On the Properties of Some Systems that Distort Signals—I

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This is the first part of a two-part paper concerned with some generalizations and extensions of the Beurling-Landau-Miranker-Zames theory of recovery of distorted bandlimited signals. We present a uniqueness proof that extends Beurling's result and study a class of functional mappings defined on Hilbert space. As an application, we show that the recovery results can be extended to cases in which a known square-integrable corrupting signal is added to the input signal and the result applied to a time-variable device which may be nonlinear. It is proved that an assumption made by the earlier writers is in fact necessary in order that stable recovery be possible. Part II will consider the more complicated situation in which a single time-variable nonlinear element is imbedded in a general linear system.

I. INTRODUCTION

A signal transmission system is a realization of an operator that maps input signals in one domain into output signals in a second domain. When the system contains energy-storage devices as well as time-variable or nonlinear elements, the mapping is usually quite complicated. Very little in the way of a general theory is known concerning the mathematical properties of such mappings.

Of course one of the important properties of a mapping is its invertability or lack of invertability. Some particularly interesting results relating to the existence of the inverse of a special mapping have been obtained by Beurling, Landau, Miranker, and Zames. They consider the situation in which a square-integrable bandlimited signal is passed through a monotonic nonlinear device. Beurling showed, by means of a nonconstructive proof,† that a knowledge of the Fourier transform of the distorted signal on the interval where the transform of the input signal does not vanish is sufficient to uniquely determine the input

† Beurling's proof is given in Refs. 1 and 3.

signal. Landau and Miranker[1] have considered a stable iteration scheme for obtaining the input signal from the bandlimited version of the distorted signal. They assume that the distortion characteristic possesses a derivative bounded above and below by positive constants. A solution of this type was found independently by G. D. Zames.[2] Some material associated with the stability of the iteration scheme and an impressive recovery experiment are discussed by Landau.[3]

This paper is concerned with some generalizations and extensions of the results mentioned above. Our primary objective is to show that the results in Refs. 1 and 2 are special cases of a quite general theory.

Section II considers some mathematical preliminaries. In Section III we discuss the solution of a class of functional equations defined on an arbitrary Hilbert space, and give a uniqueness proof that extends Beurling's result. In the next section two general signal-theoretic applications of the results in Section III are discussed. Theorem IV implies, among other things, that the recovery theory of the earlier writers can be extended to cases in which a known square-integrable corrupting signal is added to the bandlimited input signal and the result applied to a time-variable device which may be nonlinear. Section V concludes Part I with some specialized results that contribute to a deeper understanding of the character of the previous material. In particular it is proved that an assumption made by the earlier writers is in fact necessary in order that stable recovery be possible.

Part II will consider the more complicated situation in which a single time-variable nonlinear element is imbedded in a general linear system. We treat a recovery problem of the type considered by the earlier writers and prove that recovery is possible under quite general conditions. This study may have applications in improving the quality of distorted data obtained, for example, from a malfunctioning transmitter in a space satellite.

II. PRELIMINARIES

Let $\mathcal{R} = [\Theta,\rho]$ be an arbitrary metric space.[4] A mapping $\mathbf{A}$ of the space $\mathcal{R}$ into itself is said to be a contraction if there exists a number $\alpha < 1$ such that

$$\rho(\mathbf{A}x,\mathbf{A}y) \leqq \alpha\rho(x,y)$$

for any two elements $x,y \in \Theta$. The contraction-mapping fixed-point theorem[4] is basic to much of the subsequent discussion. It states that every contraction-mapping defined in a complete metric space $\mathcal{R}$ has one and only one fixed point (i.e., there exists a unique element $z \in \Theta$

such that $\mathbf{A}z = z$). Furthermore $z = \lim_{n\to\infty} \mathbf{A}^n x_0$, where x_0 is an arbitrary element of Θ.

Throughout the discussion $\mathcal{H}$ denotes a real or complex Hilbert space. If $f,g \in \mathcal{H}$, then (f,g), $\| f \| = (f,f)^{\frac{1}{2}}$, and $\| f - g \|$, respectively, denote the inner product of f with g, the norm of f, and the distance between f and g. It is not assumed that $\mathcal{H}$ is separable or that it is of infinite dimension.

The space of complex-valued square-integrable functions with inner product

$$(f,g) = \int_{-\infty}^{\infty} f\underset{\sim}{g}\, dt,$$

where $\underset{\sim}{g}$ is the complex conjugate of g, is denoted by $\mathcal{L}_2$, and $\mathcal{L}_{2R}$ denotes the intersection of the space $\mathcal{L}_2$ with the set of real-valued functions.

