Innage and Outage Intervals in Transmission Systems Composed of Links

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This note is of the nature of an addendum to a recent paper on satellite communication systems. It is concerned with the distribution and average durations of innages and outages occurring in transmission systems composed of a number of links. The links of such a composite system may be either in series, as in a radio relay system, or in parallel, as in a many-satellite system. Several results regarding composite transmission systems, including some due to D. S. Palmer, are reviewed, restated, and extended.

I. INTRODUCTION

This note is in the nature of an addendum to a recent paper of mine on satellite communication systems.[1] It is concerned with the same general problem, namely the reliability of transmission systems composed of links which fail independently. Various published results are reviewed and extended. A large part of these results is due to D. S. Palmer,[2] whose excellent work was overlooked in my satellite paper. The approach given here differs somewhat from that used by Palmer.

Incidentally, questions similar to those discussed here have also appeared in connection with coincidences in counting devices.

The notation to be followed is illustrated in Fig. 1. Suppose that a link in a transmission system is always either in one or the other of two possible states, state (a) or state (b). For example, if the link is a satellite, (a) may be taken as the state of being out of sight and (b) the state of being visible. Again, if the link is one of a series of links in tandem making up a transmission line, we may choose (a) to be the state of working order and (b) the state of breakdown. In a satellite system the links are in parallel and in the transmission line they are in series. Fig. 1 applies to both cases.

The light portions of the top line in Fig. 1 represent the intervals dur-

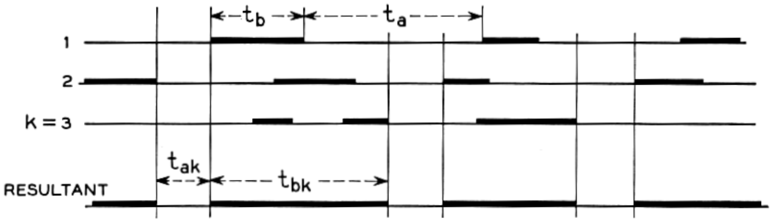

Fig. 1 — Combination of k independent alternating sequences to form the resultant alternating sequence.

ing which Link No. 1 is in state (a), and the heavy portions the intervals of state (b). Similarly, the second and third lines represent the state intervals of Links No. 2 and No. 3. This is a k-link system with $k = 3$. The last line represents the system state intervals. In system state (ak) all k links are in state (a). In state (bk) at least one link is in state (b). State (ak) corresponds to the "intersection" of type (a) intervals and state (bk) to the "union" of type (b) intervals.

For the satellite system, states (ak) and (bk) correspond to "outage" and "innage," respectively. For the transmission line they correspond to "working order" and "breakdown." This reversal of interpretation for links in parallel and for links in series has been mentioned by Palmer.[2]

The problem is to find the distributions of the durations t_{ak} and t_{bk} of states (ak) and (bk). The lengths t_a , t_b of the intervals shown in Fig. 1 are supposed to be independent random variables with given probability densities $p_a(t)$, $p_b(t)$. Usually $p_a(t)$, $p_b(t)$ will be the same for all k links, but in the more general case the densities associated with the ith link will be denoted by $p_a^{(i)}(t)$, $p_b^{(i)}(t)$. The links are assumed to operate independently of each other. It is also assumed that the system has been operating long enough to reach statistical equilibrium.

Although t_a and t_b are independent, t_{ak} and t_{bk} need not be. An example is given just below equation (25) in Section IV.

The results given here do not apply to the case where the pattern of intervals in two or more links shows periodicities. For example, if all type (a) intervals of Links No. 1 and No. 2 are of length 1 and all type (b) intervals are of length 3, then (depending on the relative phase) there may be no type (ak) intervals and just one infinitely long type (bk) interval.

The distribution of t_{ak} depends only on $p_a(t)$. It is obtained in Section II for general $p_a(t)$. The expected value $\bar{t}_{bk}$ of t_{bk} depends only on the expected values $\bar{t}_a$, $\bar{t}_b$ and is given in Section III. At present there seems to be no practicable method, other than simulation on a high-speed computer, of obtaining the distribution of t_{bk} for general $p_a(t)$, $p_b(t)$. For exponential $p_a(t)$ a method due to Palmer[2] and Takács (outlined in Ref. 1) may be used, but even this is difficult unless $p_b(t)$ is also exponential. This method is developed in Section IV and illustrated in Section V. Sections VI and VII are concerned with the special case $k = 2$ but general $p_a(t)$, $p_b(t)$. Now the determination of the distribution of t_{b2} depends upon the solution of an integral equation. A vexing problem which I have been unable to solve is to show that when $p_a(t)$ is exponential the integral equation leads to the same distribution as does setting $k = 2$ in the method of Section IV.

I am indebted to John Riordan, David Slepian, and Lajos Takács for helpful comments.

