A Functional Analysis Relating Delay Variation and Intersymbol Interference in Data Transmission

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A relationship is derived between the delay characteristic in a data transmission system and the distortion in the form of intersymbol interference created by the delay variation. The relationship is valid for small delay and involves a sequence of linear functionals, each of which has a particular significance. In addition to applications in the analysis of specific systems, problems of a more general nature may be studied using this approach. By various manipulations on the sequence of functionals, bounds on distortion in terms of rms and peak-to-peak delay are derived. On examining the problem of delay equalization, a set of virtually distortion-free delay functions is derived and related to minimum-effort and compromise equalization. Both low-pass and bandpass systems are discussed in turn, with the same general method of analysis applying to each.

I. INTRODUCTION

In this paper we will analyze and discuss one aspect of the problems associated with the transmission of digital data through an unknown linear network. The particular aspect with which we will be concerned is the effect of delay distortion on the fidelity of the transmission. Delay distortion arises generally from nonlinearity in the phase shift with frequency of the system transmission characteristic. This nonlinearity causes different frequencies of the input waveform to arrive at the receiver at different times, thereby distorting the input waveform.

The problem of delay distortion is particularly acute in the voice telephone network. Since speech is relatively insensitive to phase, the switched telephone network has not been equalized for phase shift as well as it has been for attenuation. However, the need has now arisen to make use of this network for transmission of digital data at high speeds. The digital data receiver takes the waveforms it receives quite literally and becomes hopelessly confused by delay distortion if we try to send at too high a rate. For instance, in a voice-band channel of 3 kc Nyquist's famous result tells us that it might be possible to send 6000 independent signals (representing data symbols) per second. However, the usual rate is around 1000 symbols per second (2000 bits for quaternary systems), and higher speeds are impossible because of transmission distortion.

In subsequent sections of this paper we will be concerned with quantitative effects that delay distortion has on data transmission. This does not mean that we will analyze any particular system operating in the presence of a particularly shaped delay variation. This has been done previously by a number of authors, but notably E. Sunde.[1,2,3,4] Rather, we will be more concerned with the gross features of the relationship between delay and system performance. For example, if a particular level of performance is required, what standards may be set on delay such that this minimum performance level will be guaranteed? What shapes of delay are particularly bad or good? How well does differential delay (the difference between the maximum and minimum values of delay across the band) define performance? If a channel is equalized within a certain tolerance of delay, what level of performance can be achieved?

Inasmuch as the telephone network consists of an ensemble of transmission characteristics from which the channel is randomly chosen, these questions would seem to be more meaningful than specific performance figures for particular channels. Consider the problem of comparing data systems for transmission over the voice network. It is clear that this susceptibility to delay distortion is an important factor in such a comparison. The performance of system A will be a random variable defined over the set of possible connections we could dial, likewise system B. The analytical portion of comparing the two systems would be best shown as the probability distributions of performance for the two systems. One system could only be said to be statistically "better" than the other. For example, its average performance over the ensemble of channels might be greater.

Data are now becoming available on the delay characteristics of the voice network (e.g., Alexander, Gryb and Nast).[5] Using these data, it might be possible to analyze systems for which the delay characteristic is chosen randomly from this network, or it might be possible to synthesize systems which operate well (with high probability) over this network. The latter could be accomplished by taking advantage of certain features common to the majority of delay characteristics. One way of doing this is to use a "compromise" equalizer. Quite a question exists as to how to design such an equalizer and how much would be gained by its use. Other possibilities include the use of a bank or set of equalizers from which a best choice may be made for each call or the ultimate use of automatic equalization.

In the remainder of this paper we will show an approximate method whereby the effects of delay distortion may be easily considered in answering such questions as we have asked here. The first section will be devoted to explaining what performance criterion will be used whereby a particular channel may be judged as to goodness for data transmission. In subsequent sections the criterion suggested will be manipulated to show clearly its dependence on delay for both lowpass and bandpass systems. A summary of results obtained is presented following the section on criteria.

II. A PERFORMANCE CRITERION

What we seek in this section is a criterion which may be applied to a transmission channel to determine how good the channel is for data transmission. Obviously, such a criterion should depend upon what system we intend to use over the channel as well as upon the noise environment and input data statistics and the over-all system performance criterion (such as probability of error) that is used. A channel can't be said to be "good" or "bad" irrespective of how we intend to use it. Therefore, the only exact thing which can be done is to treat each possible system separately and derive the relationship between delay and performance separately for each.

For example, Sunde[1,2,3,4] has analyzed several common systems such as AM, PM, and FM in a noiseless environment, using the deviation of the detector output voltage from its undistorted values as a measure of performance. When the details are carried out, the system performance is given as a function of sample values of the impulse response of the over-all system. We call this impulse response $h(t)$

$$h(t) = \frac{1}{\pi} \int_0^\infty A(\omega) \cos [\omega t - \beta(\omega)] \, d\omega \tag{1}$$

where $A(\omega)$ includes signal shaping at the receiver and transmitter as well as the attenuation characteristic of the channel and $\beta(\omega)$ is the channel phase characteristic. Unfortunately, the relationship between the samples of $h(t)$ and system performance is a complicated one and it is unclear as to how the shape of the delay, $\beta'(\omega)$, affects the performance.

What we shall do is to take one specific system, amplitude modulation, and show that its performance is monotonically related to a quantity

$$D = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} | h(t_0 + nT) | = \sum_n{}' | h_n | \tag{2}$$

which we shall call distortion.

Now, in later sections of this paper we will see how the shape of delay affects this distortion measure. Therefore, the results given in these sections may be interpreted as *performance for AM systems*. However, we shall argue that this distortion measure may be quite plausible even though the system being used is not AM. Indeed, many common systems may have their performance related monotonically to D. This is to say that a channel which is bad for AM is probably bad for PM too. This may be considered similar to the "What's good for General Bullmoose is good for the U.S.A." proposition, but the thought may also occur that, considering the unknown nature of the noise and input data statistics, the criterion (2) may be just as good a starting point as some arbitrary definition of environment and performance measure. At any rate, we do not intend to dwell on the difficult problem of criteria here. The criterion D is monotonically related to performance for linear systems and for those systems which can be approximated as linear.

2.1 *The Performance of a Simple Baseband System*

A mathematical model of this system is shown in Fig. 1. The transmitted signal consists of amplitude-modulated waveforms whose shape is $g_1(t)$

$$s(t) = \sum_{n=-\infty}^{+\infty} a_n g_1(t - nT) \tag{3}$$

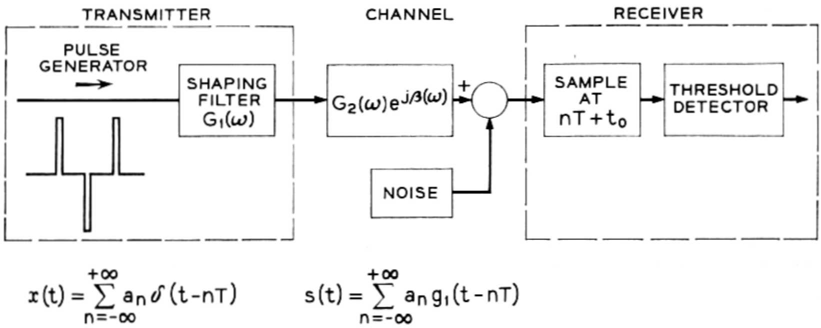

Fig. 1 — A baseband amplitude-modulated system.

and is generated in our mathematical model (not in practice) by a train of area-modulated delta functions pulsing a filter whose impulse response is $g_1(t)$. The transfer function of the channel is $G_2(\omega)e^{j\beta(\omega)}$, so that the over-all transfer function for the impulse is $A(\omega)e^{j\beta(\omega)}$, with

$$A(\omega) = G_1(\omega)G_2(\omega).$$

In the noiseless case, the received signal $y(t)$ is

$$y(t) = \int_0^{+\infty} h(\tau) \sum_{n=-\infty}^{+\infty} a_n\delta(t - nT - \tau)\, d\tau \tag{4}$$

where $h(t)$ is the over-all system impulse response

$$h(t) = \frac{1}{\pi}\int_0^{+\infty} A(\omega) \cos [\omega t - \beta(\omega)]\, d\omega. \tag{5}$$

Equation (4) may be written

$$y(t) = \sum_{n=-\infty}^{+\infty} a_n h(t - nT). \tag{6}$$

We sample this received signal at regular intervals of T seconds starting at time t_0. At time t_0 we expect the amplitude a_0, but we actually get

$$y(t_0) = \sum_{n=-\infty}^{+\infty} a_n h(t_0 - nT) = \sum_{n=-\infty}^{+\infty} a_{-n} h(t_0 + nT) \tag{7}$$

which is a function of the history of the data sequence $\{a_n\}$. For some sequences $y(t_0)$ will be more likely to be detected wrongly than for others. The error due to intersymbol interference is

$$E = a_0 - y(t_0) = a_0[1 - h(t_0)] - \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty} a_{-n} h(t_0 + nT). \tag{8}$$

Now, we are interested in the maximum value this error (which is frequently termed the eye opening) can assume. Assuming the maximum positive and negative values of the coefficients a_n are $\hat{a}$ and $-\hat{a}$ respectively, this maximum error is easily written as

$$E_{\max} = a_0[1 - h(t_0)] - \hat{a} \sum_n{}' \mid h(t_0 + nT) \mid. \tag{9}$$

The first term represents an amplification or attenuation of the signal by the channel, while the second term represents the worst possible effect of intersymbol interference. The prime in the summation sign means deletion of the $n = 0$ term

$$\sum_n{}' = \sum_{\substack{n=-\infty \\ n\neq 0}}^{+\infty}. \tag{10}$$

With the attenuation $h(t_0)$ close to unity* or normalized, we see that the distortion is proportional to

$$D = \sum{}' \mid h(t_0 + nT) \mid. \tag{11}$$

If the noise were additive with a unimodal distribution, the maximum probability of error over all sequences would be a monotonic function of D. For example, if the levels a_n are spaced equally between $+\hat{a}$ and $-\hat{a}$ and there are N levels, the distance between levels is

$$\mid a_n - a_{n-1} \mid = \frac{2\hat{a}}{N-1} \tag{12}$$

and the probability of making an error with Gaussian noise of mean zero and variance σ^2 is

$$\max_{\{a_n\}} \text{prob of error} = \text{prob}\left(\mid \text{noise} \mid > \frac{\hat{a}}{N-1} - \hat{a} \sum{}' \mid h(t_0 - nT) \mid \right) \tag{13}$$

$$P(e) = 1 - \frac{1}{\sigma} \operatorname{erf}\left[\frac{\hat{a}}{\sqrt{2}\sigma}\left(\frac{1}{N-1} - D\right)\right] \tag{14}$$

$$\operatorname{erf}\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{\sqrt{2\pi}} \int_{-x}^{+x} e^{-t^2/2}\, dt. \tag{15}$$

The distortion D is a function of the initial delay, t_0. This sampling time is optimally chosen such that the criterion D is minimized. This best time is a functional of the delay $\beta'(\omega)$ through its influence on the impulse response. Unfortunately, it is extremely difficult to optimize t_0 even for a specific impulse response. Therefore, we shall arbitrarily choose t_0 at the peak of the impulse response. This is a very good approximation to the best possible sampling instant.

2.2 *Discussion of the Criterion D*

The distortion criterion D has been written as the sum of the absolute values of the system impulse response sampled at the symbol repetition

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* This is a second-order effect for the small-delay case in which we will be interested.

rate. The zero sample of impulse response, $h_0 = h(t_0)$, is taken at the peak of the response and is deleted from the summation. We have shown that for an amplitude-modulated system this criterion is proportional to the maximum deviation in the absence of noise of the detector output voltage. The maximum is taken over all possible input symbol sequences. This performance measure is frequently termed the "eye opening," from the resemblance to an eye when the output voltage is displayed on an oscilloscope while random patterns of input symbols are transmitted.

That the criterion D is reasonable for most linear systems may be roughly shown. We recognize that $h(t)$ represents the system memory, or response from past signals. At time t_0 we are looking for symbol s_0, but unfortunately the system remembers remnants of past and future* symbols at this time. The symbols are spaced T seconds apart so that the nth past symbol is "remembered" with relative amplitude

$$| h(t_0 + nT) |.$$

It makes sense that, the larger the sum of these relative memories, the worse will be the intersymbol interference. We will show in the course of our later work how well the performance of a nonlinear system employing phase comparison detection is predicted by use of the distortion criterion D.

2.3 *A General Distortion Criterion*†

More generally, we would wish to send the signals chosen from a set $s_i(t)$, $i = 1, 2, \cdots, N$. Each symbol is chosen according to some probabilistic rule from this set in time sequence to form the signal

$$x(t) = \sum_{n=-\infty}^{+\infty} s_n(t - nT). \tag{16}$$

The signals $s_i(t)$ are sometimes viewed as vectors in a Hilbert space or in some finite-dimensional subspace. The effect of sending the sequence $x(t)$ through a linear network is to cause the received signal during a T-second interval to be a linear combination of the desired signal vector and all other unwanted signal vectors rotated and attenuated by the channel. For a given channel, if we consider the position of the resultant vector for all possible infinite sequences of symbols, we define regions of uncertainty in signal space surrounding the unperturbed vectors s_i.

For purposes of combating noise we are concerned with the distances

* The memory of future symbols is possible because of the time delay t_0 between input and output.

† This section is not essential to the understanding of subsequent material.

in signal space between the transmitted vectors, that is, with the numbers $\| s_i - s_j \|$, $i \neq j$. The greater this set of numbers is, the greater the potential noise immunity of the system. The effect of the channel is to make these distances a function of the symbol sequence. So we can say something about what the channel has done to the noise immunity by specifying the minimum protective distance $\| s_i - s_j \|$, $i \neq j$, with and without the channel. Thus, we might define a measure of distortion as

$$D_0 = 1 - \min_{\substack{i,j \\ i \neq j}} \left[\min_{y_i \subset R_i} \frac{\| y_i - s_j \|}{\| s_i - s_j \|} \right]. \tag{17}$$

R_i = region of uncertainty due to intersymbol interference surrounding the symbol s_i.

This criterion is illustrated in Fig. 2. The way the criterion was formulated did not take into effect the receiver characteristics, but rather evaluated the "loss of detectability" to an ideal maximum likelihood receiver owing to intersymbol interference. Notice also that this criterion

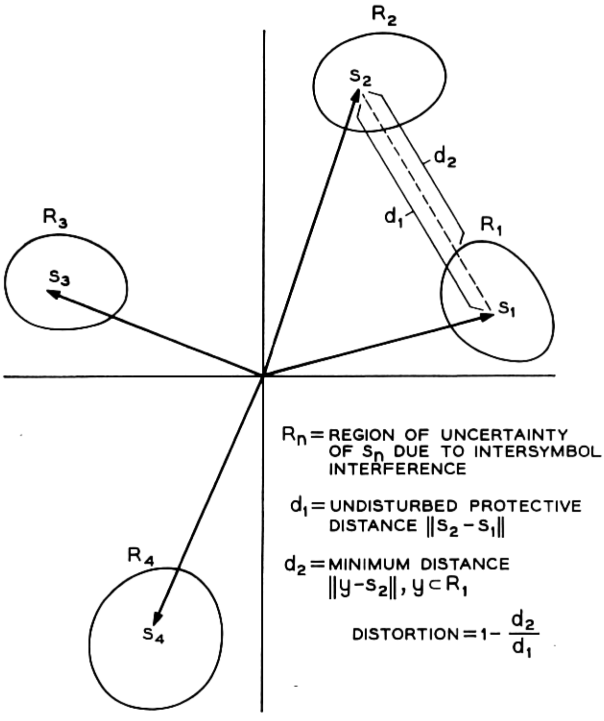

Fig. 2 — A general distortion criterion.

is dependent upon the system to the extent that it is a function of the set of possible signals, $s_i(t)$. As was previously stated, it is impossible to eliminate system dependence from a criterion and maintain usefulness for all conditions.

When this measure is applied to the AM system of Fig. 1, the result is the criterion D previously expressed in (2).

