Linear Time-Varying Circuits—Matrix Manipulations, Power Relations, and Some Bounds on Stability

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This paper is concerned with general circuits of linear, time-varying, positive, two-terminal components. It describes methods of manipulating corresponding matrix (vector) differential equations. It uses the manipulations to derive equations for power and bounds on stability. The bounds apply to the exponential factors associated with the basis functions of periodically varying circuits.

I. INTRODUCTION

Linear time-varying circuits of special kinds have been designed and analyzed with notable success. On the other hand, theoretical techniques suitable for more general linear time-varying circuits have been developing much more slowly. The development of more general techniques can be approached in various ways. One can seek to specialize the pure mathematics of linear differential equations, in order to discover the properties of those equations which can actually correspond to physical circuits. Alternatively, one can seek to apply the classical analysis of general dynamical systems. As still another alternative, one can seek to generalize, for time-varying circuits, concepts, principles, and techniques which have long been applied to fixed circuits.

This paper illustrates the circuit theory approach. After formulating matrix (vector) differential equations corresponding to circuits of linear, time-varying, two-terminal components, it describes some general methods of manipulation. These apply to combinations of time-varying matrices and the differentiation operator, and are time-varying counterparts of manipulations applied to constant matrices in the theory of fixed circuits. Thereafter, the paper uses the manipulations to derive formulas for power, and some bounds on stability.

The power equations are conventional and reflect the well known fact that time-varying capacitors and inductors can supply energy to a circuit. However, they can be derived in a way which illustrates manipulation of time-varying matrices in particularly simple terms. Furthermore, the form of the power equations suggests a starting point which leads eventually to the bounds on stability.

It is assumed throughout the paper that all circuit components (fixed or varying) are positive. The stability bounds (as derived) assume that the circuits vary in a periodic manner. Then the basis functions can be arranged as a set of exponentials, each multiplied by a periodically varying coefficient (except in singular cases which are the time-varying counterparts of fixed circuits whose frequency functions have multiple poles). The basis functions are the counterparts of the "natural modes" of fixed networks, and they play an equally important role.

The signs of the real parts of the exponents, in the basis functions, determine whether the functions grow indefinitely, or die out. The stability bounds derived herein are upper and lower bounds on the real parts of the exponents. The specific bounds depend on the composition of the circuit — whether it is composed exclusively of resistors and capacitors, resistors and inductors, or capacitors and inductors, or includes all three kinds of components. For each composition, there are two pairs of bounds, corresponding respectively to the node equation and mesh equation (which will be defined in Section II).

The form of the bounds is illustrated by the following: Let G and C be the node matrices of the conductances and capacitances of a periodically varying circuit of resistors and capacitors. Consider the zeros in terms of the scalar variable λ of the determinant of $(G + \frac{1}{2}\dot{C} + \lambda C)$. If the capacitances are positive at all times, the matrices can be so defined that C is positive definite. Then the zeros of the determinant are real, and they vary periodically with time. It will be shown that the time averages of the instantaneous maximum and minimum over the set of zeros are upper and lower bounds on the real parts of the exponents in the basis functions.

The mesh equations lead to similar bounds in terms of the zeros of the determinant of $(K - \frac{1}{2}\dot{R} + \lambda R)$, in which K and R are the mesh matrices of the stiffnesses and resistances of the capacitors and resistors. It is well known, on energy grounds, that a circuit of varying positive resistors and fixed positive capacitors cannot be unstable. These bounds show that, likewise, a circuit of fixed resistors and varying capacitors cannot be unstable. It is true even though the varying capacitors can give power gain.

Similar bounds can be obtained for circuits of resistors and inductors only, by a simple transformation of current and voltage variables.

The bounds for "RC" and "RL" circuits are at least reminiscent of known bounds on the characteristic roots of a dissymmetrical constant matrix. Let A be a real matrix. The largest and the smallest of the characteristic roots of $\frac{1}{2}(A + A^t)$ are upper and lower bounds on the real parts of the characteristic roots of A itself.

For a circuit of inductors and capacitors only, let S be the node matrix of the reciprocals of the inductances, and let C be again the node matrix of the capacitances. Consider the zeros of the determinants of the *two* matrices $(\frac{1}{2}\dot{C} + \lambda C)$ and $(-\frac{1}{2}\dot{S} + \lambda S)$. Treat the two sets of zeros as a single set of numbers. Then the time average of the maximum over the set is an upper bound on the real parts of the exponents in the basis functions, and the time average of the minimum over the set is a lower bound. The mesh analysis leads to similar bounds, except that the pertinent matrices are now $(\frac{1}{2}\dot{L} + \lambda L)$ and $(-\frac{1}{2}\dot{K} + \lambda K)$, where L and K are the mesh matrices of inductances and stiffnesses.

In some ways, the bounds for circuits of inductors and capacitors are less satisfactory than those for circuits of resistors and capacitors. When the inductors and capacitors are fixed (and positive) the damping is necessarily zero. When they are varying the damping may or may not be zero. The bounds derived here generally bracket zero damping. Thus they do not say whether or not a time-varying circuit of inductors and capacitors has any basis functions with damping different from zero. They merely say that if the damping is different from zero it is at least within the bounds.

Similar bounds are easily established for circuits of all three kinds of components. However, they tend to be weaker than the bounds for circuits of two kinds of components only, in ways which will be explained.

Some of the derivations have not been completed to the extent of proving validity for all singular, as well as normal, situations. What is reported here is the result of exploratory studies of time-varying circuits, which still leave some details to be filled in.

The author is indebted to I. W. Sandberg, whom he has consulted freely concerning properties of matrices in general and of positive definite and semidefinite matrices in particular. The circuit and power equations are formulated more carefully by C. A. Desoer and A. Paige.[1] However, their analysis tends more to pure mathematics and less to the theory of fixed circuits. They do not include the circuit theory type of manipulations or the stability bounds, which are the primary concerns of this paper. H. E. Meadows has noted somewhat similar, but weaker bounds.[2] R. A. Rohrer[3] has derived independently the same stability bounds. However, he has done so in terms of classical dynamics (generalization of the equations of Hamilton and Lagrange) rather than in the circuit theory terms used here.

The material in this paper is organized as follows: All aspects of circuits of resistors and capacitors are considered first — formulation of circuit equations, power relations, and stability conditions. Then the same analysis is shown to apply to circuits of resistors and inductors, by changes of variables. Thereafter, circuits including both inductors and capacitors are analyzed in a similar way. This appears to be less confusing than trying to develop the properties of all the kinds of circuits simultaneously.

II. CIRCUITS OF RESISTORS AND CAPACITORS

2.1 *Formulation of Circuit Equations*

Circuit equations can be formulated for linear time-varying components in almost exactly the same way as for fixed components. Basically, there are two parts to the formulation. The first defines the behavior of individual components; the second applies rules for combining the effects of the various components in a circuit.

The components with which we are concerned are of the two-terminal or one-port type. Fig. 1 represents a typical component, with terminals j and k. Voltages E_j and E_k are associated with the terminals. Current I_{kj} enters the component through terminal k and leaves it through terminal j. (Then $I_{jk} = -I_{kj}$.)

The role of the component is to perform an operation which interrelates the voltage difference $E_k - E_j$ and the current I_{kj}. The components with which we are concerned here perform only linear operations. We shall be interested sometimes in the operation which transforms

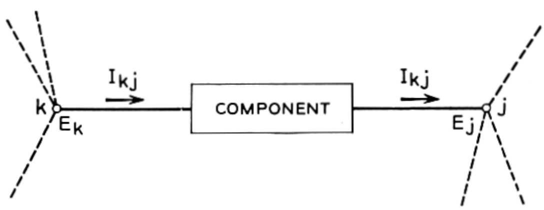

Fig. 1 — A two-terminal component.

$(E_k - E_j)$ into I_{kj} and sometimes in the inverse operation. Thus we may write

$$\begin{aligned} I_{kj} &= \Lambda_{kj}(E_k - E_j) \\ E_k - E_j &= \Lambda_{kj}^{-1} I_{kj} \end{aligned} \tag{1}$$

in which Λ_{kj} is a linear operation and Λ_{kj}^{-1} is its inverse.

Sometimes it is more convenient to consider the charge Q_{kj}, which is of course related to the current by $I_{kj} = \dot{Q}_{kj}$. We shall make extensive use of the symbol p to indicate differentiation:

$$p = \frac{d}{dt}. \tag{2}$$

Then the charge-current relation becomes

$$I_{kj} = pQ_{kj} \tag{3}$$

or, inversely

$$Q_{kj} = \int I_{kj}\, dt. \tag{4}$$

The operations performed by linear resistors and capacitors are displayed in Table I. For each of the two kinds of components, the operations are stated in the two inversely related forms, and in terms of both I_{kj} and Q_{kj}. In a time-varying circuit, the resistance R_{kj} and conductance G_{kj} of the resistor and the capacitance C_{kj} and stiffness K_{kj} of the capacitor may be functions of time.

When the coefficients vary with time they must be written in proper

TABLE I — LINEAR OPERATIONS PERFORMED BY RESISTORS AND CAPACITORS

Notation: $E_k - E_j$ = voltage across component I_{kj} = current through component Q_{kj} = charge delivered to component $I_{kj} = pQ_{kj}$	
Resistors: G_{kj} = conductance, R_{kj} = resistance $R_{kj} = G_{kj}^{-1}$	
$I_{kj} = G_{kj}(E_k - E_j),$	$Q_{kj} = \int G_{kj}(E_k - E_j)dt$
$(E_k - E_j) = R_{kj}I_{kj}\ ,$	$(E_k - E_j) = R_{kj}pQ_{kj}$
Capacitors: C_{kj} = capacitance, K_{kj} = stiffness $C_{kj} = K_{kj}^{-1}$	
$Q_{kj} = C_{kj}(E_k - E_j),$	$I_{kj} = pC_{kj}(E_k - E_j)$
$E_k - E_j = K_{kj}Q_{kj}\ ,$	$E_k - E_j = K_{kj} \int I_{kj}dt$

relationship to the differentiation operation p. For example

$$pC_{kj}(E_k - E_j) = \frac{d}{dt}\,[C_{kj}(E_k - E_j)] \neq C_{kj}p(E_k - E_j). \qquad (5)$$

Throughout the paper we shall be critically concerned with the "noncommutability of p and time-varying coefficients."