We take as the definition of the Fourier transform of $f(t) \in \mathcal{L}_2$:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt,$$

and consequently

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\, d\omega.$$

With this definition, the Plancherel identity reads:

$$2\pi \int_{-\infty}^{\infty} f(t)\underset{\sim}{g}(t)dt = \int_{-\infty}^{\infty} F(\omega)\underset{\sim}{G}(\omega)\, d\omega.$$

Except when indicated otherwise, a function and its Fourier transform are denoted, respectively, by lower and upper case versions of the same symbol.

The symbol $\mathcal{K}$ denotes an arbitrary subspace of $\mathcal{H}$. Hence $\mathcal{H} = \mathcal{K} \dotplus \mathcal{K}'$, the direct sum of $\mathcal{K}$ and $\mathcal{K}'$, where $\mathcal{K}'$ is the orthogonal complement of $\mathcal{K}$ with respect to $\mathcal{H}$. The operator that projects an arbitrary element of $\mathcal{H}$ onto $\mathcal{K}$ is denoted by $\mathbf{P}$. The subspaces of $\mathcal{L}_{2R}$ of principal interest to us are†

$$\mathcal{B}(\Omega) = \{f(t) \mid f(t) \in \mathcal{L}_{2R}; \qquad F(\omega) = 0, \omega \notin \Omega\}$$

and

$$\mathcal{D}(\Sigma) = \{f(t) \mid f(t) \in \mathcal{L}_{2R}; \qquad f(t) = 0, t \notin \Sigma\},$$

† It is a simple matter to verify that the linear manifold $\mathcal{B}(\Omega)$ is in fact a subspace. An obvious modification of the proof in Ref. 1 for the case in which Ω is a single interval suffices.

where Ω and Σ are each the union of disjoint intervals. It is hardly necessary to mention that the class of electrical signals belonging to $\mathcal{B}(\Omega)$ or $\mathcal{D}(\Sigma)$ is of considerable importance in the theory of electrical communication systems.

We shall use the fact that any projection operator defined on a Hilbert space is self adjoint [i.e., that $(f,\mathbf{P}g) = (\mathbf{P}f,g)$ for any $f,g \in \mathcal{H}$].

The symbol **I** is used throughout to denote the identity transformation.

III. INVERSION OF A CLASS OF OPERATORS DEFINED ON AN ARBITRARY HILBERT SPACE

As we have said earlier, a signal transmission system is a realization of an operator that maps input signals in one domain into output signals in a second domain. The following theorem relates to the existence of the inverse of a particularly relevant type of nonlinear mapping defined on an arbitrary Hilbert space.

Theorem I: Let **Q** *be a mapping of* $\mathcal{K}$ *into* $\mathcal{H}$ *such that for all* $f,g \in \mathcal{K}$:

$$Re(\mathbf{Q}f - \mathbf{Q}g, f - g) \geqq k_1 \| f - g \|^2$$

$$\| \mathbf{PQ}f - \mathbf{PQ}g \|^2 \leqq k_2 \| f - g \|^2$$

where k_1 *and* k_2 *are positive constants. Then for each* $h \in \mathcal{K}$, *the equation* $h = \mathbf{PQ}f$ *possesses a unique solution* $(\mathbf{PQ})^{-1}h \in \mathcal{K}$ *given by* $(\mathbf{PQ})^{-1}h = \lim_{n\to\infty} f_n$ *where*

$$f_{n+1} = \frac{k_1}{k_2}(h - \mathbf{PQ}f_n) + f_n$$

and f_0 *is an arbitrary element of* $\mathcal{K}$. *Furthermore, for all* $h_1, h_2 \in \mathcal{K}$

$$\| (\mathbf{PQ})^{-1}h_1 - (\mathbf{PQ})^{-1}h_2 \| \leqq \frac{1}{k_1} \| h_1 - h_2 \|.$$

Proof:

Let $\mathbf{A} = \mathbf{PQ}$ and note first that

$$\begin{aligned} \mathrm{Re}(\mathbf{A}f - \mathbf{A}g, f - g) &= \mathrm{Re}(\mathbf{Q}f - \mathbf{Q}g, \mathbf{P}f - \mathbf{P}g) \\ &= \mathrm{Re}(\mathbf{Q}f - \mathbf{Q}g, f - g) \geqq k_1 \| f - g \|^2 \end{aligned}$$

for all $f,g \in \mathcal{K}$ since **P** is a self-adjoint transformation.

