Delay Distributions for One Line with Poisson Input, General Holding Times, and Various Orders of Service

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At a telephone exchange, calls appear before a single trunk line in accordance with a Poisson process of density λ. If the trunk line is busy, calls are delayed. The call holding times are identically distributed, mutually independent, positive random variables with distribution function $H(x)$. In this paper the distribution function of the delay and its moments are given for a stationary process and for three orders of service: (i) order of arrival, (ii) random order, and (iii) reverse order of arrival.

I. INTRODUCTION

Let us suppose that in the time interval $(0, \infty)$ calls appear before a single trunk line at times $\tau_1, \tau_2, \cdots, \tau_n, \cdots$ where the interarrival times $\tau_n - \tau_{n-1}$ ($n = 1, 2, \cdots; \tau_0 = 0$) are identically distributed, mutually independent random variables with the distribution function

$$F(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{1}$$

that is, the input is a Poisson process of density λ. If an incoming call finds the line free, a connection is realized instantaneously. If the line is busy, the call is delayed and waits for service as long as necessary (no defection). The holding times are identically distributed, mutually independent, positive random variables with distribution function $H(x)$ and independent of the input process. Such a service system can be characterized by the symbol $[F(x),H(x),1]$ provided that the order of service is specified. In this paper three orders of service are considered: (*i*) *order of arrival* (first come-first served), (*ii*) *random order* (every waiting call, independently of the others, and of its past delay, has the same probability of being chosen for service), and (*iii*) *reverse order of arrival* (last come-first served).

We are interested in finding the distribution function of the delay for a stationary process and for the three orders of service. We shall prove that if $\lambda\alpha < 1$, where α is the average holding time, then there is a unique stationary process.

Throughout this paper we shall use the notation

$$\Psi(s) = \int_0^\infty e^{-sx}\, dH(x) \qquad (\Re(s) \geqq 0) \tag{2}$$

and

$$\alpha_k = \int_0^\infty x^k\, dH(x) \qquad (k = 0, 1, 2, \cdots). \tag{3}$$

In particular, $\alpha_1 = \alpha$ is the average holding time.

II. THE STATIONARY PROCESS

Let us denote by ξ_n the queue size at time $t = \tau_n - 0$, i.e., the nth incoming call finds ξ_n calls (either waiting or being served) in the system. Denote by χ_n the time needed to complete the current service (if any) at time $t = \tau_n - 0$. If $\xi_n = 0$ then $\chi_n = 0$. The vector sequence

$$(\xi_n, \chi_n), \qquad n = 1, 2, \cdots,$$

is a Markovian stochastic sequence and has the same stochastic behavior for each order of service. We shall prove that if $\lambda\alpha < 1$ then there exists a unique stationary distribution, whereas if $\lambda\alpha \geqq 1$ then a stationary distribution does not exist. If $\lambda\alpha < 1$ and (ξ_1, χ_1) has the stationary distribution, then every (ξ_n, χ_n) has the same distribution as the initial distribution. For the stationary process, let us introduce the following notation

$$\mathbf{P}\{\xi_n = j\} = P_j \qquad (j = 0, 1, \cdots) \tag{4}$$

$$\mathbf{P}\{\chi_n \leqq x, \xi_n = j\} = P_j(x) \qquad (x \geqq 0, j = 1, 2, \cdots) \tag{5}$$

and

$$\Pi_j(s) = \int_0^\infty e^{-sx}\, dP_j(x) \qquad (\Re(s) \geqq 0, j = 1, 2, \cdots). \tag{6}$$

We shall prove the following theorem, due to D. M. G. Wishart:[1]

Theorem 1: If $\lambda\alpha < 1$, then the stochastic sequence (ξ_n, χ_n), $n = 1, 2, \cdots$, has a unique stationary distribution which is given by $P_0 = 1 - \lambda\alpha$ and

$$U(s,z) = \sum_{j=1}^{\infty} \Pi_j(s)z^j = \frac{(1 - \lambda\alpha)\lambda z(1 - z)}{z - \Psi(\lambda(1 - z))} \left(\frac{\Psi(s) - \Psi(\lambda(1 - z))}{s - \lambda(1 - z)}\right) \tag{7}$$

for $\Re(s) \geqq 0$ *and* $|z| \leqq 1$.

Proof: If we express the distribution of (ξ_{n+1}, χ_{n+1}) with the aid of the distribution of (ξ_n, χ_n), and assume that both (ξ_{n+1}, χ_{n+1}) and (ξ_n, χ_n) have the same stationary distribution, and if we form Laplace-Stieltjes transforms, then we obtain that P_0 and $\Pi_j(s)$ $(j = 1, 2, \cdots)$ must satisfy the following system of linear equations:

$$\begin{aligned} P_0 &= P_0\Psi(\lambda) + \sum_{k=1}^{\infty} \Pi_k(\lambda)[\Psi(\lambda)]^k, \\ \Pi_1(s) &= \frac{\lambda[\Psi(\lambda) - \Psi(s)]}{s - \lambda} \left\{P_0 + \sum_{k=1}^{\infty} \Pi_k(s)[\Psi(\lambda)]^{k-1}\right\}, \\ \Pi_j(s) &= \frac{\lambda[\Pi_{j-1}(\lambda) - \Pi_{j-1}(s)]}{s - \lambda} \\ &\quad + \frac{\lambda[\Psi(\lambda) - \Psi(s)]}{s - \lambda} \sum_{k=j}^{\infty} \Pi_k(s)[\Psi(\lambda)]^{k-j} \end{aligned} \tag{8}$$