III. SUMMARY OF RESULTS

The problem now is to explore the functional dependence of the distortion, D, upon the delay characteristic, $\beta'(\omega)$. As we have previously shown, the distortion D is a measure related to the eye opening for most linear systems. Specifically, for an N-level AM system we have

$$I_{N\text{-level AM}} = 1 - (N - 1)D. \tag{18}$$

However, by ignoring second-order effects the criterion may be applied to some nonlinear systems. In examples used subsequently in the text, results are obtained for a four-phase data system using phase comparison detection. These results are in close agreement with published data on this system. The eye opening for this system is approximately

$$I_{4\text{-phase}} \approx 1 - D. \tag{19}$$

Similar expressions may be obtained for other systems.

3.1 *The Fundamental Equation*

In Section 4.1 an approximation of small delay is made. The validity of this approximation is explored in a later section, where it is shown to hold for all delay such that the peak delay is limited to 1.1 pulse intervals. For most delay curves the range is wider than this figure, however, and accuracy is generally maintained when $D \leqq 0.6$.

The fundamental equation obtained with the aid of this approximation relates the distortion to the delay variation through a sequence of linear functionals

$$D = \frac{2}{\pi} \sum_{n=1}^{\infty} |(\beta', f_n)| \tag{20}$$

where

$$(\beta', f_n) = \int_0^w \beta'(\omega) f_n(\omega)\, d\omega. \tag{21}$$

Each linear functional yields the intersymbol interference from a particular range of symbols. For example, the adjacent symbol interference is

$$| h_1 | + | h_{-1} | = \frac{2}{\pi} | (\beta', f_1) |. \tag{22}$$

For real delay curves only the first few terms of (20) are usually significant.

The linear functionals (β', f_n) are defined by a sequence of functions $\{f_n\}$, independent of delay, obtained from the amplitude shaping of the system, $A(\omega)$, by the following operation

$$f_n(\omega) = \int_\omega^w [a_n x - \sin nxT] \, A(x) \, dx \tag{23}$$

$$a_n = \frac{\int_0^w \omega A(\omega) \sin n\omega T \, d\omega}{\int_0^w \omega^2 A(\omega) \, d\omega}. \tag{24}$$

3.2 *Application*

Examples are given of the use of (20) in the analysis of system performance when raised cosine amplitude shaping is employed. By reformulating the equation to

$$D = \frac{2}{\pi} \max_{\{\epsilon_n\}} \left(\beta', \sum_{n=1}^{\infty} \epsilon_n f_n \right) \tag{25}$$

$$\epsilon_n = \pm 1$$

bounds on distortion in terms of delay may be derived. We find

$$D \leqq 1.15 \times (\text{rms delay}) \tag{26}$$

$$D \leqq 0.412 \times (\text{peak-to-peak delay}) \tag{27}$$

with delay normalized so that the bandwidth $w = \pi$. The delay curves which achieve equality in the bounds (26) and (27) are illustrated. Other bounds are considered.

In computer simulations, experimental testing, and analysis it is frequently necessary to consider only finite-length input sequences resulting in an effective truncation of the system memory. The possible error in such results is examined and bounded.

The effect of changing the input symbol rate upon distortion is analyzed for a particular example where a binary system is compared with a quaternary system operating at half speed (thus having the same information rate). Depending upon the particular delay function, either system may perform better than the other. However, it is shown that the quaternary system is ultimately the more sensitive to delay variation.

Problems connected with delay equalization are approached with the aid of (20). In the equalization of a specific delay to achieve zero distortion, it is necessary and sufficient that the resultant delay variation be orthogonal to all f_n. For raised cosine shaping there are an infinite number of nonconstant delay curves which have this property. An orthonormal basis for this space of distortionless delay is derived, and examples of projections yielding minimum effort equalization are given. [All curves so derived have, of course, zero distortion only to the order of approximation involved in (20)]. Optimum compromise equalization to match an ensemble of delay variations is also considered.

3.3 *Bandpass Modifications*

For bandpass systems the criterion D is reformulated using the sum of samples of the envelope of the impulse response

$$D = \sum{}' P(t_0 + nT) \tag{28}$$

$$h(t) = P(t) \cos [\omega_c t - \psi(t)] \tag{29}$$

$$\omega_c = \text{carrier or reference frequency.}$$

A fundamental equation for bandpass systems analogous to (20) is derived involving quadrature components

$$D = \sqrt{D_r^2 + D_q^2}. \tag{30}$$

The distortion component D_r results from even components (for symmetrical amplitude shaping) of delay variation, and the component D_q results from odd components of delay variation. Each may be treated as in the low-pass analysis by a sequence of linear functionals

$$D_r = \frac{2}{\pi} \sum_{n=1}^{\infty} | (\varphi', f_{rn}) | \tag{31}$$

$$D_q = \frac{2}{\pi} \sum_{n=1}^{\infty} | (\varphi', f_{qn}) | \tag{32}$$

where $\varphi'(\omega)$ is the bandpass delay and the sequences $\{f_{rn}\}$ of even functions and $\{f_{qn}\}$ of odd functions are derived from the amplitude shaping by operations similar to (23).

All results obtained for low-pass systems may also be obtained for bandpass systems. For example, it is shown that for raised cosine amplitude shaping

$$\begin{aligned} D &\leqq 1.337 \times (\text{rms delay}) \\ D &\leqq 0.467 \times (\text{peak-to-peak delay}). \end{aligned} \tag{33}$$

Thus the bandpass system is slightly more sensitive to delay distortion than its baseband equivalent.

IV. THE RELATIONSHIP OF DISTORTION TO DELAY FOR LOW-PASS SYSTEMS

4.1 *Derivation of a Sequence of Linear Functionals Relating Distortion, D, to Delay, $\beta'(\omega)$*

$$D = \sum_n{}' \mid h(t_0 + nT) \mid = \sum_n{}' \epsilon_n h(t_0 + nT) \tag{34}$$

$$\epsilon_n = \begin{cases} +1 & h(t_0 + nT) \geqq 0 \\ -1 & h(t_0 + nT) < 0 \end{cases} \tag{35}$$

$$D = \sum_n{}' \epsilon_n \int_0^w A(\omega) \cos [\omega(t_0 + nT) - \beta(\omega)]\, d\omega \tag{36}$$

$$D = \frac{1}{\pi} \sum_n{}' [\epsilon_n C_n - \epsilon_n S_n] \tag{37}$$

$$C_n = \int_0^w A(\omega) \cos n\omega T \cos [\omega t_0 - \beta(\omega)]\, d\omega \tag{38}$$

$$S_n = \int_0^w A(\omega) \sin n\omega T \sin [\omega t_0 - \beta(\omega)]\, d\omega. \tag{39}$$

Equations (34) to (39) are self-explanatory reformulations of the criterion D. Equation (37) is summed over all n, $-\infty$ to $+\infty$, except $n = 0$. Because of the obvious symmetry properties of C_n and S_n, namely $C_n = C_{-n}$ and $S_n = -S_{-n}$, we can rewrite (37) as a sum over positive integers only

$$D = \frac{1}{\pi} \sum_{n=1}^{\infty} [C_n(\epsilon_n + \epsilon_{-n}) - S_n(\epsilon_n - \epsilon_{-n})]. \tag{40}$$

Since $\epsilon_n = \pm 1$, one of the pair $(\epsilon_n + \epsilon_{-n})$ and $(\epsilon_n - \epsilon_{-n})$ is zero and the other must be ± 2. Therefore the criterion becomes

$$D = \frac{2}{\pi} \sum_{n=1}^{\infty} \max\, (\mid C_n \mid , \mid S_n \mid). \tag{41}$$

What we are doing here is evaluating terms of the sum D of (34) two at a time. The integral C_n represents $[h(t_0 + nT) + h(t_0 - nT)]$, while the integral S_n represents $[h(t_0 + nT) - h(t_0 - nT)]$. Since what we want is $| h(t_0 + nT) | + | h(t_0 - nT) |$, this is equivalent to $| C_n |$ if these two terms are of the same sign and is $| S_n |$ if they are of opposite sign. We shall now argue that, for small delay, $| S_n | > | C_n |$ and (41) may be summed over the S_n terms alone.

Specifically, what we mean by small delay is that the sine and cosine of $[\omega t_0 - \beta(\omega)]$ may be approximated by the first terms of their expansions. Later we will investigate the conditions under which this approximation is valid. Making these approximations in (38) and (39) gives

$$C_n = \int_0^w A(\omega) \cos n\omega T \, d\omega = 0 \tag{42}$$

(since we must assume that $A(\omega)$ is such that transmission is perfect in the absence of delay, distortion; i.e., $h(nT) = 0, n \neq 0$).

$$S_n = \int_0^w [\omega t_0 - \beta(\omega)] A(\omega) \sin n\omega T \, d\omega. \tag{43}$$

Thus we see that, to this order of approximation, terms linear in $[\omega t_0 - \beta(\omega)]$, $| S_n | > | C_n |$,* and

$$D = \frac{2}{\pi} \sum_{n=1}^{\infty} | S_n | . \tag{44}$$

Now, as we have previously explained, the initial delay t_0 is ideally chosen so as to minimize D for a given $h(t)$. Unfortunately this is impossible to do analytically. We recognize that for zero delay distortion $t_0 = 0$ and that the presence of delay increases t_0. As a good approximation to the ideal sampling time, we are using t_0 as the time of the peak value of the impulse response $h(t)$. An additional, and extremely important, consideration in this choice is that the approximation of $[\omega t_0 - \beta(\omega)]$ small has been made. To choose t_0 at the peak of the impulse response results in the smallest possible values for the function $[\omega t_0 - \beta(\omega)]$. We shall see this more clearly later on.

* The second term in the cosine expansion is

$$-\int_0^w \frac{1}{2} [\omega t_0 - \beta(\omega)]^2 A(\omega) \cos n\omega T \, d\omega.$$

Since $[\omega t_0 - \beta(\omega)]^2 < | \omega t_0 - \beta(\omega) |$ we would generally expect $| C_n |$ to be smaller than $| S_n |$. However, this is not necessarily true; e.g., $[\omega t_0 - \beta(\omega)]$ may be orthogonal to $\sin n\omega T$ on the $[0,w]$ interval.

We now solve for the time t_0 as a functional of $\beta(\omega)$ using the equation $h'(t_0) = 0$

$$h'(t_0) = 0 = \frac{-1}{\pi} \int_0^w \omega A(\omega) \sin [\omega t_0 - \beta(\omega)] \, d\omega \tag{45}$$

$$\int_0^w [\omega t_0 - \beta(\omega)] \omega A(\omega) \, d\omega \approx 0 \tag{46}$$

$$t_0 = \frac{\int_0^w \omega A(\omega) \beta(\omega) \, d\omega}{\int_0^w \omega^2 A(\omega) \, d\omega}. \tag{47}$$

Notice that t_0 is a linear functional of $\beta(\omega)$ and observe that consequently S_n, (43), is linear in $\beta(\omega)$. Therefore, according to the theorem of Riesz,[6] S_n is expressible in the succinct form

$$S_n = \int_0^w f_n(\omega) \beta'(\omega) \, d\omega \tag{48}$$

since $\beta(\omega)$ is linear in $\beta'(\omega)$. We now proceed to put S_n into the form of (48).

Combining (48) and (43), we write S_n as

$$S_n = \int_0^w \beta(\omega)[a_n \omega - \sin n\omega T] A(\omega) \, d\omega \tag{49}$$

where a_n does not depend on β, i.e.

$$a_n = \frac{\int_0^w \omega A(\omega) \sin n\omega T \, d\omega}{\int_0^w \omega^2 A(\omega) \, d\omega}.$$

Integrate (49) by parts to yield

$$S_n = \beta(\omega) g_n(\omega) \Big|_0^w - \int_0^w \beta'(\omega) g_n(\omega) \, d\omega \tag{50}$$

$$g_n(\omega) = \int_0^\omega [a_n x - \sin nxT] A(x) \, dx \tag{51}$$

which may finally be manipulated to give

$$S_n = \int_0^w f_n(\omega) \beta'(\omega) \, d\omega \tag{52}$$

where

$$f_n(\omega) = \int_\omega^w [a_n x - \sin nxT]A(x)\,dx. \tag{53}$$

We have now written the distortion D as

$$D = \frac{2}{\pi} \sum_{n=1}^{\infty} | S_n | \tag{54}$$

where each term S_n is a linear functional of delay and represents the distortion arising from intersymbol interference from symbols $\pm n$ symbols away. Obviously the terms S_n become quite insignificant for large n. For many delay curves, the principal interference is from adjacent symbols and only S_1 is of major importance. We shall demonstrate this when $A(\omega)$ is the commonly used raised cosine shaping and the delay is parabolic.

Using the Schwarz inequality in (52) gives a useful bound on S_n.

$$| S_n | \leqq \| f_n \| \, \| \beta' \| \tag{55}$$

$$\| f_n \| = \sqrt{\int_0^w f_n^2(\omega)\,d\omega}$$

$$\| \beta' \| = \sqrt{\int_0^w [\beta'(\omega)]^2\,d\omega} = \text{rms delay} \times \sqrt{w}. \tag{56}$$

Thus we can see how fast the successive terms in (54) must approach zero. The total distortion is of course bounded by

$$D \leqq \frac{2}{\pi} \| \beta' \| \sum_{n=1}^{\infty} \| f_n \|. \tag{57}$$

The norm $\| f_n \|$ may be thought of as the sensitivity of a system to intersymbol interference at a distance of $\pm n$ symbols. The greater $\| f_n \|$, the more sensitive the system is to delay distortion.

The effect of the shape of the delay curve is clearly illustrated in (52), which is abbreviated

$$S_n = (\beta', f_n) \tag{58}$$

and represents the inner (or scalar, or dot) product of the functions (vectors) f_n and β'. S_n is less than or equal to the product of the lengths of the two vectors, which is what the Schwarz inequality in (55) says, and the equality occurs when the delay $\beta'(\omega)$ has the same shape as $f_n(\omega)$.

While expression (57) represents an upper bound on the distortion as a function of rms delay, this bound is generally not realizable. In fact, this bound is generally useless, since the sum of the norms $\| f_n \|$ frequently diverges. This does not mean the distortion can diverge, since this would require the delay to simultaneously have appreciable components in the direction of each of the vectors f_n. In the two examples we will study, it is shown that this divergence is possible in one case, where all the f_n are approximately in the same direction, whereas it is impossible in the other, where the f_n are nearly orthogonal.

To find a least upper bound on distortion as a function of rms delay we write

$$D = \frac{2}{\pi} \sum_{n=1}^{\infty} \epsilon_n S_n \qquad \epsilon_n = \begin{cases} +1 & S_n \geqq 0 \\ -1 & S_n < 0 \end{cases} \tag{59}$$

$$D = \frac{2}{\pi} \sum_{n=1}^{\infty} \epsilon_n (\beta', f_n) = \frac{2}{\pi} \left(\beta', \sum_{n=1}^{\infty} \epsilon_n f_n \right). \tag{60}$$

The distortion D is thus a scalar product of delay and some combination of the functions f_n. The sequence of sign coefficients $\{\epsilon_n\}$ is chosen so as to maximize this scalar product. The resulting value of D is the same as forming the sum of absolute values of the individual scalar products (β', f_n).

Using the Schwarz inequality in (60) we obtain

$$D \leqq \frac{2\sqrt{w}}{\pi} \times (\text{rms delay}) \times \max_{\{\epsilon_n\}} \left\| \sum_{n=1}^{\infty} \epsilon_n f_n \right\| \tag{61}$$

$$D \leqq \frac{2\sqrt{w}}{\pi} \times (\text{rms delay}) \times \max_{\{\epsilon_n\}} \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \epsilon_n \epsilon_m (f_n, f_m) \right]^{\frac{1}{2}}. \tag{62}$$

Expression (62) is the least upper bound on distortion for a given value of rms delay, and the equality is obtained when

$$\beta'_{\text{worst}} = \sum_{n=1}^{\infty} \epsilon_n f_n . \tag{63}$$

The inequality (62) may be used to define the over-all *sensitivity* of a system to delay distortion

$$D \leqq (\text{rms delay}) \times (\text{sensitivity}) \tag{64}$$

$$\text{sensitivity} = \frac{2\sqrt{w}}{\pi} \max_{\{\epsilon_n\}} \left[\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \epsilon_n \epsilon_m (f_n, f_m) \right]^{\frac{1}{2}}. \tag{65}$$

The sensitivity is equal to $2\sqrt{w}/\pi$ times the length of the longest vector which can be obtained by summing the vectors $\pm f_n$.