II. THE DISTRIBUTION OF t_{ak}

It is convenient to set

$$F_a(t) = \int_t^\infty p_a(\tau)\, d\tau \tag{1}$$

$$A_a(t) = \int_t^\infty F_a(\tau)\, d\tau/\bar{t}_a \tag{2}$$

$$\bar{t}_a = \int_0^\infty \tau p_a(\tau)\, d\tau = \int_0^\infty F_a(\tau)\, dt. \tag{3}$$

Here $F_a(t)$ is the probability that $t_a > t$, $\bar{t}_a$ is the expected value of t_a, and $A_a(t)$ is closely related to C. Palm's[3] "next-arrival" distribution. If $\alpha(s)$ is the Laplace transform of $p_a(t)$, i.e.

$$\alpha(s) = \int_0^\infty e^{-st} p_a(t)\, dt \tag{4}$$

then

$$\begin{aligned} \int_0^\infty e^{-st} F_a(t)\, dt &= \frac{1 - \alpha(s)}{s} \\ \int_0^\infty e^{-st} A_a(t)\, dt &= \frac{1}{s}\left[1 - \frac{1 - \alpha(s)}{s\bar{t}_a}\right]. \end{aligned} \tag{5}$$

For the special case $p_a(t) = ae^{-at}$

$$\begin{aligned} F_a(t) &= A_a(t) = e^{-at}, \qquad \bar{t}_a = 1/a \\ \alpha(s) &= a/(a + s). \end{aligned} \tag{6}$$

To interpret $A_a(t)$, consider Fig. 2, which shows a line in Fig. 1 corresponding to a typical link. Choose a point $t = x$ at random [this means that when the choice is from the very long interval $(0,T)$ the chance that x falls between t, $t + dt$ is dt/T]. Let l be the distance to the end of the interval (which may be of either type) in which x falls. Then[2,4,5] $A_a(\tau)$ is the probability that $l > \tau$, given that x fell in an (a) interval.

It should be noted that expression (2) for $A_a(t)$ holds even when successive t_a's and t_b's are correlated. This point is important in the proof of (7), since the intervals t_{ak}, t_{bk} may be correlated. The only requirement is that as $T \to \infty$ the distribution of the lengths of the (a) intervals in $(0,T)$ approaches a definite distribution $F_a(t)$ possessing an average $\bar{t}_a$ which is neither zero nor infinite. For emphasis we sketch a proof of (2) which is tailored to Fig. 2, in which, for the moment, t_a and t_b may be correlated. The chance that $\tau < l < \tau + d\tau$ is the limit as $T \to \infty$ of the ratio

$$\frac{[\text{number of } (a) \text{ intervals longer than } \tau \text{ in } (0,T)](d\tau)}{\text{total length of } (a) \text{ intervals in } (0,T)}.$$

In the limit this ratio approaches $NF_a(\tau)d\tau/N\bar{t}_a$, where N is the number of (a) intervals in $(0,T)$. Cancelling the N's and integrating τ from t to ∞ then gives (2).

To find the probability $F_{ak}(t)$ that $t_{ak} > t$, suppose that all links are in state (a) at the randomly chosen time x. Since the links are independent, the chance that none has changed to state (b) by time $x + t$ is $[A_a(t)]^k$. Hence the function $A_{ak}(t)$ corresponding to the complete system is $[A_a(t)]^k$. This $A_{ak}(t)$ is related to $F_{ak}(t)$ by an equation obtained from (2) by replacing the subscripts "a" by "ak." Differentiation gives

$$\begin{aligned} F_{ak}(t) &= -\bar{t}_{ak} \frac{d}{dt} [A_a(t)]^k \\ &= (\bar{t}_{ak}/\bar{t}_a) k [A_a(t)]^{k-1} F_a(t) \end{aligned} \tag{7}$$

where $\bar{t}_{ak}$ denotes the expected length of intervals of type (ak).

Setting $t = 0$ in (7) and using $F_{ak}(0) = A_a(0) = F_a(0) = 1$ leads to

$$\bar{t}_{ak} = \bar{t}_a/k. \tag{8}$$

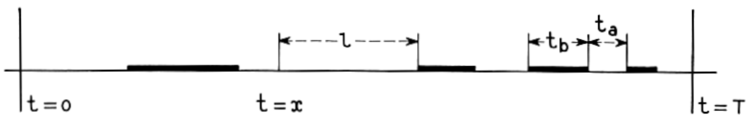

Fig. 2 — $A_a(t)$ is the chance that $l > t$.

When the individual links have probability densities $p_a^{(i)}(t)$, $p_b^{(i)}(t)$, $i = 1, 2, \cdots, k$, the chance that the length of a type (ak) interval exceeds t is

$$F_{ak}(t) = -\bar{t}_{ak} \frac{d}{dt} \prod_{i=1}^{k} A_a^{(i)}(t) \tag{9}$$

which implies

$$(\bar{t}_{ak})^{-1} = \sum_{i=1}^{k} (\overline{t_a^{(i)}})^{-1} \tag{10}$$

just as (7) implies (8). The A_a's and $\bar{t}_a$'s are related by equations corresponding to (1), (2) and (3). These results are due to Palmer,[2] who obtains (7) by a different argument.