A circuit is formed by interconnecting a number of components, for example as illustrated in Fig. 2(a). The interconnections may be represented by the corresponding linear graph, as in Fig. 2(b). The interactions between the various components are determined by Kirchoff's two laws. The voltage law says that the same voltage E_k can be assigned to each node (graph vertex) k, in forming the voltage differences for all components connected to k. Then the sum of the voltage differences around any mesh (graph cycle) must be zero. The current law states that the sum of the currents into any node must be zero. These remarks are exactly the same whether or not the components vary with time.

We shall consider separately the two different forms of circuit equa-

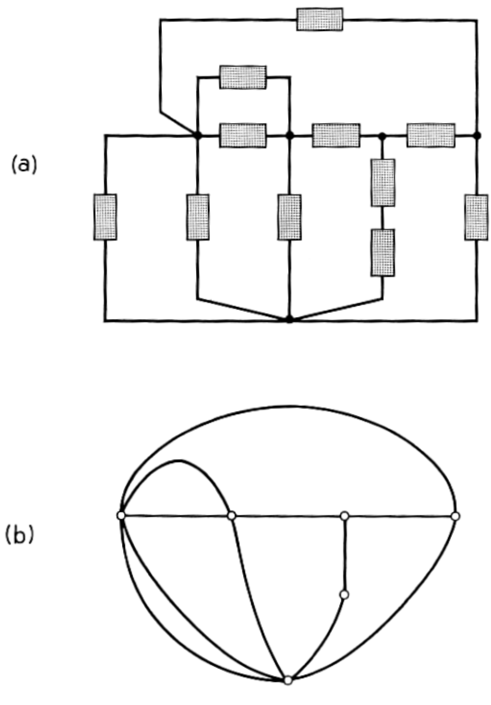

Fig. 2 — A typical circuit: (a) circuit diagram; (b) linear graph.

tions which are commonly used — the node equations and the mesh equations. For the node equations one node is chosen as datum, and the excitation (forcing function) is described as currents fed into other nodes as in Fig. 3(a). Then the node voltages (relative to the datum) are

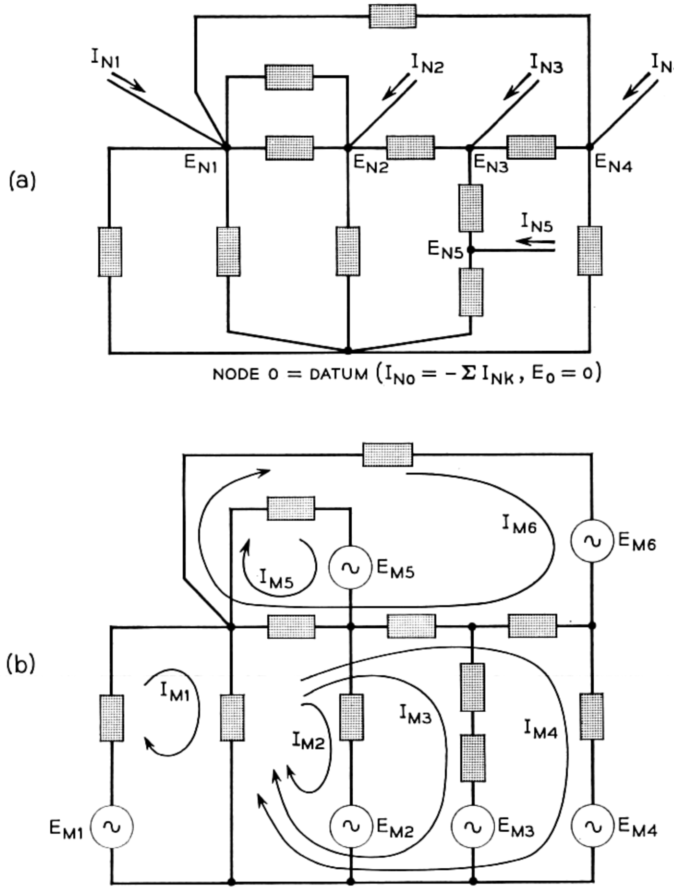

Fig. 3 — Node and mesh currents and voltages: (a) node analysis; (b) mesh analysis.

related to the currents by a vector differential equation. The procedure is exactly the same for time-varying circuits as for fixed circuits, provided one is careful to preserve the correct order of the differentiation operator p and time-varying coefficients.

The vector equation may be written

$$I_N = (G + pC)E_N\,. \tag{6}$$

Here E_N and I_N are column matrices, or vectors, whose elements are the voltages and excitation currents associated with the various nodes. If only certain of the nodes are externally accessible, for excitation currents, the elements of I corresponding to the other ("internal") nodes are simply constrained to be zero.

G and C are square matrices defining the specific relation between I_N and E_N. Their elements are easily determined from the instantaneous conductances and capacitances of the resistors and capacitors of the circuit, and the usual relations of elementary circuit theory apply even though the conductances and capacitances vary with time.

It follows that G and C, at each instant of time, have the properties of the matrices usually associated with fixed networks. Thus G and C are *symmetrical*, and if the conductances and capacitances of the components are nonnegative, G and C are *positive definite or semidefinite.* The symmetry and the nonnegative character of the matrices, together, lead to an important part of the specialization in circuit theory, relative to the usual pure mathematics of differential equations.

When the differentiation operator p appears in front of a matrix, it signifies differentiation of each element of the matrix. When it is followed by a matrix product, it signifies differentiation of each element in the single matrix equal to the product of matrices. Thus (6) is merely a compact way of writing

$$\begin{aligned} I_N &= GE_N + \frac{d}{dt} Q_N \\ Q_N &= CE_n = (q_{ij}) \\ \frac{d}{dt} Q_N &= \left(\frac{d}{dt} q_{ij}\right). \end{aligned} \tag{7}$$

For the mesh equations, meshes are chosen in a somewhat arbitrary way, and the excitation is described as voltage generators inserted in the meshes as in Fig. 3(*b*). The meshes correspond to cycles in the linear graph, and the number of meshes is the maximum number of independent cycles permitted by the topology of the graph. The mesh currents are related to the excitation voltages by a vector differential equation. As before, the procedure is the same for time-varying as for fixed circuits, provided p and time-varying coefficients are written in the proper order.

For a network of resistors and capacitors, it is more convenient to use charges in place of currents. Then

$$\begin{aligned} E_M &= (K + Rp)Q_M \\ I_M &= pQ_M . \end{aligned} \tag{8}$$

Here E_M, Q_M, I_M are column matrices or vectors. E_M and I_M are *not* the same as E_N and I_N, although they are related to them in a quite complicated way.

R and K are square matrices defining the specific relation between E_M and Q_M. Their elements are determined from the instantaneous resistances and stiffnesses of the resistors and capacitors of the circuit, and the usual relations of elementary circuit theory again apply. R and K are related to G and C of (6), but in a quite complicated way. The elements of R and K are *not* simply reciprocals of elements of G and C nor are matrices R and K the inverses of G and C.

Like G and C, R and K have properties of matrices usually associated with fixed networks. They are *symmetrical*, and if the resistances and stiffnesses of the components are nonnegative, the matrices are *positive definite or semidefinite.*

A further point should be mentioned. If (8) is to be valid for all circuits, the constants of integration implicit in the Q_{kj}, as defined in (4), must be consistent in the following sense: They must be such that the indefinite integral in (4) can be replaced by a definite integral, say

$$\int_{t_0}^{t} I_{kj}\, dt,$$

with t_0 the same for all kj. This is the same as requiring that all Q_{kj} must be zero at some one time t_0.

When the condition is not met, the superposition theorem can be invoked to express the complete circuit relations in two parts. The first assumes that all capacitors are completely discharged at time t_0, and relates the charges accumulated at $t > t_0$ to the generator (mesh) voltages at $t > t_0$. The second starts with the actual charges at $t = t_0$ and determines their later values in the absence of generator voltages. When the capacitances vary with time, the initial charges may have a much more important effect than is possible in a fixed circuit. This will be discussed in physical terms in Section 2.4.1.

2.2 *An Algebra of p and Matrix Coefficients*

This section introduces manipulations of a sort which we shall use extensively in this and later papers. In the manipulations it is convenient to use both p and a dot over a symbol to indicate differentiation with respect to time:

$$px = \dot{x} = \frac{d}{dt} x. \tag{9}$$

Generally (but not quite always) the dot will be used for rates of change of coefficients and p for differentiation of primary variables (such as voltages or currents) or products of coefficients and primary variables.

By way of introduction, consider the following scalar expression:

$$pax = \frac{d}{dt}(ax) = \dot{a}x + apx = (\dot{a} + ap)x. \tag{10}$$

Suppose x is a principal variable and a is a time-varying coefficient. Then a may be regarded as a linear operator which multiplies x by a function of time. In the same way, p is an operator which differentiates x, pa is an operator which multiplies x by a and differentiates the product, and ap is an operator which differentiates x and multiplies the derivative by a.

Equation (10) may be said to state a commutation rule, which may may also be stated as an "operator identity" (both sides of which are operators):

$$pa = \dot{a} + ap. \tag{11}$$

Thus p and a commute without change if and only if a is constant, so that $\dot{a} = 0$.

The concept is easily extended to more complicated combinations, for example

$$(a + p)(b + p) = ab + \dot{b} + (a + b)\,p + p^2. \tag{12}$$

An algebra of this sort, in p and scalar coefficients, is useful for the manipulation of scalar differential equations. It is a principal tool in, for example, Ref. 4.