We also have the obvious bounds

$$\frac{2\sqrt{w}}{\pi}\left[\sum_{n=1}^{\infty} \| f_n \|^2\right]^{\frac{1}{2}} \leqq \text{sensitivity} \leqq \frac{2\sqrt{w}}{\pi}\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} | (f_n, f_m) |\right]^{\frac{1}{2}}. \tag{66}$$

4.2 *Example — Perfect Low-Pass System Operating at the Nyquist Rate*

As our first example we choose the "ideal" system, a perfect low-pass channel operating at a rate of $2W$ symbols per second

$$\begin{aligned} A(\omega) &= 1 \qquad 0 \leqq \omega \leqq \pi \\ A(\omega) &= 0 \qquad \pi \leqq \omega \\ T &= 1 \\ w &= \pi. \end{aligned} \tag{67}$$

We now evaluate the function $f_n(\omega)$, using these values

$$f_n(\omega) = \int_\omega^w [a_n x - \sin nxT]\, dx \tag{68}$$

$$a_n = \frac{\int_0^\pi \omega \sin n\omega \, d\omega}{\int_0^\pi \omega^2 \, d\omega} = \frac{-3}{n\pi^2}(-1)^n \tag{69}$$

$$\begin{aligned} f_n(\omega) &= \int_\omega^\pi [a_n x - \sin nx]\, dx \\ &= \frac{(-1)^n}{n}\left[\frac{3}{2\pi^2}\omega^2 - \frac{(-1)^n \cos n\omega}{n} - \frac{1}{2}\right]. \end{aligned} \tag{70}$$

If the delay is constant, say $\beta'(\omega) = c$, we have

$$S_n = (\beta', f_n) = c\int_0^\pi f_n(\omega)\, d\omega \tag{71}$$

$$S_n = \frac{1}{n}\left(\frac{\pi^3}{3}\right)\left(\frac{3}{2\pi^2}\right)(-1)^n - \frac{1}{n}\frac{(-1)^n}{2}\pi = 0. \tag{72}$$

Thus there is no distortion when the delay is constant. Of course we already knew this, but it serves as a useful check on the method.

Looking now at the system sensitivity, we compute the following products

$$
\begin{aligned}
(f_n, f_m) &= \frac{1}{nm}\left[\frac{\pi}{5} - \frac{3}{\pi}\left(\frac{1}{m^3} + \frac{1}{n^3}\right)\right] \qquad n \neq m \\
&= \frac{1}{n^2}\left[\frac{\pi}{5} + \frac{\pi}{2n^2} - \frac{6}{\pi n^3}\right] \qquad n = m.
\end{aligned} \tag{73}
$$

Choose all the coefficients ϵ_n as positive and it is immediately seen that

$$
\text{sensitivity} \geqq \frac{2}{\sqrt{\pi}}\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}(f_n, f_m)\right]^{\frac{1}{2}} \to \infty \tag{74}
$$

Therefore the perfect low-pass channel is infinitely sensitive to delay distortion, so that the smallest increment of delay can result in divergence of the eye picture. This result is not entirely unexpected, and is a good reason for not using "perfect" channels even if they were physically realizable.

4.3 *The Raised Cosine System*

4.3.1 *Derivation and Discussion of the Functions $f_n(\omega)$*

Now, instead of the flat amplitude shaping of the previous example which was so sensitive to delay distortion, we use a more gradual cutoff. The raised cosine shaping is the shaping most frequently used in practice, since it retains the proper zero crossings at the Nyquist rate and, as we will show, is less sensitive to delay distortion. Of course the penalty one pays for this protection against delay distortion is a doubling of the bandwidth for a given symbol rate as compared to the flat shaping previously discussed.

For the raised cosine shaping we have

$$
\begin{aligned}
A(\omega) &= \cos\omega + 1 \qquad 0 \leqq \omega \leqq \pi \\
A(\omega) &= 0 \qquad \omega \geqq \pi \\
T &= 2 \qquad w = \pi
\end{aligned} \tag{75}
$$

$$
a_n = \frac{\displaystyle\int_0^{\pi} \omega \sin 2n\omega(\cos\omega + 1)\,d\omega}{\displaystyle\int_0^{\pi} \omega^2(\cos\omega + 1)\,d\omega} \tag{76}
$$

$$
a_n = \frac{1}{\left(\dfrac{\pi^2}{3} - 2\right)2n(4n^2 - 1)}. \tag{77}
$$

Notice that a_n falls off as $1/n^3$ as contrasted with the previous example where $a_n \sim 1/n$.

$$f_n(\omega) = \int_\omega^\pi [a_n x - \sin 2nx](\cos x + 1)\, dx \tag{78}$$

$$f_n(\omega) = a_n \cos \omega + \frac{1}{2(2n+1)} \cos (2n-1)\omega + \frac{1}{2n} \cos 2n\omega + \frac{1}{2(2n+1)} \cos (2n+1)\omega + a_n\omega \sin \omega + \frac{a_n \omega^2}{2} - a_n \left[1 + \frac{\pi^2}{6}\right]. \tag{79}$$

The first few functions $f_1(\omega)$, $f_2(\omega)$, and $f_3(\omega)$ are shown in Fig. 3. Observe that $f_1(\omega)$, representing adjacent symbol interference, has the greatest energy of these functions. Its shape in crude terms might be described as one cycle of cosine exponentially attenuated. The next function, $f_2(\omega)$, consists of about 2 cycles of cosine with less exponential attenuation, and this trend continues for the higher-order functions. The effect of delay distortion on the raised cosine system can be visualized with the aid of these functions. About the worst form of delay consists of one cycle of delay looking like $f_1(\omega)$. When the residual delay consists of a large number of ripples, say n cycles, then its distortion is not so great and comes largely from intersymbol interference at a distance of n symbols. When the delay is a slowly varying function of ω,

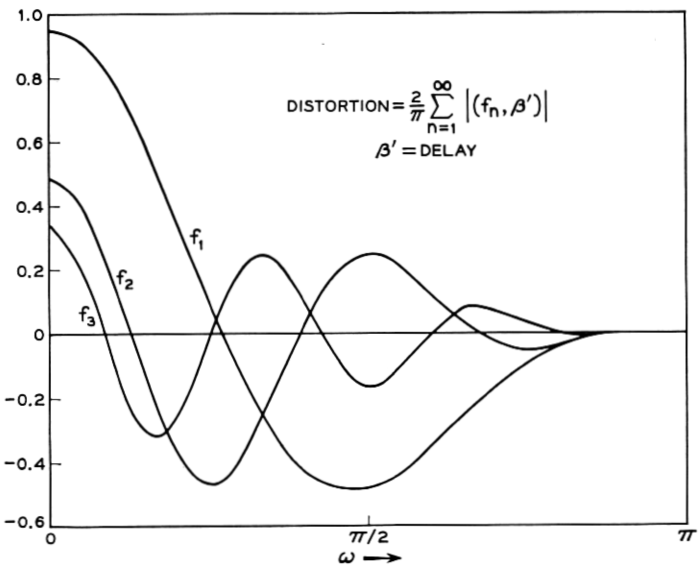

Fig. 3 — The functions $f_n(\omega)$ for raised cosine shaping.

the higher-order terms $S_n = (f_n, \beta')$ $n > 1$ become insignificant, and only adjacent symbol interference is of importance.

4.3.2 *Use of the Functions $f_n(\omega)$ in Computing Distortion*

All the functions $f_n(\omega)$ integrate to zero over the $[0,\pi]$ interval, so there is no distortion when the delay is constant. Now suppose we have parabolic delay

$$\beta'(\omega) = k\omega^2. \tag{80}$$

This is the general shape of delay to be expected in an unequalized voice channel.

Carrying out the relevant integrations gives

$$\begin{aligned} S_1 &= \int_0^\pi k\omega^2 f_1(\omega)\, d\omega = -0.411\pi k \\ S_2 &= 0.025\pi k \\ S_n &\approx \frac{0.282\pi k}{n^3}. \end{aligned} \tag{81}$$

In Ref. 2, Sunde computes the impulse response of a raised cosine network with parabolic delay distortion. In terms of the parameter m, the maximum delay in pulse intervals used in this reference, we find

$$k = 2m/\pi^2. \tag{82}$$

Using a value of $m = 2$ we read from Sunde's curve*

$$|h_1| + |h_{-1}| = 0.31 \tag{83}$$

while from (81) we have

$$|h_1| + |h_{-1}| = \frac{2}{\pi}|S_1| = 0.33. \tag{84}$$

The agreement is good, although the value of delay $m = 2$ is somewhat outside the range where the approximations are entirely valid.

To compute the distortion arising from parabolic delay we form the sum

$$D = \frac{2}{\pi}\sum_{n=1}^{\infty}|S_n| = 0.912k. \tag{85}$$

---

* We occasionally abbreviate $h(t_0 + nT)$ as simply h_n.

Notice that some 90 per cent of this distortion is due to the term S_1 (adjacent symbol interference). For this general shape of delay the term $S_1 = (\beta',f_1)$ would seem to be sufficiently indicative of system performance.

As another example of the computation of the effect of delay, we consider delay of the form

$$\beta' = a \cos \nu\omega. \tag{86}$$

This cosinusoidal delay is of the type frequently encountered as residual delay after partial equalization or in wider band systems. Depending on the number of cycles of delay across the band ν, only one or two of the terms S_n are of importance. These are the terms $n \approx \nu$. These various products (β',f_n) are shown as a function of ν in Fig. 4. Each product (β',f_n) peaks for ν a little less than n cycles and is very small elsewhere.

Rappeport[7] has studied the effect of this type of delay on the 4-phase data set using phase comparison detection. This study was effected using a digital computer simulation of the system. Rappeport plots curves of eye opening versus the number of cycles of delay in the passband, ν. Since the cosine is symmetrical, our low-pass results carry over directly to the passband in this case. The 4-phase system is essentially nonlinear because of the multiplication in the detection process. An exact expression relating eye opening to impulse response is not derivable for this system. An approximate expression for the eye opening is

$$I = 1 - D. \tag{87}$$

This expression neglects terms involving products such as $h_n h_m$. When the first four curves in Fig. 4 are summed to form D, the curve relating eye opening to delay frequency may be drawn as shown in Fig. 5. This curve is compared with the curve computed by Rappeport using $a = 0.5$ in each case, and it is seen that the general agreement is quite good except for somewhat more oscillation in the latter than in the former.

The exact eye opening for the 4-phase system depends not only on D, but on the relative magnitudes and signs of the samples h_n which sum to form D.

4.3.3 *Sensitivity and Bounds on Distortion for Raised Cosine Shaping*

The sensitivity of the raised cosine system to delay distortion may be calculated from consideration of the functions f_n. For $n > 4$ the terms involving a_n become approximately negligible and f_n consists of the terms

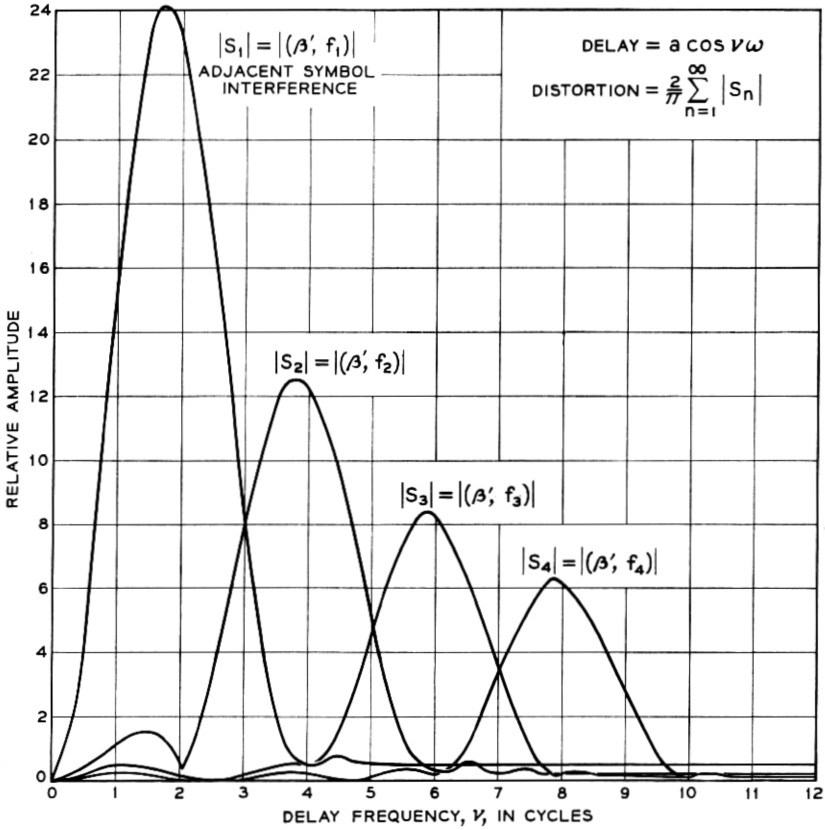

Fig. 4 — Various components of distortion for cosine delay.

$$f_n \approx \frac{1}{2(2n-1)} \cos (2n-1)\omega + \frac{1}{2n} \cos 2n\omega + \frac{1}{2(2n-1)} \cos (2n+1)\omega. \tag{88}$$

Thus f_n becomes approximately orthogonal to all f_m except for $m = n$ and $m = n \pm 1$. There is an overlap between f_n and f_{n+1} in the term $1/2(2n+1) \cos (2n+1)\omega$ and similarly an overlap between f_n and f_{n-1} in the term $1/2(2n-1) \cos (2n-1)\omega$. Obviously, to construct the sequence $\{\epsilon_n f_n\}$ of greatest energy we choose the signs ϵ_n such that all these shared cosine terms (the other terms are orthogonal) add in phase and thus reinforce each other. Thus it seems reasonable that $\epsilon_n \equiv +1$ for $n \geqq M$.

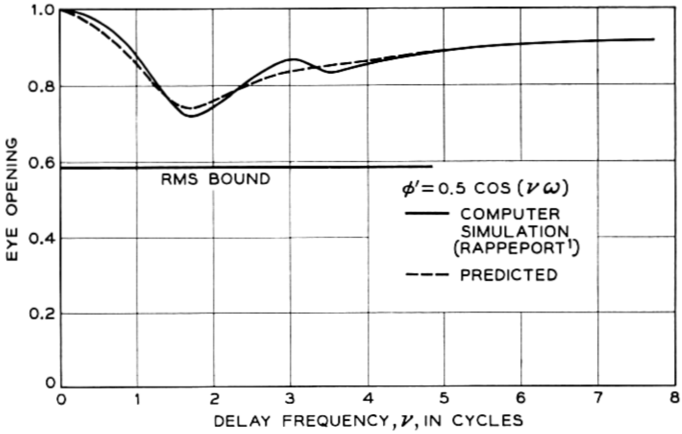

Fig. 5 — Eye opening for 4-phase data set for cosinusoidal delay.