III. THE EXPECTED LENGTH OF INTERVALS OF VARIOUS TYPES, INCLUDING TYPE (bk)

The expected value $\bar{t}_{bk}$ of the length of the type (bk) intervals is related to $\bar{t}_{ak}$ by the equation

$$\bar{t}_{bk} = \left(\frac{1}{p_{ak}} - 1\right) \bar{t}_{ak} \tag{11}$$

where p_{ak} [not to be confused with the probability density $p_{ak}(t)$] is the chance that the random point x will fall in an interval of type (ak). This follows almost immediately from

$$\bar{t}_{bk}/\bar{t}_{ak} = p_{bk}/p_{ak}, \tag{12}$$

$$p_{ak} + p_{bk} = 1. \tag{13}$$

A careful discussion of the probability p_{ak} has been given by Weiss.[6]

A relation similar to (11) also holds for $\bar{t}_b$, $\bar{t}_a$, and the probability p_a that the random point x will fall in a type (a) interval. Solving for p_a gives

$$p_a = \bar{t}_a(\bar{t}_a + \bar{t}_b)^{-1} \tag{14}$$

and when this is combined with $p_{ak} = p_a^k$, which follows from the independence of the links, (11) becomes

$$\bar{t}_{bk} = (p_a^{-k} - 1)\bar{t}_{ak} = \left[\left(1 + \frac{\bar{t}_b}{\bar{t}_a}\right)^k - 1\right] \frac{\bar{t}_a}{k}. \tag{15}$$

Palmer's generalization of (15) can be written as

$$\begin{aligned} \bar{t}_{bk} &= [(\prod p_a^{(i)})^{-1} - 1]\bar{t}_{ak} \\ p_a^{(i)} &= \bar{t}_a^{(i)}(\bar{t}_a^{(i)} + \bar{t}_a^{(i)})^{-1} \end{aligned} \tag{16}$$

where $\bar{t}_{ak}$ is given by (10).

Einhorn[7] has given the instances $k = 2$ of (10) and (16). He also gives a generalization in which all k links are alike and attention is fixed on the average length $\bar{t}_{ar,k}'$ of the periods during which r or more of the links are in state (a).* He takes both $p_a(t)$ and $p_b(t)$ to be exponential, but his expression for $\bar{t}_{ar,k}'$ appears to hold for general distributions. Thus, let state j be the state of the system in which exactly j of the k links are in state (a), and let

$$p_{aj,k} = \binom{k}{j} p_a^{\,j} p_b^{\,k-j} \tag{17}$$

be the fraction of time the system spends in state j. Here $\binom{k}{j}$ is a binomial coefficient, $p_b = 1 - p_a$, and p_a is given by (14). Then Einhorn's results may be stated as

$$\begin{aligned} \bar{t}_{ar,k}' &= \bar{t}_{ar} \sum_{j=r}^{k} p_{aj,k}/p_{ar,k} \qquad (18) \\ &= \sum_{j=r}^{k} \binom{k}{j} (\bar{t}_a)^j(\bar{t}_b)^{k-j} \Big/ r \binom{k}{r} (\bar{t}_a)^{r-1}(\bar{t}_b)^{k-r} \\ \bar{t}_{ar,k}'' &= \bar{t}_{ar} \sum_{j=0}^{r-1} p_{aj,k}/p_{ar,k} \qquad (19) \end{aligned}$$

where $\bar{t}_{ar} = \bar{t}_a/r$ and $\bar{t}_{ar,k}''$ is the average length of the intervals during which $j < r$. Setting $r = k$ in (18) and (19) gives (8) and (15), respectively.

To establish (18) note that, in the very long interval $(0,T)$, the amount of time the system spends in states for which $j \geqq r$ is $T \sum_r^k p_{aj,k}$.

The number of periods in $(0,T)$ during which $j \geqq r$ is equal (to within one) to the number of periods during which $j < r$, and both are equal to the number of times the system jumps from state r to state $r - 1$. As shown in the next paragraph, the number of these jumps is $Tp_{ar,k}/\bar{t}_{ar}$, and (18) follows by division.

Divide the links into two groups, Group I consisting of the first r

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* A similar problem has been considered in unpublished work by my colleague H. Coo, in which account is also taken of repairs at periodic intervals.

links and Group II of the last $k - r$ links. The fraction of time all $k - r$ links in II are in state (b) is p_b^{k-r}. The number of times all links in I are in state (a) is $p_a^r T/\bar{t}_{ar}$. This is also the number of times Group I jumps from state r to state $r - 1$. Since the two groups operate independently and no periodicities exist, we assume that p_b^{k-r} gives the fraction of these jumps occurring while all links in II are in state (b). Thus $[p_a^r T/\bar{t}_{ar}]p_b^{k-r}$ is the number of jumps the complete system makes from state r to state $r - 1$ when a specified set of $k - r$ links (namely the last $k - r$) remain in state (b). Since the set may be chosen in $\binom{k}{k-r}$ ways, the complete number of jumps is $Tp_{ar,k}/\bar{t}_{ar}$, as stated.