The equation $h = \mathbf{A}f$ is equivalent to $f = \tilde{\mathbf{A}}f$, where $\tilde{\mathbf{A}}f = ch + f - c\mathbf{A}f$ and c is any nonzero constant. The following calculation shows that $\tilde{\mathbf{A}}$, a mapping of $\mathcal{K}$ into $\mathcal{K}$, is a contraction when $c = k_1(k_2)^{-1}$:

$$\begin{aligned}\|\tilde{\mathbf{A}}f - \tilde{\mathbf{A}}g\|^2 &= \|f - g - c\mathbf{A}f + c\mathbf{A}g\|^2 \\ &= \|f - g\|^2 - 2c\,\mathrm{Re}(\mathbf{A}f - \mathbf{A}g, f - g) + c^2\|\mathbf{A}f - \mathbf{A}g\|^2 \\ &\leqq (1 - 2ck_1 + c^2k_2)\|f - g\|^2, \qquad c > 0.\end{aligned}$$

Since $(1 - 2ck_1 + c^2k_2) \geqq 0$ for all $c > 0$, it follows that† $k_1^2 \leqq k_2$. Hence

$$\|\tilde{\mathbf{A}}f - \tilde{\mathbf{A}}g\|^2 \leqq \left(1 - \frac{k_1^2}{k_2}\right)\|f - g\|^2, \qquad 0 \leqq \left(1 - \frac{k_1^2}{k_2}\right) < 1.$$

The last inequality stated in the theorem follows from an application of the Schwarz inequality. For all $f,g \in \mathcal{K}$

$$\|\mathbf{A}f - \mathbf{A}g\| \cdot \|f - g\| \geqq |(\mathbf{A}f - \mathbf{A}g, f - g)| \geqq k_1\|f - g\|^2.$$

Thus

$$\|\mathbf{A}f - \mathbf{A}g\| \geqq k_1\|f - g\|.$$

In particular, with $f = \mathbf{A}^{-1}h_1$ and $g = \mathbf{A}^{-1}h_2$,

$$\|h_1 - h_2\| \geqq k_1\|\mathbf{A}^{-1}h_1 - \mathbf{A}^{-1}h_2\|.$$

3.1 *Uniqueness Theorem*

We show here that the uniqueness property of solutions to equations of the type considered in Theorem I is implied by much weaker hypotheses than those stated in the theorem.

Theorem II: Let $f,g \in \mathcal{K}$ and let $\mathbf{Q}$ be a mapping of $\mathcal{K}$ into $\mathcal{H}$ such that $(\mathbf{Q}f - \mathbf{Q}g, f - g)$ vanishes only if $f = g$. Then if the equation $h = \mathbf{PQ}z$ has a solution $z \in \mathcal{K}$, it is unique.

Proof:

Assume that $\mathbf{PQ}z_1 = \mathbf{PQ}z_2$ where $z_1, z_2 \in \mathcal{K}$. Since $\mathbf{P}$ is self-adjoint,

$$\begin{aligned}(\mathbf{Q}z_1 - \mathbf{Q}z_2, z_1 - z_2) &= (\mathbf{Q}z_1 - \mathbf{Q}z_2, \mathbf{P}z_1 - \mathbf{P}z_2) \\ &= (\mathbf{PQ}z_1 - \mathbf{PQ}z_2, z_1 - z_2) \\ &= 0.\end{aligned}$$

Hence $z_1 = z_2$.

Theorem II is a generalization of the uniqueness theorem due to A. Beurling.[1,3]

† Alternatively, the hypotheses and an application of the Schwarz inequality yields:

$$k_1^2\|f - g\|^4 \leqq |(\mathbf{A}f - \mathbf{A}g, f - g)|^2 \leqq \|\mathbf{A}f - \mathbf{A}g\|^2 \cdot \|f - g\|^2 \leqq k_2\|f - g\|^4.$$

IV. APPLICATIONS

We present two theorems that have specific signal-theoretic interpretations.