By an exhaustive search on a digital computer of the effect of the first twelve coefficients ϵ_n, it was found that the maximum energy combination occurred for all $\epsilon_n = +1$ except for $\epsilon_1 = \epsilon_2 = \epsilon_3 = -1$. We designate this combination $f_{mec}(\omega)$ (*m*aximum *e*nergy *c*ombination). This function has the worst shape a delay can assume for a given rms value, and the sensitivity of the system is proportional to the norm of this function

$$f_{mec}(\omega) = \sum_{n=1}^{\infty} f_n(\omega) - 2[f_1(\omega) + f_2(\omega) + f_3(\omega)]. \tag{89}$$

We first perform the infinite summation involved here

$$\sum_{n=1}^{\infty} f_n(\omega) = \left[\cos\omega + \omega\sin\omega + \frac{\omega^2}{2} - \left(1 + \frac{\pi^2}{6}\right)\right]\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty}\left[\frac{1}{2(2n-1)}\cos(2n-1)\omega + \frac{1}{2n}\cos 2n\omega + \frac{1}{2(2n+1)}\cos(2n+1)\omega\right]. \tag{90}$$

Both sums can be put in closed form. The first sum is

$$\sum_{n=1}^{\infty} a_n = \frac{3}{2(\pi^2 - 6)}\sum_{n=1}^{\infty}\frac{1}{n(4n^2-1)} = 0.14973 \tag{91}$$

while the last sum in (90) may be recognized as simply

$$\sum_{n=1}^{\infty} \frac{1}{n} \cos n\omega - \frac{1}{2} \cos \omega \tag{92}$$

which converges to

$$-\log \left| 2 \sin \frac{\omega}{2} \right| - \frac{1}{2} \cos \omega. \tag{93}$$

So that we finally find

$$\begin{aligned} f_{\text{mec}}(\omega) = {} & 0.14973 \left[\omega \sin \omega + \frac{\omega^2}{2} - 2.64493 \right] - 0.35027 \cos \omega \\ & - \log \left| 2 \sin \frac{\omega}{2} \right| - 2f_1(\omega) - 2f_2(\omega) - 2f_3(\omega). \end{aligned} \tag{94}$$

This function is shown in Fig. 6. A delay curve of this shape has maximum detrimental effect on the raised cosine shaped system. The norm of $f_{\text{mec}}(\omega)$ was computed numerically to be

$$\| f_{\text{mec}} \| = 1.02 \tag{95}$$

and so the sensitivity of the raised cosine system is

$$\text{sensitivity} = \| f_{\text{mec}} \| \frac{2\sqrt{w}}{\pi} = 1.15 \tag{96}$$

$$D \leqq 1.15 \times (\text{rms delay}) \tag{97}$$

In the previous paragraphs we investigated the effect of cosinusoidal delay. For this shape and for an amplitude $a = 0.5$, the bound (97) gives $D \leqq 1.15 \times 0.5 \times 0.707 = 0.407$, so the eye opening $(1 - D)$ must be greater than or equal to 0.593. This value is shown on Fig. 5 along with the curves representing actual and computed performance for the cosine delay. At the lowest dips in these curves the distortion is about $\frac{2}{3}$ of the bound (97). The distortion computed for the parabolic delay distortion, however, is only about $\frac{1}{4}$ of its corresponding bound. As might be anticipated, the parabolic shape is a relatively weak form of delay distortion.

It is also of interest to compute bounds on samples of the impulse response.

$$| h_n | + | h_{-n} | = \frac{2}{\pi} | (\beta', f_n) | \leqq \frac{2}{\sqrt{\pi}} \| f_n \| \times (\text{rms delay}) \tag{98}$$

$$| h_1 | + | h_{-1} | \leqq 0.829 \times (\text{rms delay}) \tag{99}$$

$$| h_2 | + | h_{-2} | \leqq 0.443 \times (\text{rms delay}) \tag{100}$$

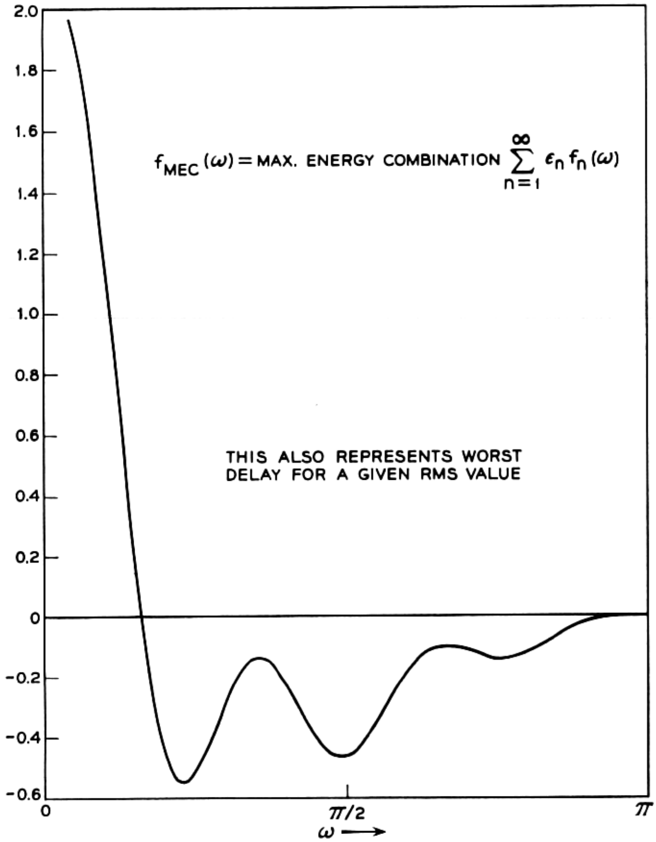

Fig. 6 — RMS bound on distortion; $D = 1.15 \times$ (rms delay).

$$|h_3| + |h_{-3}| \leqq 0.289 \times \text{(rms delay)}. \tag{101}$$

For large n we have asymptotically

$$|h_n| + |h_{-n}| \leqq \frac{\sqrt{3}}{2n} \times \text{(rms delay)}. \tag{102}$$

Thus the individual samples of the impulse response are only bounded inversely proportionally to their distance from the peak time of the impulse response. In each case the maximum value is obtained when the delay is some constant times $f_n(\omega)$. However, the sum of all these terms can never exceed $1.15 \times$ (rms delay), which paradoxically is less than the sum of the attainable bounds on only the first two terms.

Since adding or subtracting any constant delay does not affect the distortion D, the bounds given here are most effectively used by first subtracting the mean value of delay and dealing only with the variational component.

We can find similar bounds in terms of peak-to-peak constraints on the delay. The "differential" delay is frequently taken as the difference between the maximum and minimum values of delay across the band. We shall now find the shape of delay which maximizes distortion for a given peak-to-peak constraint and the corresponding value of distortion.

From (60) we have

$$D = \frac{2}{\pi} \max_{\{\epsilon_n\}} \left(\beta', \sum_{n=1}^{\infty} \epsilon_n f_n \right). \tag{103}$$

If $\beta'(\omega)$ is peak-limited then the maximum value of (103) is obtained when $\beta'(\omega)$ is chosen as $+\beta'_{\max}$ when $\sum_{n=1}^{\infty} \epsilon_n f_n$ is positive and $-\beta'_{\max}$ when $\sum_{n=1}^{\infty} \epsilon_n f_n$ is negative. The resulting distortion is

$$D_{\max} = \frac{2\beta'_{\max}}{\pi} \max_{\{\epsilon_n\}} \int_0^w \left| \sum_{n=1}^{\infty} \epsilon_n f_n(\omega) \right| d\omega. \tag{104}$$

The problem reduces to finding the combination of signs $\{\epsilon_n\}$ such that the absolute integral of $\sum_{n=1}^{\infty} \epsilon_n f_n(\omega)$ is maximized. We call this maximizing combination $f_{\mathrm{mai}}(\omega)$ (*m*aximum *a*bsolute *i*ntegral). By trial and error on a digital computer, the following sequence of signs for raised cosine shaping was found

$$\epsilon_n = +1 \quad \text{except} \quad \epsilon_1 = \epsilon_4 = -1 \tag{105}$$

$$f_{\mathrm{mai}}(\omega) = \sum_{n=1}^{\infty} f_n(\omega) - 2[f_1(\omega) + f_4(\omega)]. \tag{106}$$

This function is shown in Fig. 7. The worst peak-to-peak delay is positive when $f_{\mathrm{mai}}(\omega)$ is positive and negative when $f_{\mathrm{mai}}(\omega)$ is negative. This worst delay curve is also shown in this figure. It is rectangular in shape with the single axis crossing at $\omega = 0.32\pi$. The integral of $| f_{\mathrm{mai}}(\omega) |$ was computed numerically, so that from (104) we have

$$D \leqq \frac{2}{\pi} \times 1.293 \times \frac{1}{2} \text{ (peak-to-peak delay)} \tag{107}$$

$$D \leqq 0.412 \times \text{(peak-to-peak delay)}. \tag{108}$$

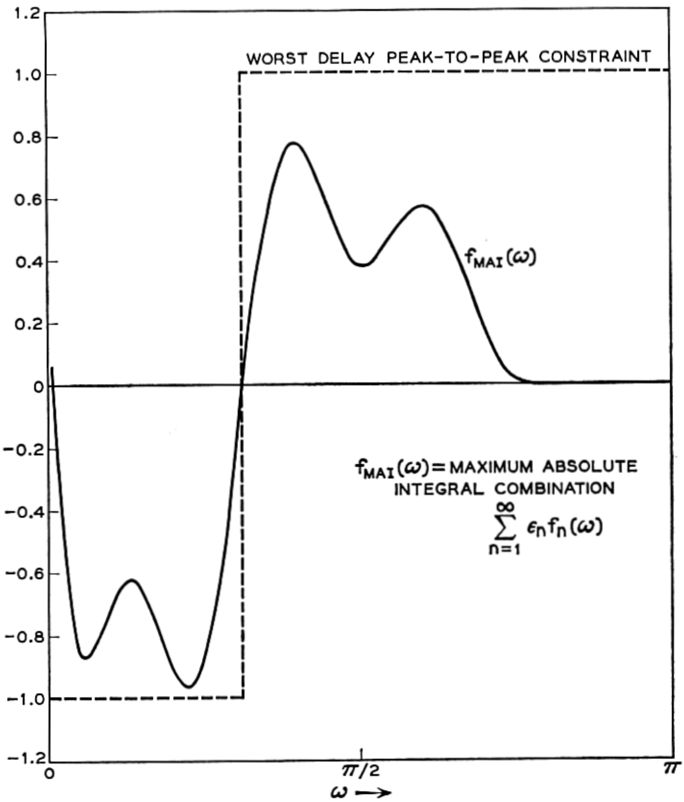

Fig. 7 — Peak bound on distortion; D = 0.412 × (peak-to-peak delay).

For example, the peak-to-peak delay of the cosine delay example was 1.0 and so the bound on D is 0.412 for this particular class of delay waveforms.

4.3.4 *The Effect of Increasing the Period T on Distortion*

It is possible to decrease the distortion due to intersymbol interference by sending symbols at a slower rate. If the same information rate is to be retained, the amount of information a given symbol conveys must be proportionally increased. With more symbols to be distinguished at the receiver, the smaller amount of distortion may be even more troublesome than before the rate was diminished, so there is a question as to whether or not the system performance for a given information rate can be improved by sending at a slower rate.

We consider a binary AM system. We previously found that the distortion from the normalized quiescent values of $+1$ and -1 is limited by the inequality

$$D \leqq 1.15 \times \text{rms delay}. \tag{109}$$

Suppose that we now send at half speed and, in order to maintain a constant information rate, change from a binary system to quaternary. First we compute a new value for sensitivity using the period $T = 4$.

Obviously the same functions $f_n(\omega)$ that we computed before still apply, with the change that we now use $f_{2n}(\omega)$ instead of $f_n(\omega)$. Now there is no overlap in the successive cosine terms in $f_n(\omega)$ and $f_{n\pm1}(\omega)$ and the terms $f_n(\omega), f_m(\omega)$, $n \neq m$ are very nearly orthogonal.

Assuming orthogonality we can easily compute the sensitivity from (65)

$$\text{sensitivity} = \frac{2}{\sqrt{\pi}}\left[\sum_{n=1}^{\infty} \| f_{2n} \|^2\right]^{\frac{1}{2}} \tag{110}$$

$$\text{sensitivity} = 0.628$$

$$D(\text{half speed}) \leqq 0.628 \times (\text{rms delay}). \tag{111}$$

However, in an n-level AM system the amount of distortion necessary to cause an error is

$$D(\text{error}) = \frac{1}{n-1}. \tag{112}$$

The eye opening is defined as unity minus the ratio of distortion to the amount necessary to cause an error

$$I_{n \text{ level AM}} = 1 - (n-1)D. \tag{113}$$

Therefore we have for the same information rate

$$I_{\text{binary}} \geqq 1 - 1.15 \times (\text{rms delay}) \tag{114}$$

$$I_{\text{quaternary}} \geqq 1 - 1.884 \times (\text{rms delay}). \tag{115}$$

It is seen that the system has been made more susceptible to delay distortion by sending at a slower speed with proportionally more information per symbol. However, for any particular delay curve either system may perform better than the other. In comparing the two systems the statistics of the particular ensemble of delays to be encountered should be taken into consideration. Lacking any such statistics, the binary system is the obvious "minimax" choice in that it has less sensitivity to delay than the quaternary system.

4.3.5 *Zero-Distortion Delay Functions*

We have derived upper bounds on distortion as a function of rms and peak-to-peak delay. Now we might ask for some corresponding lower bounds. Since distortion is a positive quantity, we might wonder if it is possible to have zero distortion for some nonconstant delay curves. The distortion has been shown to be the sum of absolute values of certain linear functionals (β', f_n). Thus, to achieve zero distortion each of these functionals must be identically zero. In other words, the delay $\beta'(\omega)$ must be orthogonal to each of the functions f_n. This leads naturally to consideration of the completeness of the infinite sequence of functions $\{f_n\}$. If this sequence is complete in the space $L^2(0,\pi)$ then there exist no delay functions in this space for which the distortion is identically zero (see, for example, Ref. 6).

Actually, we have already demonstrated that constant functions are orthogonal to all f_n, so trivially the sequence is not complete. However, we are not particularly interested in constant-delay functions, so we might as well append a constant function to the sequence $\{f_n\}$ and consider the augmented sequence. That this sequence is not complete either may be easily proved by finding a function which is orthogonal to all f_n. For this purpose we write

$$f_n(\omega) = a_n\mu(\omega) + \frac{1}{2(n-1)} \cos(2n-1)\omega + \frac{1}{2n} \cos 2n\omega + \frac{1}{2(n+1)} \cos(2n+1)\omega \tag{116}$$

$$\mu(\omega) = \cos\omega + \omega \sin\omega + \frac{\omega^2}{2} - \left(1 + \frac{\pi^2}{6}\right). \tag{117}$$

Now observe that a function of the form

$$\psi_n(\omega) = \cos 2n\omega + b_1^n \cos(2n+1)\omega + b_2^n \cos(n+2)\omega + b_3^n \cos(2n+3)\omega + b_4^n \cos(2n+4)\omega \tag{118}$$

can be made orthogonal to all $f_n(\omega)$ by proper choice of the coefficients b^n. The four simultaneous conditions on these coefficients are

$$\begin{aligned} (\psi_n, \mu) &= 0 \\ (\psi_n, f_n) &= 0 \\ (\psi_n, f_{n-1}) &= 0 \\ (\psi_n, f_{n+1}) &= 0. \end{aligned} \tag{119}$$

These four conditions insure the orthogonality of ψ_n and f_m, since for $m > n + 1$ and $m < n - 1$ there is no overlap in the cosine terms and we made ψ_n orthogonal to μ, which constitutes the remaining portion of f_m.

Inserting (116), (117) and (118) into the four simultaneous equations (119) yields the result

$$b_1^n = \frac{4n^2 - 1}{n(2n - 1)} \tag{120}$$

$$b_2^n = \frac{-6(n + 1)}{n(2n - 1)} \tag{121}$$

$$b_3^n = \frac{(2n + 3)(2n + 5)}{n(2n - 1)} \tag{122}$$

$$b_4^n = \frac{-(n + 2)(2n + 5)}{n(2n - 1)}. \tag{123}$$

Some special considerations come in when solving for $\psi_0(\omega)$, which is a little different from the others. We merely quote the result here

$$\psi_0(\omega) = \cos \omega - 6 \cos 2\omega + 15 \cos 3\omega - 10 \cos 4\omega. \tag{124}$$

Thus, we have derived an infinite sequence of functions all of which are orthogonal to $\{f_n\}$ and therefore are distortionless. What we would really like to do, however, is to find all the functions which are distortionless in the space $L^2(0,\pi)$. We designate the subspace consisting of all distortionless functions as G. We call the linear manifold spanned by the sequence $\{f_n\}$ the distortion subspace, F. Each delay function in $L^2(0,\pi)$ can be expressed as the sum of two orthogonal functions $f \subset F$ which causes distortion and $g \subset G$ which is distortionless.

$$L^2(0,\pi) = F \oplus G. \tag{125}$$

We can form a sequence of orthonormal basis functions for F by orthonormalizing the sequence $\{f_n\}$. Unfortunately the sequence $\{\psi_n\}$ is not complete in G. For the purposes of analysis a sequence of approximate basis functions for G may be derived by the following procedure.

(*i*) Approximate f_n to the desired accuracy by

$$f_n = \sum_{m=1}^{M} a_m^{(n)} \cos m\omega. \tag{126}$$

(*ii*) Orthonormalize the functions f_n; $n = 1, 2, \cdots, (M - 1)/2$ giving $(M - 1)/2$ orthonormal basis functions in the M-dimensional space of the approximation.

