussian field distribution in the perturbed medium. We will call D a pseudo-period because the fields at abscissas $z_{\mu o}$, in general, do not have the same position with respect to the axis as the input field. Only if $4L/3a\lambda$ is an integer those fields are identical and the pseudo-period becomes a true period.

Do we know about the field close to $z_{\mu o}$, say within a few L? The perturbed medium differs only slightly from the quadratic one, therefore close to $z_{\mu o}$, the field in both media must be quite similar. The similarity of fields in both media in the neighborhood of abscissas $(\mu + 1)D$ for μ equal to -1 and 0 is depicted in Figs. 1(a) and 1(b).

For $\nu = 1$, that is at abscissas

$$z_{\mu 1} = (\mu + \tfrac{1}{2})D \tag{32}$$

half way between the previous ones (29), the field derived again from (26) and (25) is

$$F(z_{\mu 1}) = \frac{\exp\left[i\frac{\pi}{2}(\mu + 1)\right][I(\zeta_{\mu 10}) - iI(\zeta_{\mu 11})]}{\sqrt{2}}. \tag{33}$$

Ignoring the phases the field in the perturbed medium $F(z_{\mu 1})$ is made of the superposition of the fields $I(\zeta_{\mu 10})$ and $I(\zeta_{\mu 11})$ in the unperturbed medium, reduced in amplitude by $\sqrt{2}$ and found at abscissas derived from (28) to be

$$\zeta_{\mu 10} = (\mu + \tfrac{1}{2})D + \frac{L}{2} \tag{34}$$

and

$$\zeta_{\mu 11} = (\mu + \tfrac{1}{2})D - \frac{L}{2}. \tag{35}$$

Again, since the medium is only slightly perturbed, the two Gaussians can be treated independently. Therefore, within a few L from $z_{\mu 1}$, the field in the perturbed medium consists of two Gaussian beams interleaving. In Fig. 1(a), for $\mu = 0$, the fields at abscissas $\zeta_{010} = (D+L)/2$ and $\zeta_{011} = (D - L)/2$ are located, and then used in Fig. 1(b) to construct the field in the perturbed medium around $z_{01} = D/2$.

For $\nu = 2$ we find that the field at $z_{\mu 2} = (\mu + \frac{1}{4})D$, can be synthesized by the superposition of four fields that, except for the amplitude and phase, are found in the unperturbed medium at abscissas

$$\begin{aligned} \zeta_{\mu 20} &= z_{\mu 2} + \frac{L}{4}, \\ \zeta_{\mu 21} &= z_{\mu 2} - \frac{L}{4}, \\ \zeta_{\mu 22} &= z_{\mu 2} - \frac{3L}{4}, \end{aligned} \tag{36}$$

and

$$\zeta_{\mu 23} = z_{\mu 2} - \frac{5L}{4}.$$

The field close to $z_{\mu 2}$ consists of four Gaussian beams weaving as shown in Fig. 1(b).

Within a pseudo-period the field in the perturbed medium differs substantially from the Gaussian input, and at a single cross-section it can exhibit several maximas if the peaks of the several Gaussians that depict the field at that cross-section are resolved. Furthermore, the cross-section of the beam becomes at most as wide as twice the half-width of the input beam plus twice the amplitude of the beam oscillation in the unperturbed medium.

Even though the fields in the perturbed medium at distances close to $D/2$ and $D/4$, Fig. 1(b), are made of symmetrically interleaving beams it must not be concluded that the fields at $D/8$, $D/16$, etc. must be symmetric with respect to the $x = 0$ plane. As soon as the interleaving beams overlap, the relative phases become important and the apparent symmetry breaks up. As a matter of fact, close to $z = 0$, the field (26) can be expressed in two ways. Either choosing $\mu = -1$ and $\nu = 0$ in which case the field is that of a single beam or else picking $\mu = 0$ and $\nu \rightarrow \infty$, in which case the same field is made by the superposition of 2^{ν} interleaving beams with deceiving symmetry.