For present purposes we need to extend the concept to an algebra of p and *matrix* coefficients, as a tool for manipulating vector differential equations. Suppose X is a matrix variable and A is a matrix coefficient. As a first example, it is easily established that

$$pAX = \dot{A}X + ApX = (\dot{A} + Ap)X. \tag{13}$$

The relation follows at once from the fact that each element of AX is a sum of terms, each of which is a product of one element from A and one from X. Equation (11) can be applied term by term and the results can then be sorted into contributions to $\dot{A}X$ and ApX.

Corresponding to (13) is the "operator identity"

$$pA = \dot{A} + Ap. \tag{14}$$

The concept is quickly extended to more complicated operations. Some "operators equations" are collected in Table II, together with some familiar algebraic matrix identities. In the table

$$\begin{aligned} A^t &= \text{transpose of } A \\ A^{-1} &= \text{inverse of } A \\ U &= \text{unit or identity matrix.} \end{aligned} \tag{15}$$

Constant scalar and matrix coefficients commute with p without change. Time-varying coefficients do not. (But of course matrix factors, constant or not, generally do not commute with each other.) This may be regarded as the most important distinction between the theories of linear time-varying and fixed circuits. If it were not for the difference in the commutation rules, most of the familiar techniques applied to fixed circuits would apply directly to time-varying circuits. As it is in fact, the more complicated commutation rules lead to numerous complications, as we shall see.

2.2.1 *Matrices of Order One*

Much of this paper is concerned with scalar quantities (for example, net input power) derived from vector circuit equations. For some pur-

Table II — Some Matrix Relations

A, B, X, Y = matrices, φ = a scalar
In relations which involve inverses, pertinent matrices are assumed to be square and nonsingular.
SOME ALGEBRAIC IDENTITIES
$X(A + B)Y = XAY + XBY$
$\varphi XY = X\varphi Y = XY\varphi$
$(XY)^t = Y^tX^t$, $(XY)^{-1} = Y^{-1}X^{-1}$
SOME OPERATOR IDENTITIES
$pA = \dot{A} + Ap$, $\quad p^2A = \ddot{A} + 2\dot{A}p + Ap^2$
$pAp = \dot{A}p + Ap^2 = -p\dot{A} + p^2A$
$pA\varphi = \dot{\varphi}A + \varphi pA$, $\quad pAe^{\varphi} = e^{\varphi}(\dot{\varphi}A + pA)$
DERIVATIVES OF SOME MATRIX FUNCTIONS
$pX^2 = \dot{X}X + X\dot{X} \neq 2\dot{X}X$
$pXYZ = \dot{X}YZ + X\dot{Y}Z + XY\dot{Z}$
$\dot{X}p(X^{-1}) = -\dot{X}X^{-1}$, $\quad pX^{-1} = -X^{-1}\dot{X}X^{-1} \neq -(X^{-1})^2\dot{X}$
$pX^t = (pX)^t$

poses, scalars may be represented by matrices of order one, and the scalars in question will be derived in that form. Certain operations will be used repeatedly in this connection.

Suppose W and V are $n \times 1$ column matrices and Y is an $n \times n$ square matrix. Then

$$\begin{aligned} YV &= \text{a column matrix} \\ W^tYV &= \text{a matrix of order one.} \end{aligned} \tag{16}$$

A matrix of order one is always symmetrical, for there are no off-diagonal terms to interchange in forming the transpose. Thus, using a transpose rule from Table II,

$$W^tYV = (W^tYV)^t = V^tY^tW. \tag{17}$$

Certain special cases are particularly important.

If $Y^t = Y$ and W, V are column matrices,

$$W^tYV = V^tYW. \tag{18}$$

The differentiation rules in Table II require

$$p(V^tYV) = \dot{V}^tYV + V^tY\dot{V} + V^t\dot{Y}V. \tag{19}$$

Applying (18) gives

if $Y^t = Y$ and V is a column matrix

$$\dot{V}^tYV = V^tY\dot{V} = \tfrac{1}{2}\, p(V^tYV) - \tfrac{1}{2}\, V^t\dot{Y}V. \tag{20}$$

Finally, suppose the transpose of Y is the negative of Y. Then (17) requires

if $Y^t = -Y$ and V is a column matrix

$$V^tYV = V^tY^tV = -\, V^tYV = 0. \tag{21}$$

(If a quantity is equal to its negative it must be zero.)

2.3 *Power and Stability in Terms of the Node Equation*

2.3.1 *Instantaneous and Average Powers*

We return now to the circuit equation of the nodal analysis:

$$I_N = (G + pC)E_N. \tag{22}$$

The power P supplied to the circuit by the excitation currents is

$$P = \sum_k E_{Nk} I_{Nk} = E_N{}^t I_N . \tag{23}$$

Multiplying the circuit equation by E^t,

$$P = E_N{}^t (G + pC) E_N . \tag{24}$$

Expanding in terms of identities in Table II gives

$$P = E_N{}^t G E_N + E_N{}^t \dot{C} E_N + E_N{}^t C \dot{E}_N . \tag{25}$$

Applying (20) to the last term leaves

$$P = E_N{}^t (G + \tfrac{1}{2}\dot{C}) E_N + \tfrac{1}{2} p (E_N{}^t C E_N). \tag{26}$$

The power relation confirms, in circuit equation terms, what must be expected on physical arguments. Thus $\frac{1}{2}p(E_N{}^t C E_N)$ is the rate at which energy is being stored electrically in the capacitors. Then $E_N{}^t G E_N$ is the rate at which energy is being dissipated in the resistors, and $\frac{1}{2} E_N{}^t \dot{C} E_N$ is the rate at which energy is being removed from the circuit by whatever means are used to vary the capacitances. (Recall that *increasing* a capacitance *decreases* the stored energy per unit charge.)

The average power $\bar{P}$ is frequently of interest as well as the instantaneous power P. For the average over a *finite* interval, say t_1 to t_2, integration of (26) gives

$$\underset{t_2 \text{ to } t_1}{\text{Ave}}\ P = \underset{t_2 \text{ to } t_1}{\text{Ave}}\ [E_N{}^t (G + \tfrac{1}{2}\dot{C}) E_N] + \frac{(E_N{}^t C E_N)\big|_{t_1}^{t_2}}{2(t_2 - t_1)} . \tag{27}$$

When $E_N{}^t C E_N$ is bounded at all times, the last term approaches zero as $(t_2 - t_1)$ approaches infinity. Thus, for long time averages,

$$\text{if } E_N{}^t C E_N \text{ is bounded at all times}$$

$$\bar{P} = \text{Ave}\,[E_N{}^t (G + \tfrac{1}{2}\dot{C}) E_N] . \tag{28}$$

2.3.2 *Linear Transformations on E_N and I_N*

The power equation suggests rearranging the current equation so as to emphasize the matrix $(G + \frac{1}{2}\dot{C})$. Operator identities in Table II yield

$$I_N = [(G + \tfrac{1}{2}\dot{C}) + \tfrac{1}{2}(pC + Cp)] E_N . \tag{29}$$

It is now time to introduce a linear transformation of a sort which will be used extensively in this and later papers. In particular, let

$$\begin{aligned} E_N &= N \hat{E} \\ \hat{I} &= N^t I_N . \end{aligned} \tag{30}$$

Linear transformations of this sort are of course standard means for diagonalizing matrices. They are also well known as means for generating equivalent circuits, when components are fixed.[5] Their appropriateness for time-varying circuits is by no means obvious, and they do in fact lead to serious complications not encountered in connection with fixed circuits. As usual, the complications stem from the commutation rules.

Multiplying (29) by N^t and using (30) gives

$$\hat{I} = [N^t(G + \tfrac{1}{2}\dot{C})N + \tfrac{1}{2}(pN^tCN + N^tCNp) + \tfrac{1}{2}(N^tC\dot{N} - \dot{N}^tCN)]\hat{E}. \tag{31}$$

Note that

$$(N^tC\dot{N} - \dot{N}^tCN)^t = -(N^tC\dot{N} - N^tCN). \tag{32}$$

When C is positive definite, there is a transformation N such that:

$$\begin{aligned} N^tCN &= U \\ N^t(G + \tfrac{1}{2}\dot{C})N &= -D. \end{aligned} \tag{33}$$

Here U is the unit matrix of suitable order and D is a diagonal matrix. Defining D with negative sign simplifies the later discussion. The existence of a suitable transformation matrix, at each instant, follows from elementary circuit and matrix theory (for positive components). When C is only positive semidefinite, (6) can be transformed into a new equation, in fewer dimensions, with a positive *definite* C. One such procedure is outlined in the Appendix.

With the negative sign in the second equation of (33), the (diagonal) elements d_{kk} of D are identical with the zeros λ_k of the determinant of matrix $(G + \frac{1}{2}\dot{C} + \lambda C)$.

$$\begin{aligned} d_{kk} &= \lambda_k \\ \det\,(G + \tfrac{1}{2}\dot{C} + \lambda_k C) &= 0. \end{aligned} \tag{34}$$

In our applications, because G and C are functions of time, the λ_k are functions of time. If the circuit components are always nonnegative, the λ_k are all real.

When the transformation N is fixed by (33), (31) and (32) become

$$\begin{aligned} \hat{I} &= (-D + pU + J)\hat{E} \\ J &= \tfrac{1}{2}[N^{-1}\dot{N} - (N^{-1}\dot{N})^t] \\ J^t &= -J. \end{aligned} \tag{35}$$

The power equation (26) becomes

$$P = E^tI = \hat{E}^tN^tI = \hat{E}^t\hat{I} = -E^tD\hat{E} + \tfrac{1}{2}p(\hat{E}^t\hat{E}). \tag{36}$$

(The product $\hat{E}^tJ\hat{E} = 0$ because $J^t = -J$ and $\hat{E}$ is a column matrix.) The power equation could have been transformed directly, but we shall find it informative to have also the transformed current equation (35).