TABLE I—AN APPROXIMATE SET OF ORTHONORMAL BASIS FUNCTIONS FOR THE SPACE G OF DISTORTIONLESS FUNCTIONS

$$\left(\sqrt{\frac{\pi}{2}}\, g_n = a_1 \cos \omega + a_2 \cos 2\omega + a_3 \cos 3\omega + \cdots + a_{11} \cos 11\, \omega\right)$$

	g_1	g_2	g_3	g_4	g_5	g_6
a_1	0.8478					
a_2	−0.5053	−0.2887	−0.0016	−0.0004	−0.0001	
a_3	−0.1042	0.8087				
a_4	0.1136	−0.4837	−0.3296			
a_5	0.0435	−0.1483	0.8233			
a_6	−0.0016	0.0766	−0.4322	−0.3540		
a_7	0.0044	0.0251	−0.1445	0.8259		
a_8	0.0092	−0.0131	0.0713	−0.4078	−0.3687	
a_9	0.0066	−0.0049	0.0252	−0.1443	0.8295	
a_{10}	0.0047	0.0018	−0.0116	0.0664	−0.3819	−0.4138
a_{11}	0.0021	0.0008	−0.0053	0.0302	−0.1736	0.9104

(*iii*) Derive the missing $(M + 1)/2$ orthonormal basis functions in this M-dimensional space. These are approximately g_n ; $n = 1, 2, \cdots, (M + 1)/2$.

(*iv*) $g_0 = 1$.

The reason this procedure works well is that each successive function f_n adds strong components of $\cos 2n\omega$ and $\cos (2n + 1)\omega$ as the sequence $\{f_n\}$ is orthonormalized. The g_n functions "interleave" to form a Fourier series. For $M = 11$ the six g functions thus generated are given in Table I.

Now any linear combination of the functions $g_n(\omega)$ has zero distortion. In particular, for any given delay curve we can find the closest distortion-free curve. This would indicate the minimum amount of equalization necessary to eliminate intersymbol interference. This nearest distortion-free function may be found by taking the projection of the particular delay $\beta'(\omega)$ onto the subspace G

$$P_G(\beta') = \sum_{n=1}^{\infty} (\beta', g_n) g_n . \tag{127}$$

In Figs. 8 and 9 an example of the use of (127) is shown. In Fig. 8 we consider a cosine delay, $\beta' = \cos 3\omega$, which we have previously considered (see Fig. 4). The projection of this delay on G using (127) is also shown in Fig. 8 and their corresponding impulse responses are shown in Fig. 9. Notice that the samples h_n of the impulse response of the uncorrected delay are poor. There is a peak in the response between h_1 and h_2 which we expect from our previous considerations in Section II. The corrected

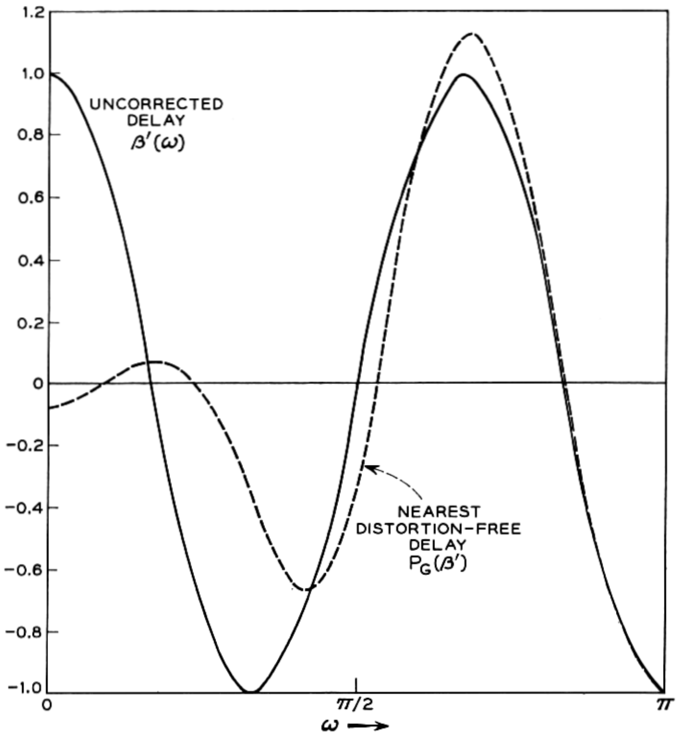

Fig. 8 — An example delay function and the nearest distortion-free delay.

delay seems to be a good fit in the interval $[\pi/2,\pi]$, and its response is well behaved, as is evidenced by the impulse response shown in Fig. 9.

Thus, we have apparently found an infinite set of delay functions corresponding to a particular amplitude characteristic such that the impulse responses satisfy the Nyquist criterion of regularly spaced zero crossings. Note that this was not possible for the case of flat amplitude characteristic mentioned in Section 4.2, since the set $\{f_n\}$ for this shaping is complete in the system bandwidth. For the raised cosine shaping we use more bandwidth for the same rate of transmission and consequently have more leeway in selection of good delay characteristics.

Actually the responses $h(t)$ corresponding to delays in G need not go exactly through zero at time $t_0 + nT$, but only approach zero to the order of approximation employed in our original assumptions. Since ordinarily* the approximation is good to terms cubic in $[\omega t_0 - \beta(\omega)]$, the

* So long as $S_n \geqq C_n$; see Section 4.1.

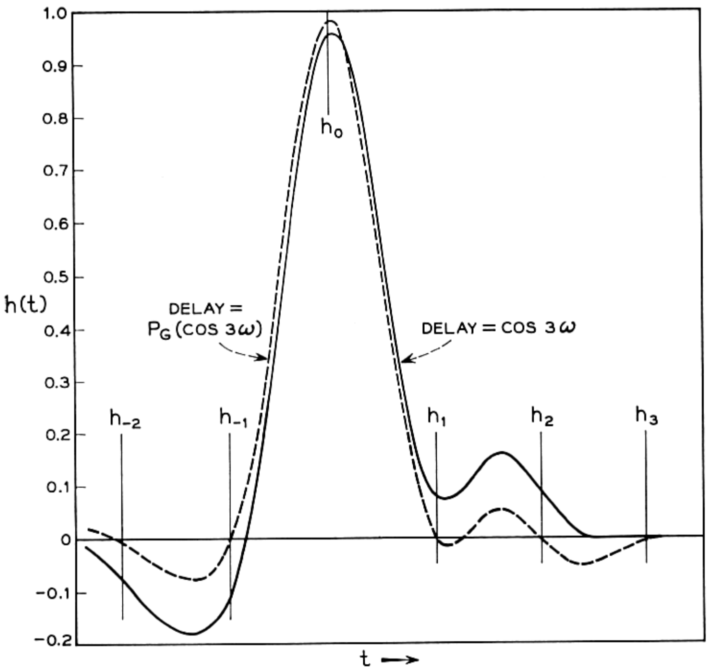

Fig. 9 — Impulse responses of uncorrected and corrected channels.

difference between $h(t_0 + nT)$ and zero becomes insignificant for small delays.

4.3.6 *Equalization*

There are several alternatives available when dealing with delay distortion. One alternative is the use of automatic equalization, whereby channel characteristics are measured and automatically equalized at the transmitter or receiver. Another alternative is the use of compromise equalization, in which a fixed network or a choice of fixed networks is designed to provide for average correction over a particular range of channels. Finally, one can do nothing to the system while amusing oneself with calculations of degradations in performance. We always assume that the particular channel to be used for transmission is chosen randomly for each call, so that it is not economically feasible to design an equalizer for each channel to be used.

For a fixed delay characteristic, the last section shows that there are an infinite number of all-pass networks which will provide near perfect equalization. The network with least rms delay has delay $\beta_c' = -\beta' + P_G(\beta')$, and in general any function $g(\omega) \subset G$ may be added to this delay so long as the approximation remains valid. Thus, the particular function easiest to realize physically may be chosen from this class of functions.

Now suppose we desire to design a delay equalizer to work over a certain class B of delay functions. For each call, the delay $\beta'(\omega)$ is to be chosen randomly from this ensemble. The optimum compromise equalizer $\beta_c'(\omega)$ is to be chosen such that the average distortion over the ensemble B is minimized. Since the delays $\beta'(\omega)$ (random) and $\beta_c'(\omega)$ (fixed) simply add, we have for the resulting distortion

$$D = \frac{2}{\pi} \sum_{n=1}^{\infty} \mid (\beta' + \beta_c', f_n) \mid \tag{128}$$

$$D = \frac{2}{\pi} \sum_{n=1}^{\infty} \mid (\beta', f_n) + (\beta_c', f_n) \mid . \tag{129}$$

The expected value of D averaged over the ensemble B is written

$$E[D] = \frac{2}{\pi} \sum_{n=1}^{\infty} E[\mid (\beta', f_n) + (\beta_c', f_n) \mid]. \tag{130}$$

This is the expression to be minimized by choice of β_c'. Knowing the statistics of the ensemble of channels we can derive the joint distribution of the variables $S_n = (\beta', f_n)$ and the marginal distributions $p(S_n)$. In terms of the latter distributions we have

$$E[D] = \frac{2}{\pi} \sum_{n=1}^{\infty} \int_{-\infty}^{+\infty} \mid S_n + (\beta_c', f_n) \mid p(S_n) dS_n . \tag{131}$$

Each term of the summation is positive, and it is possible to specify independently each component (β_c', f_n) of β_c'.

Therefore, we simply choose each component (β_c', f_n) so as to minimize the corresponding term of the summation (131). Each integral may be written

$$\begin{aligned} I_n = & \int_{-\infty}^{-(\beta_c', f_n)} [-S_n - (\beta_c', f_n)] p(S_n) dS_n \\ & + \int_{-(\beta_c', f_n)}^{\infty} [S_n + (\beta_c', f_n)] p(S_n) dS_n . \end{aligned} \tag{132}$$

Differentiation with respect to (β_c', f_n) yields the stationary point (β_c', f_n) chosen such that

$$\int_{-\infty}^{-(\beta_c',f_n)} p(S_n)dS_n = \frac{1}{2}. \tag{133}$$

Therefore, each component (β_c',f_n) of the compromise delay is optimally chosen such that it is the *negative of the median value of* S_n. The compromise delay β_c' is not uniquely specified by these components; only its projection onto the space F has been determined. As before, any function $g(\omega) \subset G$ may be added without affecting the optimality of the resulting equalizer.

As a somewhat trivial example, suppose we desire to equalize a set of channels bounded by the narrow "ribbon" of width 2Δ.

$$\beta_0'(\omega) - \Delta \leqq \beta'(\omega) \leqq \beta_0'(\omega) + \Delta. \tag{134}$$

Now suppose that each of the scalar products (β',f_n) is equally likely to be greater or less than (β_0',f_n). In this event the completely trivial solution is to use $\beta_c' = -\beta_0' + g$. In particular, the smallest rms function of this sort is $\beta_c' = -\beta_0' + P_G(\beta_0')$. The residual delay in using this equalizer is bounded by $\pm\Delta$ across the band (plus a harmless g function), and our peak-to-peak distortion bound derived previously may be used to give

$$D \text{ residual} \leqq 0.824\Delta. \tag{135}$$

4.3.7 *The Range of the Approximation*

The key approximation made in the analysis thus far has been that $[\omega t_0 - \beta(\omega)]$ is small enough to use

$$\sin [\omega t_0 - \beta(\omega)] \approx [\omega t_0 - \beta(\omega)] \tag{136}$$

where t_0 is chosen to be the time of the peak value of the impulse response. We will now briefly examine the range of delay for which this approximation is valid.

By setting $h'(t_0) = 0$, we were able to derive an expression relating t_0 and the phase shift, $\beta(\omega)$. This equation, (47), was

$$t_0 = \frac{\int_0^w \omega A(\omega)\beta(\omega)\, d\omega}{\int_0^w \omega^2 A(\omega)\, d\omega} \tag{137}$$

and this was the value used for t_0 in (136).

Now we will demonstrate that the resulting function $[\omega t_0 - \beta(\omega)]$ is the *error in a least-squares straight-line fit to* $\beta(\omega)$. Hence, consider fitting a straight line $y = c\omega$ to the phase curve $\beta(\omega)$. The integral square error in the fit weighted, as all our integral expressions are, by the amplitude shaping $A(\omega)$ is

$$\text{integral squared error} = \int_0^w [c\omega - \beta(\omega)]^2 A(\omega)\, d\omega. \qquad (138)$$

By minimizing the error (138) with respect to c and using (137) we obtain $c_{\min} = t_0$.* Thus, the use of the peak time of the impulse response for t_0 results in the smallest values of $[\omega t_0 - \beta(\omega)]$ in a mean-square sense, which was an assertion previously made in connection with the choice of t_0. Also, we see that the time t_0 is the slope of a best fit straight line to the phase $\beta(\omega)$. This state of affairs is depicted in Fig. 10.

In order to be assured of, say, 10 per cent accuracy in the use of approximation (136), we might guarantee that $[\omega t_0 - \beta(\omega)] \leqq \pi/4$. Thus, the phase should not deviate from a straight line by more than $\pi/4$ radians. (Of course, these are *sufficient* but *not necessary* conditions for 10 per cent accuracy.) Now we ask what limits we may put on *peak delay* such that the *phase* will meet this condition no matter what the exact shape of the delay happens to be. Remember that $\beta(0) = 0$ and $\beta'(\omega) \geqq 0$ for physical realizability.

This problem is best suited for the semi-mathematical method called "common sense." Since the delay is to be peak limited between $\beta'_{\max}$ and 0, we allow only use of these two extreme values in finding the shape of delay such that the deviation of its integral (phase) from a straight line is maximized. Furthermore, it is evident that only one transition from $\beta' = \beta'_{\max}$ to $\beta' = 0$ should be used in the interval $[0,w]$. The reader may convince himself that more transitions in delay would result in a better straight-line fit to the phase. Therefore, the shape of the phase which for a given peak delay results in the poorest use of the approximation (136) is specified except for the transition point ω_0. This situation is shown in Fig. 11.

For raised cosine amplitude shaping, the error near the edge of the band is very lightly weighted, and so we take the error at $\omega = \omega_0$ as the largest important error. This error is equal to $\omega_0 t_0$ and is to be maximized with respect to the transition point ω_0.

$$E = \omega_0 t_0 = \frac{\omega_0}{m} \int_0^w \omega A(\omega)\beta(\omega)\, d\omega \qquad (139)$$

* This expression with $c = t_0$ may be used as a useful distortion measure relating performance and phase shift. For an AM system operating at rate $2W$ symbols/second this may be shown to be proportional to the mean-square estimation error at the receiver due to intersymbol interference.