for $j = 2, 3, \cdots$ and

$$P_0 + \sum_{j=1}^{\infty} \Pi_j(0) = 1. \tag{9}$$

To prove (8) we use the following two facts: First, the probability that during a holding time no call arrives is given by

$$\Psi(\lambda) = \int_0^{\infty} e^{-\lambda x}\, dH(x).$$

Second, let ρ and θ be mutually independent, positive random variables with distribution functions $\mathbf{P}\{\rho \leqq x\} = P(x)$, and $\mathbf{P}\{\theta \leqq x\} = F(x)$ defined by (1). Write $A = \{\theta < \rho\}$. Then

$$\mathbf{P}\{A\}\mathbf{E}\{e^{-s(\rho-\theta)} \mid A\} = \int_0^{\infty} \int_0^{x} e^{-(x-y)s-\lambda y}\, dy dP(x) = \frac{\lambda[\Pi(\lambda) - \Pi(s)]}{s - \lambda}$$

where

$$\Pi(s) = \int_0^{\infty} e^{-sx}\, dP(x).$$

Forming generating functions in (8) we obtain

$$[s - \lambda(1 - z)]U(s,z) = \lambda z U(\lambda,z) + \lambda z[\Psi(\lambda) - \Psi(s)] \left\{P_0 + \frac{U(\lambda,\Psi(\lambda)) - U(\lambda,z)}{\Psi(\lambda) - z}\right\}. \tag{10}$$

If $s = \lambda(1 - z)$ in (10) then we get

$$U(\lambda,z) + [\Psi(\lambda) - \Psi(\lambda(1 - z))] \left\{P_0 + \frac{U(\lambda,\Psi(\lambda)) - U(\lambda,z)}{\Psi(\lambda) - z}\right\} = 0. \tag{11}$$

The comparison of (10) and (11) gives

$$U(s,z) = \frac{\lambda z[\Psi(s) - \Psi(\lambda(1 - z))]}{[s - \lambda(1 - z)][\Psi(\lambda) - \Psi(\lambda(1 - z))]} U(\lambda,z). \tag{12}$$

By the first equation of (8), $U(\lambda,\Psi(\lambda)) = P_0[1 - \Psi(\lambda)]$, and if we put this into (11) we get

$$U(\lambda,z) = \frac{P_0(1 - z)[\Psi(\lambda) - \Psi(\lambda(1 - z))]}{z - \Psi(\lambda(1 - z))}. \tag{13}$$

Thus by (12) and (13)

$$U(s,z) = \frac{\lambda P_0 z(1 - z)[\Psi(s) - \Psi(\lambda(1 - z))]}{[z - \Psi(\lambda(1 - z))][s - \lambda(1 - z)]}. \tag{14}$$

Since by (9) $P_0 + U(0,1) = 1$, it follows from (14) that $P_0 = 1 - \lambda\alpha$. Thus if $\lambda\alpha \geqq 1$, then the assumption that a stationary distribution exists leads to a contradiction, i.e., a stationary distribution cannot exist if $\lambda\alpha \geqq 1$. If $\lambda\alpha < 1$, then there exists one and only one stationary distribution which is given by $P_0 = 1 - \lambda\alpha$ and by (14). This proves (7).

Remark. From (7) we obtain by inversion that for $x \geqq 0$

$$\sum_{j=1}^{\infty} P_j(x)z^j = \frac{(1 - \lambda\alpha)\lambda z(1 - z)}{z - \Psi(\lambda(1 - z))} \int_0^{\infty} e^{-\lambda(1-z)u}[H(u + x) - H(u)]\, du. \tag{15}$$

Hence for $x \geqq 0$

$$\sum_{j=1}^{\infty} P_j(x) = \lambda \int_0^x [1 - H(u)]\, du. \tag{16}$$

Accordingly if (ξ_n, χ_n), $n = 1, 2, \cdots$, is a stationary sequence, then we have for $x \geqq 0$ that

$$\mathbf{P}\{\chi_n \leqq x \mid \xi_n \geqq 1\} = \frac{1}{\alpha}\int_0^x [1 - H(u)]\, du, \tag{17}$$

i.e., if an incoming call finds the line busy, then the distribution function of the time needed to complete the current service is given by

$$H^*(x) = \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^x [1 - H(u)]\, du & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{18}$$

Finally we also remark that the stationary distribution of ξ_n, $n = 1, 2, \cdots$, is given by the following generating function

$$U(z) = \sum_{j=0}^{\infty} P_j z^j = \frac{(1 - \lambda\alpha)(1 - z)\Psi(\lambda(1 - z))}{\Psi(\lambda(1 - z)) - z}. \tag{19}$$

This follows from (7), because $U(z) = P_0 + U(0,z)$.