Incidentally, it may be shown that

$$F_{aj,k}(t) = -\bar{t}_{aj,k} \frac{d}{dt} A_a^j(t) A_b^{k-j}(t) \tag{20}$$

$$(\bar{t}_{aj,k})^{-1} = j(\bar{t}_a)^{-1} + (k - j)(\bar{t}_b)^{-1}$$

give the distribution and average length of the state j intervals.

IV. THE DISTRIBUTION OF t_{bk}

In the first part of this section several auxiliary distributions are discussed. They correspond to arbitrary $p_a(t)$, $p_b(t)$ and are more general than needed here, where ultimately $p_a(t)$ is required to be exponential. However, they are used in Section VI.

Consider the probability $Q_{aa}(t)$ that $x + t$ falls in a type (a) interval, given that the random point x falls in a type (a) interval. We have

$$Q_{aa}(t) = A_a(t) - \int_0^t P_{ba}(t - \tau) \frac{d}{d\tau} A_a(\tau)\, d\tau \tag{21}$$

in which $A_a(t)$ is the chance that the original (a) interval lasts beyond $x + t$ and $[-dA_a(\tau)/d\tau]\, d\tau$ the chance that it ends in $x + \tau, x + \tau + d\tau$. The end of the original (a) interval marks the beginning of a type (b) interval, and $P_{ba}(t')$ is the probability that a type (a) interval exists at time t', given that a type (b) interval began at time 0.

Let the Laplace transforms of $p_a(t)$, $p_b(t)$ be $\alpha(s)$, $\beta(s)$. Then the transforms of $F_a(t)$, $A_a(t)$ are given by (5). Weiss,[6] and Brooks and Diamantides,[8] have shown that the transform of $P_{ba}(t)$ is

$$s^{-1}[1 - \alpha(s)]\beta(s)/[1 - \alpha(s)\beta(s)].$$

This result is also developed in my paper[1] in ignorance of the earlier work of Weiss. Since the integral in (21) represents a convolution, its transform is the product of the transforms of $dA_a(t)/dt$ and $P_{ba}(t)$. When the transform of $Q_{aa}(t)$ is computed from (21), it is found to be

$$\begin{aligned} \phi_a(s) &= \int_0^\infty e^{-st} Q_{aa}(t)\, dt \\ &= \frac{1}{s} - \frac{[1 - \alpha(s)][1 - \beta(s)]}{s^2 \bar{t}_a [1 - \alpha(s)\beta(s)]}. \end{aligned} \tag{22}$$

This result is given by Palmer[2] and, independently, by Brooks and Diamantides.[8] It is also given, together with a number of related results, by Cox (Ref. 5, Ch. 7).

The argument in the two preceding paragraphs is concerned with type (a) intervals. It applies equally well to intervals of type (ak) *when the* (ak) *and* (bk) *intervals are independent*. When the links are alike, in place of $Q_{aa}(t)$ we have $[Q_{aa}(t)]^k$ for the chance that a type (ak) interval exists at time $x + t$, given that one exists at the randomly chosen time x. In place of the probability densities $p_a(t)$, $p_b(t)$ and their transforms $\alpha(s)$, $\beta(s)$ we have $p_{ak}(t)$, $p_{bk}(t)$ and their transforms $\alpha_k(s)$, $\beta_k(s)$. Equation (22) goes into an expression for the Laplace transform of $[Q_{aa}(t)]^k$

$$\frac{1}{s} - \frac{[1 - \alpha_k(s)][1 - \beta_k(s)]}{s^2 \bar{t}_{ak} [1 - \alpha_k(s)\beta_k(s)]} = \int_0^\infty e^{-st} [Q_{aa}(t)]^k\, dt. \tag{23}$$

Since $\alpha_k(s)$ may be computed from (7) and $Q_{aa}(t)$ from (22), $\beta_k(s)$ is the only unknown in (23). In principle, if not in practice, (23) may be solved for $\beta_k(s)$ and then $p_{bk}(t)$ obtained by inversion.

It should be remembered that (23) is based on the assumption that the (ak) and (bk) intervals are independent. They are independent when $p_a(t)$ is exponential, since then $p_{ak}(t)$ is also exponential, and a knowledge of the lengths of the (ak) intervals tells us nothing about the (bk) intervals and vice versa.