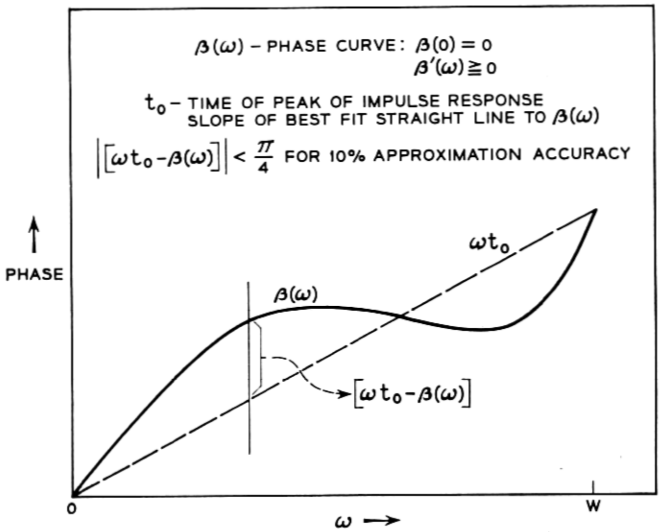

Fig. 10 — Factors involved in the approximation.

where m is a constant

$$m = \int_0^w \omega^2 A(\omega)\, d\omega. \tag{140}$$

Using the curve $\beta(\omega)$ shown in Fig. 11 gives

$$E = \frac{\omega_0}{m} \int_{\omega_0}^{w} \omega \beta'_{\max} (\omega - \omega_0) A(\omega)\, d\omega \tag{141}$$

$$\frac{dE}{d\omega_0} = \frac{\beta'_{\max}}{m} \left\{ \int_{\omega_0}^{w} \omega^2 A(\omega)\, d\omega - 2\omega_0 \int_{\omega_0}^{w} \omega A(\omega)\, d\omega \right\}. \tag{142}$$

For the raised cosine shaping the maximum point of (141) may be found by a solution of the resulting transcendental equation when $dE/d\omega_0 = 0$ and $A(\omega) = \cos \omega + 1$ are used in (142). This procedure yields

$$\omega_0 = 0.255\pi. \tag{143}$$

Notice that the corresponding delay curve is very similar to the worst delay from the standpoint of distortion, which is shown in Fig. 7. For this choice of ω_0 we may evaluate the maximum error from (141).

$$E_{\max} = 0.37\beta'_{\max}. \tag{144}$$

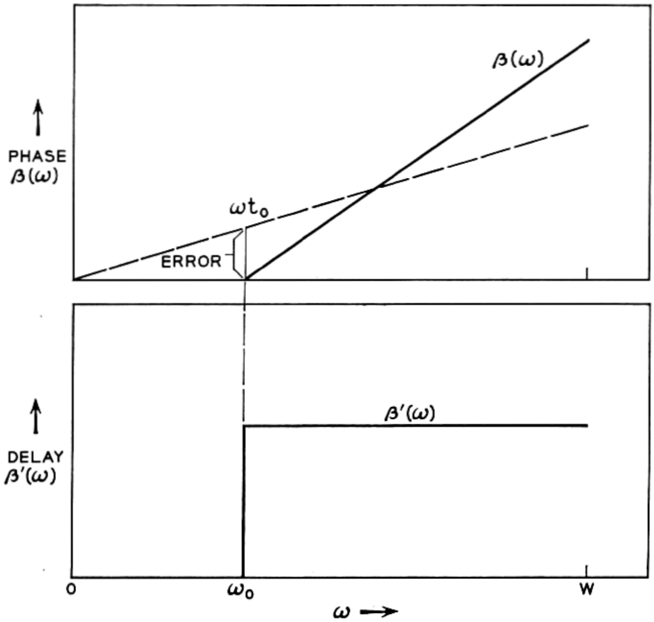

Fig. 11 — Most unfavorable (approximation poorest) delay curve for a given peak value.

For this error to be less than $\pi/4$ for an assured 10 per cent accuracy, we finally arrive at

$$\beta'_{\max} \leqq 2.12. \tag{145}$$

Thus, as long as the delay variation does not exceed 2.12 seconds across the band we are assured of at least 10 per cent accuracy in results regardless of the actual shape of the delay. This is, of course, normalized for the choice of $\omega = \pi$ and $T = 2$ seconds, so that 2.12 seconds of delay variation corresponds to 1.06 pulse intervals.

When the delay differs from the worst form we have just derived, the approximation holds for greater ranges. For example, we found that for the parabolic delay discussed in Section 4.2.2 the approximation was accurate to within 10 per cent in spite of a total delay variation of 4 seconds across the band. The important consideration is that the distortion is quite appreciable before the approximation breaks down. From the peak distortion bound derived previously, we find that the distortion corresponding to a variation of 2.12 seconds may be as large as 0.873. There is a definite connection between the value of the distortion and the goodness of the approximation through the validity of (138) as a distortion measure. Thus, we would expect that the techniques would be accurate in nearly all cases of, roughly, $D \leqq 0.6$. For this value of distortion, the eye in a binary system is more than half closed, and the channel may be unsuitable for higher alphabet size transmission.

4.3.8 *Channel Memory Truncation Error*

Theoretically, the response from a band-limited network lasts to infinity, so that in calculating distortion an infinite number of terms must be used for the criterion D

$$D = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} | h_n |. \tag{146}$$

In many analysis problems and particularly in experimental work and computer simulations it is necessary to neglect the channel memory for $| t | > NT$ seconds. In experimental runs and computer simulations this corresponds to using all possible patterns of length $2N + 1$ symbols. The problem arises as to how much of an error can be made in computing the distortion D using a finite number of terms.

Assume that the terms $| h_n |$ for $| n | > N$ are to be neglected in the summation. The error in computing D is

$$E(N) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} | h_n | - \sum_{\substack{n=-N \\ n \neq 0}}^{+N} | h_n | \tag{147}$$

$$E(N) = \sum_{n=-\infty}^{-N-1} | h_n | + \sum_{n=N-1}^{\infty} | h_n |. \tag{148}$$

As we have previously shown, this expression is approximately equal to

$$E(N) = \frac{2}{\pi} \sum_{n=N+1}^{\infty} | (f_n , \beta') | \tag{149}$$

and thus the error can be bounded using the Schwarz inequality on the maximum energy combination of the functions f_n, $n = N + 1, \cdots, \infty$. For $N > 3$ we may as well neglect the terms in f_n involving the fast-vanishing constant a_n, leaving only the three cosine terms. Obviously, the maximum energy combination of these is all terms adding in-phase

$$E(N) \leqq \frac{2}{\sqrt{\pi}} \times (\text{rms delay}) \times \| \xi_N \| \tag{150}$$

$$\xi_N(\omega) = \sum_{n=N+1}^{\infty} f_n(\omega) \tag{151}$$

$$\xi_N(\omega) \approx \sum_{n=N+1}^{\infty} \left[\frac{1}{2(2n-1)} \cos (2n-1)\omega + \frac{1}{2n} \cos 2n\omega + \frac{1}{2(2n+1)} \cos (2n+1)\omega \right] \tag{152}$$

$$\xi_N(\omega) = \sum_{n=2N+1}^{\infty} \frac{1}{n} \cos n\omega - \frac{1}{2(2N+1)} \cos (2N+1)\omega \tag{153}$$

$$\| \xi_N \|^2 = \frac{\pi}{2} \sum_{n=2N+1}^{\infty} \frac{1}{n^2} - \frac{3\pi}{8(2N+1)^2}. \tag{154}$$

We finally write the maximum error as

$$E(N) \leqq e_N \times (\text{rms delay}) \tag{155}$$

$$e_N = \sqrt{\frac{\pi^2}{3} - \frac{3}{2(2N+1)^2} - 2\sum_{n=1}^{2N} \frac{1}{n^2}}. \tag{156}$$

As shown in Fig. 12, this bound drops rapidly for N small and then levels out to a very slow descent, so that some 20 per cent of the original distortion bound can still remain after consideration of 16 pulse intervals on each side of the peak. This rather negative result tells us only that there exist mathematical delay functions that have considerable distor-

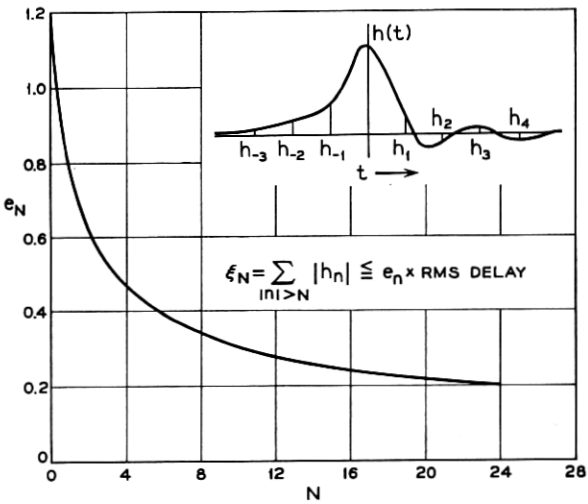

Fig. 12 — An upper bound on distortion arising from symbols at a distance greater than N.

tion at great distances from the reference (peak response) time. These functions, however, are high-frequency waveforms which would not ordinarily be encountered.

Consider a computer simulation in which 40 samples across the band $[0,\pi]$ are used to specify the delay. The highest delay frequency which need be considered is then 20 cycles of delay across the $[0,\pi]$ band. Consequently, only the functions f_n for $n \leqq 20$ will contribute distortion components. In order to test distortion over this interval it would be necessary to test 2^{40} sequences of binary symbols, which is, of course, quite unreasonable. Therefore, the effect of intersymbol interference is usually only measured to the extent of, say, four symbols in either direction.

We can find the maximum error now by computing the norm of the sum of the functions $f_n(\omega)$, $n = 5, 6, \cdots, 20$

$$\xi_{5,20}(\omega) = \sum_{n=11}^{40} \frac{1}{n} \cos n\omega - \frac{1}{22} \cos 11\omega \tag{157}$$

$$\| \xi_{5,20} \|^2 = \frac{\pi}{2} \left\{ \sum_{n=11}^{40} \frac{1}{n^2} - \frac{3}{22^2} \right\}. \tag{158}$$

The sums involved in the expressions are conveniently computed using a Euler-Maclaurin expansion for the integral of $1/x^2$. This gives

$$E_{5,20} \leqq 0.352 \times \text{rms delay} \tag{159}$$

which is still a considerable error, even though the delay is bandwidth limited.

In all our computations of distortion bounds we have been using the maximum energy combination of the functions $\pm f_n(\omega)$. For each particular delay it will be one of the combinations of functions $\pm f_n(\omega)$ which defines the distortion functional, not necessary the maximum energy combination. However, it is interesting to note that the combination with *least* energy would only result in a factor of $\sqrt{2}$ in the bounds calculated.

V. THE RELATIONSHIP OF DISTORTION TO DELAY FOR BANDPASS SYSTEMS

5.1 *Derivation of the Sequence of Functionals Involved for Bandpass Systems*

In dealing with bandpass systems, the system impulse response is most conveniently given in terms of its envelope and phase with respect to a carrier or other reference frequency within the bandwidth of the system

$$h(t) = P(t) \cos [\omega_c t - \psi(t)]. \tag{160}$$

Alternately, the response may be written in terms of in-phase and quadrature components at the carrier frequency

$$h(t) = R(t) \cos \omega_c t - Q(t) \sin \omega_c t \tag{161}$$

$$P(t) = \sqrt{R^2(t) + Q^2(t)}. \tag{162}$$

As discussed by Sunde,[1] the in-phase and quadrature components of the impulse response may be related to the amplitude and phase characteristics of the channel's frequency domain description by a simple transformation of the defining Fourier integral. This transformation to passband coordinates gives

$$R(t) = \frac{1}{\pi} \int_{-\omega_c}^{\infty} \mathcal{A}(\omega) \cos [\omega t - \varphi(\omega)]\, d\omega \tag{163}$$

$$Q(t) = \frac{1}{\pi} \int_{-\omega_c}^{\infty} \mathcal{A}(\omega) \sin [\omega t - \varphi(\omega)]\, d\omega. \tag{164}$$

In this section we use $\mathcal{A}(\omega)$ and $\varphi(\omega)$ for amplitude and phase instead of $A(\omega)$ and $\beta(\omega)$ since these functions are now defined with respect to the carrier frequency, ω_c. That is

$$\mathcal{A}(\omega) = A(\omega_c + \omega) \tag{165}$$

$$\varphi(\omega) = \beta(\omega_c + \omega). \tag{166}$$

Now we will work under the hypothesis that a suitable criterion of distortion for bandpass systems is

$$D = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} P(nT + t_0) = \sum_n{}' P_n . \tag{167}$$

This criterion is similar to the low-pass criterion, except we now assume that the receiver makes use of the envelope properties of the impulse response.

In terms of samples of the quadrature components we have

$$D = \sum_n{}' \sqrt{R_n^2 + Q_n^2}. \tag{168}$$

Unfortunately this is a fairly hopelessly nonlinear criterion to work with, so we shall make the judicious approximation shown in Fig. 13. Here we take the distortion D as the length of the vector formed by summing all vectors of the form

$$P_n \angle \tan^{-1} \frac{|Q_n|}{|R_n|}.$$

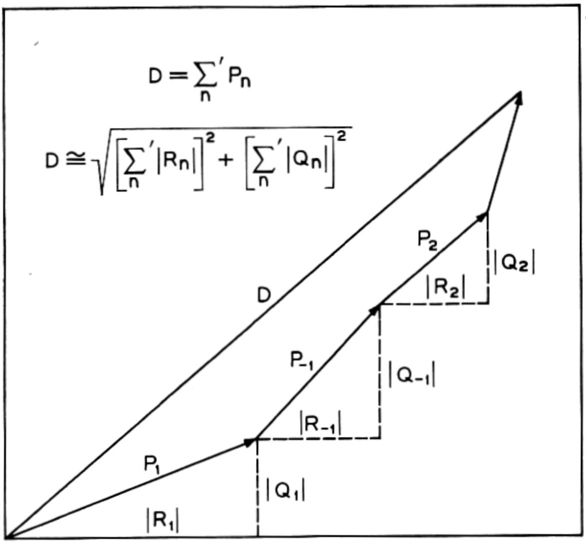

Fig. 13 — The approximate distortion criterion for bandpass systems.

Thus we have

$$D \approx \sqrt{D_r^2 + D_q^2} \tag{169}$$

$$D_r = \sum_n{}' \mid R_n \mid \tag{170}$$

$$D_q = \sum_n{}' \mid Q_n \mid . \tag{171}$$

Now, the component distortions D_r and D_q are of the same form as the low-pass distortion treated previously. We take D_q as an example in what follows

$$\begin{aligned} Q_n \pm Q_{-n} &= \frac{1}{\pi} \int_{-\omega_c}^{\infty} \mathcal{A}(\omega) \{\sin [\omega(t_0 + nT) - \varphi(\omega)] \\ &\qquad \pm \sin [\omega(t_0 - nT) - \varphi(\omega)]\} \, d\omega \end{aligned} \tag{172}$$

$$Q_n + Q_{-n} = \frac{2}{\pi} \int_{-\omega_c}^{\infty} \mathcal{A}(\omega) \sin [\omega t_0 - \varphi(\omega)] \cos n\omega T \, d\omega \tag{173}$$

$$Q_n - Q_{-n} = \frac{2}{\pi} \int_{-\omega_c}^{\infty} \mathcal{A}(\omega) \cos [\omega t_0 - \varphi(\omega)] \sin n\omega T \, d\omega . \tag{174}$$

Using the approximation $[\omega t_0 - \varphi(\omega)]$ small gives

$$Q_n + Q_{-n} \approx \frac{2}{\pi} \int_{-\omega_c}^{\infty} \mathcal{A}(\omega)[\omega t_0 - \varphi(\omega)] \cos n\omega T \, d\omega \tag{175}$$

$$Q_n - Q_{-n} \approx 0 \tag{176}$$

$$| Q_n | + | Q_{-n} | \approx \frac{2}{\pi} \left| \int_{-\omega_c}^{\infty} \mathcal{A}(\omega)[\omega t_0 - \varphi(\omega)] \cos n\omega T \, d\omega \right|. \tag{177}$$

Notice that in (177) the quantity $(Q_n - Q_{-n})$ need not be identically zero as in the approximation (176), but should only be smaller in absolute value than the quantity $(Q_n + Q_{-n})$.

We find the time of the peak value of the impulse response, t_0, by requiring $R'(t_0) = 0$. The quadrature component goes through zero at $= 0$ and is small enough at t_0 to be neglected

$$R'(t_0) = 0 = \frac{-1}{\pi} \int_{-\omega_c}^{\infty} \omega \mathcal{A}(\omega) \sin [\omega t_0 - \varphi(\omega)] \, d\omega \tag{178}$$

$$t_0 \approx \frac{\int_{-\omega_c}^{\infty} \omega \mathcal{A}(\omega) \varphi(\omega) \, d\omega}{\int_{-\omega_c}^{\infty} \omega^2 \mathcal{A}(\omega) \, d\omega}. \tag{179}$$

This is incidental to the development of the quantity $| Q_n | + | Q_{-n} |$, because for symmetric shaping of $\mathcal{A}(\omega)$ the terms involving t_0 in (177) integrate to zero. This shows that the antisymmetric portion of the delay $\varphi'(\omega)$ does not have a first-order influence on t_0. However, in solving for $| R_n | + | R_{-n} |$ the equations do involve t_0 in first-order terms.