Theorem III: Let $\mathcal{H} = \mathfrak{L}_2$ *and let*

$$\mathbf{L}f = \int_{-\infty}^{\infty} l(t - \tau)\, f(\tau)\, d\tau$$

where $l(t) \; \varepsilon \; \mathfrak{L}_2$ *and* $f \; \varepsilon \; \mathcal{K}$. *Suppose that*

$$\sup_{\omega} | L(\omega) | < \infty, \qquad Re\, L(\omega) \geqq -\alpha$$

where $\alpha < 1$. *Then for any* $h \; \varepsilon \; \mathcal{K}$, $h = f + \mathbf{PL}f$ *has a unique solution* $f \; \varepsilon \; \mathcal{K}$. *Suppose alternatively that*

$$\sup_{\omega} | L(\omega) | < \infty, \qquad Re\, L(\omega) > 0 \text{ a.e.}$$

Then **PL** *is a mapping of* $\mathcal{K}$ *into itself such that the equation*

$$h = \mathbf{PL}f, \quad h \; \varepsilon \; \mathcal{K}$$

possesses at most one solution $f \; \varepsilon \; \mathcal{K}$.

Proof:

Let $\mathbf{Q} = \mathbf{I} + \mathbf{L}$ and let $z \; \varepsilon \; \mathcal{K}$. Using the Plancherel identity

$$\begin{aligned} \mathrm{Re}(\mathbf{Q}z,z) &= \| z \|^2 + \mathrm{Re}(\mathbf{L}z,z) \\ &= \| z \|^2 + \frac{1}{2\pi} \mathrm{Re} \int_{-\infty}^{\infty} L(\omega) \,| Z(\omega) |^2 \, d\omega \\ &\geqq (1 - \alpha) \| z \|^2 . \end{aligned}$$

Also,

$$\begin{aligned} \| \mathbf{PQ}z \|^2 &\leqq \| z \|^2 + 2\, \mathrm{Re}(\mathbf{L}z,z) + \| \mathbf{L}z \|^2 \\ &\leqq (1 + 2\delta + \delta^2) \| z \|^2 \end{aligned}$$

where $\delta = \sup_{\omega} | L(\omega) |$. Hence the hypotheses of Theorem I are satisfied. This establishes the first part of Theorem III. The second part is a direct application of Theorem II since,† in view of the Plancherel identity, it is clear that here $\mathrm{Re}(\mathbf{L}z,z)$ vanishes only if $z = 0$.

If $\mathcal{K} = \mathfrak{D}(\Sigma)$, Theorem III implies that under either of the stated conditions only a knowledge of the output for $t \; \varepsilon \; \Sigma$ of a known linear filter is necessary to completely determine the input to the filter, if it is known that the input vanished for $t \not\varepsilon \; \Sigma$. In addition, if $h(t)$ is any ele-

† The boundedness of $| L(\omega) |$ is required in order that $\mathbf{L}f \; \varepsilon \; \mathfrak{L}_2$ whenever $f \; \varepsilon \; \mathcal{K}$.

ment of $\mathfrak{D}(\Sigma)$, there exists in the first case a unique input signal in $\mathfrak{D}(\Sigma)$ such that the projection of the output signal is $h(t)$, and this input signal, which can be computed in accordance with Theorem I, depends continuously on $h(t)$. Some related results are discussed in the Appendix.

Definition I: It is assumed throughout that $\varphi(x) = \varphi(x,t)$ *is a real-valued function of the real variables* x *and* t.

Theorem IV: Let $\mathfrak{H} = \mathfrak{K} \dotplus \mathfrak{K}'$ *be a real Hilbert space in which* $| f(t) | \geqq | g(t) |$ *for all* t *implies that* $\| f \| \geqq \| g \|$ *whenever* $f,g \; \varepsilon \; \mathfrak{H}$. *Let* $\varphi(x,t)$ *satisfy*

$$m(x - y) \leqq \varphi(x,t) - \varphi(y,t) \leqq M(x - y) \text{ when } x \geqq y$$

where m *and* M *are positive constants. Let* $\varphi[f] \; \varepsilon \; \mathfrak{H}$, $f \; \varepsilon \; \mathfrak{H}$. *Then for any* $u(t) \; \varepsilon \; \mathfrak{K}$, $v(t) \; \varepsilon \; \mathfrak{K}'$, *there exists a unique* $w(t) \; \varepsilon \; \mathfrak{K}$ *such that*

$$\mathbf{P}\varphi[w(t) + v(t)] = u(t).$$

In fact, $w(t) = \lim_{n \to \infty} w_n$ *where*

$$w_{n+1} = \frac{m}{M^2} \{u - \mathbf{P}\varphi[v + w_n]\} + w_n$$

and w_0 *is an arbitrary element of* $\mathfrak{K}$. *In addition,*

$$\| \mathbf{P}\varphi[v + f] - \mathbf{P}\varphi[v + g] \| \geqq m \| f - g \| ; \qquad f,g \; \varepsilon \; \mathfrak{K}, \; v \; \varepsilon \; \mathfrak{K}'$$