If instead of Gaussian the input field has another shape such as that in Fig. 2(a), the field in the unperturbed medium reproduces the input at even multiples of L, while it repeats the input mirrored in the plane

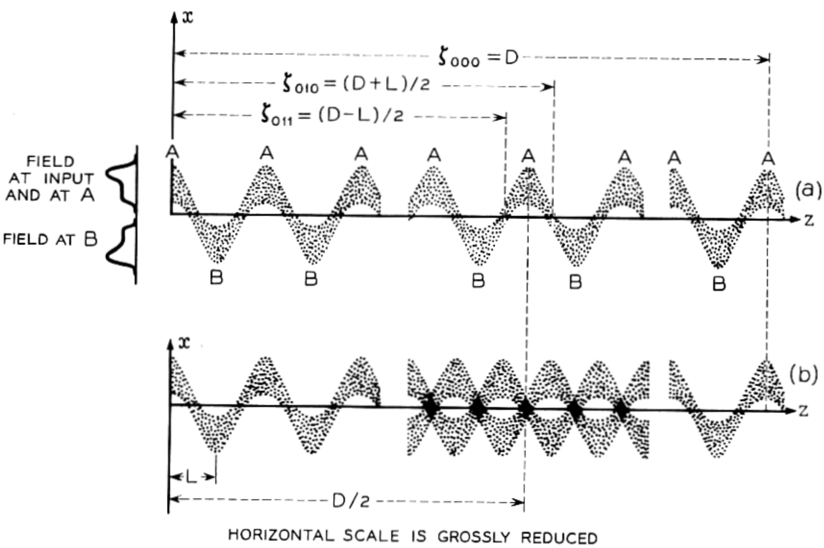

Fig. 2—(a) Off-axis beam in ideal medium, $n = n_o[1 - (\pi x/L)^2]^{\frac{1}{2}}$. (b) Off-axis beam in perturbed medium, $n = n_o[1 - (\pi x/L)^2 - a(\pi x/L)^4]^{\frac{1}{2}}$. $D = 8L^2/3a\lambda$ and $4L/3a\lambda$ integer.

$x = 0$, at odd multiples of L. In between the beamwidth varies periodically with period L and we represent it qualitatively in Fig. 2(a).

The field in the perturbed medium at $z_{0\infty}$, z_{00}, z_{01}, and their neighborhood is depicted in Fig. 2(b) following similar steps to those taken for the previous example.

If the perturbation of the medium were of sixth-order, instead of fourth, the pseudo-period, deduced from (18) and Table I, would be $32/5\ L^3/a_6\lambda^2$ and as before, an off-axis beam would periodically deform itself exhibiting several maxima and the largest beam cross-section would be roughly twice the input beam half-width plus twice the input beam displacement from the axis.

V. LIMITS OF APPLICABILITY

Over what length z is the field in the unperturbed medium (20) valid? That length, calculated from the inequality (17), is

$$z \ll \frac{4L^2}{\pi\lambda P^2}. \tag{37}$$

We need the value of P, that is the highest-order mode of significant amplitude.

Continuing with the example, the input

$$F(0) = D_0(\xi - \xi_i)$$

is a Gaussian beam of half-width W, displaced x_i (normalized ξ_i) from the z axis. Then according to (14) the amplitude of the pth mode is

$$B_p = \frac{\displaystyle\int_{-\infty}^{\infty} \exp\left[-\frac{(\xi - \xi_i)^2}{4}\right] D_p(\xi)\, d\xi}{\displaystyle\int_{-\infty}^{\infty} D_p^2(\xi)\, d\xi} = \left(\frac{\xi_i}{2}\right)^p \frac{\exp\left(-\frac{\xi_i^2}{8}\right)}{!}. \tag{38}$$

For

$$p \gg 1$$

the asymptotic value of B_p results[13] in

$$B_p = \frac{\exp\left(-\frac{\xi_i^2}{8}\right)}{\sqrt{2\pi p}} \left(\frac{e\xi_i}{2p}\right)^p. \tag{39}$$

Obviously B_p decreases rapidly for p such that the second parenthesis is smaller than one. Therefore, we select the highest-order mode of significant amplitude to be

$$P = \frac{e\xi_i}{2}$$

or substituting ξ_i by its equivalent in (3)

$$P = \frac{ex_i}{W}. \tag{40}$$

With this value and that for W given in (4) the range of validity (37) results

$$z \ll \frac{4}{\pi^3 e^2} \frac{L^3}{x_i^2}. \tag{41}$$

It depends on the characteristic of the medium L and the beam input displacement x_i but is independent of wavelength. Let us put some typical numbers for a sequence of closely spaced gas lenses.[3] For

$$L = 1m$$

$$x_i = 2mm$$

the range of validity results

$$z \ll 4300m.$$

The inequality (8) determines the amount of perturbation for which the calculations are applicable. For $\alpha = 4$, that inequality reads

$$a \frac{\lambda}{L} \ll \frac{1}{P}$$

and with the help of (40)

$$a \frac{\lambda}{L} \ll \frac{W}{ex_i}. \tag{42}$$

The physical significance of $a\lambda/L$ can be derived from (1). With only fourth-order aberration the refractive index of the perturbed medium is

$$n = n_0\left[1 - \left(\frac{\pi x}{L}\right)^2 - a\left(\frac{\pi x}{L}\right)^4\right]^{\frac{1}{2}}.$$