III. THE DISTRIBUTION FUNCTION OF THE LENGTH OF A BUSY PERIOD

A busy period is defined as a time interval during which the line is continuously busy. The stochastic law of a busy period is obviously independent of the order of service. Every busy period (except the initial one, if the line is busy at time $t = 0$) independently of the others has the same stochastic law. Denote by $G(x)$ the probability that the length of a busy period (other than the initial one, if any) is $\leqq x$ and define

$$\gamma(s) = \int_0^\infty e^{-sx}\, dG(x) \qquad (\Re(s) \geqq 0). \tag{20}$$

In Ref. 2 it is proved that $\gamma(s)$ is the root with smallest absolute value in z of the equation

$$z = \Psi(s + \lambda(1 - z)). \tag{21}$$

By Lagrange's expansion (cf. Ref. 3, p. 132) we obtain that

$$\gamma(s) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} \int_0^\infty e^{-(\lambda+s)x} x^{n-1}\, dH_n(x) \tag{22}$$

where $H_n(x)$ denotes the nth iterated convolution of $H(x)$ with itself. From (22) it follows by inversion that

$$G(x) = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}}{n!} \int_0^x e^{-\lambda u} u^{n-1}\, dH_n(u). \tag{23}$$

If $\lambda\alpha \leqq 1$ then $G(\infty) = 1$, whereas if $\lambda\alpha > 1$ then $G(\infty) < 1$.

In the case of $\lambda\alpha \leqq 1$ the rth moment of $G(x)$ is defined by

$$\Gamma_r = \int_0^\infty x^r \, dG(x) \qquad (r = 0, 1, 2, \cdots). \tag{24}$$

If $\lambda\alpha < 1$ and α_r is finite then $\Gamma_0, \Gamma_1, \cdots, \Gamma_r$ are also finite and we have $\Gamma_0 = 1$, $\Gamma_1 = \alpha/(1 - \lambda\alpha)$, and

$$\Gamma_{n+1} = \sum_{\nu=1}^{n} \frac{(n + \nu)! \, \lambda^{\nu-1}}{n! \, (1 - \lambda\alpha)^{n+\nu+1}} Y_{n,\nu} \tag{25}$$

for $n = 1, 2, \cdots$, where

$$Y_{n,\nu} = \sum_{\substack{j_1+j_2+\cdots+j_n=\nu \\ j_1+2j_2+\cdots+nj_n=n}} \frac{n! \, {\alpha_2}^{j_1} {\alpha_3}^{j_2} \cdots {\alpha_{n+1}}^{j_n}}{j_1! j_2! \cdots j_n! (2!)^{j_1} (3!)^{j_2} \cdots ((n + 1)!)^{j_n}}. \tag{26}$$

If, in particular, $H(x) = 1 - e^{-x/\alpha}$ $(x \geqq 0)$, then $\alpha_r = r!\alpha^r$ and

$$Y_{n,\nu} = \alpha^{n+\nu} \sum_{\substack{j_1+j_2+\cdots+j_n=\nu \\ j_1+2j_2+\cdots+nj_n=n}} \frac{n!}{j_1! j_2! \cdots j_n!} = \frac{n!}{\nu!} \binom{n-1}{\nu-1} \alpha^{n+\nu}.$$

Formula (26) can be proved as follows. If we define

$$u = s + \lambda[1 - \gamma(s)],$$

then by (21) $s = u - \lambda[1 - \Psi(u)]$, whence by Bürmann's theorem (cf. Appendix) for $n = 0, 1, \cdots$ we have

$$\left(\frac{d^{n+1}u}{ds^{n+1}}\right)_{s=0} = \left[\frac{d^n}{du^n}\left(\frac{u}{s}\right)^{n+1}\right]_{u=0} = \left[\frac{d^n}{du^n}\left(\frac{1}{1 - \lambda \dfrac{1 - \Psi(u)}{u}}\right)^{n+1}\right]_{u=0} \tag{27}$$

and the nth derivative can be calculated by using Faa di Bruno's formula (cf. Appendix). On the other hand

$$\left(\frac{du}{ds}\right)_{s=0} = 1 - \lambda\gamma'(0) = 1 + \lambda\Gamma_1, \tag{28}$$

and

$$\left(\frac{d^{n+1}u}{ds^{n+1}}\right)_{s=0} = -\lambda\gamma^{(n+1)}(0) = (-1)^n \lambda\Gamma_{n+1} \qquad (n = 1, 2, \cdots). \tag{29}$$

Comparing the above formulas we obtain Γ_n for every n.