and if $\mathbf{P}\varphi[v_a + w_a] = u_a$, $\mathbf{P}\varphi[v_b + w_b] = u_b$ *where* $w_a, w_b, u_a, u_b \; \varepsilon \; \mathfrak{K}$ *and* $v_a, v_b \; \varepsilon \; \mathfrak{K}'$,

$$\| w_a - w_b \| \leqq \frac{1}{m} \| u_a - u_b \| + \frac{M}{m} \| v_a - v_b \|$$

$$\| u_a - u_b \| \leqq M \| v_a - v_b \| + M \| w_a - w_b \|.$$

Proof:

We first show that the hypotheses of Theorem I are satisfied when $\mathbf{Q}$ is defined by $\mathbf{Q}w = \varphi[w + v]$. Let $\hat{\eta} = (\eta - m)$ where

$$\frac{\varphi[v + f] - \varphi[v + g]}{f - g} = \eta; \qquad f,g \; \varepsilon \; \mathfrak{K}.$$

Observe that an application of a well known identity yields (with $z = f - g$):

$$\begin{aligned} (\eta z,z) - m(z,z) &= (\hat{\eta} z,z) \\ &= \tfrac{1}{4} \| (\hat{\eta} + 1)z \|^2 - \tfrac{1}{4} \| (\hat{\eta} - 1)z \|^2 \\ &\geqq 0. \end{aligned}$$

Hence $(\varphi[v + f] - \varphi[v + g], f - g) \geqq m \| f - g \|^2$. Since, in addition,

$$\| \mathbf{P}\varphi[v + f] - \mathbf{P}\varphi[v + g] \| \leqq \| \varphi[v + f] - \varphi[v + g] \|$$
$$\leqq M \| f - g \|,$$

the hypotheses are satisfied. The bound on $\| w_a - w_b \|$ is obtained from the inequality:

$$\| w_a - w_b \| \leqq \frac{1}{m} \| \mathbf{P}\varphi[w_a + v_a] - \mathbf{P}\varphi[w_b + v_a] \|.$$

Specifically, the right-hand side is equal to

$$\frac{1}{m} \| \mathbf{P}\varphi[w_a + v_a] - \mathbf{P}\varphi[w_b + v_b] + \mathbf{P}\varphi[w_b + v_b] - \mathbf{P}\varphi[w_b + v_a] \|$$
$$\leqq \frac{1}{m} \| u_a - u_b \| + \frac{1}{m} \| \mathbf{P}\varphi[w_b + v_b] - \mathbf{P}\varphi[w_b + v_a] \|$$
$$\leqq \frac{1}{m} \| u_a - u_b \| + \frac{M}{m} \| v_a - v_b \|.$$

With $\mathcal{H} = \mathcal{L}_{2R}$ and $\mathcal{K} = \mathcal{B}(\Omega)$, Theorem IV implies that if a function of time $w(t)$ having frequency components which vanish outside Ω is added to a second function $v(t)$ with frequency components which vanish inside Ω, and if the result is applied to a quite general type of time-variable nonlinear amplifier in cascade with an ideal linear filter having only passbands coincident with the intervals contained in Ω, then the output is sufficient to uniquely determine the signal $w(t)$, assuming of course that $v(t)$, Ω, and the function $\varphi(x,t)$ are known. Furthermore, for each signal $v(t) \varepsilon \mathcal{K}'$, there exists a unique input $w(t) \varepsilon \mathcal{K}$ such that the output is any prescribed element of $\mathcal{K}$. In particular, $w(t)$ depends continuously on the prescribed output and $v(t)$.

If $\mathcal{H}$ is the usual space of real-valued periodic functions of t, and $\varphi(x,t)$ is similarly periodic in t, the theorem possesses a similar interpretation. Of course, all of the results are valid for the interesting special case in which $\varphi(x,t) = x\varphi(1,t)$ (i.e., when the physical operation corresponding to this function is product modulation).

The inequality: $\| \mathbf{P}\varphi[v + f] - \mathbf{P}\varphi[v + g] \| \geqq m \| f - g \|$ in the conclusion of Theorem IV is quite interesting from an engineering viewpoint. For example, let $\mathcal{H} = \mathcal{L}_{2R}$, $\mathcal{K} = \mathcal{B}(\Omega)$, and suppose that $f \varepsilon \mathcal{B}(\Omega)$ is the input to a time-variable nonlinear amplifier with transfer characteristic $\varphi(x,t)$ which satisfies the assumptions stated and for simplicity $\varphi(0,t) = 0$. Then $\| \mathbf{P}\varphi[f] \| \geqq m \| f \|$, a *lower bound* on that part of the energy of the output signal which is associated with the frequency bands occupied by the input signal.