Thus when $p_a(t) = ae^{-at}$, so that $\alpha_k(s) = ka[ka + s]^{-1}$, (23) reduces to

$$\frac{1}{s + ka - ka\beta_k(s)} = \int_0^\infty e^{-st} [Q_{aa}(t)]^k\, dt \tag{24}$$

where $Q_{aa}(t)$ has the transform $[s + a - a\beta(s)]^{-1}$. More generally, when $p_a^{(i)}(t) = a_i \exp(-a_i t)$ and $p_b^{(i)}(t)$ is arbitrary the equation for $\beta_k(a)$ becomes

$$\frac{\bar{t}_{ak}}{s\bar{t}_{ak} + 1 - \beta_k(s)} = \int_0^\infty e^{-st} \prod_{i=1}^{k} Q_{aa}^{(i)}(t)\, dt \tag{25}$$

where $\bar{t}_{ak} = [\sum a_i]^{-1}$ and $Q_{aa}^{(i)}(t)$ has the transform $[s + a_i - a_i\beta^{(i)}(s)]^{-1}$. This is essentially the result obtained by Palmer[2] and Takács. (Takács' version is given in my paper.[1])

The only case in which the (ak) and (bk) intervals are obviously independent seems to be that for exponential $p_a(t)$. On the other hand, the following example shows that successive (ak) and (bk) intervals may be correlated even though the (a) and (b) intervals for the individual links are not. Let t_a and t_b be uniformly distributed between 1.00, 1.01 and 2.00, 2.01 respectively. Let $k = 2$. Then, given an (ak) interval of length $t_{a2} = 0.5$, we can infer that the length of the following (bk) interval lies between 2.5 and 2.52.

Hence, so far as the discussion given in this section goes, (23) is no more general than (24). The question now arises as to the form taken by $p_{bk}(t)$ when $p_a(t)$ and $p_b(t)$ are arbitrary. Some information on this is given in Section VI for the case $k = 2$.

V. DISCUSSION AND EXAMPLES

When k tends to infinity, with $p_a(t)$, $p_b(t)$ fixed but arbitrary, $\bar{t}_{ak}$ tends to zero and $\bar{t}_{ba}$ to infinity. When t becomes small, (2) shows that $A_a(t)$ tends to $1 - t/\bar{t}_a$ and its kth power to $\exp(-tk/\bar{t}_a)$. It follows that when t is small and k is large, the chance that the length of an (ak) interval exceeds t is

$$F_{ak}(t) \approx \exp(-t/\bar{t}_{ak}). \tag{26}$$

It may be conjectured that when $k \to \infty$ the chance $F_{bk}(t)$ that $t_{bk} > t$ also tends to an exponential

$$F_{bk}(t) \to \exp(-t/\bar{t}_{bk}). \tag{27}$$

Here t is supposed to be many times larger than $\bar{t}_a$ and $\bar{t}_b$. Indeed, consider a (bk) interval to be in progress and k to be large. Over all of the interval, except for a negligibly small fraction near the beginning, the process will be uncorrelated with the initial conditions, and the chance that the interval will end in $(t, t + dt)$ is independent of t. This leads to the exponential form (27). When $\bar{t}_b/\bar{t}_a \ll 1$, k may have to be extremely large before (27) begins to hold. This is because the argument pictures a great deal of overlap of (b) type intervals.

Next we turn to the case where the k links are different and

$$\begin{gathered} p_a^{(i)}(t) = a_i e^{-a_i t}, \qquad p_b^{(i)} = b_i e^{-b_i t}, \qquad i = 1, 2, \cdots, k \\ \bar{t}_a^{(i)} = a_i^{-1}, \qquad \bar{t}_b^{(i)} = b_i^{-1}. \end{gathered} \tag{28}$$

This is one of the few cases in which the work may be carried forward to even a moderate degree. From Section II

$$\begin{aligned} F_a^{(i)}(t) &= A_a^{(i)}(t) = e^{-a_i t} \\ \bar{t}_{ak} &= [\textstyle\sum a_i]^{-1} \\ F_{ak}(t) &= \exp[-t/\bar{t}_{ak}] = \exp[-t\textstyle\sum a_i] \\ p_{ak}(t) &= [\textstyle\sum a_i]\exp[-t\textstyle\sum a_i]. \end{aligned} \tag{29}$$

These expressions for $F_{ak}(t)$ and $p_{ak}(t)$ also hold when $p_b^{(i)}(t)$ is arbitrary. From Section III

$$\begin{aligned} \bar{t}_{bk} &= \left(\frac{1}{p_{ak}} - 1\right)\bar{t}_{ak}, \\ p_{ak} &= \prod_{i=1}^{k} p_a^{(i)}, \qquad p_a^{(i)} = b_i(a_i + b_i)^{-1}. \end{aligned} \tag{30}$$

The first step in using (25) is to compute $Q_{aa}^{(i)}(t)$ by inverting its Laplace transform. The result is given in the last line of Table I in Section VI and leads to

$$\prod_{i=1}^{k} Q_{aa}^{(i)}(t) = \prod_{i=1}^{k}\left[\frac{b_i + a_i e^{-(a_i+b_i)t}}{b_i + a_i}\right] = \sum_j c_j e^{-d_j t} \tag{31}$$

where in the general case the sum on j contains 2^k terms. The right-hand member of (25) becomes $\sum c_j(s + d_j)^{-1}$, and it follows that the Laplace transform of $F_{bk}(t)$ is equal to

$$\frac{1 - \beta_k(s)}{s} = \frac{\bar{t}_{ak}}{s\sum_j c_j(s + d_j)^{-1}} - \bar{t}_{ak}. \tag{32}$$