Finally we remark that, by (25)

$$\Gamma_2 = \frac{\alpha_2}{(1 - \lambda\alpha)^3}, \tag{30}$$

$$\Gamma_3 = \frac{\alpha_3}{(1 - \lambda\alpha)^4} + \frac{3\lambda\alpha_2^2}{(1 - \lambda\alpha)^5}, \tag{31}$$

$$\Gamma_4 = \frac{\alpha_4}{(1 - \lambda\alpha)^5} + \frac{10\lambda\alpha_2\alpha_3}{(1 - \lambda\alpha)^6} + \frac{15\lambda^2\alpha_2^3}{(1 - \lambda\alpha)^7}. \tag{32}$$

IV. THE DISTRIBUTION FUNCTION OF THE DELAY

Let us denote by η_n the delay of the nth call. If the order of service is specified then the distribution function of η_n is uniquely determined by the distribution of (ξ_n, χ_n). If (ξ_n, χ_n), $n = 1, 2, \cdots$, is a stationary stochastic sequence, then η_n has the same distribution for every n. In the case of the stationary process write $\mathbf{P}\{\eta_n \leqq x\} = W(x)$ and

$$\mathbf{E}\{e^{-s\eta_n}\} = \Omega(s)$$

for each order of service. Define

$$W_n = \int_0^\infty x^n \, dW(x) \qquad (n = 0, 1, 2, \cdots). \tag{33}$$

In each case W_n is finite if α_{n+1} is finite. For each order of service

$$W_1 = \frac{\lambda\alpha_2}{2(1 - \lambda\alpha)}. \tag{34}$$

For service in order of arrival

$$W_2 = \frac{\lambda\alpha_3}{3(1 - \lambda\alpha)} + \frac{\lambda^2\alpha_2^2}{2(1 - \lambda\alpha)^2}, \tag{35}$$

for service in random order

$$W_2 = \frac{2\lambda\alpha_3}{3(1 - \lambda\alpha)(2 - \lambda\alpha)} + \frac{\lambda^2\alpha_2^2}{(1 - \lambda\alpha)^2(2 - \lambda\alpha)}, \tag{36}$$

and for service in reverse order of arrival

$$W_2 = \frac{\lambda\alpha_3}{3(1 - \lambda\alpha)^2} + \frac{\lambda^2\alpha_2^2}{2(1 - \lambda\alpha)^3}. \tag{37}$$

(*i*) *Service in Order of Arrival.* This case was first investigated by F. Pollaczek[4] and A. Y. Khintchine.[5] Cf. also D. V. Lindley.[6]

Theorem 2: If $\lambda\alpha < 1$, *if the process is stationary and if service is in order of arrival, then the distribution function of the delay of a call is given by*

$$W(x) = (1 - \lambda\alpha) \sum_{k=0}^{\infty} (\lambda\alpha)^k H_k^*(x) \tag{38}$$

where $H_k^*(x)$ denotes the k-th iterated convolution of

$$H^*(x) = \begin{cases} \dfrac{1}{\alpha}\displaystyle\int_0^x [1 - H(u)]\,du & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0, \end{cases} \tag{39}$$

with itself; $H_0^*(x) = 1$ if $x \geqq 0$ and $H_0^*(x) = 0$ if $x < 0$.

Proof: Evidently

$$W(x) = P_0 + \sum_{j=1}^{\infty} P_j(x)^*H_{j-1}(x)$$

where $H_j(x)$ $(j = 1, 2, \cdots)$ denotes the jth iterated convolution of $H(x)$ with itself; $H_0(x) = 1$ if $x \geqq 0$ and $H_0(x) = 0$ if $x < 0$. The symbol * denotes convolution. Hence

$$\Omega(s) = P_0 + \sum_{j=1}^{\infty} \Pi_j(s)[\Psi(s)]^{j-1} = P_0 + \frac{U(s,\Psi(s))}{\Psi(s)} \tag{41}$$

where $U(s,z)$ is defined by (7) and $P_0 = 1 - \lambda\alpha$. Thus

$$\Omega(s) = \frac{1 - \lambda\alpha}{1 - \lambda\dfrac{1 - \Psi(s)}{s}}, \tag{42}$$

whence (38) follows by inversion. Formula (38) was found by V. E. Beneš.[7]

If α_{n+1} is finite then W_n is also finite and is given by

$$W_n = \sum_{\nu=1}^{n} \frac{\lambda^\nu \nu!}{(1 - \lambda\alpha)^\nu} Y_{n,\nu} \tag{43}$$

where $Y_{n,\nu}$ is defined by (26). For,

$$W_n = (-1)^n \left(\frac{d^n\Omega(s)}{ds^n}\right)_{s=0} \qquad (n = 0, 1, \cdots) \tag{44}$$

and the nth derivative of $\Omega(s)$ can be obtained by Faa di Bruno's formula (cf. Appendix).

In this case W_1 is given by (34), W_2 by (35), and

$$W_3 = \frac{\lambda\alpha_4}{4(1 - \lambda\alpha)} + \frac{\lambda^2\alpha_2\alpha_3}{(1 - \lambda\alpha)^2} + \frac{3\lambda^3\alpha_2^3}{4(1 - \lambda\alpha)^3}. \tag{45}$$

Remark. Let $T(x) = W(x)^*H(x)$, i.e., $T(x)$ is the distribution function of the sum of the delay and the holding time of a call for a stationary process. Define

$$T_n = \int_0^\infty x^n \, dT(x) \qquad (n = 0, 1, 2, \cdots). \tag{46}$$

If α_{n+1} is finite then T_n is also finite and is given by $T_1 = W_1 + \alpha$ and

$$T_n = W_n + \frac{n}{\lambda} W_{n-1} \qquad (n = 2, 3, \cdots) \tag{47}$$

where W_n is defined by (43). Conversely, if we know T_j for $j = 1, 2, \cdots, n$ then we can obtain W_n by the following formula

$$W_n = \frac{(-1)^n \, n!}{\lambda^n} \left[\lambda\alpha + \sum_{j=1}^{n} \frac{(-1)^j \lambda^j T_j}{j!} \right] \qquad (n = 1, 2, \cdots). \tag{48}$$