Remark: It can be shown that Theorem IV remains valid if the words "Hilbert space" are replaced with "Banach space" (and $\mathcal{K}$ denotes an arbitrary subspace of the Banach space with $\mathbf{P}$ the corresponding projection operator). In particular, the existence and uniqueness of the function $w(t)$ follows from an application of the contraction-mapping fixed-point theorem to the equation $w = (\mathbf{P} - c\mathbf{PQ})w + cu$ in which $\mathbf{Q}$ is defined by $\mathbf{Q}w = \varphi[w + v]$ and c is a real constant. Using the fact that $\| \mathbf{P} \| \leqq 1$, it is not difficult to show that there exists a c for which $(\mathbf{P} - c\mathbf{PQ})$ is a contraction.

V. SOME SPECIAL RESULTS

In this section we present some results that contribute to a deeper understanding of the character of the material already described. We shall be concerned throughout with the space $\mathcal{L}_2$.

In the proof of Theorem IV the hypotheses concerning $\varphi(x,t)$ is used to establish the applicability of Theorem I. The following theorem asserts that, for this purpose, the hypotheses can be relaxed somewhat if $\mathcal{H} = \mathcal{L}_{2R}$ and $\mathcal{K} = \mathcal{B}$, where $\mathcal{B}$ denotes $\mathcal{B}(\Omega)$ when Ω is a single fixed finite interval centered at the origin. The orthogonal complement of $\mathcal{B}$ is denoted by $\mathcal{B}^*$.

Theorem V: Let $\mathcal{H} = \mathcal{L}_{2R}$ *and* $\mathcal{K} = \mathcal{B}$. *Let* $f \varepsilon \mathcal{B}$, $v \varepsilon \mathcal{B}^*$. *The operator* $\mathbf{Q}$ *defined by* $\mathbf{Q}f = \varphi[f + v]$ *satisfies the hypotheses of Theorem I assuming that*

$$m(x - y) \leqq \varphi(x,t) - \varphi(y,t) \leqq M(x - y) \text{ when } x \geqq y$$

for all $t \varepsilon \Pi$, *where* m *and* M *are positive constants,* Π *is a subset of the real line, and*

$$\delta < \frac{m[1 - \lambda(\Pi)]}{\lambda(\Pi)}$$

in which

$$\delta = \sup_{\substack{t \varepsilon \Pi \\ x,y}} \left| \frac{\varphi(x,t) - \varphi(y,t)}{x - y} \right|$$

and

$$\lambda(\Pi) = \sup_{f \varepsilon \mathcal{B}} \frac{\int_\Pi |f|^2 \, dt}{\|f\|^2} .$$

Proof:

Clearly, $\| \mathbf{P}\varphi[f + v] - \mathbf{P}\varphi[g + v] \| \leqq \| \varphi[f + v] - \varphi[g + v] \| \leqq \max(\delta,M) \| f - g \|$.

Let Π^* be the complement of Π with respect to the real line. Observe that

$$(\varphi[f + v] - \varphi[g + v], f - g) = \int_{\Pi^*} (\varphi[f + v] - \varphi[g + v])(f - g)\, dt$$

$$+ m \int_{\Pi} (f - g)^2\, dt + \int_{\Pi} (\varphi[f + v] - \varphi[g + v])(f - g)\, dt$$

$$- m \int_{\Pi} (f - g)^2\, dt \geqq m \int_{-\infty}^{\infty} (f - g)^2 - (m + \delta) \int_{\Pi} (f - g)^2\, dt$$

$$\geqq [m - (m + \delta)\lambda(\Pi)]\, \| f - g \|^2.$$

When Π is any set of finite measure, $\lambda(\Pi)$ is less than unity.†

At this point it is convenient to introduce

Definition II: An operator $\mathbf{A}$ *defined on a Banach space is said to be bounded if there exists a constant k such that* $\| \mathbf{A}f - \mathbf{A}g \| \leqq k \| f - g \|$ *for all* f,g *in the domain of* $\mathbf{A}$.