2.3.3 *Some Bounds on the Basis Functions*

Carrying the analysis only a little further establishes some interesting bounds on the basis functions of circuits which vary periodically. This subsection outlines a derivation, but simplifies the argument by means of some somewhat restrictive assumptions. The next subsection reviews the derivation and removes most of the restrictions.

The basis functions are counterparts, for time-varying circuits, of the familiar natural modes of fixed circuits. In node terms, they are a set of linearly independent solutions of (22) for E_N with $I_N = 0$. Thus, if E_σ is a vector basis function,

$$0 = (G + pC)E_\sigma . \tag{37}$$

When $I_N = 0$, $P = 0$ and (26) becomes

$$0 = E_\sigma{}^t(G + \tfrac{1}{2}\dot{C})E_\sigma + \tfrac{1}{2}p(E_\sigma{}^tCE_\sigma). \tag{38}$$

Also, when $I_N = 0$, $\hat{I} = 0$ and the transformed equations (35) and (36) become

$$\begin{aligned} 0 &= (-D + Up + J)\hat{E}_\sigma \\ 0 &= \hat{E}_\sigma{}^tD\hat{E}_\sigma + \tfrac{1}{2}p(\hat{E}_\sigma{}^t\hat{E}_\sigma) \\ E_\sigma &= N\hat{E}_\sigma . \end{aligned} \tag{39}$$

If the circuit has n degrees of freedom there are n basis functions in the set. They may be chosen in many different ways, but each choice is a linear transformation on every other choice.

If the circuit varies periodically, G and C in (37) and (38) vary periodically [and also D and J in (39)]. It is well known that the basis functions of a linear differential equation with periodically varying coefficients can usually be so chosen that they are exponentials with periodic coefficients.* Any exceptions are singular cases which we can best take care of in the next subsection.

* See the discussion of the Floquet-Poincaré theorem in a text on differential equations, such as pages 78–81 in Coddington and Levinson.[6]

Thus for a periodically varying network, we can use

$$E_\sigma = H_\sigma \exp(s_\sigma t). \tag{40}$$

Here s_σ is a constant, the exponential is a scalar factor, and H_σ is a periodically varying vector. For the purposes of this subsection, we can best restrict ourselves to circuits for which s_σ and H_σ are real. The restriction will be removed in the next subsection.

We shall find it convenient to replace H_σ by another periodic vector F_σ, related to H_σ by

$$H_\sigma = F_\sigma e^\theta \tag{41}$$

in which θ is an arbitrary periodic real function of t (with the same period as the time-varying circuit components). In terms of F_σ,

$$E_\sigma = F_\sigma \exp(s_\sigma t + \theta). \tag{42}$$

Because the exponential is simply a scalar factor, (42) and the last equation of (39) require

$$\begin{aligned} \hat{E}_\sigma &= \hat{F}_\sigma \exp(s_\sigma t + \theta) \\ F_\sigma &= N\hat{F}_\sigma . \end{aligned} \tag{43}$$

For any matrix A,

$$pAe^{(s_\sigma t+\theta)} = [\exp(s_\sigma t + \theta)][(s_\sigma + \dot{\theta})A + pA]. \tag{44}$$

Using this relation in (39) and then cancelling out the exponential factor gives

$$\begin{aligned} 0 &= [(s_\sigma + \dot{\theta})U - D + J]\hat{F}_\sigma + p\hat{F}_\sigma \\ 0 &= \hat{F}_\sigma{}^t[(s_\sigma + \dot{\theta})U - D]\hat{F}_\sigma + \tfrac{1}{2}p(\hat{F}_\sigma{}^t\hat{F}_\sigma). \end{aligned} \tag{45}$$

Now $\hat{E}_\sigma$ does not remain bounded over an infinite time interval. Hence averaging the second equation of (39) does not eliminate the second term. On the other hand, $\hat{F}_\sigma$ *is* bounded at all times, and thus the second equation of (45) implies:

$$0 = \text{Ave } \hat{F}_\sigma{}^t[(s_\sigma + \dot{\theta})U - D]\hat{F}_\sigma . \tag{46}$$

Since $\hat{F}_\sigma$, D and $\dot{\theta}$ are all periodic, while s_σ is constant, the long time average is the same as the average over any one period.

The matrix $(s\sigma + \dot{\theta})U - D$ is diagonal. That is, all its elements are zero except on the main diagonal, where the typical element is

$$\begin{aligned} a_{kk} &= s_\sigma + \dot{\theta} - \lambda_k \\ \text{Det}\,(G + \tfrac{1}{2}\dot{C} + \lambda_k C) &= 0. \end{aligned} \tag{47}$$

Then multiplying out the matrix product in (46) gives

$$0 = \text{Ave} \sum_k (s_\sigma + \dot{\theta} - \lambda_k)\hat{F}_{\sigma_k}{}^2 \tag{48}$$

in which $\hat{F}_{\sigma_k}$ is the kth element in the column matrix $\hat{F}_\sigma$.

When all the quantities are real (as assumed), condition (48) cannot be true if all the coefficients $(s_\sigma + \dot{\theta} - \lambda_k)$ are positive at all times, or if all are negative at all times. We can use this circumstance to set bounds on the exponent s_σ of the basis function. The arbitrary function θ has been introduced as a means of strengthening these bounds, as explained below.

Let us assume temporarily that the λ_k are at all times distinct. Fig. 4(a) illustrates a plot of the λ_k over one period of their periodic variations. Let λ_M be the largest λ_k, and let θ_M be a choice of θ such that

$$\lambda_M - \dot{\theta}_M = \text{constant}. \tag{49}$$

Fig. 4(b) illustrates a plot of the corresponding $\lambda_k - \dot{\theta}_M$. Every coefficient $(s_\sigma + \dot{\theta} - \lambda_k)$ in (48) will now be positive unless s_σ is no greater than $\lambda_M - \dot{\theta}_M$.

Now $\dot{\theta}_M$ is actually determined uniquely by (49). Because θ_M is periodic, the average of $\dot{\theta}_M$ over any period is zero, and hence also the long-time average. Then, when $\lambda_M - \dot{\theta}_M$ is constant

$$\begin{aligned} \text{Ave}\,(\lambda_M - \dot{\theta}_M) &= \lambda_M - \dot{\theta}_M = \text{Ave}\,\lambda_M \\ \dot{\theta}_M &= \lambda_M - \text{Ave}\,\lambda_M . \end{aligned} \tag{50}$$

Thus all terms in (48) will have positive coefficients $(s_\sigma + \dot{\theta}_M - \lambda_k)$ unless

$$s_\sigma \leqq \text{Ave}\,\lambda_M . \tag{51}$$

Exactly the same sort of procedure leads also to a lower bound on s_σ. Let λ_m be the smallest λ_k, and θ_m a choice of θ which makes $\lambda_m - \dot{\theta}_m$ a constant. Fig. 4(c) illustrates the corresponding $\lambda_k - \dot{\theta}_m$. The bound on s_σ can be written

$$s_\sigma \geqq \text{Ave}\,\lambda_m . \tag{52}$$

2.3.4 *Recapitulation, Discussion, and Removal of Restrictions*

This subsection reviews the derivation of the stability bounds, and removes most of the restrictions.

Our diagonalization of the matrices $G + \frac{1}{2}\dot{C}$ and C assumed that C is positive definite. The Appendix describes how an equation with a positive semidefinite C can be transformed into one of lower dimensionality

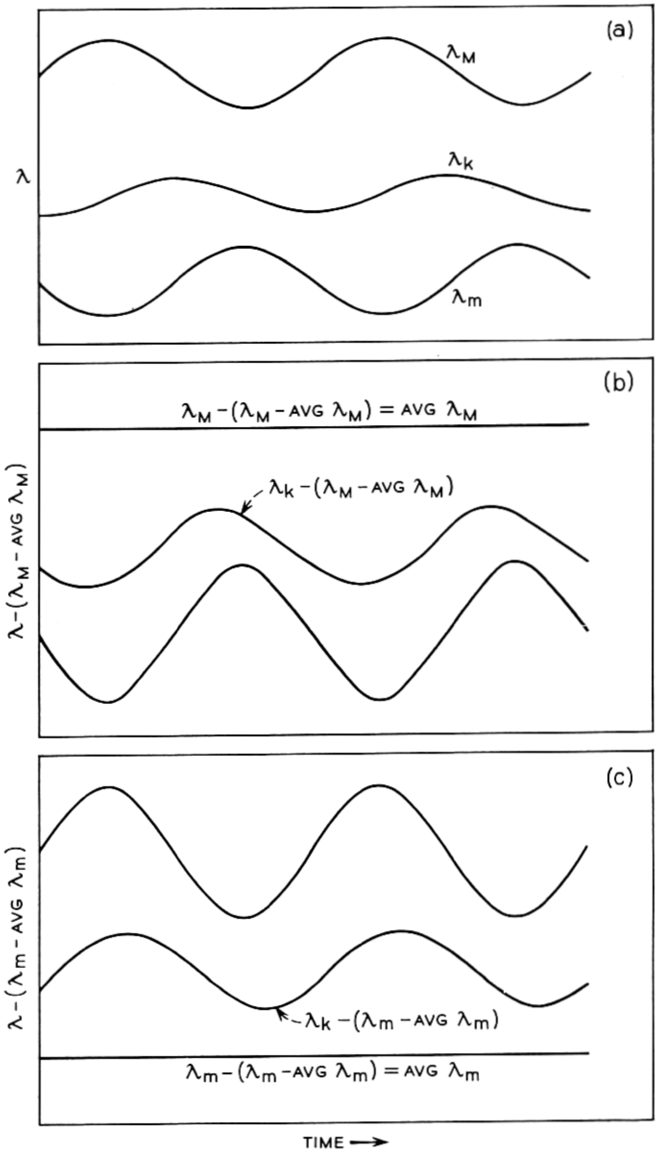

Fig. 4 — Characteristic roots: (a) λ; (b) $\lambda - \dot{\theta}$, with $\dot{\theta}$ to make maximum = constant; (c) $\lambda - \dot{\theta}$, with $\dot{\theta}$ to make minimum = constant.

with a positive definite C. More exactly, the transformation is such that the components of the transformed current and voltage vectors may be divided into two autonomous parts. If n and m are the order and rank of the original matrix C, m components of the transformed current and voltage vectors are related by a differential equation of our standard form, in m dimensions, with a positive definite C matrix, and m linearly independent basis function subject to our stability bounds. The remaining $n - m$ components of the current and voltage vectors are related by a purely algebraic equation, which raises no questions of stability.

