A Further Discussion of Stimulated Emission of Bremsstrahlung

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In an earlier paper[1] we proved the existence of stimulated emission of bremsstrahlung for electrons moving in the vicinity of nuclei. However, no estimate of the available power was given at that time.

This paper extends the theory to the fourth order of perturbation theory, which allows one to estimate the available power from this process. We find that the available power increases proportional to the fifth power of the frequency, and that one might obtain power in the order of one watt at a frequency of 1000 gigacycles. The oscillation condition at these high frequencies is met by the passage of many slow electrons through a dense assembly of ions or atoms. Although these conditions are uncommon, stimulated emission of bremsstrahlung may play a role at microwave frequencies in very high current semiconductor experiments.

I. INTRODUCTION

In an earlier paper[1] we showed that stimulated emission of bremsstrahlung exists. This statement has the following meaning. Bremsstrahlung is the radiation which an electron emits by passing in the vicinity of a nucleus. We demonstrated that the emitted power into a specific mode of the radiation field is proportional to the energy density in that mode, which shows that we have indeed stimulated emission of radiation. It has to be expected that the emitted power can not be strictly proportional to the energy density, but that it must depend on the energy density in some nonlinear fashion. If this were not the case, the oscillation would not reach saturation and the energy density in the cavity of the oscillator would build up indefinitely. If we knew the nonlinear dependence of the emitted power on the energy density, we could predict the power output of a practical device.

Since it is very hard to find an exact solution of the problem, we will give an approximation by finding the next higher approximation of the perturbation theory, which will give us a term proportional to the square

of the energy density in addition to the already known linear term. This approximation will allow us to calculate the available power for electron currents which just barely exceed the current necessary to satisfy the oscillation condition. For higher electron currents this approximation will give not more than an order of magnitude estimate of the available power.

The result so obtained shows that the available power is proportional to f^5 (f = frequency) and is very low below 10 gigacycles (gc). However, the available power increases very rapidly, and a circuit designed to satisfy the oscillation condition at 1000 gc would deliver power in the order of one watt.

Since the use of a Coulomb potential appears to be an unnecessary restriction, we use a potential $V = e^{-\gamma r}/r$ which is an approximation to the potential of a neutral atom. The shielding effect of the electrons orbiting around the nucleus is taken into account by the factor $e^{-\gamma r}$. It turns out that stimulated emission will occur if $\hbar\gamma/mv \ll 1$ (with $2\pi\hbar = h$ = Planck's constant, m and v mass and velocity of the incident electron). This treatment neglects the interaction of the incident electron with the bound electrons in the atom. It is conceivable that additional emission or perhaps absorption of radiation may occur which is not included in this analysis. The bound electrons will certainly decelerate the free electron in flying through solid matter so that their presence poses additional serious problems.

Finally, in Section VII we discuss the advantages of using periodically distributed scattering centers. Periodic structures have the advantage that the emitted power becomes proportional to the square of the number of elements in the periodic arrangement. However, to utilize these at frequencies of 100 gc or higher requires the use of monoenergetic electrons, which are not easily available.

The utilization of stimulated emission of bremsstrahlung requires an arrangement which allows slow electrons to pass through dense assemblies of atoms or ions. This effect, most likely, plays a roll in semiconductor crystals to which high dc currents or current pulses are applied.

The following sections II to IV outline the fourth-order perturbation theory. Instead of the second quantized relativistic theory of the electron used by the authors of most textbooks, the problem is simplified by using a nonrelativistic approximation and first quantization only. The theory is presented with the purpose of showing the particular approximation used in deriving (34).

The reader who is not interested in the derivation of the theory may skip over sections II to IV and continue with section V. The expression

for the available power in section V is derived under the assumption that the incident electrons move parallel to the electric vector of the stimulating radiation field.

II. PERTURBATION THEORY

We will use quantum electrodynamics to derive the equation for the emitted power.

We simplify the problem by assuming that only one mode of the radiation field exists. The interaction of the electron with the photon vacuum will be neglected. This is justified as long as we are only interested in the stimulation effects and not in the spontaneous emission of bremsstrahlung. The state of the system including the electron and the radiation field will be described by a state function $\Phi(n,k)$; n designates the number of photons in the radiation mode while k refers to the electron propagation vector, which is related to the momentum $m\mathbf{v}$ of the electron by

$$\hbar\mathbf{k} = m\mathbf{v}. \tag{1}$$

The system is described by the Hamiltonian

$$H = H_0 + H_i\,. \tag{2}$$

H_0 is the Hamiltonian of the electromagnetic field plus the free electron. H_i is the interaction Hamiltonian, which is related to the interaction energy between the field and the electron.