This definition obviously reduces to the usual one in the event that $\mathbf{A}$ is a linear operator. From the viewpoint of implementing a signal recovery scheme (i.e., of constructing a device that reverses the effect of some known operator), it is highly desirable that the inverse operator be known to be bounded, since this situation guarantees that an error in the input signal to the recovery device would produce at most a proportional error in the recovered signal, assuming that the device functions as an ideal realization of the inverse operator. We shall consider the existence of two situations in which a mapping of the type considered earlier does not possess a bounded inverse.

Theorem VI: Let $m(x - y) \leqq \varphi(x,t) - \varphi(y,t) \leqq M(x - y)$ *for* $x \geqq y$ *when* $t \varepsilon \Pi$ *and* $\varphi(x,t) = 0$ *when* $t \not\varepsilon \Pi$, *where m and M are positive constants and* Π *is a set of finite measure. Let* $\mathbf{A}$ *be the mapping of* $\mathfrak{B}$ *into* $\mathfrak{B}$ *defined by* $\mathbf{A}f = \mathbf{P}\varphi[f]$, $f \varepsilon \mathfrak{B}$. *Then* $\mathbf{A}$ *does not possess a bounded inverse.*

Proof:

If $\mathbf{A}^{-1}$ existed and satisfied $\| \mathbf{A}^{-1}f - \mathbf{A}^{-1}g \| \leqq k \| f - g \|$ for all $f,g \varepsilon \mathfrak{B}$, it would follow that $\| \mathbf{A}f - \mathbf{A}g \| \geqq (k)^{-1} \| f - g \|$. However, since for any $\epsilon > 0$ there exists a $z \varepsilon \mathfrak{B}$ such that

† This is proved in Ref. 5 for the case in which Π is a single interval. H. J. Landau has pointed out to the writer in a private conversation that the published argument can be extended to apply to an arbitrary set of finite measure.

$$\|z\| = 1 \qquad \text{and} \qquad \int_{\Pi} z^2 \, dt < \epsilon,$$

the following calculation shows that the inequality cannot hold for any finite k:

$$\|\mathbf{P}\varphi[f] - \mathbf{P}\varphi[g]\|^2 \leqq \|\varphi[f] - \varphi[g]\|^2 = \int_{\Pi} (\varphi[f] - \varphi[g])^2 \, dt$$
$$\leqq M^2 \int_{\Pi} (f - g)^2 \, dt.$$

Recall that the mapping described in Theorem IV possesses a bounded inverse and that $\varphi(x,t)$ is assumed to satisfy the Lipschitz condition: $m(x - y) \leqq \varphi(x,t) - \varphi(y,t)$ when $x \geqq y$, where m is a positive constant. The assumption that m does not vanish is essential; the result is obviously not valid if $\varphi(x)$ vanishes throughout a neighborhood of the origin of the x-axis for all t. The following theorem focuses attention on some restrictions imposed on the derivative of $\varphi(x)$ by the requirement that the mapping possess a bounded inverse.

Theorem VII: Let $\varphi(x,t)$ be independent of t and continuously differentiable with respect to x on the interval Ξ. Let $|\varphi(x,t) - \varphi(y,t)| \leqq M|x - y|$ and

$$\inf_{x \varepsilon \Xi} \left| \frac{d\varphi(x)}{dx} \right| = 0.$$

Then the mapping $\mathbf{A}$, of $\mathcal{B}$ into $\mathcal{B}$, defined by $\mathbf{A}f = \mathbf{P}\varphi[f]$, $f \varepsilon \mathcal{B}$ does not possess a bounded inverse.

Proof:

As in the proof of Theorem VI it suffices to show that for any $\epsilon > 0$ there exist functions $f,g \varepsilon \mathcal{B}$ such that $\|f - g\| = 1$ and $\|\mathbf{P}\varphi[f] - \mathbf{P}\varphi[g]\| < \epsilon$. We need the following result.

Lemma I: Let τ and ϵ be positive constants and let k be a real number. Then there exists a function $g \varepsilon \mathcal{B}$ such that

$$|g(t) - k| < \epsilon, \qquad |t| < \tau.$$

The proof of the lemma is very simple. Let $\hat{g}(t) \varepsilon \mathcal{B}$ such that $\hat{g}(0) \neq 0$. Since $\hat{g}(t)$ is continuous, $|a\hat{g}(t) - k| < \epsilon$, $|t| < b\tau$ for some constants a and b where $b > 0$. If $b < 1$, set $g(t) = a\hat{g}(bt)$. This proves the lemma.