In our derivation of the bounds on s_k, we restricted ourselves to periodically varying circuits such that all the basis functions (corresponding to a particular choice) are exponentials with periodic coefficients. In singular instances such a choice is not possible. However, for periodically varying circuits, the basis functions can all be exponentials with coefficients which are at most polynomials in t with periodic coefficients. Furthermore, the coefficients are polynomials in t only when more than one basis function has the same coefficient s_k in the exponential factor $\exp(s_k t)$. Out of the set of basis functions corresponding to a single s_k, *one* can always be assigned a periodic coefficient H_k. This is sufficient to establish our bound on s_k, without regard for the possible existence of other basis function with the same s_k.

We also assumed that the constant s_k and the periodic function H_k are real. The assumption is not actually necessary. With real circuit components, complex basis functions can be chosen in conjugate pairs. Then equally valid choices are the real and imaginary parts of the complex functions. Thus we can write, for a complex exponential basis function,

$$\begin{aligned} (X_\sigma + iY_\sigma)e^{(s_\sigma + i\omega_\sigma)t} &= Z_{1\sigma} \exp(s_\sigma t) + iZ_{2\sigma} \exp(s_\sigma t) \\ Z_{1\sigma} &= X_\sigma \cos \omega_\sigma t - Y_\sigma \sin \omega_\sigma t \\ Z_{2\sigma} &= X_\sigma \sin \omega_\sigma t + Y_\sigma \cos \omega_\sigma t. \end{aligned} \tag{53}$$

We can now use either $Z_{1\sigma} \exp(s_\sigma t)$ or $Z_{2\sigma} \exp(s_\sigma t)$ as a basis function, in place of $H_\sigma \exp(s_\sigma t)$. The only difference is that the Z_σ's are not periodic, as is H_σ. However, the Z_σ's *are* bounded at all times, and so are F_σ and $\hat{F}_\sigma$, which are now defined by

$$\begin{aligned} Z_{1\sigma} \text{ or } Z_{2\sigma} &= F_\sigma e^\theta \\ F_\sigma &= N\hat{F}_\sigma . \end{aligned} \tag{54}$$

Furthermore, the *boundedness* of $\hat{F}_\sigma$ is all that is required, for passing from (45) to (46), provided the average in (46) is a long-time average. Thus the real part s_σ, of a complex coefficient $s_\sigma + i\omega_\sigma$, also obeys our bounds.

An alternate proof uses the complex basis function itself, replaces E_σ^t by its conjugate in (38) and (39) and sorts out real and imaginary parts of the equations.

The exponents corresponding to circuits of constant resistors and capacitors are necessarily real. However, when both the resistors and capacitors vary with time, the exponents may be complex. A number of specific examples are known, including examples cited by Desoer and Paige[1] and by Meadows.[2]

Finally, we assumed that the zeros λ_k of the determinant of $(G + \frac{1}{2}\dot{C} + \lambda C)$ were distinct at all times. This is not necessary, provided we make a simple change in the statement of our bounds on s_σ. Suppose the various λ_k's crisscross, as in Fig. 5 (which may be contrasted with Fig. 4a). The variables λ_M and λ_m are now defined as the instantaneous maxima and minima over the set of variables $\lambda_1, \cdots, \lambda_n$. Otherwise the bounds are the same as before.

Thus, for general circuits of periodically varying positive resistors and capacitors, if s_σ is the real part of the coefficient in the exponential factor associated with a basis function,

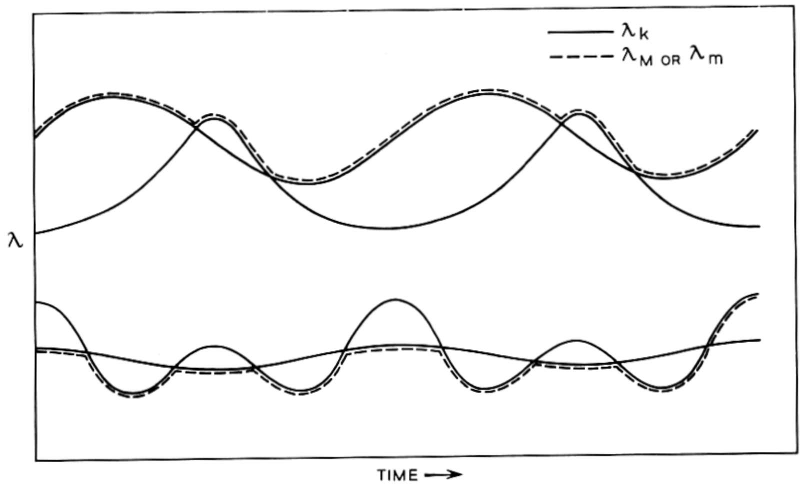

Fig. 5 — λ_M and λ_m when λ_k's crisscross.

$$\text{Ave}\,\lambda_m \leqq s_\sigma \leqq \text{Ave}\,\lambda_M \tag{55}$$

$$\lambda_m, \lambda_M = \text{instantaneous min. and max. over the set } \lambda_1, \cdots, \lambda_n.$$

$$\text{Det}\,[G + \tfrac{1}{2}\dot{C} + \lambda_k C] = 0.$$

When the circuit components are constants, the λ_k are constant, and also each s_σ is exactly equal to one of the λ_k. Why do the averages of the λ_k only furnish bounds when the components vary with time?

The answer lies in the joint implications of the vector and scalar equations

$$\begin{aligned} 0 &= [(s_\sigma + \dot{\theta})U - D + J]\hat{F}_\sigma + p\hat{F}_\sigma \\ 0 &= \text{Ave}\,\{\hat{F}_\sigma{}^t[(s_\sigma + \dot{\theta})U - D]\hat{F}_\sigma\} \\ J &= N^{-1}\dot{N} - (N^{-1}\dot{N})^t \\ J^t &= -J \end{aligned} \tag{56}$$

[equations (45) and (46) of Section 2.3.3]. It will be sufficient to consider only the second-order case, for which (56) represents the following collection of equations

$$\begin{aligned} 0 &= (s_\sigma + \dot{\theta} - \lambda_1)\hat{F}_{\sigma 1} + \dot{\hat{F}}_{\sigma 1} + J_{12}\hat{F}_{\sigma 2} \\ 0 &= (s_\sigma + \dot{\theta} - \lambda_2)\hat{F}_{\sigma 2} + \dot{\hat{F}}_{\sigma 2} - J_{12}\hat{F}_{\sigma 1} \\ 0 &= \text{Ave}\,[(s_\sigma + \dot{\theta} - \lambda_1)\hat{F}_{\sigma 1}{}^2 + (s_\sigma + \dot{\theta} - \lambda_2)\hat{F}_{\sigma 2}{}^2]. \end{aligned} \tag{57}$$

($J_{11} = J_{22} = 0$ and $J_{21} = -J_{12}$ because $J^t = -J$.)

With constant coefficients, $J_{12} = 0$, $\hat{F}_\sigma$ is constant, and one suitable solution is

$$\begin{aligned} (s_\sigma + \dot{\theta} - \lambda_1) &= 0, && \text{to satisfy the 1st eq.} \\ \hat{F}_{\sigma 2} &= 0, && \text{to satisfy the 2nd eq.} \\ (s_\sigma + \dot{\theta} - \lambda_1) \text{ and } \hat{F}_{\sigma 2} &= 0, && \text{to satisfy the 3rd eq.} \end{aligned} \tag{58}$$

With time-varying components, making $(s_\sigma + \dot{\theta} - \lambda_1)$ zero in (57) leaves

$$\begin{aligned} 0 &= \dot{\hat{F}}_{\sigma 1} + J_{12}\hat{F}_{\sigma 2} \\ 0 &= (\lambda_1 - \lambda_2)\hat{F}_{\sigma 2} + \dot{\hat{F}}_{\sigma 2} - J_{12}F_{\sigma 1} \\ 0 &= \text{Ave}\,[(\lambda_1 - \lambda_2)\hat{F}_{\sigma 2}{}^2]. \end{aligned} \tag{59}$$

These equations are (at least usually) incompatible when $J_{12} \neq 0$. Thus the nondiagonal matrix J, which appears only when components are time-varying, is what destroys the identity of the exponents s_σ and the determinant zeros λ_k (as here defined). Recall that J stems from the commutation rules applied to combinations of E^t, p, C, and E.

When the resistors are time-varying but the capacitors are fixed, $\dot{C} = 0$ and $(G + \frac{1}{2}\dot{C} + \lambda C)$ becomes $(G + \lambda C)$, which has been studied extensively in connection with fixed networks. When C is positive definite and G is positive definite or semidefinite, none of the zeros λ_k can be positive, and the same is true of their averages when they are time-varying. Then our bounds exclude any positive s_σ in the exponential factors $\exp(s_\sigma t)$ associated with our basis functions. But an unstable basis function (defined as one which grows indefinitely) requires a positive s_σ. Thus our bounds confirm the stability of circuits in which only the resistors vary.