Formulas (47) and (48) follow from the relationship

$$\int_0^\infty e^{-sx} \, dT(x) = \Omega(s)\Psi(s) = \frac{(1 - \lambda\alpha)s}{\lambda} + \left(1 - \frac{s}{\lambda}\right) \Omega(s). \tag{49}$$

Finally we also note that the rth binomial moment of the stationary distribution of the queue size, i.e., that of $\{P_j\}$ defined by (19), is given by

$$B_r = \sum_{j=r}^{\infty} \binom{j}{r} P_j = \frac{\lambda^r T_r}{r!} \qquad (r = 0, 1, \cdots). \tag{50}$$

For we can easily see that

$$P_j = \int_0^\infty e^{-\lambda x} \frac{(\lambda x)^j}{j!} \, dT(x), \tag{51}$$

whence

$$B_r = \int_0^\infty \frac{(\lambda x)^r}{r!} \, dT(x) = \frac{\lambda^r T_r}{r!}. \tag{52}$$

(*ii*) *Service in Random Order*. The case of exponentially distributed holding times was investigated by many authors (cf. Ref. 8), the case of constant holding time by P. J. Burke,[9] and the general case by J. F. Kingman.[10] The following theorem is due to J. F. Kingman.[10]

Theorem 3: If $\lambda\alpha < 1$, if the process is stationary and if service is in random order, then the distribution function of the delay of a call has the following Laplace-Stieltjes transform:

$$\Omega(s) = (1 - \lambda\alpha) \left\{ 1 + \frac{\lambda}{s} \int_{\gamma(s)}^{1} \exp\left[- \int_u^1 \frac{dv}{v - \Psi(s + \lambda(1 - v))} \right] \cdot \left[1 + \frac{u - 1}{u - \Psi(\lambda(1 - u))} \right] du \right\} \tag{53}$$

where $\gamma(s)$ *is the root with smallest absolute value in* z *of the equation*

$$z = \Psi(s + \lambda(1 - z)) \tag{54}$$

and is given by (22).

Proof: Under the condition that $j(j = 1, 2, \cdots)$ calls are waiting in the system when a service is about to start, denote by $W_j(x)$ the probability that the service of a given call among the j calls starts within time x if time is measured from this instant. Define

$$\Omega_j(s) = \int_0^\infty e^{-sx}\, dW_j(x) \qquad (\Re(s) \geqq 0). \tag{55}$$

The distribution functions $W_j(x)$ $(j = 1, 2, \cdots)$ can be obtained by using the following relationships: $W_1(x) = 1$ and

$$W_j(x) = \frac{1}{j} + \left(1 - \frac{1}{j}\right) \sum_{k=0}^{\infty} \left[\int_0^x e^{-\lambda u} \frac{(\lambda u)^k}{k!}\, dH(u)\right] * W_{j+k-1}(x) \tag{56}$$

for $j = 2, 3, \cdots$ and for $x \geqq 0$. To prove (56) we take into consideration that if the given call will be chosen for service among the j waiting calls, then its service starts immediately; if the given call is not chosen for service at this time then it must wait during the holding time of the call chosen for service, and if during this holding time k new calls arrive, then there is an additional delay which has the distribution function $W_{j+k-1}(x)$. Forming the Laplace-Stieltjes transform of (56) we obtain the following system of linear equations for the determination of $\Omega_j(s)$ $(j = 1, 2, \cdots)$: $\Omega_1(s) = 1$ and

$$j\Omega_j(s) = 1 + (j - 1) \sum_{k=0}^{\infty} \Omega_{j+k-1}(s) \int_0^\infty e^{-(\lambda+s)x} \frac{(\lambda x)^k}{k!}\, dH(x) \tag{57}$$

for $j = 2, 3, \cdots$. The solution of this system is given by J. F. Kingman[10] in the following form:

$$\Omega_j(s) = \int_{\gamma(s)}^1 \exp\left[-\int_u^1 \frac{dv}{v - \Psi(s + \lambda(1 - v))}\right] \cdot \frac{u^{j-1}}{u - \Psi(s + \lambda(1 - u))}\, du \tag{58}$$

where $\gamma(s)$ is defined by (22). By integrating by parts (58) can be written in the following equivalent form:

$$\Omega_j(s) = 1 - (j - 1) \int_{\gamma(s)}^1 \cdot \exp\left[-\int_u^1 \frac{dv}{v - \Psi(s + \lambda(1 - v))}\right] u^{j-2}\, du. \tag{59}$$

Now the distribution function of the delay is given by

$$W(x) = P_0 + \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \left[\int_0^x e^{-\lambda u} \frac{(\lambda u)^k}{k!} \, dP_j(u) \right] * W_{j+k}(x). \quad (60)$$