According to Heitler[2]

$$H_i = H_1 + H_2 + V \tag{3a}$$

$$H_1 = -\frac{e}{mc}\,pA\,, \qquad H_2 = \frac{e^2}{2mc^2}\,A^2\,. \qquad V = -Ze^2\,\frac{e^{-\gamma r}}{r}\,. \tag{3b}$$

The meanings of the symbols used are explained in the list of symbols at the end of this paper, Section VIII.

The vector potential is given by [3]

$$A = \frac{2\sqrt{\pi}\,c}{\sqrt{L^3}}\,(qe^{i\beta z} + q^*e^{-i\beta z})\,. \tag{4}$$

It is assumed that the x-direction coincides with the direction of the incident electron and the direction of the vector $\mathbf{A}$, which means that the electrons are incident parallel to the electric vector of the radiation field. The symbols q and q^* are the absorption and emission operators, respectively.

We assume that box normalization is being used. That means that the propagation constant k of the electron wave can assume only the values

$$k = \frac{2\pi\mu}{L} \qquad (\mu = \text{integer}). \tag{5}$$

The electron wave function is given by

$$\psi = \frac{1}{\sqrt{L^3}} e^{i\mathbf{k}\cdot\mathbf{r}}. \tag{6}$$

Using the unperturbed wave functions of plane electron waves as zero-order approximation corresponds to the Born approximation. The state function Φ is the product of the photon state function and the electron wave function ψ. The time dependence of the state function is given by the Schroedinger equation.

$$i\hbar \frac{d\Phi}{dt} = (H_0 + H_i)\Phi. \tag{7}$$

It is more convenient to introduce interaction representation by making the transformation

$$\Phi' = \exp\left(\frac{i}{\hbar} H_0 t\right) \Phi \tag{8}$$

$$H_i' = \exp\left(\frac{i}{\hbar} H_0 t\right) H_i \exp\left(-\frac{i}{\hbar} H_0 t\right) \tag{9}$$

With these transformations (7) goes over into

$$i\hbar \frac{d\Phi'}{dt} = H_i'\Phi'. \tag{10}$$

The interaction energy is small compared to the energy of the noninteracting fields. We can therefore use an approximate iteration solution of (10). (We write henceforth H instead of H_i)

$$\Phi_F(t) = S\Phi_0(0) = (S_0 + S_1 + S_2 + \cdots)\Phi_0(0) \tag{11}$$

where

$$S_0 = 1$$

$$S_1 = \frac{1}{i\hbar}\int_0^t d\tau_1 H'(\tau_1) \qquad S_2 = \frac{1}{(i\hbar)^2}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 H'(\tau_1)H'(\tau_2)$$

$$S_3 = \frac{1}{(i\hbar)^3}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 H'(\tau_1)H'(\tau_2)H'(\tau_3)$$

$$S_4 = \frac{1}{(i\hbar)^4}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \int_0^{\tau_2} d\tau_3 \int_0^{\tau_3} d\tau_4 H'(\tau_1)H'(\tau_2)H'(\tau_3)H'(\tau_4).$$

The operator S is called the scattering matrix. The probability of finding a system, which at $t = 0$ is described by the wave function Φ_0, after time t in the final state described by Φ_F, is given by the absolute square of the matrix element between these two state functions:

$$P = |\,(\Phi_F^*, S\Phi_0)\,|^2. \tag{12}$$

In order to evaluate the matrix element of the S-matrix it is necessary to convert the operator products $H'(\tau_1)H'(\tau_2)$ into matrix products. This can be done with the help of a complete set of state functions which are assumed to be eigenfunctions Φ_r of the unperturbed Hamiltonian H_0 with the eigenvalues

$$E_r = \frac{\hbar^2}{2m} k_r^2 + n_r\hbar\omega. \tag{13}$$

Making use of

$$(\Phi_r^*, H'(\tau)\Phi_s) = \exp\left[\frac{i}{\hbar}(E_r - E_s)\tau\right](\Phi_r^*, H\Phi_s) \tag{14}$$

which follows from equation (9), we obtain, for example, for S_2

$$\begin{aligned}(\Phi_F^*, S_2\Phi_0) = \frac{1}{(i\hbar)^2}\sum_r (\Phi_F^*, H\Phi_r)(\Phi_r^*, H\Phi_0)\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \\ \cdot\exp\left[\frac{i}{\hbar}(E_F - E_r)\tau_1\right]\exp\left[\frac{i}{\hbar}(E_r - E_0)\tau_2\right].\end{aligned} \tag{15}$$

Corresponding expansions hold for all the other S_i. The summation extends over all possible combinations of the products of all the free photon and free electron states.