2.3.5 *Comparison with a Known Property of Constant Matrices*

The bounds on s_σ are at least reminiscent of known bounds on the characteristic roots of a dissymmetrical constant matrix. Consider the roots s_σ of

$$\text{Det}\,(A + s_\sigma U) = 0 \tag{60}$$

in which A is a dissymmetrical matrix. For the closest parallel to our analysis assume that the characteristic roots are all real. The equation may be rewritten as follows:

$$\begin{aligned} &\text{Det}\,(S + J + s_\sigma U) = 0 \\ &S = \tfrac{1}{2}(A + A^t), \qquad S^t = S \\ &J = \tfrac{1}{2}(A - A^t), \qquad J^t = -J. \end{aligned} \tag{61}$$

When the determinant is zero, there are nonzero vectors X such that:

$$(S + J + s_\sigma U)X = 0. \tag{62}$$

Because $J^t = -J$, this implies the scalar equation

$$X^t(S + s_\sigma U)X = 0. \tag{63}$$

Diagonalizing S leaves

$$\begin{aligned} &\sum_k (s_\sigma - \lambda_k)\hat{X}_k^2 = 0 \\ &\text{Det}\,(S + \lambda_k U) = 0. \end{aligned} \tag{64}$$

The J term in (62) excludes (in general) an $\hat{X}$ in which all elements are zero except one. Then $s_\sigma \neq \lambda_k$, but the largest and smallest λ_k are upper and lower bounds on s_σ. Simple changes in the analysis establish the same bounds for the real parts of complex characteristic roots.

2.4 *Stability in Terms of the Mesh Equation*

Section 2.3 dealt exclusively with the node equation. However, the mesh equation can be manipulated in almost exactly the same way. Recall the mesh equation [(8) of Section 2.1],

$$E_M = (K + Rp)Q_M . \tag{65}$$

Note the order of operations Rp here, and contrast it with pC in the node equation.

Define a scalar M by

$$M = \sum_k Q_{Mk}E_{Mk} = Q_M{}^t E_M . \tag{66}$$

While M does not have the dimensions of power, it has many of the mathematical properties of the power function P which we associated with the node equations. Multiplying E_M by $Q_M{}^t$ and applying operator identities gives

$$M = Q_M{}^t(K - \tfrac{1}{2}\dot{R})Q_M + \tfrac{1}{2}p(Q_M{}^t R Q_M). \tag{67}$$

Compare this with the power equation (26) in Section 2.3. The appearance of $-\frac{1}{2}\dot{R}$ in M, as opposed to $+\frac{1}{2}\dot{C}$ in P, reflects the difference of the order of operations in Rp and pC. The quantity $\frac{1}{2}(Q_M{}^t R Q_M)$ does not represent stored energy, but it is mathematically similar to the stored energy function $\frac{1}{2}E_N{}^t C E_n$. Both these quantities may be regarded as Lyaponov functions.

Proceeding from here exactly as in the analysis of the node equations leads eventually to similar, but not identical, bounds. The basis functions are now mesh charges Q_σ instead of node voltages E_σ. When circuits vary periodically they can be chosen as exponentials with periodic coefficients:

$$Q_\sigma = H_\sigma \exp(s_\sigma t). \tag{68}$$

The bounds on the s_σ (or the real parts when complex) may be written

$$\begin{gathered} \text{Ave } \lambda_m \leqq s_\sigma \leqq \text{Ave } \lambda_M \\ \lambda_m, \lambda_M = \text{instantaneous min. and max.} \\ \text{over the set } \lambda_1, \cdots, \lambda_n \\ \text{Det }[K - \tfrac{1}{2}\dot{R} + \lambda_k R] = 0. \end{gathered} \tag{69}$$

Now the coefficients s_σ must be identical in the exponentials associated with the node and mesh analyses. This is because the ratios of charges and voltages remain bounded. Thus the two sets of bounds apply to the same set of s_σ. Then one can form a single pair of bounds by choosing the lesser of the two upper bounds and the greater of the two lower bounds.

When the capacitors are time varying but the resistors are fixed, $\dot{R} = 0$ and $(K - \frac{1}{2}\dot{R} + \lambda_k R)$ becomes the familiar $(K + \lambda_k R)$ of the theory of fixed circuits. Then the bounds require that such a circuit cannot be unstable (with positive circuit components) even though the time-varying capacitors give power gain.

2.4.1 *A Complication in Degenerate Special Cases*

In the discussion of the mesh formulation we have ignored a degenerate special situation, which complicates a more nearly general analysis. The complications arise when there is a node within the network to which two or more capacitors are connected, but no resistors.

Consider first the simplest example, as illustrated in Fig. 6(a). Here, two capacitors are connected in series between nodes j and k, and no other components are connected to their common node c. If the capacitances are constant, one can simply replace the two capacitors by a single equivalent. When they are time varying, the substitution may have to be more complicated.

The reason is briefly as follows: Suppose there is a positive charge on

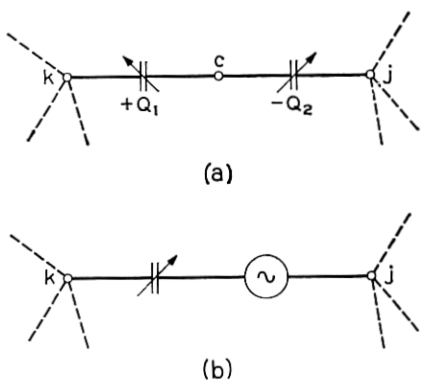

Fig. 6 — Illustration of a degenerate special case: (a) time-varying capacitors with a common node to which no resistors are connected; (b) the Thevenin equivalent.

one capacitor and a negative charge on the other. Then there will be a ratio of charges (at any one time) which will produce a zero voltage difference between terminals j and k. If the ratio of capacitances varies with time, the required ratio of charges varies with time. Conversely, the only combination of constant charges which yields zero voltage difference $(E_k - E_j)$ at all times is zero charge on each capacitor. Furthermore, if there is initially a positive charge on one and a negative charge on the other, both charges cannot be reduced to zero through nodes j and k alone. On the other hand, when the capacitances vary periodically there is a ratio of constant charges such that the voltage difference varies periodically and with zero average.

For the mesh analysis one can invoke Thevenin's theorem, and replace the two capacitors by an equivalent capacitor and a periodic, zero average voltage source, as illustrated in Fig. 6(b). When there are several capacitors and no resistors to a single node, or a combination of resistorless nodes, several Thevenin voltage generators are called for. A general characteristic of circuits which can lead to this sort of complication is a resistance matrix R which is singular (positive semidefinite instead of positive definite).

It follows that the initial charges may be much more important in time-varying than in fixed circuits. For example, suppose there is only one resistorless node and only two capacitors connected thereto. In a fixed circuit, the effects of initial charges will eventually die out except for the voltage at the single resistorless node. In a time-varying circuit, however, the varying Thevenin voltage may produce voltages at all nodes forever.

For the node analysis, one can ignore these complications by simply not eliminating the resistorless nodes. When there are resistorless nodes, the conductance matrix G is positive semidefinite. When G has order n and rank m (and C is positive definite), $n - m$ of the basis functions have zero damping. This can be proved by applying to G, with only minor alterations, the analysis applied to a semidefinite C as outlined in the Appendix. The Thevenin voltages of the mesh analysis set up undamped oscillations within the circuit ($s_\sigma = 0$) and correspond to the undamped modes of the node analysis. The validity of the derivation of our stability bounds, which does not preclude a singular G matrix, implies that the bounds will include zero when G is positive semidefinite.

One way to avoid these complications is to avoid ideal capacitors. If a resistor is connected across each capacitor, to represent the leakage through any actual component, there are no resistorless nodes.

III. CIRCUITS OF RESISTORS AND INDUCTORS

The theory of circuits of resistors and capacitors can be applied to circuits of resistors and inductors by interchanging currents and voltages and also the mesh and node formulations. Since currents and voltages are interchanged, one needs variables which are related to voltages in the same way that charges are related to currents. Corresponding to node voltage E_k, let

$$E_k = \dot{\Phi}_k \tag{70}$$

or, inversely

$$\Phi_k = \int E_k \, dt. \tag{71}$$

If an inductor is connected between nodes j and k, $E_k - E_j$ is the voltage across the inductor, and $\Phi_k - \Phi_j$ is proportional to the flux linkages within it. The fact that the definition of Φ_k in (70) or (71) leaves undetermined a constant of integration reflects the fact that constant flux linkages produce no voltage across an inductor.

Table III states the linear operations performed by inductors, in two inversely related forms, and in terms of both $E_k - E_j$ and $\Phi_k - \Phi_j$. The inductance L_{kj} and its reciprocal S_{kj} can vary with time, provided the order of the differentiation symbol p and the coefficients is preserved.

For resistors and inductors, the *mesh* equation corresponds to the *node* equation (6) of our previous circuits, and is

$$E_M = (R + pL)I_M . \tag{72}$$

The mesh voltage and current vectors E_M and I_M are defined as before, and also the matrix R of the mesh resistances. Then L is the mesh matrix of the inductances of the inductors. If one properly chooses constants of integration stemming from (71), the *node* equation is

Table III — Linear Operations Performed by Inductors

$$I_{kj} = \text{current through component}$$
$$E_k - E_j = \text{voltage across component}$$
$$\Phi_k - \Phi_j = \text{variable related to flux}$$
$$E_k = p\Phi_k$$

$$L_{kj} = \text{inductance}, \; S_{kj} = \text{reciprocal inductance}$$
$$S_{kj} = L_{kj}^{-1}$$

$\Phi_k - \Phi_j = L_{kj} I_{kj}$,	$E_k - E_j = pL_{kj} I_{kj}$
$I_{kj} = S_{kj}(\Phi_k - \Phi_j)$,	$I_{kj} = S_{kj} \int (E_k - E_j)dt$

$$\begin{aligned} I_N &= (S + Gp)\Phi_N \\ E_N &= p\Phi_N \end{aligned} \tag{73}$$

and corresponds to the old *mesh* equation (8). Here vector I_N represents the node currents as before, and G is again the matrix of node conductances. The elements of vector Φ_N are the Φ_k and S is the node matrix of the reciprocals of the inductances.

The formulation of (73) requires an assumption regarding constants of integration, implicit in the definition of the Φ_k, exactly like the assumptions regarding the Q_{kj} in the formulation of (8).