From the hypotheses there exists for any $\epsilon_1 > 0$ an $x_0 \varepsilon \Xi$ such that

$$\left| \frac{d\varphi(x)}{dx} \right| < \epsilon_1, \qquad |x - x_0| < \delta_1$$

where δ_1 is a positive constant that depends on ϵ_1. Choose† h such that‡ $h \varepsilon \mathcal{B}$, $\| h \| = 1$, and $| h(t) | < \frac{1}{2}\delta_1$; and then, for any $\epsilon_2 > 0$, determine T such that

$$\int_{|t|>T} h^2 \, dt \leqq \epsilon_2^2 .$$

Through Lemma I, choose $g \varepsilon \mathcal{B}$ such that $| g - x_0 | < \frac{1}{2}\delta_1$ when $| t | < T$, and set $f = g + h$. Observe that $\| \mathbf{P}\varphi[f] - \mathbf{P}\varphi[g] \|^2 \leqq \| \varphi[f] - \varphi[g] \|^2$ and that the right-hand side is equal to

$$\int_{|t| \leqq T} \{\varphi[g + h] - \varphi[g]\}^2 \, dt + \int_{|t|>T} \{\varphi[g + h] - \varphi[g]\}^2 \, dt$$

$$\leqq \sup_{|t| \leqq T} \left| \frac{\varphi[g + h] - \varphi[g]}{h} \right|^2 \int_{|t| \leqq T} h^2 \, dt$$

$$+ M^2 \int_{|t|>T} h^2 \, dt \leqq \epsilon_1^2 + M^2 \epsilon_2^2 .$$

Since ϵ_1^2 and ϵ_2^2 are arbitrary positive constants, our proof is complete. *Remark:* The proof can easily be extended to cover some situations in which the variation of $\varphi(x,t)$ with t plays an important role. One such situation is that in which

$$\left. \frac{\partial \varphi(x,t)}{\partial x} \right|_{x=\zeta(t)} = 0$$

where $\zeta(t)$ is continuous for all finite t and $\partial\varphi/\partial x$ is uniformly continuous in a neighborhood of the curve $x = \zeta(t)$.

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APPENDIX

Some Results Related to the Previously Mentioned Application of the First Part of Theorem III

Suppose that $\mathbf{L}$ is redefined by

$$\mathbf{L}f = \int_{\Sigma} l(t,\tau) \, f(\tau) \, d\tau, \quad f \varepsilon \mathfrak{D}(\Sigma)$$

† The writer is indebted to H. J. Landau for suggesting this approach.

‡ The function $(\sin kt)/\sqrt{k\pi}\, t$ satisfies the unit norm condition and for sufficiently small k satisfies the other two requirements.

where

$$\int_\Sigma \int_\Sigma | l(t,\tau) |^2 \, d\tau \, dt < 1.$$

Then **PL** is a mapping of $\mathfrak{D}(\Sigma)$ into itself such that for any $h \varepsilon \mathfrak{D}(\Sigma)$, the equation $h = f + \mathbf{PL}f$ possesses a unique solution $f \varepsilon \mathfrak{D}(\Sigma)$. The proof of this result follows from Theorem I, a two-fold application of the Schwarz inequality which shows that

$$| \mathrm{Re}(\mathbf{L}z,z) | \leqq || z ||^2 \int_\Sigma \int_\Sigma | l(t,\tau) |^2 \, d\tau \, dt$$

for all $z \varepsilon \mathfrak{D}(\Sigma)$, and a similar calculation using the Schwarz inequality which establishes that $\mathbf{L}z \varepsilon \mathfrak{L}_2$ whenever $z \varepsilon \mathfrak{D}(\Sigma)$ and that there exists a constant k such that $\| \mathbf{P}(\mathbf{I} + \mathbf{L})z \| \leqq k \| z \|$ for all $z \varepsilon \mathfrak{D}(\Sigma)$.

The result mentioned above can be obtained also from a direct consideration of the pertinent Fredholm integral equation:[6]

$$h(t) = f(t) + \int_\Sigma e(t) \, l(t,\tau) \, f(\tau) \, d\tau, \tag{1}$$

where

$$\begin{aligned} e(t) &= 1, \qquad t \,\varepsilon\, \Sigma, \\ &= 0, \qquad t \not\varepsilon\, \Sigma. \end{aligned}$$

In addition when

$$l(t,\tau) = 0, \qquad t < \tau$$

[i.e., when $l(t,\tau)$ is a Volterra kernel], it is known[6] that (1) possesses a solution f if

$$\sup_{t,\tau} | l(t,\tau) | < \infty, \quad \int_\Sigma | h(t) | \, dt < \infty,$$

and Σ is a bounded set.

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