III. EVALUATION OF THE MATRIX ELEMENTS OF THE S-MATRIX

It can easily be appreciated that for higher-order approximations the matrix elements of S_i become very complicated. Not only will an increasing number of nonvanishing terms appear in the summation but also the products of the individual matrix elements of H become increasingly involved since each H according to (3a) consists of three

terms, so that each term in the sum corresponding to S_i will contain 3^i terms.

Luckily, however, not all of these terms give a contribution. Let us, for the moment, assume that we are interested in the process of the emission of one photon. We see immediately that all terms containing only products of matrix elements of V give no contribution, since V does not change the photon number and since the scalar product of state functions with different photon number is zero. Also, all those products disappear which contain one term with the matrix element of H_2 and all others with V. H_2 changes the photon number either by 2 or 0, so that no state with one additional photon can result. These considerations, if applied to all possible products, reduce the number of those terms contributing to the matrix element S_i considerably.

In addition we will make one more approximation. Since we are interested in obtaining the nonlinear effects (nonlinear with respect to n), we do not want to improve the approximation of the result given in Ref. 1 as far as the potential V is concerned. In other words, we have obtained previously a result which was proportional to e^6nV. By considering all terms in the higher-order approximations which contain one factor H_1 and an arbitrary number of factors containing V, we could obtain improvements in the previous calculation in the (symbolic) form

$$e^6n(V + V^2 + V^3 + \cdots).$$

However, these terms improving the previous approximation as far as the potential V is concerned will not give any information concerning nonlinear effects with respect to n. What we want instead is an approximation which results in

$$e^6Vn(1 + e^2n + e^4n^2 + \cdots).$$

Therefore, we will consider only terms containing one factor V and neglect all those containing more than one.

Without going further into the details of the calculation, we will immediately quote the results of the calculation of the matrix elements which differ by one photon in the initial and final states. This fact will be indicated by a subscript 1 on the matrix elements. The matrix elements for the operators H_1, H_2 and V are given by:

$$\left(\Phi^*(n \pm 1, \kappa), H_1\Phi(n,k)\right) = -\frac{\sqrt{n}}{\sqrt{L^3}} \cdot \frac{e}{m} \sqrt{\frac{2\pi\hbar}{\omega}}\, \hbar k \delta_{\kappa_x,k}\delta_{\kappa_y,0}\delta_{\kappa_z,\mp\beta} \tag{16}$$

$$\left(\Phi^*(n \pm 2, \kappa), H_2\Phi(n,k)\right) = \frac{\pi}{L^3}\frac{e^2}{m}\frac{\hbar}{\omega}\, n\delta_{\kappa_x,k}\delta_{\kappa_y,0}\delta_{\kappa_z,\mp 2\beta} \tag{17}$$

$$(\Phi^*(n,\kappa), H_2\Phi(n,k)) = 2\,\frac{\pi}{L^3}\,\frac{e^2}{m}\,\frac{\hbar}{\omega}\,n\delta_{\kappa_x,k}\delta_{\kappa_y,0}\delta_{\kappa_z,0} \tag{18}$$

$$(\Phi^*(n,\kappa), V\Phi(n,k)) = -\frac{Ze^2}{L^3}\,\frac{4\pi}{|\boldsymbol{\kappa}-\mathbf{k}|^2+\gamma^2}. \tag{19}$$

With these matrix elements we obtain

$$(\Phi_F{}^*, S_2\Phi_0)_1 = \mp\frac{4\pi}{(L^3)^{\frac{1}{2}}}\,\frac{Ze^3n^{\frac{1}{2}}}{m\omega}\,\sqrt{\frac{2\pi\hbar}{\omega}}\,\frac{\kappa_x-k}{|\boldsymbol{\kappa}-\mathbf{k}|^2+\gamma^2} \cdot \frac{\exp\left[\frac{i}{\hbar}(E_F-E_0)t\right]-1}{E_F-E_0} \tag{20a}$$