One can now proceed exactly as before to obtain power equations and bounds on the exponential factors associated with the basis functions of periodically varying circuits.

Section 2.4.1 described degeneracies which occur in the *mesh* analysis of circuits of resistors and capacitors which have resistorless *nodes*. The counterparts for circuits of resistors and inductors occur in the *node* analysis of circuits with resistorless *meshes*.

IV. CIRCUITS WHICH CONTAIN BOTH CAPACITORS AND INDUCTORS

4.1 *Circuits of Capacitors and Inductors Only*

For circuits which contain both capacitors and inductors, but no resistors, the node equation contains the capacitor and inductor terms from our previous equations (6) and (73). Thus

$$\begin{aligned} I_N &= S\Phi_N + pCE_N \\ E_N &= p\Phi_N \end{aligned} \tag{74}$$

or, replacing E_N by $p\Phi_N$,

$$I_N = (S + pCp)\Phi_N . \tag{75}$$

As before, S is the node matrix of the reciprocals of the inductances, and C is the node matrix of the capacitances.

To obtain the input power P from the excitation, multiply by $E_N{}^t$, which is the same as $\dot{\Phi}_N{}^t$

$$P = \dot{\Phi}_N{}^t(S + pCp)\Phi_N . \tag{76}$$

Manipulating in terms of operator identities, again identifying $\dot{\Phi}_N$ with E_N, and using the symmetry of matrices S and C gives

$$P = -\tfrac{1}{2}\Phi_N{}^t\dot{S}\Phi_N + \tfrac{1}{2}E_N{}^t\dot{C}E_N + \tfrac{1}{2}p(\Phi_N{}^tS\Phi_N) + \tfrac{1}{2}p(E_N{}^tCE_N). \quad (77)$$

The first two terms are the rates at which energy is removed from the system by the means used to vary the inductances and capacitances. The last two terms are the rates of increase of the electrical energy stored in the inductors and capacitors.

The basis functions Φ_σ are solutions of

$$0 = (S + pCp)\Phi_\sigma . \quad (78)$$

The corresponding power equation is

$$\begin{aligned} 0 &= -\tfrac{1}{2}\Phi_\sigma{}^t\dot{S}\Phi_\sigma + \tfrac{1}{2}E_\sigma{}^t\dot{C}E_\sigma + \tfrac{1}{2}p(\Phi_\sigma{}^tS\Phi_\sigma) + \tfrac{1}{2}p(E_\sigma{}^tCE_\sigma) \\ E_\sigma &= p\Phi_\sigma . \end{aligned} \quad (79)$$

If the circuit components vary periodically, we can define a bounded function F_σ by

$$\Phi_\sigma = F_\sigma \exp(s_\sigma t + \theta). \quad (80)$$

There will be a corresponding function F_σ' defined by

$$E_\sigma = F_\sigma' \exp(s_\sigma t + \theta). \quad (81)$$

Because $E_\sigma = p\Phi_\sigma$, F_σ' is related to F_σ by

$$F_\sigma' = (s_\sigma + \dot{\theta})F_\sigma + \dot{F}_\sigma . \quad (82)$$

Under reasonable circuit conditions (which exclude, for example, discontinuous changes in inductances), F_σ' is bounded, as well as F_σ. Using (80) and (81) in (79), eliminating the exponentials and averaging gives

$$0 = \text{Ave } F_\sigma{}^t[(s_\sigma + \dot{\theta})S - \tfrac{1}{2}\dot{S}]F_\sigma + \text{Ave } F_\sigma{}'t[(s_\sigma + \dot{\theta})C + \tfrac{1}{2}\dot{C}]F_\sigma'. \quad (83)$$

If both S and C are positive definite, all matrices can be diagonalized. This merely requires a different transformation on F_σ and F_σ'. Thus let

$$F_\sigma = N_\sigma\hat{F}_\sigma , \qquad F_\sigma' = N_\sigma'\hat{F}_\sigma' \quad (84)$$

and choose N_σ and N_σ' in such a way that

$$\begin{aligned} N_\sigma{}^tSN_\sigma &= U, & N_\sigma{}'^tCN_\sigma' &= U \\ N_\sigma{}^t\dot{S}N_\sigma &= 2D_\sigma , & N_\sigma{}'^t\dot{C}N_\sigma' &= -2D_\sigma'. \end{aligned} \quad (85)$$

Here U is again the unit matrix and D_σ and D_σ' are diagonal matrices.

After transformation, (83) can be written

$$0 = \text{Ave} \sum_k \{(s_\sigma + \dot{\theta} - \lambda_b)\hat{F}_{\sigma k}{}^2 + (s_\sigma + \dot{\theta} - \lambda_k)\hat{F}_{\sigma k}{}'^2\}. \quad (86)$$

Here, $\hat{F}_{\sigma k}$ and $\hat{F}_{\sigma k}'$ are elements of the column matrices $\hat{F}_\sigma$ and $\hat{F}_\sigma'$, and λ_k and λ_k' are the corresponding (like-rowed) diagonal elements of D_σ and D_σ'. As before, all quantities can be made real (provided $\hat{F}_{\sigma k}$, $\hat{F}_{\sigma k}'$ are only required to be bounded, not necessarily periodic). Then s_σ cannot be so large that all terms in the sum are positive, or so small that all terms in the sum are negative.

If one makes no use of the implicit relation between $\hat{F}_\sigma$ and $\hat{F}_\sigma'$, one can consider the two sets of constants λ_k and λ_k' as simply two parts of a single set. Then bounds on s_σ can be obtained exactly as before. Since the λ_k depend only on C and the λ_k' only on S, while C and S represent different circuit components, the two sets of bounds are likely to criss-cross as in Fig. 7. Thus, including also the relation of λ_k and λ_k' to S and C [defined by (85)], the bounds may be written

$$\begin{aligned} &\text{Ave (min. over } \lambda) \leqq s_\sigma \leqq \text{Ave (max. over } \lambda) \\ &\text{set } \lambda = \lambda_k\text{'s and } \lambda_k'\text{'s} \\ &\text{Det } (-\tfrac{1}{2}\dot{S} + \lambda_k S) = 0 \\ &\text{Det } (\tfrac{1}{2}\dot{C} + \lambda_k' C) = 0. \end{aligned} \tag{87}$$

The mesh analysis differs only in the specific quantities involved. The mesh circuit equation is

$$\begin{aligned} E_M &= (K + pLp)Q_M \\ I_M &= pQ_M . \end{aligned} \tag{88}$$

As before, K is the mesh matrix of stiffnesses and L is the mesh matrix of inductances.

The input power from the excitation is

$$P = -\tfrac{1}{2}Q_M{}^t\dot{K}Q_M + \tfrac{1}{2}I_M{}^t\dot{L}I_M + \tfrac{1}{2}p(Q_M{}^tKQ_M) + \tfrac{1}{2}p(I_M{}^tLI_M). \tag{89}$$

The bounds on s_σ may be written

$$\begin{aligned} &\text{Ave (min. over } \lambda) \leqq s_\sigma \leqq \text{Ave (max. over } \lambda) \\ &\text{set } \lambda = \lambda_k\text{'s and } \lambda_k'\text{'s} \\ &\text{Det } (-\tfrac{1}{2}\dot{K} + \lambda_k K) = 0 \\ &\text{Det } (\tfrac{1}{2}\dot{L} + \lambda_k L) = 0. \end{aligned} \tag{90}$$

4.2 *Circuits of all Three Kinds of Components*

For circuits which contain capacitors, inductors and resistors, the

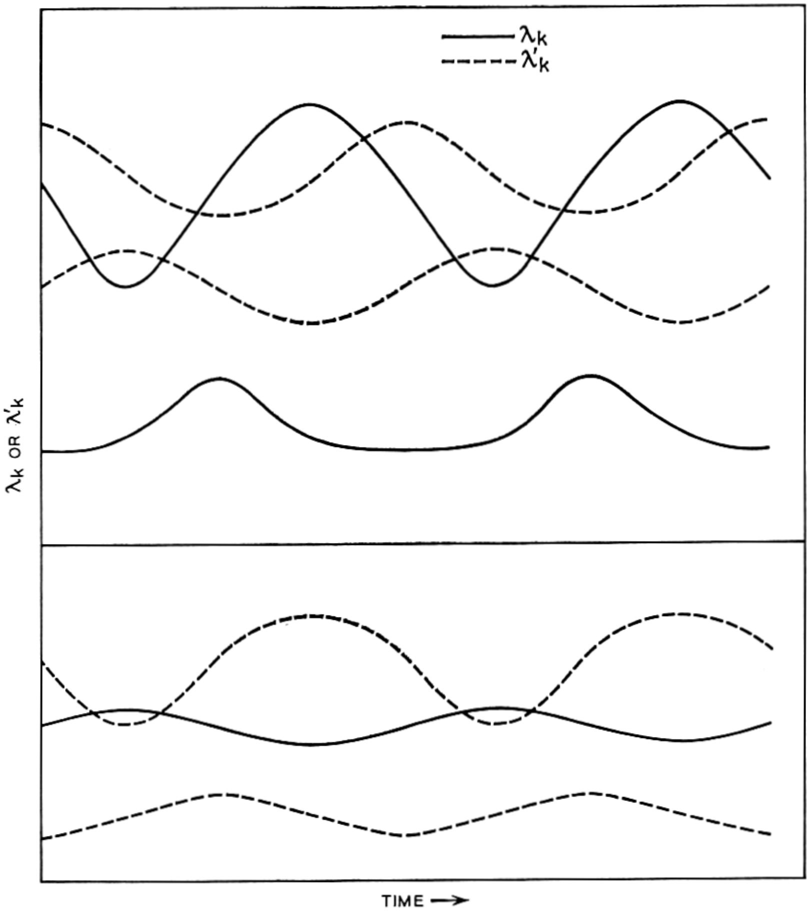

Fig. 7 — Characteristic roots of $(-\frac{1}{2}\dot{S} + \lambda S)$ and $(\frac{1}{2}\dot{C} + \lambda C)$.

node equation may be written

$$\begin{aligned} I_N &= (S + Gp + pCp)\Phi_N \\ E_N &= p\Phi_N . \end{aligned} \tag{91}$$

The corresponding power equation may be written

$$\begin{aligned} P = -\tfrac{1}{2}\Phi_N{}^t\dot{S}\Phi_N + \tfrac{1}{2}E_N{}^tGE_N + \tfrac{1}{2}E_N{}^t\dot{C}E_N \\ + \tfrac{1}{2}p(\Phi_N{}^tS\Phi_N) + \tfrac{1}{2}p(E_N{}^tCE_N) \end{aligned} \tag{92}$$

in which the new term, $\frac{1}{2}E_N{}^tGE_N$, represents the power dissipated in the resistors.