To maintain notational continuity with low-pass results as much as possible we designate

$$S_{qn} = - \int_{-\omega_c}^{\infty} \mathcal{A}(\omega)[\omega t_0 - \varphi(\omega)] \cos n\omega T \, d\omega \tag{180}$$

$$| Q_n | + | Q_{-n} | = \frac{2}{\pi} | S_{qn} |. \tag{181}$$

For $\mathcal{A}(\omega)$ symmetrical about zero (the carrier frequency) (180) becomes

$$S_{qn} = \int_{-\omega_c}^{\infty} \varphi(\omega) \mathcal{A}(\omega) \cos n\omega T \, d\omega. \tag{182}$$

We integrate by parts and make the arbitrary assignment of zero phase shift at the reference frequency to obtain*

$$S_{qn} = \int_{-w/2}^{+w/2} f_{qn}(\omega)\varphi'(\omega)\,d\omega = (f_{qn}, \varphi') \tag{183}$$

$$f_{qn}(\omega) = \int_0^{\omega} \mathcal{A}(x) \cos nxT\,dx. \tag{184}$$

A similar development holds for the terms $|R_n| + |R_{-n}|$, and the resulting expressions are exactly the same as the low-pass equations except they are translated to the reference and are defined for both negative and positive deviations from this reference

$$|R_n| + |R_{-n}| = \frac{2}{\pi} |S_{rn}| \tag{185}$$

$$S_{rn} = \int_{-w/2}^{+w/2} f_{rn}(\omega)\varphi'(\omega)\,d\omega = (f_{rn}, \varphi') \tag{186}$$

$$f_{rn}(\omega) = \int_{\omega}^{w/2} [\alpha_n x - \sin nxT]\mathcal{A}(x)\,dx \tag{187}$$

$$\alpha_n = \frac{\displaystyle\int_{-w/2}^{w/2} \omega\mathcal{A}(\omega) \sin n\omega T\,d\omega}{\displaystyle\int_{-w/2}^{w/2} \omega^2\mathcal{A}(\omega)\,d\omega}. \tag{188}$$

To briefly summarize, we have written the distortion in a bandpass system as the length of a vector whose two quadrature components are D_r and D_q

$$D = \sqrt{D_r^2 + D_q^2}. \tag{189}$$

Each of these components is the sum of the absolute values of a sequence of linear functionals of delay

$$D_r = \frac{2}{\pi} \sum_{n=1}^{\infty} |(f_{rn}, \varphi')| \tag{190}$$

$$D_q = \frac{2}{\pi} \sum_{n=1}^{\infty} |(f_{qn}, \varphi')|. \tag{191}$$

The functions f_{rn} and f_{qn} are of course independent of delay and are obtained from the amplitude characteristics by operations (187) and (184).

We are working with symmetric amplitude characteristics, and consequently it may be seen that

* Another choice of reference phase may easily be made here.

$$f_{qn}(\omega) = -f_{qn}(-\omega) \tag{192}$$

$$f_{rn}(\omega) = f_{rn}(-\omega). \tag{193}$$

The quadrature functions f_{qn} are odd functions of frequency and the in-phase functions f_{rn} are even functions. The two parts into which the distortion was divided therefore arise separately from the odd and even portions of the delay. For example, if the delay is an even function, $D_q = 0$ and the only distortion is D_r. Since D_r is defined identically except for a translation as the low-pass distortion D, we have the necessary result that the system may be treated as low-pass with identical results in the event of even delay. Obviously also

$$(f_{rn}, f_{qm}) = 0 \qquad \text{all } n \text{ and } m. \tag{194}$$

The delay function $\varphi'(\omega)$ may be divided into its even and odd components, $\varphi_r'(\omega)$ and $\varphi_q'(\omega)$, and the analysis of the distortion properties of each of these components proceeds exactly as in the low-pass analysis. For the even component of delay, we use the functionals defined by the sequence $\{f_{rn}\}$ and for the odd components we use the sequence $\{f_{qn}\}$. The two distortions D_r and D_q are then added root-sum-square.

5.2 *The Raised Cosine System*

5.2.1 *The Functions $f_{rn}(\omega)$ and $f_{qn}(\omega)$*

We now consider the use of an amplitude shaping of the raised cosine form

$$\mathcal{A}(\omega) = \tfrac{1}{2}(1 + \cos \omega) \qquad -\pi \leqq \omega \leqq +\pi. \tag{195}$$

By substitution into equations (188), (187) and (184) the following results are obtained

$$\alpha_n = \frac{1}{\left(\frac{\pi^2}{3} - 2\right) 2n(4n^2 - 1)} \tag{196}$$

$$\begin{aligned} f_{rn}(\omega) = \frac{\alpha_n}{2}\Bigg[&\cos \omega + \omega \sin \omega + \frac{\omega^2}{2} - \left(1 + \frac{\pi^2}{6}\right)\Bigg] \\ &+ \frac{1}{4(2n-1)} \cos (2n-1)\omega + \frac{1}{4n} \cos 2n\omega \\ &+ \frac{1}{4(2n-1)} \cos (2n+1)\omega \end{aligned} \tag{197}$$

$$f_{qn}(\omega) = \frac{1}{4(2n-1)} \sin (2n-1)\omega + \frac{1}{4n} \sin 2n\omega + \frac{1}{4(2n+1)} \sin (2n+1)\omega. \tag{198}$$

The first few pairs of these functions are shown in Fig. 14.

5.2.2 *Sample Distortion Calculations*

Having assumed any particular shape of delay curve, one may easily compute the resultant distortion to the desired accuracy by computing a number of the linear functionals. Although the primary use of these functionals is in understanding the effect of shape in delay on the distortion and in ascertaining bounds and other factors in this relationship, it is quite necessary that when confronted with the reality of an actual system the mathematics be able to predict specific results.

First, we consider a check on the mathematical methods and approxi-

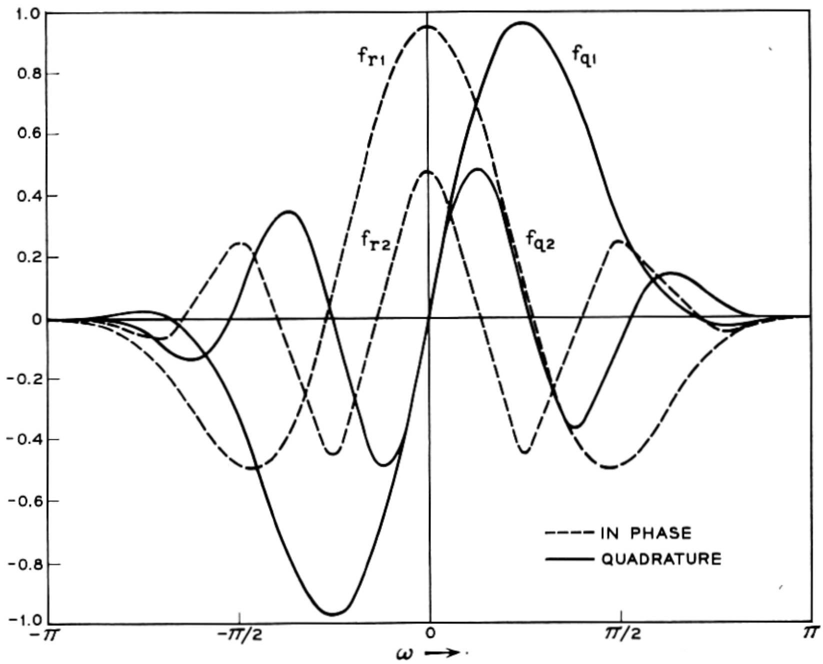

Fig. 14 — Some of the functions f_{rn} and f_{qn} for raised cosine shaping.

mations used. Consider the effect of linear delay on the raised cosine response

$$\varphi'(\omega) = k\omega. \tag{199}$$

Since this delay function is odd, the products (f_{rn}, φ') vanish and the only distortion is D_q. For adjacent symbol interference we have

$$P_1 + P_{-1} = \frac{2}{\pi} | S_{q1} | = \frac{2}{\pi} | (f_{q1}, k\omega) | \tag{200}$$

$$(f_{q1}, k\omega) = \int_{-\pi}^{+\pi} k\omega f_{q1}(\omega)\, d\omega = \frac{11\pi}{36} k \tag{201}$$

$$P_1 + P_{-1} = 11k/18. \tag{202}$$

For a specific example we take $k = 2/\pi$ which gives

$$P_1 + P_{-1} = 0.389 \qquad \text{(predicted)} \tag{203}$$

while from Sunde[2] the computed impulse response for this value of slope is

$$P_1 + P_{-1} = 0.387 \qquad \text{(computed)}. \tag{204}$$

The agreement here is probably better than should ordinarily be expected.

Now we turn to predicting the performance, measured by the eye opening, of the four-phase data subset. As explained in the previous chapter, this system is inherently nonlinear and using $I = 1 - D$ as the eye aperture for this system involves a certain approximation. In particular, we will examine the performance of this system for delay of the form

$$\varphi'(\omega) = \alpha \sin \nu\omega \tag{205}$$

since there are published results for this choice of delay. Again we are dealing for the moment with an odd delay function and need only evaluate D_q. As a function of the number of delay ripples in the band, ν, the various products (f_{qn}, φ') are easily visualized, since f_{qn} consists of only three sine terms itself. The behavior of these products is very nearly like the behavior of its cosine counterpart shown previously in Fig. 4 and will not be depicted here. In Fig. 15 the distortion for $\alpha = 0.5$ calculated by summing these products is illustrated as a function of ν along with the corresponding curve from Rappeport.[7] The latter curve was obtained by use of a computer simulation of the system, and the agreement between this simulation and actual results is claimed to be

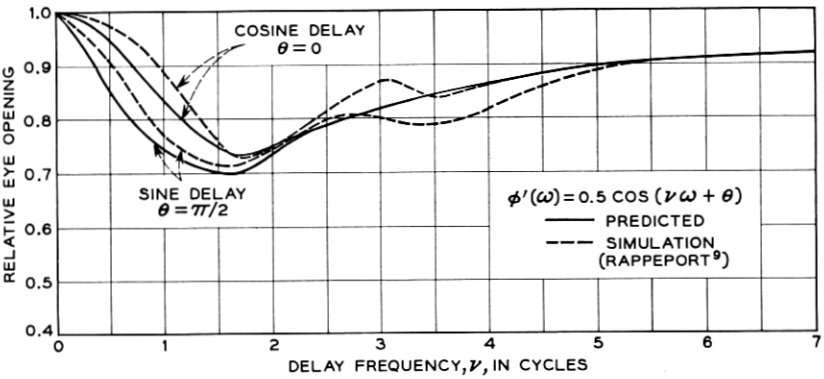

Fig. 15 — The performance of a four-phase system for sine and cosine delay.

excellent. Two other curves are also drawn in Fig. 15 representing the system performance for cosine delay variation previously shown in Fig. 5. Notice that the slight difference in rate of deterioration of performance at low frequency between sine and cosine delay is correctly predicted by the mathematical model.

To test the mathematical model with delay which has both odd and even components, we turn to published results concerning a different data system. This system is an amplitude-modulated system investigated by computer simulation by R. A. Gibby in Ref. 8. Again our criterion is not expected to hold exactly, since Gibby's binary system is an on-off system rather than bipolar. Gibby considers delay of the form

$$\varphi'(\omega) = \alpha \cos (b\omega + \theta) \tag{206}$$

and plots loci of constant eye aperture (constant distortion) on a polar diagram of delay amplitude α and phase θ for a given value of delay frequency b. The quadrature components of the distortion for the delay (206) are

$$D_r = \frac{2}{\pi} \alpha \cos \theta \sum_{n=1}^{\infty} | (f_{rn}, \cos b\omega) | \tag{207}$$

$$D_q = \frac{2}{\pi} \alpha \sin \theta \sum_{n=1}^{\infty} | (f_{qn}, \sin b\omega) | . \tag{208}$$

The integrations are performed and summed to give (for $b = 1.5$)

$$D_r = 0.508 \, \alpha \cos \theta \tag{209}$$

$$D_q = 0.590\ \alpha \sin \theta \tag{210}$$

$$D^2 = 0.258\ [\alpha \cos \theta]^2 + 0.348\ [\alpha \sin \theta]^2. \tag{211}$$

Therefore, lines of constant distortion are ellipses on a polar chart of α and θ. Fig. 16 shows two of these ellipses of constant distortion with $b = 1.5$ along with the corresponding curves obtained by Gibby.[8]

Figs. 15 and 16 demonstrate that the mathematical model has provided a good description of the behavior of two diverse modulation systems under the influence of delay distortion.

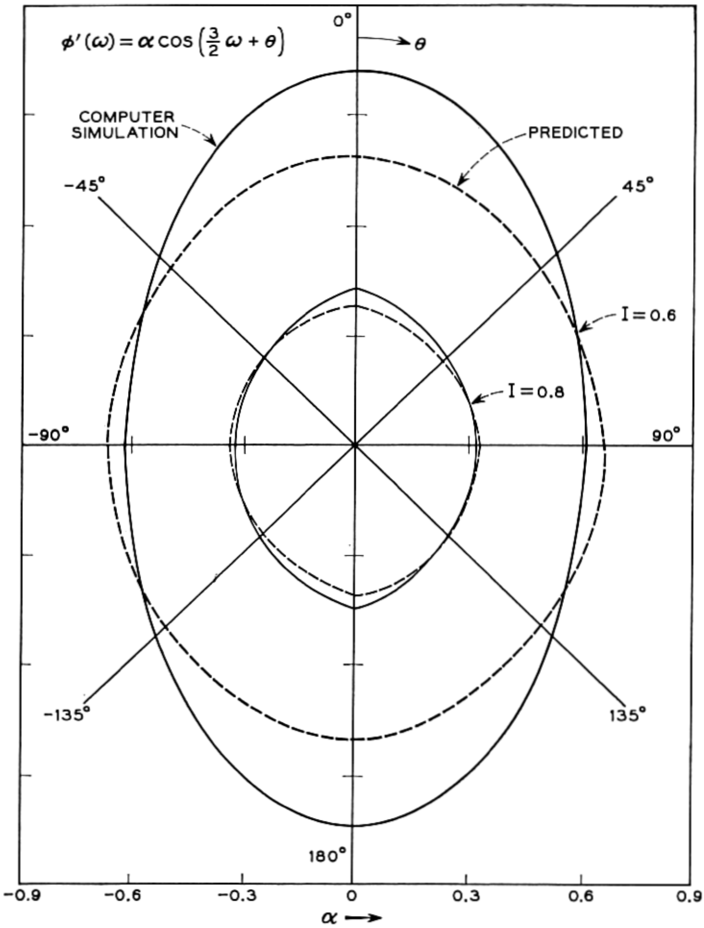

Fig. 16 — Loci of constant eye aperture for cosine delay.

5.2.3 *Sensitivity and Bounds on Distortion*

In the low-pass analysis we found the system sensitivity, which we defined as the maximum achievable distortion for one rms unit of delay, by summing the functions $\pm f_n(\omega)$ in such a fashion as to produce a combination of greatest energy. Obviously, in the bandpass case we can bound both distortion components in like fashion. The system will have a certain sensitivity to even delay and a certain sensitivity to odd delay. Any delay function can be divided into its odd and even components and these components are orthogonal. The contribution to the distortion from each is bounded by the system sensitivities to odd and even delay. Now we ask for a given rms value of delay, how should the delay energy be divided between odd and even components such that the distortion is maximized? Naturally, all the delay energy should be put into the component (odd or even) which has greatest sensitivity to delay distortion. Therefore the over-all system sensitivity is the maximum of the pair of odd and even delay sensitivities.