$$(\Phi_F{}^*, S_3\Phi_0)_1 = -\frac{4\pi^2}{(L^3)^{\frac{3}{2}}}\,\frac{Ze^5n^{\frac{3}{2}}}{m^2\omega^3\hbar}\,\sqrt{\frac{2\pi\hbar}{\omega}}\,\frac{1}{|\boldsymbol{\kappa}-\mathbf{k}|^2+\gamma^2} \cdot \left\{\mp 2i\omega t(\kappa_x-k)\,\frac{\exp\left[\frac{i}{\hbar}(E_F-E_0)t\right]-1}{E_F-E_0} - 2i\,\frac{t}{\hbar}\,(\kappa_x+k-\kappa_x e^{\pm i\omega t}-ke^{\mp i\omega t})\right\} \tag{20b}$$

$$(\Phi_F{}^*, S_4\Phi_0)_1 = \pm\frac{4\pi^2}{(L^3)^{\frac{3}{2}}}\,\frac{Ze^5n^{\frac{3}{2}}}{m^3\omega^4}\,\sqrt{\frac{2\pi\hbar}{\omega}}\,\frac{\hbar}{|\boldsymbol{\kappa}-\mathbf{k}|^2+\gamma^2} \cdot \left\{(5k^2\kappa_x-3k^3-\kappa_x{}^2k-\kappa_x{}^3) \cdot \frac{\exp\frac{i}{\hbar}\left[(E_F-E_0)t\right]-1}{E_F-E_0} + 2i\,\frac{t}{\hbar}\,[2\kappa_x{}^3-2k\kappa_x{}^2+2k\kappa_x(k-\kappa_x)\cos\omega t]\right\}. \tag{20c}$$

Of the two signs given in these equations, the upper sign refers to the case of emission and the lower sign of absorption of one photon. The value of $\kappa = |\boldsymbol{\kappa}|$ is also dependent on whether emission or absorption of one photon is being considered

$$\kappa = k\sqrt{1\mp 2\epsilon} \tag{21a}$$

with

$$\epsilon = \frac{\hbar\omega}{mv^2}. \tag{21b}$$

The equations (21) follow from the conservation of energy $E_F = E_0$ with the help of (13).

The matrix elements (20b) and (20c) include the process of the virtual emission and absorption of one photon. While the final state Φ_F differs from the initial state Φ_0 by one photon, states with two additional photons appear in the intermediate steps of the calculation leading to (20b) and (20c). One may say, therefore, that these matrix elements correspond to a process either by which one photon is being absorbed and two photons are emitted, or by which two photons are emitted and one is absorbed. The final state has gained a photon in either case. In contrast to this, the matrix element (20a) corresponds to the simple emission or absorption of one photon without virtual emission and absorption processes taking place.

However, we have also to consider the case that two real photons are emitted or absorbed. This process leads to transition probabilities which are of the same order in e as the contributions which (20b) and (20c) will give to the emission or absorption probability of one photon.

The matrix elements corresponding to the emission or absorption of two real photons are given by:

$$(\Phi_F^*, S_2\Phi_0)_2 = \pm \frac{2\pi^2}{L^6} \frac{Ze^4 n\hbar\sigma_z}{m^2\omega^2 c} \frac{1}{|\mathfrak{d} - \mathbf{k}|^2 + \gamma^2} \cdot \frac{\exp\left[\frac{i}{\hbar}(E_F - E_0)t\right] - 1}{E_F - E_0} \tag{22a}$$

$$(\Phi_F^*, S_3\Phi_0)_2 = -\frac{4\pi^2}{L^6} \frac{Ze^4 n\hbar}{m^2\omega^3} \frac{(\sigma_x - k)^2}{|\mathfrak{d} - k|^2 + \gamma^2} \cdot \frac{\exp\left[\frac{i}{\hbar}(E_F - E_0)t\right] - 1}{E_F - E_0}. \tag{22b}$$

The subscript 2 on the matrix elements indicates that two photons have been emitted or absorbed. In (22) $\mathfrak{d}$ is the propagation vector of the final electron with the energy

$$E_F = \frac{\hbar^2}{2m}\sigma^2 + (n \pm 2)\hbar\omega. \tag{23}$$

From $E_F = E_0$ follows

$$\sigma = k\sqrt{1 \mp 4\epsilon} \tag{24}$$

with ϵ of equation (21b).

IV. THE PHOTON CREATION RATE

We can now calculate the probabilities for the emission of one and also for the simultaneous emission of two photons. More important than the transition probability is the transition probability per unit time. According to Heitler[4] this is obtained by summing the transition probability P over all possible values of the energy of the final state and dividing by the time t.

$$w = \frac{1}{t}\sum_F P = \frac{1}{t}\int_0^\infty P\rho_F \, dE_F. \tag{25}$$

ρ_F is the number of states per unit energy. We keep the energy of the radiation field fixed and allow the final energy of the electron to vary, disregarding conservation of energy for the moment. It turns out that conservation of energy is automatically assured since P contains a δ-function, as will be shown.