The stability bounds may be written

$$\begin{aligned} \text{Ave (min. over } \lambda) \leqq s_\sigma &\leqq \text{Ave (max. over } \lambda) \\ \text{set } \lambda &= \lambda_k\text{'s and } \lambda_k'\text{'s} \\ \text{Det } (-\tfrac{1}{2}\dot{S} + \lambda_k S) &= 0 \\ \text{Det } (G + \tfrac{1}{2}\dot{C} + \lambda_k' C) &= 0. \end{aligned} \tag{93}$$

Note that λ_k' is the same as λ_k' for a circuit of capacitors and resistors only [see (55)]. On the other hand, the present λ_k do not depend at all on the resistors.

The corresponding mesh equation is

$$\begin{aligned} E_M &= (K + Rp + pLp)Q_M \\ I_M &= pQ_M . \end{aligned} \tag{94}$$

Then the power equation may be written

$$\begin{aligned} P = -\tfrac{1}{2}Q_M{}^t\dot{K}Q_M + \tfrac{1}{2}I_M{}^tRI_M + \tfrac{1}{2}I_M{}^t\dot{L}I_M \\ + \tfrac{1}{2}p(Q_M{}^tKQ_M) + \tfrac{1}{2}p(I_M{}^tLI_M). \end{aligned} \tag{95}$$

Finally, the stability conditions may be written

$$\begin{aligned} \text{Ave (min. over } \lambda) \leqq s_\sigma &\leqq \text{Ave (max. over } \lambda) \\ \text{set } \lambda &= \lambda_k\text{'s and } \lambda_k'\text{'s} \\ \text{Det } (-\tfrac{1}{2}\dot{K} + \lambda_k K) &= 0 \\ \text{Det } (R + \tfrac{1}{2}\dot{L} + \lambda_k' L) &= 0. \end{aligned} \tag{96}$$

4.3 *Discussion*

Let us return temporarily to circuits of capacitors and inductors only. Suppose all the components are constant (and positive). Then it is well known that the basis functions (natural modes) are undamped. Since $\dot{S}$ and $\dot{C} = 0$, etc., when the components are constant, all λ_k and λ_k' in (88) or (90) are zero. Then the upper and lower bounds must be zero. Thus, in the special case of fixed components, our bounds collapse onto the actual s_σ.

Suppose the inductors are fixed but the capacitors are time-varying. Then, for example, in the bounds (88), $\lambda_1 = 0$ but generally $\lambda_k' \neq 0$.

The zero coefficients λ_k will affect the maximum or minimum of the combined set λ at any times when all λ_k' have the same sign.

When components vary periodically, C and S, etc., vary periodically and $\dot{C}$ and $\dot{S}$, etc., have zero averages. Then any nonzero elements in $\dot{C}$ and $\dot{S}$, etc., must be positive some of the time and negative some of the time. This suggests (but does not prove) that the bounds on s_σ include zero. It is confirmed by some quite different analysis, which is outlined in a short paper in this issue of the B.S.T.J.[7] The paper points out that the s_σ's of nondissipative circuits occur in pairs of the form $+s_\sigma$, $-s_\sigma$. Thus when any s_σ has a nonzero real part, another s_σ will have the negative of that real part, and any true bounds are consistent therewith. It also follows that the bounds described above can be tightened (for nondissipative networks) by using the bound closer to zero and its negative.

The basis functions of a circuit of capacitors and inductors may all have zero damping even though the components are periodically time-varying. This remark is supported by well known properties of the Mathieu-Hill equation, which corresponds to the one-dimensional special case of our vector equations. A weakness of our bounds is that they do not give meaningful sufficient conditions for basis functions with (positive or negative) damping different from zero. The bounds will (at least almost) always be different from zero, but by bracketing zero they will not exclude it.

Suppose now the circuit includes resistors as well as inductors and capacitors. Consider first the bounds (93) derived from the node equation. If the node conductances (represented by matrix G) are sufficiently large, every characteristic root λ_k' will be more negative than every characteristic root λ_k (which does not involve G). Then the lower (more negative) bound on s_σ will be set by the λ_k' and the upper bound by the λ_k. If the conductances are further increased, the lower bound will become more negative, but the upper bound will remain unchanged. On the other hand, increasing the conductances will usually increase the damping of all the basis functions.

The same remarks apply to the bounds derived from the mesh equation except that mesh resistances replace node conductances. Thus when resistors add substantial damping in circuits containing both capacitors and inductors, our *upper* (more positive) bounds may be too weak to have much significance.

Our node analysis assumes that both S and C are positive definite, although the initial formulation may make one or both positive semidefinite. As before, however, the procedure can easily be modified for positive semidefinite matrices. Recall that two different transformations are used to diagonalize S and C, in the derivation of our bounds. When both are positive semidefinite, they can be separately transformed to positive definite submatrices bordered by zeros. Because *constant* transformations can be used (in accordance with the Appendix), the transformed $\dot{S}$ and $\dot{C}$ retain the derivative relationship to the transformed S and C. Then the number of characteristic roots λ_k, λ_k' depends on the ranks of S and C. When only capacitors are connected to some nodes, the mesh analysis will involve some zero damped Thevenin voltages; when only inductors appear in some meshes, the node analysis will involve some Thevenin currents.

APPENDIX

Positive Semidefinite Capacitance Matrices

When C is only positive semidefinite, one can transform equation (6) as follows: The rank m of C is now less than the order n. There exist transformation matrices N_1 such that

$$H_1{}^t C N_1 = \begin{vmatrix} 0 & 0 \\ 0 & \hat{C} \end{vmatrix}. \tag{97}$$

The zeros represent submatrices of zero elements, bordering the complete matrix with $n - m$ rows and $n - m$ columns of zeros. Then $\hat{C}$ is an $m \times m$ positive *definite* submatrix.

These remarks apply to time-varying as well as to fixed circuits. Furthermore, the matrix N_1 can be so chosen that it is *constant*, provided the time-varying capacitances are always positive (>0, not $\geqq 0$). In fact, N_1 can be chosen as a matrix of 0's and 1's only. A detailed demonstration is beyond the scope of this paper; briefly it derives from the fact that the rank of C is determined by the topology of the capacitor part of the circuit, without regard to the values of the capacitances (provided they remain >0, so that the topology cannot be a function of time).

Because $\dot{N}_1 = 0$, $N_1{}^t p = p N_1{}^t$, and (6) can be transformed (as for fixed circuits) into

$$\begin{aligned} \hat{I}_1 &= \left[\hat{G}_1 + p \begin{vmatrix} 0 & 0 \\ 0 & \hat{C} \end{vmatrix} \right] \hat{E}_1 \\ E = N_1 \hat{E}_1, &\qquad \hat{I}_1 = N_1{}^t I. \end{aligned} \tag{98}$$

A further transformation, N_2, is now easily found, of the form:

$$\begin{vmatrix} U & 0 \\ N_{12}{}^t & U \end{vmatrix} \begin{vmatrix} \hat{G}_1 \end{vmatrix} \begin{vmatrix} U & N_{12} \\ 0 & U \end{vmatrix} = \begin{vmatrix} G_g & 0 \\ 0 & \hat{G} \end{vmatrix}. \tag{99}$$

Here U represents a unit, or identity submatrix of appropriate order, and N_{12} is an $m \times (n - m)$ submatrix. Because of the zero submatrix in the upper right-hand corner of the complete matrix,

$$\begin{vmatrix} U & 0 \\ N_{12}{}^t & U \end{vmatrix} \begin{vmatrix} 0 & 0 \\ 0 & \hat{C} \end{vmatrix} \begin{vmatrix} U & N_{12} \\ 0 & 0 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ 0 & \hat{C} \end{vmatrix}. \tag{100}$$

Thus the capacitance matrix is unaffected by the second transformation.

The elements of the N_{12} portion of N_2 may be time-varying. However, because the time-varying elements combine with no nonzero elements of $\hat{C}$, $N_2{}^t p\hat{C} = pN_2{}^t\hat{C}$. Thus the second transformation yields

$$\hat{I} = \left[\begin{vmatrix} G_g & 0 \\ 0 & \hat{G} \end{vmatrix} + p \begin{vmatrix} 0 & 0 \\ 0 & \hat{C} \end{vmatrix} \right] \hat{E}_2 . \tag{101}$$

We can now divide the components of the transformed current and voltage vectors into two parts, as follows:

$$\begin{aligned} \hat{I}_g &= G_g \hat{E}_g , && \text{in } n - m \text{ dimensions} \\ \hat{I} &= (\hat{G} + p\hat{C})\hat{E}, && \text{in } m \text{ dimensions.} \end{aligned} \tag{102}$$

The first equation is algebraic, and its diagonalization is routine. The second equation is in our standard form (6), with $\hat{C}$ positive *definite*, and it may be transformed further in accordance with (29) to (36).

We arrived at this formulation by starting with a one-to-one correspondence between modes and components of the current and voltage vectors, and then transformed the corresponding differential equation. Desoer and Paige[1] arrive at the same conclusion by *starting* with more general choices of the current and voltage components, taking account of the circuit topology.

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