The sensitivity to even delay is the same as the sensitivity calculated previously for low-pass systems:

$$\text{even sensitivity} = 1.15. \tag{212}$$

For the sensitivity to odd delay, we find the maximum energy combination of the functions $\pm f_{qn}(\omega)$. This is found trivially as the sum of the functions $f_{qn}(\omega)$, since all the terms add in phase in this sum

$$f_{q\text{mec}}(\omega) = \sum_{n=1}^{\infty} f_{qn}(\omega) \tag{213}$$

$$f_{q\text{mec}}(\omega) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\sin n\omega}{n} - \frac{\sin \omega}{4} \tag{214}$$

$$f_{q\text{mec}}(\omega) = \begin{cases} \frac{1}{4}(\pi - \omega - \sin \omega) & \omega \geqq 0 \\ \frac{1}{4}(-\pi - \omega - \sin \omega) & \omega < 0. \end{cases} \tag{215}$$

This particular form of delay is the worst odd delay for a given rms value and is shown in Fig. 17. The norm of this function is

$$\| f_{q\text{mec}} \| = \frac{1}{2} \sqrt{\frac{\pi^3}{6} - \frac{3\pi}{4}} = 0.835 \tag{216}$$

and the sensitivity becomes

$$\text{odd sensitivity} = 2 \sqrt{\frac{2}{\pi}} \, \| f_{q\text{mec}} \| = 1.337. \tag{217}$$

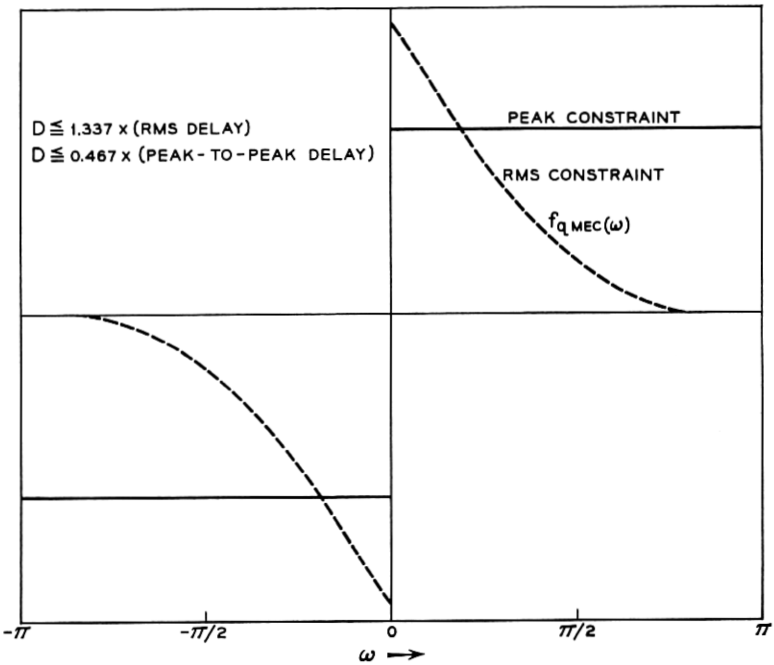

Fig. 17 — Bounds on distortion for bandpass, raised cosine systems.

Thus we see that the bandpass, raised cosine system is somewhat more sensitive to odd delay than it is to even delay. The over-all system sensitivity is therefore equal to the odd sensitivity and consequently we have

$$D \leqq 1.337 \times (\text{rms delay}). \tag{218}$$

Also, the delay curve $f_{q\text{mec}}(\omega)$ in Fig. 17 becomes the worst shape a delay can assume for a given rms value of delay.

For a peak-to-peak constraint on delay, the technique for bounding the distortion is less obvious. Clearly we can find the combinations of signs $\{\epsilon_{rn}\}$ and $\{\epsilon_{qn}\}$ such that the resulting functions

$$f_r(\omega) = \sum_{n=1}^{\infty} \epsilon_{rn} f_{rn}(\omega) \qquad \epsilon_{rn} = \pm 1 \tag{219}$$

$$f_q(\omega) = \sum_{n=1}^{\infty} \epsilon_{qn} f_{qn}(\omega) \qquad \epsilon_{qn} = \pm 1 \tag{220}$$

have maximum absolute integrals and the distortion may be as large as

TABLE II—AN APPROXIMATE SET OF ORTHONORMAL BASIS FUNCTIONS FOR THE SPACE G OF DISTORTIONLESS FUNCTIONS FOR BANDPASS, RAISED COSINE SYSTEMS

$$(\sqrt{\pi}\, g_{rn} = a_1 \cos \omega + a_2 \cos 2\omega + \cdots + a_{11} \cos 11\,\omega)^*$$
$$(\sqrt{\pi}\, g_{qn} = b_1 \sin \omega + b_2 \sin 2\omega + \cdots + b_{11} \sin 11\,\omega)$$

	g_1	g_2	g_3	g_4	g_5	g_6
b_1	0.7205					
b_2	−0.6674	−0.2715				
b_3	−0.1593	0.8145				
b_4	0.0947	−0.4841	−0.3294			
b_5	0.0288	−0.1473	0.8234			
b_6	−0.0151	0.0773	−0.4321	−0.3540		
b_7	−0.0051	0.0258	−0.1444	0.8260		
b_8	0.0025	−0.0128	0.0713	−0.4078	−0.3687	
b_9	0.0009	−0.0045	0.0252	−0.1443	0.8294	
b_{10}	−0.0004	0.0021	−0.0116	0.0664	−0.3819	−0.4138
b_{11}	−0.0002	0.0009	−0.0053	0.0302	−0.1737	0.9104

* For values of a_n see Table I.

the greater of these absolute integrals. The question is, can we do better than this by using a delay with both odd and even components?

If we were to use both odd and even components in the delay, the only acceptable strategy for maximizing distortion would be to use odd delay $(\varphi'(\omega) = -\varphi'(-\omega))$ when $|f_q(\omega)| > |f_r(\omega)|$ and even delay $(\varphi'(\omega) = \varphi'(-\omega))$ when $|f_r(\omega)| > |f_q(\omega)|$.* In addition we would have to run through all possible sequences of the signs $\{\epsilon_{rn}\}$ and $\{\epsilon_{qn}\}$. This procedure was carried out to the extent of time limitations on the IBM 7090 digital computer with the result that the best such delay has distortion less than a delay using an all odd strategy.

The maximum absolute integral of $f_q(\omega)$ is obtained by using $\epsilon_{qn} = +1$, i.e., by simply adding all the functions $f_{qn}(\omega)$. We found this function previously as $f_{q\mathrm{mec}}(\omega)$

$$D = D_q \leqq \frac{2}{\pi}\varphi'_{\max}\int_{-\pi}^{+\pi}\left|\sum_{n=1}^{\infty} f_{qn}(\omega)\right| d\omega \tag{221}$$

$$D \leqq \frac{2}{\pi}\varphi'_{\max}\int_{0}^{+\pi}\left|\tfrac{1}{2}(\pi - \omega - \sin\omega)\right| d\omega \tag{222}$$

$$D \leqq \left(\frac{\pi}{2} - \frac{2}{\pi}\right)\varphi'_{\max} \tag{223}$$

* This is not exactly true, but an exact proof here does not seem to be worth the considerable effort involved.

$$D \leqq 0.467 \times (\text{peak-to-peak delay}). \tag{224}$$

The worst peak-to-peak delay is simply positive when $f_{qmec}(\omega)$ is positive and negative when $f_{qmec}(\omega)$ is negative. This is a particularly simple delay function which is $+\varphi'_{\max}$ for $\omega \geqq 0$ and $-\varphi'_{\max}$ for $\omega < 0$. This function is also shown in Fig. 17.

5.2.4 *Zero-Distortion Delay Functions*

A space G of distortion-free delay functions for raised cosine systems may be obtained using techniques similar to the low-pass methods. After orthonormalizing the sequences $\{f_{rn}\}$ and $\{f_{qn}\}$, we find the orthonormal

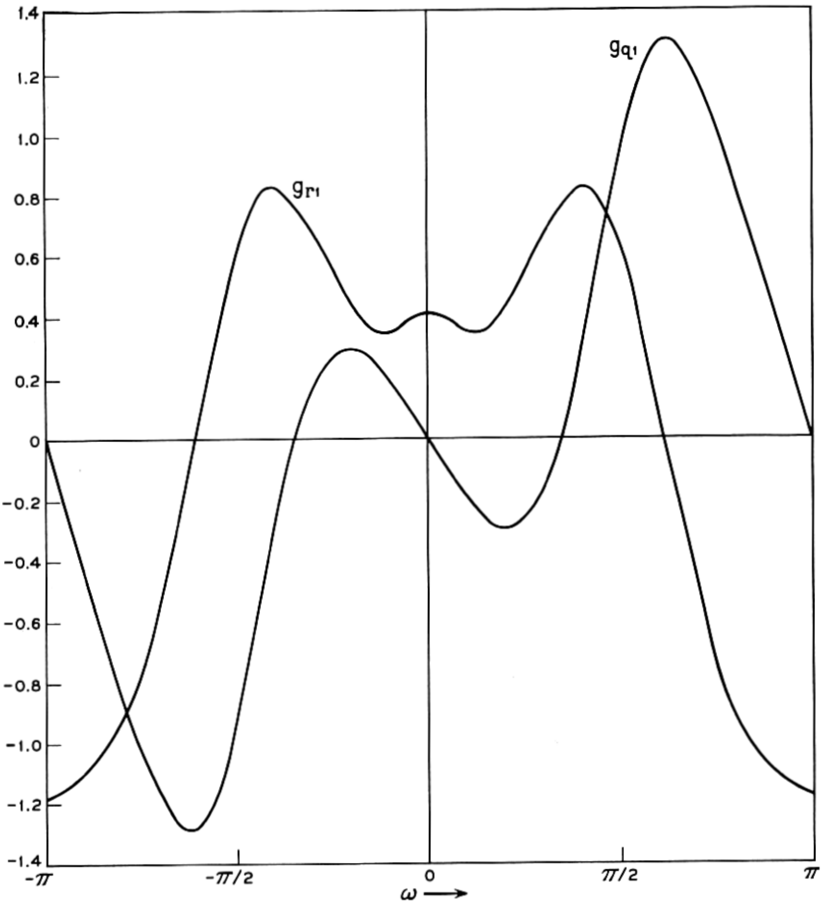

Fig. 18 — Bandpass distortionless delays.

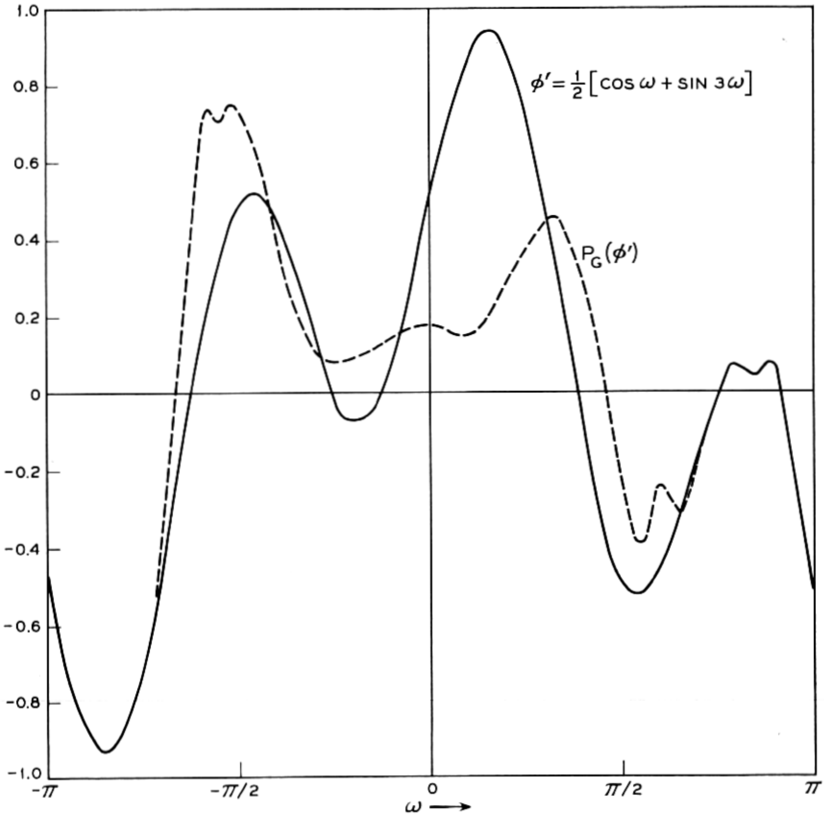

Fig. 19 — An example delay and the nearest distortion-free delay.

functions which complete these sequences in their respective subspaces of even and odd square-integrable functions. Thus we derive the sequences $\{g_{rn}\}$ and $\{g_{qn}\}$ of even and odd functions which span the distortionless space G. Any delay φ' in $L^2(-\pi, +\pi)$ can then be expanded in terms of the functions f_{rn}, f_{qn}, g_{rn}, and g_{qn} with the terms involving g functions comprising the projection of φ' upon G and yielding zero distortion and the terms involving f functions containing all the distortion content of φ'

$$\varphi' = \sum_{\substack{n=1 \\ \text{even}}}^{\infty} b_{rn} f_{rn} + \sum_{\substack{n=1 \\ \text{odd}}}^{\infty} b_{qn} f_{qn} + \sum_{n=1}^{\infty} c_{rn} g_{rn} + \sum_{n=1}^{\infty} c_{qn} g_{qn} \tag{225}$$

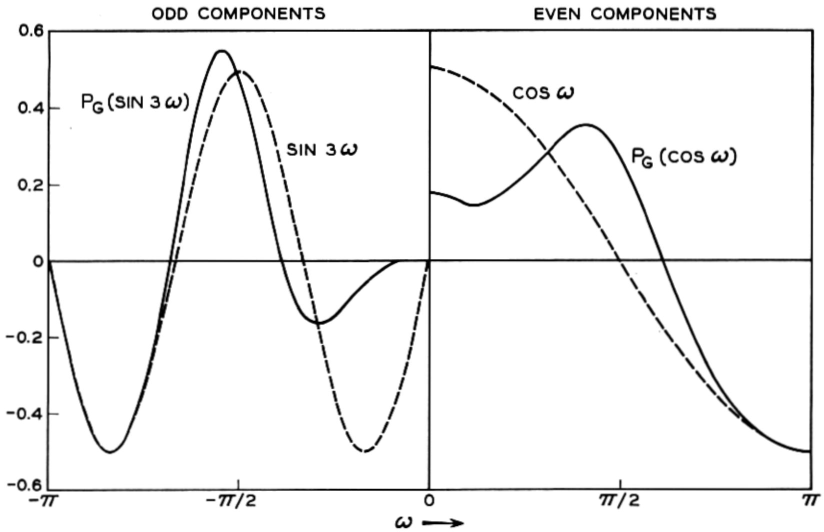

Fig. 20 — Resolution of the example delay into odd and even components.

$$P_G(\varphi')$$

$$c_{rn} = (\varphi', g_{rn}) \qquad (g_{rn}, g_{rm}) = 0 \; n \neq m \tag{226}$$

$$c_{qn} = (\varphi', g_{qn}) \qquad\qquad = 1 \; n = m.$$

A list of the functions g_{rn} and g_{qn} obtained for raised cosine shaping is given in Table II, and the first few functions are illustrated in Fig. 18.

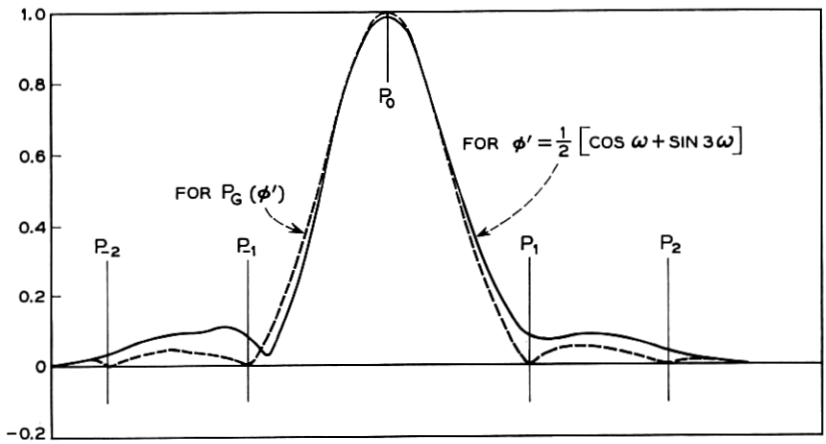

Fig. 21 — Impulse response envelopes for the corrected and uncorrected delays.

It is necessary here to reiterate the fact that functions $g \subset G$ have zero distortion only to the extent of the approximations employed in obtaining the fundamental relationship for distortion in terms of the sequence of linear functionals. As an example computation a delay $\varphi' = \frac{1}{2}[\cos \omega + \sin 3\omega]$ is shown in Fig. 19. The odd and even components of this delay and their respective projections on G are shown in Fig. 20. Combining these projection components the total projection is shown in Fig. 19 back with the original delay function. The envelopes of the impulse responses of the corrected and uncorrected delays are illustrated finally in Fig. 21. As may be seen from this figure, the correction is near perfect. The corrected envelope approaches zero at each sample point as close as the numerical integration techniques employed permit.

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