The ratio between the perturbing term and the quadratic term at an ordinate x is

$$r_x = a\left(\frac{\pi x}{L}\right)^2.$$

For x equal to a beam half-width $W = \sqrt{\lambda L/\pi}$,

$$r_W = a \frac{\lambda}{L}. \tag{44}$$

With (42) the following self-explanatory inequality is derived

$$r_W \ll \frac{W}{ex_i}. \tag{45}$$

Let us continue with the previous example. For

$$\lambda = 1\mu$$

and

$$r_W = 0.1 \frac{W}{ex_i},$$

we obtain

$$r_W = 0.0058.$$

The pseudo-period (22) results

$$D = \frac{8}{3} \frac{L}{r_W}, \tag{46}$$

that is $D = 460\ m$.

VI. CONCLUSIONS

An off-axis beam oscillates and reproduces itself periodically for any arbitrary length only when it propagates in the hyperbolic secant square lens-like medium described by A. H. Carter.[6] Nevertheless, within some approximations the quadratic law yields similar results with much simpler mathematics. For example, a Gaussian beam which is properly matched to maintain a constant spot size will oscillate periodically with period $2L$ and amplitude x_i. We have found that this approximation holds for a distance z that obeys the inequality

$$z \ll \frac{4}{\pi^3 e^2} \frac{L^3}{x_i^2}.$$

We surmise that the inequality is valid also if the width of the beam is not perfectly matched and if the guiding medium is not continuous but made of a sequence of square law lenses. For a typical sequence of gas lenses with half natural period $L = 1$ m and beam displacement $x_i = 2$ mm the range of validity is

$$z \ll 4300\text{m}.$$

A more realistic lens-like medium though, is one in which the quadratic law is perturbed by aberration terms of higher order. Again, a Gaussian beam of half-width $W = \sqrt{\lambda L/\pi}$ entering parallel to the axis at a distance x_i from it oscillates at the beginning with period $2L$, but changes shape slowly as it travels along, increasing the width of the beam up to $2(x_i + W)$, after which the beamwidth shrinks back to the starting value $2W$. This process repeats at intervals D that depend on the nature of the aberration. For fourth-order aberration

$$D = \frac{8}{3}\frac{L}{r_W},$$

where r_W is the ratio between the fourth-order term and the quadratic term of the refractive index at a distance from the axis equal to the natural half beamwidth $W = \sqrt{\lambda L/\pi}$ of the medium. These results hold as long as

$$r_W \ll \frac{W}{ex_i}.$$

Where the beam is large and distorted, the field intensity at one cross-section exhibits several maxima. The number of them and their resolution varies along the trajectory. Both increase with the ratio of x_i/W.

We saw that one of the effects of the aberrations is to smear the beam size to roughly twice the displacement that the same beam would have in an ideal quadratic medium; furthermore, it is known that in a sequence of perfect but randomly misaligned lenses the rms deviation of a beam from the axis grows proportionally to the square root of the number of lenses.[14] Therefore, if the lenses have aberrations we conclude tentatively that the rms beam size grows with the same law. If redirectors[15] are to be used to compensate for misalignment of the lenses the corrections probably must be made before a large break-up of the beam occurs.

APPENDIX

Approximate Solution of

$$\frac{d^2E_p}{d\xi^2} + \left\{p + \frac{1}{2} - \left(\frac{\xi}{2}\right)^2 + \sum_\alpha a_\alpha\left(\frac{\lambda}{L}\right)^{\alpha/2-1}\left[f_\alpha - \left(\frac{\xi}{2}\right)^\alpha\right]\right\}E_p = 0.$$

Provided that

$$\sum_\alpha a_\alpha(\lambda/L)^{\alpha/2-1}f_\alpha \ll p + \tfrac{1}{2} \tag{47}$$

and

$$\sum_\alpha a_\alpha(\lambda/L)^{\alpha/2-1}(\xi/2)^{\alpha-2} \ll 1 \tag{48}$$

up to a value of ξ to be defined later, the differential equation can be solved using the stationary perturbation theory.[12]

The solution given in terms of parabolic cylinder functions is

$$E_p = D_p(\xi) + \sum_{n \neq p} c_n D_n(\xi). \tag{49}$$

The summation is a perturbation on $D_p(\xi)$ and it extends from 0 to ∞ excluding the pth term.