For, if a call arrives and finds the line free, then its service starts without delay. If an arriving call finds j ($j = 1, 2, \cdots$) calls in the system, then its delay is composed of the time needed to complete the current service, and if during this time k new calls arrive, then there is an additional delay that has the distribution function $W_{j+k}(x)$. Forming the Laplace-Stieltjes transform of (60) we get

$$\Omega(s) = P_0 + \sum_{j=1}^{\infty} \sum_{k=0}^{\infty} \left[\int_0^{\infty} e^{-(\lambda+s)x} \frac{(\lambda x)^k}{k!} \, dP_j(x) \right] \Omega_{j+k}(s). \quad (61)$$

Putting (58) into (61), we obtain

$$\Omega(s) = P_0 + \int_{\gamma(s)}^1 \exp\left[- \int_u^1 \frac{dv}{v - \Psi(s + \lambda(1 - v))} \right] \cdot \frac{U(s + \lambda(1 - u), u)}{u[u - \Psi(s + \lambda(1 - u))]} \, du \quad (62)$$

where $P_0 = 1 - \lambda\alpha$ and $U(s,z)$ is defined by (7). Thus

$$\Omega(s) = (1 - \lambda\alpha) \left\{ 1 + \frac{\lambda}{s} \int_{\gamma(s)}^1 \cdot \exp\left[- \int_u^1 \frac{dv}{v - \Psi(s + \lambda(1 - v))} \right] \cdot \left[\frac{u - 1}{u - \Psi(\lambda(1 - u))} - \frac{u - 1}{u - \Psi(s + \lambda(1 - u))} \right] du \right\}, \quad (63)$$

whence (53) follows by integrating by parts and by using the fact that $\gamma(s)$ satisfies (54) in z.

If α_{n+1} is finite, then W_n is also finite and can be expressed by

$$\int_0^{\infty} x^r \, dW_j(x) = (-1)^r \left(\frac{d^r \Omega_j(s)}{ds^r} \right)_{s=0} \quad (64)$$

for $r = 1, 2, \cdots, n$ and $j = 1, 2, \cdots$, and by

$$U_{jk} = (-1)^j \left(\frac{\partial^{j+k} \, U(s, z)}{\partial s^j \, \partial z^k} \right)_{s=0,\ z=1} \quad (65)$$

for $j + k \leqq n$. By using the following formulas we obtain (34) and (36):

$$\int_0^{\infty} x \, dW_j(x) = \frac{\alpha(j - 1)}{2 - \lambda\alpha}, \quad (66)$$

$$\int_0^\infty x^2\,dW_j(x) = \frac{2\alpha^2(j-1)(j-2)}{(2-\lambda\alpha)(3-2\lambda\alpha)} + \frac{(6-\lambda\alpha)\alpha_2(j-1)}{(2-\lambda\alpha)^2(3-2\lambda\alpha)}, \tag{67}$$

and further $U_{00} = \lambda\alpha$, $U_{10} = \lambda\alpha_2/2$, $U_{20} = \lambda\alpha_3/3$,

$$U_{01} = \lambda\alpha + \frac{\lambda^2\alpha_2}{2(1-\lambda\alpha)},$$

$$U_{11} = \frac{\lambda^2\alpha_3}{6} + \frac{\lambda\alpha_2}{2} + \frac{\lambda^3\alpha_2^2}{4(1-\lambda\alpha)},$$

and

$$U_{02} = \frac{\lambda^3\alpha_3 + 3\lambda^2\alpha_2}{3(1-\lambda\alpha)} + \frac{\lambda^4\alpha_2^2}{2(1-\lambda\alpha)^2}.$$

(iii) Service in Reverse Order of Arrival. The case of exponentially distributed service times was investigated by E. Vaulot[11] and the general case by J. Riordan[12] and D. M. G. Wishart.[13] Now we shall prove

Theorem 4: If $\lambda\alpha < 1$, if the process is stationary and if service is in reverse order of arrival, then the distribution function of the delay of a call for $x \geqq 0$ is given by

$$W(x) = (1-\lambda\alpha) + \lambda \sum_{j=1}^{\infty} e^{-\lambda x} \frac{(\lambda x)^{j-1}}{j!} \int_0^x [1 - H_j(u)]\,du \tag{68}$$

where $H_j(x)$ denotes the jth *iterated convolution of $H(x)$ with itself.*

Proof: Denote by $G(x)$ the probability that the length of a busy period in the queueing process considered is $\leqq x$. $G(x)$ is given by (23). Then we can write that

$$W(x) = P_0 + (1-P_0) \sum_{k=0}^{\infty} \left[\int_0^x e^{-\lambda u} \frac{(\lambda u)^k}{k!}\,dH^*(u) \right] * G_k(x) \tag{69}$$

where $G_k(x)$ denotes the kth iterated convolution of $G(x)$ with itself; $G_0(x) = 1$ if $x \geqq 0$ and $G_0(x) = 0$ if $x < 0$; $H^*(x)$ is defined by (18) and $P_0 = 1 - \lambda\alpha$. For, if an arriving call finds the line free, which has probability P_0, then its service starts without delay; if the line is busy, which has probability $1 - P_0$, then its delay is composed of the remaining holding time of the call being served, which has the distribution function $H^*(x)$, and if during this time interval k new calls join the queue, then there is an additional delay that has the same distribution function as the total length of k independent busy periods. Thus we obtain (69). Forming the Laplace-Stieltjes transform of (69) we obtain