The transform of $F_{bk}(t)$ is thus a rational function of s. Its poles are at the zeros of $\sum c_j(s + d_j)^{-1}$, and these zeros lie on the negative real axis between the points $s = -d_j$, the rightmost of which is $s = 0$. The nth moment, $\overline{t_{bk}^n}$, of t_{bk} is $n!(-1)^{n-1}$ times the coefficient of s^{n-1} in the power series expansion of (32). Hence for $n > 0$

$$\overline{t_{bk}^n} = n\bar{t}_{ak}\left[\left(-\frac{d}{ds}\right)^{n-1}\left\{\frac{1}{s\sum c_j(s + d_j)^{-1}} - 1\right\}\right]_{s=0}. \tag{33}$$

When the links are alike, some of the d_j's are equal, and

$$\sum_j c_j e^{-d_j t} = \left[\frac{b + ae^{-(a+b)t}}{b + a}\right]^k = \sum_{n=0}^{k}\binom{k}{n}\frac{b^{k-n}a^n}{(b + a)^k}\, e^{-n(a+b)t}.$$

In this case, the results are those used in Ref. 1, namely

$$F_{ak}(t) = e^{-tka}, \qquad \bar{t}_{ak} = 1/ka$$
$$F_{bk}(t) = \frac{(1+\rho)^{k+1}}{k\rho} \sum_{m=0}^{k-1} \frac{\exp[(a+b)z_m t]}{z_m f'(z_m)} \tag{34}$$
$$\bar{t}_{bk} = [(1+\rho)^k - 1]/ka, \qquad \rho = a/b$$

where $f_k'(z) = df_k(z)/dz$, and $z_0, z_1, \cdots, z_{k-1}$ are the zeros of

$$f_k(z) = \sum_{n=0}^{k} \binom{k}{n} \frac{\rho^n}{z+n}. \tag{35}$$

These zeros lie between the poles at $0, -1, -2, \cdots, -k$. The first few terms in the power series for $F_{bk}(t)$ are given by (14) of Ref. 1. In present notation

$$\begin{aligned} F_{bk}(t) = 1 - \frac{bt}{1!} + \frac{[(k-1)a+b]bt^2}{2!} \\ - [(k-1)^2a^2 + 4(k-1)ab + b^2]\frac{bt^3}{3!} + \cdots. \end{aligned} \tag{36}$$

The nth moment, $n > 0$, is

$$\overline{t_{bk}^{\,n}} = \frac{n(1+\rho)^{k-n+1}}{k\rho b^n}\left[\left(-\frac{d}{dz}\right)^{n-1}\left\{\frac{1}{zf_k(z)} - (1+\rho)^{-k}\right\}\right]_{z=0}. \tag{37}$$

In particular

$$\overline{t_{bk}^{\,2}} = \frac{2(1+\rho)^{k-1}}{akb}\sum_{n=1}^{k}\binom{k}{n}\frac{\rho^n}{n} \tag{38}$$

a result given by Palmer[2] for $a = b = 1$.

When $k = 2$ and the links are alike, (32) gives

$$\frac{1-\beta_2(s)}{(s)} = \frac{s+a+2b}{s^2+(3b+a)s+2b^2}. \tag{39}$$

Results of the sort given here have occurred in the reliability studies of R. S. Dick[9] and others.

VI. THE DISTRIBUTION OF t_{bk} FOR $k = 2$

Here we consider the determination of $p_{bk}(t)$ for the case of $k = 2$ links when $p_a(t)$ and $p_b(t)$ are arbitrary. The following probabilities, which are related to those mentioned in the first part of Section IV, will be needed:

$P(t)$ = chance that an (a) interval exists at time t, given that an (a) interval starts at time 0;
$Q(t)$ = chance that an (a) interval exists at time $x + t$, given that the random x of Fig. 2 falls in an (a) interval. $Q(t)$ is equal to the $Q_{aa}(t)$ of Section IV;
$R(t)\,dt$ = chance that an (a) interval ends in $t, t + dt$, given that an (a) interval starts at time 0; and
$S(t)\,dt$ = chance that an (a) interval ends in $x + t, x + t + dt$, given that the random x falls in an (a) interval.

Some information regarding these probabilities is summarized in Table I. Here $\alpha(s)$ and $\beta(s)$, the Laplace transforms of $p_a(t)$ and $p_b(t)$, are written for brevity as α and β. The entries may be obtained by the methods indicated in Section IV. Closely related results have been given in unpublished work by my colleague H. E. Rowe.

An expression for the distribution of the length of a $(b2)$ interval which consists of, say, three (b) intervals can be obtained by examining Fig. 3. The probability that the $(b2)$ interval shown in Fig. 3 has a length between t and $t + dt$ can be obtained by integrating

$$\begin{gathered} p_b(t - x_2)\,dx_2 \cdot R(x_2 - x_3)\,dx_3 \cdot p_b(x_3)\,dt \\ S(t - x_1)\,dx_1 \cdot p_b(x_1 - x_4)\,dx_4 \cdot P(x_4) \end{gathered} \tag{40}$$

over the permissible values of the x's, namely $0 \leqq x_4 \leqq x_3 \leqq x_2 \leqq x_1 \leqq t$.