The eigenvalue corresponding to the pth eigenfunction E_p is

$$p + \sum_\alpha a_\alpha (\lambda/L)^{\alpha/2-1} f_\alpha .$$

In order to find c_n and f_α we substitute E_p given by (49) in the differential equation. Neglecting terms containing products of $a_\alpha c_n$ and knowing that the parabolic cylinder equation is

$$\frac{d^2 D_p(\xi)}{d\xi^2} + \left[p + \frac{1}{2} - \left(\frac{\xi}{2}\right)^2\right] D_p(\xi) = 0, \tag{50}$$

we obtain

$$\sum_{n \neq p} (p - n) c_n D_n(\xi) + \sum_\alpha a_\alpha \left(\frac{\lambda}{L}\right)^{\alpha/2-1} \left[f_\alpha - \left(\frac{\xi}{2}\right)^\alpha\right] D_p(\xi) = 0. \tag{51}$$

The functions $D_q(\xi)$ with integer index q, are orthogonal, therefore, multiplying (51) by $D_q(\xi)$ and integrating from $-\infty$ to $+\infty$ one derives

$$c_n = \sum_\alpha a_\alpha \left(\frac{\lambda}{L}\right)^{\alpha/2-1} \frac{\int_{-\infty}^{\infty} \left(\frac{\xi}{2}\right)^\alpha D_p(\xi) D_n(\xi)\, d\xi}{(p - n) \int_{-\infty}^{\infty} D_n^2(\xi)\, d\xi} \tag{52}$$

and

$$f_\alpha = \frac{\int_{-\infty}^{\infty} \left(\frac{\xi}{2}\right)^\alpha D_p^2(\xi)\, d\xi}{\int_{-\infty}^{\infty} D_p^2(\xi)\, d\xi}. \tag{53}$$

After integration[16] the explicit results of the last two expressions are

$$c_n = \begin{cases} \displaystyle\sum_\alpha a_\alpha \left(\frac{\lambda}{L}\right)^{\alpha/2-1} \frac{2^{(p+n-3\alpha)/2} \alpha!\, p!}{p - n} \\ \displaystyle\quad \cdot \sum_{m=0}^{n} \frac{2^{-m}}{(p - m)!\,(n - m)!\,m!\left(m - \dfrac{p + n - \alpha}{2}\right)!} \\ \qquad\qquad\qquad \text{if} \quad p + n + \alpha \text{ is even}, \\ 0 \qquad\qquad\qquad \text{if} \quad p + n + \alpha \text{ is odd} \end{cases} \tag{54}$$

and

$$f_\alpha = \begin{cases} 2^{-3\alpha/2}\alpha!\,p!\sum_{m=0}^{\alpha/2}\dfrac{2^m}{(m!)^2(p-m)!\left(\dfrac{\alpha}{2}-m\right)!} & \text{if } \alpha \text{ is even,} \\ 0 & \text{if } \alpha \text{ is odd.} \end{cases} \tag{55}$$

These results are applicable as long as the inequalities (47) and (48) are satisfied. Both are harder to satisfy for large values of f_α and ξ which we proceed to find next.

According to (55), f_α increases with the order of the mode p and for large va ues of p

$$f_\alpha \cong \frac{\alpha!}{2^\alpha\left(\dfrac{\alpha}{2}\,!\right)^2}\,p^{\alpha/2}. \tag{56}$$

The function E_p given in (49) has significant amplitude essentially in the same range of ξ than the unperturbed solution $D_p(\xi)$. Since[11]

$$D_p(\xi) = (-1)^p \exp\left(\frac{\xi^2}{4}\right)\frac{d^p}{d\xi^p}\exp\left(-\frac{\xi^2}{2}\right) \tag{57}$$

a good approximation for $\xi \gg 1$ is

$$D_p(\xi) \cong \exp\,(-\xi^2/4)\xi^p. \tag{58}$$

This function has significant amplitude for values of ξ smaller than that of the second inflexion point. We find it by making the second derivative of (58) equal to zero and by finding the largest solution of that equation. The result, again for large p is

$$\xi_{\max} = 2\sqrt{p}. \tag{59}$$

Substituting f_α and $\xi_{\max}$ of (56) and (59) in (47) and (48) we obtain

$$\sum_\alpha a_\alpha\left(\frac{\lambda}{L}\,p\right)^{\alpha/2-1}\frac{\alpha!}{2^\alpha\left(\dfrac{\alpha}{2}\,!\right)^2} \ll 1 \tag{60}$$

and

$$\sum_\alpha a_\alpha\left(\frac{\lambda}{L}\,p\right)^{\alpha/2-1} \ll 1. \tag{61}$$

Furthermore,

$$\frac{\alpha!}{2^\alpha\left(\frac{\alpha}{2}\,!\right)^2} < 1$$

for

$$\alpha > 0.$$

Therefore, condition (61) is the most stringent. Given a_α, λ, and L it establishes which is the highest-order mode p for which the perturbation calculations apply.

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