$$\begin{aligned}\Omega(s) &= P_0 + \frac{(1 - P_0)}{\alpha} \sum_{k=0}^{\infty} [\gamma(s)]^k \int_0^{\infty} e^{-(\lambda+s)x} \frac{(\lambda x)^k}{k!} [1 - H(x)]\, dx \\ &= (1 - \lambda\alpha) + \lambda \frac{1 - \Psi(s + \lambda(1 - \gamma(s)))}{s + \lambda[1 - \gamma(s)]}\end{aligned} \tag{70}$$

where $\gamma(s)$ is defined by (22). Since $\gamma(s)$ satisfies (21) in z, we get from (70) that

$$\Omega(s) = (1 - \lambda\alpha) + \frac{\lambda[1 - \gamma(s)]}{s + \lambda[1 - \gamma(s)]}. \tag{71}$$

By using Lagrange's expansion (cf. Ref. 3, p. 132) we obtain

$$\begin{aligned}\Omega(s) = (1 - \lambda\alpha) &+ \frac{\lambda}{\lambda + s} \\ &+ s \sum_{j=1}^{\infty} \frac{(-1)^j \lambda^j}{j!} \frac{d^{j-1}}{ds^{j-1}} \left(\frac{[\Psi(s + \lambda)]^j}{(s + \lambda)^2} \right),\end{aligned} \tag{72}$$

whence (68) follows by inversion.

If α_{n+1} is finite then W_n is also finite and we have for $n = 2, 3, \cdots$ that

$$W_n = \sum_{\nu=1}^{n} \frac{(n - 2 + \nu)!\, \lambda^{\nu}}{(n - 1)!\, (1 - \lambda\alpha)^{n-1+\nu}} Y_{n,\nu} \tag{73}$$

where $Y_{n\nu}$ is defined by (26). If $n = 1$ then W_n is given by (34). For,

$$W_n = (-1)^n \left(\frac{d^n \Omega(s)}{ds^n} \right)_{s=0} \qquad (n = 0, 1, 2, \cdots). \tag{74}$$

If we use the notation $u = s + \lambda[1 - \gamma(s)]$ and $s = u - \lambda[1 - \Psi(u)]$, then we can write that

$$\Omega(s) = (1 - \lambda\alpha) + \lambda \frac{1 - \Psi(u)}{u}, \tag{75}$$

whence by using Bürmann's theorem (cf. Appendix) we obtain for $n = 2, 3, \cdots$ that

$$W_n = \frac{(-1)^{n-1}}{n - 1} \left(\frac{d^n}{du^n} \left[\frac{1}{1 - \lambda \dfrac{1 - \Psi(u)}{u}} \right]^{n-1} \right)_{u=0} \tag{76}$$

and the nth derivative can be calculated by Faa di Bruno's formula (cf. Appendix).

In this case W_1 is given by (34), W_2 by (37) and

$$W_3 = \frac{\lambda\alpha_4}{4(1 - \lambda\alpha)^3} + \frac{3\lambda^2\alpha_2\alpha_3}{2(1 - \lambda\alpha)^4} + \frac{3\lambda^3\alpha_2^3}{2(1 - \lambda\alpha)^5}. \tag{77}$$

Remark. If $P_0(t)$ denotes the probability that the line is free at time t given that it was free at time $t = 0$, then we can write that

$$W(x) = 1 - [P_0(x) - P_0(\infty)] \tag{78}$$

where $P_0(\infty) = 1 - \lambda\alpha$.

If $G^*(x)$ denotes the probability that the length of a busy period is $\leq x$ for the dual process $[H(x),F(x),1]$, i.e., when the interarrival times and holding times are interchanged, then we can write that

$$W(x) = 1 - [G^*(\infty) - G^*(x)] \tag{79}$$

where $G^*(\infty) = \lambda\alpha$.

(iv) An Extreme Case. Suppose that in the stationary process the service of a particularly chosen call starts when and only when no other calls are in the system, i.e., its service is delayed until it becomes the only call in the system. Denote by $W^*(x)$ the distribution function of the delay of this particular call.

Theorem 5: If $\lambda\alpha < 1$, if the process is stationary, and if a particularly chosen call will be served only when no other calls are in the system, then the distribution function of the delay of this call is given by $W^*(0) = 1 - \lambda\alpha$ and for $x > 0$

$$\frac{dW^*(x)}{dx} = (1 - \lambda\alpha)\lambda[1 - G(x)], \tag{80}$$

where $G(x)$ is the distribution function of the length of a busy period and is given by (23).