Starting with the probability $p_b(t)Q(t)\,dt$ that a $(b2)$ interval consists of one (b) interval and is of length $t, t + dt$, one can write a series for $p_{b2}(t)\,dt$. The first term is $p_b(t)Q(t)\,dt$, and the later terms are multiple integrals of the type obtained by integrating (40). An examination of the series shows that

$$p_{b2}(t) = p_b(t)Q(t) + \int_0^t dx_1 S(t - x_1)q(t,x_1) \tag{41}$$

where $q(t,x)$ satisfies the relation

$$\begin{aligned} q(t,x_1) = \int_0^{x_1} & dx_2 p_b(t - x_2) \\ & \cdot \left[p_b(x_1)P(x_2) + \int_0^{x_2} dx_3\, R(x_2 - x_3)q(x_1, x_3) \right]. \end{aligned} \tag{42}$$

The series for $p_{b2}(t)$ obtained by repeated substitution of (42) in (41) is sometimes useful for small values of t.

TABLE I — INFORMATION ON AUXILIARY DISTRIBUTIONS

	$P(t)$	$Q(t)$	$R(t)$	$S(t)$
Laplace transforms	$\frac{1-\alpha}{s(1-\alpha\beta)}$	$\frac{1}{s} - \frac{(1-\alpha)(1-\beta)}{s^2\bar{t}_a(1-\alpha\beta)}$	$\frac{\alpha}{1-\alpha\beta}$	$\frac{1-\alpha}{s\bar{t}_a(1-\alpha\beta)}$
Exponential $p_a(t)$, $\alpha = a(s+a)^{-1}$	$\varphi_P = \frac{1}{s+a-a\beta}$	φ_P	$a\varphi_P$	$a\varphi_P$
Exponential $p_b(t)$, $\beta = b(s+b)^{-1}$	$\frac{(b+s)(1-\alpha)}{s(s+b-b\alpha)}$	$\frac{1}{s} - \frac{1-\alpha}{s\bar{t}_a(s+b-b\alpha)}$	$\frac{(s+b)\alpha}{(s+b-b\alpha)}$	$\frac{(s+b)(1-\alpha)}{s\bar{t}_a(s+b-b\alpha)}$
Exponential $p_a(t)$ and $p_b(t)$	$\varphi_P = \frac{s+b}{s(s+a+b)}$	φ_P	$a\varphi_P$	$a\varphi_P$
$P(t)$, $\cdots$ for exponential $p_a(t)$ and $p_b(t)$	$P = \frac{b+ae^{-(a+b)t}}{b+a}$	P	aP	aP

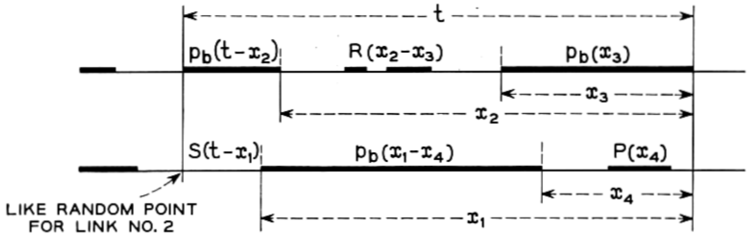

Fig. 3 — Sketch illustrating quantities in expression (40).

Equation (42) is of the form

$$q(t,x) = g(t,x) + \int_0^x K(t - x, x - y)q(x,y)\, dy, \tag{43}$$

where g and K are known functions, and is an integral equation to determine $q(t,x)$ in the region $0 \leqq x \leqq t$. In this region the values of $q(t,x)$ along the line $x = x_1$ are expressed in terms of its values on the line $t = x_1$.

It appears difficult to solve (42) for general $p_a(t)$ and $p_b(t)$. Two special cases are discussed in Section VII.

It should be possible to verify that when $p_a(t) = a \exp(-at)$, (41) and (42) lead to the same result as does (24) with $k = 2$. However, I have been unable to do this for general $p_b(t)$. The problem may be stated as follows: show that the $\beta_2(s)$ defined by setting $k = 2$ and $Q_{aa}(t) = P(t)$ in (24) is the Laplace transform of

$$p_{b2}(t) = p_b(t)P(t) + a \int_0^t P(t - x_1)q(t,x_1)\, dx_1 \tag{44}$$

where $P(t)$ has the transform $[s + a - a\beta(s)]^{-1}$ and $q(t,x_1)$ satisfies

$$q(t,x_1) = \int_0^{x_1} dx_2 p_b(t - x_2)$$
$$\cdot \left[p_b(x_1)P(x_2) + a \int_0^{x_2} dx_3 P(x_2 - x_3)q(x_1, x_3) \right].$$

When the two links have different statistics, the distribution of the lengths of ($b2$) intervals which begin with a (b) interval of Link No. 1, say, may be expressed as an integral similar to (41). However, there are now two integral equations similar to (42) which must be solved simultaneously.