Proof: Denote by $G_n(x)$ the nth iterated convolution of $G(x)$ with itself. Now we have

$$W^*(x) = P_0 + \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\left[\int_0^x e^{-\lambda u}\frac{(\lambda u)^k}{k!}\,dP_j(u)\right] * G_{j+k}(x). \tag{81}$$

For, if the particularly chosen call finds the line to be free, then its service starts without delay; if it finds j calls ($j = 1, 2, \cdots$) in the system and if during the remaining part of the current service k ($k = 0, 1, \cdots$) more calls arrive, then its delay is composed of the remaining holding time of the call being served at its arrival and an additional delay which has the same distribution as the sum of $j + k$ mutually independent random variables each of which has the same distribution as a busy period.

Denote by $\Omega^*(s)$ the Laplace-Stieltjes transform of $W^*(x)$. By (81)

$$\begin{aligned}\Omega^*(s) &= P_0 + \sum_{j=1}^{\infty} \Pi_j\,(s+\lambda - \lambda\gamma(s))[\gamma(s)]^j \\ &= P_0 + U(s+\lambda-\lambda\gamma(s),\gamma(s)) = P_0\left[1 - \lambda\,\frac{1-\gamma(s)}{s}\right],\end{aligned} \tag{82}$$

where $P_0 = 1 - \lambda\alpha$ and $\gamma(s)$ is given by (22). We obtain (80) by inversion.

If α_{n+1} is finite then

$$W_n^* = \int_0^\infty x^n\,dW^*(x) \tag{83}$$

is also finite and is given by

$$W_n^* = \frac{\lambda(1-\lambda\alpha)\Gamma_{n+1}}{n+1} \qquad (n = 1, 2, \cdots) \tag{84}$$

where Γ_{n+1} is defined by (25). This follows immediately from (82). In particular we have

$$W_1^* = \frac{\lambda\alpha_2}{2(1-\lambda\alpha)^2}, \tag{85}$$

$$W_2^* = \frac{\lambda\alpha_3}{3(1-\lambda\alpha)^3} + \frac{\lambda^2\alpha_2^2}{(1-\lambda\alpha)^4}, \tag{86}$$

$$W_3^* = \frac{\lambda\alpha_4}{4(1-\lambda\alpha)^4} + \frac{5\lambda^2\alpha_2\alpha_3}{2(1-\lambda\alpha)^5} + \frac{15\lambda^3\alpha_2^3}{4(1-\lambda\alpha)^6}. \tag{87}$$

APPENDIX

A.1 *Bürmann's Theorem*

(Cf. Ref. 3, p. 128.) Suppose that the first N derivatives of $f(z)$ and the first $N-1$ derivatives of $g(z)$ exist at $z = 0$. If $s = u/g(u)$ and $g(0) \neq 0$, then

$$f(u) = f(0) + \sum_{n=1}^{N} \frac{s^n}{n!}\left(\frac{d^{n-1}f'(v)[g(v)]^n}{dv^{n-1}}\right)_{v=0} + o(s^N). \tag{88}$$

A.2 *Faa di Bruno's Formula*

(Cf. Ref. 14, p. 33.) If $z = f(y)$ where $y = g(x)$, then the nth derivative of $z = f(g(x))$ with respect to x at $x = 0$ is given by the following

TABLE I

n	ν	j_1	j_2	j_3	j_4	j_5	$C_{j_1,j_2,\cdots,j_n}$
1	1	1					1/2
2	1	0	1				1/3
2	2	2	0				1/4
3	1	0	0	1			1/4
3	2	1	1	0			1/2
3	3	3	0	0			1/8
4	1	0	0	0	1		1/5
4	2	1	0	1	0		1/2
4	2	0	2	0	0		1/3
4	3	2	1	0	0		1/2
4	4	4	0	0	0		1/16
5	1	0	0	0	0	1	1/6
5	2	1	0	0	1	0	1/2
5	2	0	1	1	0	0	5/6
5	3	2	0	1	0	0	5/8
5	3	1	2	0	0	0	5/6
5	4	3	1	0	0	0	5/6
5	5	5	0	0	0	0	1/32

formula

$$\left(\frac{d^n f(g(x))}{dx^n}\right)_{x=0} = \sum_{\nu=1}^{n} Y_{n,\nu} \left(\frac{d^\nu f(y)}{dy^\nu}\right)_{y=g(0)} \tag{89}$$

where

$$Y_{n,\nu} = \sum_{\substack{j_1+j_2+\cdots+j_n=\nu \\ j_1+2j_2+\cdots+nj_n=n}} \frac{n![g^{(1)}(0)]^{j_1}[g^{(2)}(0)]^{j_2} \cdots [g^{(n)}(0)]^{j_n}}{j_1!j_2! \cdots j_n!(1!)^{j_1}(2!)^{j_2} \cdots (n!)^{j_n}} \tag{90}$$

provided that the derivatives in question exist.

For $n \leqq 5$ Table I contains all the n-tuples $(j_1, j_2, \cdots, j_n)$ satisfying the requirements $j_1 + j_2 + \cdots + j_n = \nu$ and $j_1 + 2j_2 + \cdots + nj_n = n$, and in addition the coefficients

$$C_{j_1,j_2,\cdots,j_n} = \frac{n!}{j_1!j_2! \cdots j_n!(2!)^{j_1}(3!)^{j_2} \cdots ((n+1)!)^{j_n}}, \tag{91}$$

which we need in using formula (26).

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