VII. SPECIAL CASES FOR $p_{b2}(t)$

Here two special cases are given in which the integral equation (42) may be solved. The second case has been used to study the problem mentioned in connection with (44).

(*i*) *Exponential* $p_b(t)$ *and general* $p_a(t)$. When $p_b(t) = b \exp(-bt)$, (42) shows that $q(t,x_1)$ is of the form $f(x_1) \exp(-bt)$ where

$$f(x_1) = b \int_0^{x_1} dx_2 e^{-b(x_1-x_2)} \left[bP(x_2) + \int_0^{x_2} R(x_2 - x_3) f(x_3)\, dx_3 \right]. \quad (45)$$

This goes into

$$F(s) = \frac{b}{s+b} [b\varphi_P(s) + \varphi_R(s) F(s)] \quad (46)$$

where $F(s)$, $\varphi_P(s)$, and $\varphi_R(s)$ are the Laplace transforms of $f(t)$, $P(t)$, and $R(t)$. Similarly, the transform of (41) is

$$\beta_2(s) = b\varphi_Q(s+b) + \varphi_S(s+b) F(s+b). \quad (47)$$

From (46), (47), and Table I

$$F(s) = \frac{b^2[1 - \alpha(s)]}{s[s + b - 2b\alpha(s)]}, \quad (48)$$

$$\beta_2(s) = \frac{b}{s+b} - \frac{sb[1 - \alpha(s+b)]}{(s+b)^2 \bar{t}_a [s + 2b - 2b\alpha(s+b)]}. \quad (49)$$

Inversion of $\beta_2(s)$ and $[1 - \beta_2(s)]s^{-1}$ now gives $p_{b2}(t)$ and $F_{b2}(t)$.

When the two links have different exponential $p_b(t)$'s and general $p_a(t)$'s, the two simultaneous integral equations mentioned at the end of Section VI may be solved by a somewhat similar procedure.

(*ii*) $p_a(t)$ *exponential and* $p_b(t)$ *the sum of two exponentials.* For

$$p_a(t) = ae^{-at}, \qquad p_b(t) = c_1 e^{-b_1 t} + c_2 e^{-b_2 t}, \qquad c_1 b_1^{-1} + c_2 b_2^{-1} = 1 \quad (50)$$

(42) shows that $q(t,x)$ is of the form

$$c_1 e^{-b_1 t} f_1(x) + c_2 e^{-b_2 t} f_2(x).$$

Instead of $f_j(x)$, $j = 1, 2$, it is more convenient to deal with

$$g_j(x) = P(x) + a \int_0^x P(x - y) f_j(y)\, dy$$

having the Laplace transform $G_j(s)$. When Laplace transforms are introduced, (42) goes into two simultaneous equations for $G_1(s + b_1)$, $G_2(s + b_2)$. Solving these leads to

$$\beta_2(s) = c_1G_1(s + b_1) + c_2G_2(s + b_2)$$
$$= \frac{c_2B_1 + c_1B_2 + 2ac_1c_2(s + b_1 + b_2)^{-1}}{B_1B_2 - a^2c_1c_2(s + b_1 + b_2)^{-2}},$$
$$B_j = \frac{1}{\varphi_P(s + b_j)} - \frac{ac_j}{s + 2b_j}, \qquad \varphi_P(s) = \frac{1}{s + a - a\beta(s)}, \tag{51}$$
$$\beta(s) = \frac{c_1}{s + b_1} + \frac{c_2}{s + b_2}.$$

The equation for $\beta_2(s)$ obtained from (24) in this case is

$$[s + 2a - 2a\beta_2(s)]^{-1} = \int_0^\infty e^{-st}P^2(t)\,dt$$
$$= \frac{1}{2\pi i}\int_L \varphi_P(s - s')\varphi_P(s')\,ds'$$
$$= \frac{\varphi_P(s)b_1b_2}{s_1s_2} + \frac{\varphi_P(s - s_1)(s_1 + b_1)(s_1 + b_2)}{s_1(s_1 - s_2)} \tag{52}$$
$$+ \frac{\varphi_P(s - s_2)(s_2 + b_1)(s_2 + b_2)}{s_2(s_2 - s_1)}$$

where the path of integration L runs from $-i\infty$ to $+i\infty$ so that the singularities of $\varphi_P(s - s')$ lie on its right and those of $\varphi(s')$ on its left. The integral has been evaluated by writing $\varphi_P(s)$ as

$$\varphi_P(s) = \frac{(s + b_1)(s + b_2)}{s(s - s_1)(s - s_2)},$$
$$(s - s_1)(s - s_2) = s^2 + (a + b_1 + b_2)s + b_1b_2 + a(b_1 + b_2 - c_1 - c_2)$$

closing L by an infinite semicircle on the left and evaluating the residues at the poles $s' = 0, s_1, s_2$.

The task of verifying that (51) and (52) give the same value of $\beta_2(s)$ appears to be a lengthy one and has not been carried out. Several numerical checks have been made and show no discrepancy.

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