The number of states in the box of size L^3 (box normalization) turns out to be

$$\rho_F = \frac{mL^3 k_F}{\hbar^2(2\pi)^3}\, d\Omega. \tag{26}$$

k_F is the magnitude of the electron propagation vector, which is either κ or σ depending on whether we consider the one- or two-photon process. $d\Omega$ is the element of solid angle into which the electron is scattered.

The probability P_1 for the emission of one photon is given by (12) and (20)

$$P_1 = |\,(\Phi_F^*, S_2\Phi_0)_1 + (\Phi_F^*, S_3\Phi_0)_1 + (\Phi_F^*, S_4\Phi_0)_1\,|^2. \tag{27}$$

We neglect all terms of orders higher than e^8. Two important terms will occur in (27). One is

$$\begin{aligned} t\,\frac{\exp\left[\frac{i}{\hbar}(E_F - E_0)t\right] - 1}{E_F - E_0} &= 2it\exp\left[\frac{i}{2\hbar}(E_F - E_0)t\right] \\ &\quad \cdot \frac{\sin(E_F - E_0)\frac{t}{2\hbar}}{E_F - E_0} \\ &= 2\pi i t\delta(E_F - E_0). \end{aligned} \tag{28}$$

The other one is

$$\left(\frac{\exp\left[\frac{i}{\hbar}(E_F - E_0)t\right] - 1}{E_F - E_0}\right)^2 = 2\pi i \frac{\exp\left[\frac{i}{\hbar}(E_F - E_0)t\right] - 1}{E_F - E_0} \cdot \exp\left[\frac{i}{2\hbar}(E_F - E_0)t\right] \cdot \delta(E_F - E_0) = 2\pi i \frac{t}{\hbar}\delta(E_F - E_0). \tag{29}$$

The limiting process

$$\lim_{K\to\infty} \frac{\sin xk}{x} = \pi\delta(x)$$

can be taken since, even for relatively short times, we will have $E_0 \cdot \frac{t}{\hbar} \gg 1$. The somewhat daring calculation in (29) is suggested by Heitler[4] and Mandl.[5]

The same calculations have to be performed with P_2, the probability for the emission of two photons.

We see that P is proportional to t, so that $w = 1/t(\sum P)$ will be independent of time. P is also proportional to a δ-function which guarantees the conservation of energy. If we write

$$P = 2\pi \frac{t}{\hbar} |K_{F0}|^2 \delta(E_F - E_0)$$

we obtain

$$w = \frac{2\pi}{\hbar} |K_{F0}|^2 \rho_F . \tag{30}$$

This equation was taken as the starting point of the previous paper [Ref. 1, equation (1)].

We will not write down the explicit expressions for w_1 and w_2 but will go immediately to the photon creation rate $\Delta N'$. $\Delta N'$ denotes the number of emitted photons per second and, according to equation (15) of Ref. 1, is given by:

$$\Delta N' = (w_1 + 2w_2) \frac{L^3}{v} N_e N_n . \tag{31}$$

By substituting (30) and (27) and its equivalent for the two-photon process in (31), we obtain:

$$\Delta N_{\substack{e\\a}}' = \frac{8\pi Z^2 e^6 N}{vm\omega^3\hbar^2}\, d\Omega \left\{ \frac{\kappa(\kappa_x - k)^2}{(|\boldsymbol{\kappa} - \mathbf{k}|^2 + \gamma^2)^2} + \frac{2\pi e^2 N\hbar}{m^2\omega^3} \left[\frac{\kappa(k - \kappa_x)[3(\kappa_x^3 - k^3) + 5k\kappa_x(k - \kappa_x)]}{(|\boldsymbol{\kappa} - \mathbf{k}|^2 + \gamma^2)^2} \mp \frac{2m\omega}{\hbar} \frac{\kappa(\kappa_x^2 - k^2)}{(|\boldsymbol{\kappa} - \mathbf{k}|^2 + \gamma^2)^2} + \frac{1}{2} \frac{\sigma(\sigma_x - k)^4}{(|\boldsymbol{\sigma} - \mathbf{k}|^2 + \gamma^2)^2} \mp \frac{1}{2} \frac{\beta\sigma\sigma_2(\sigma_x - k)^2}{(|\boldsymbol{\sigma} - \mathbf{k}|^2 + \gamma^2)^2} \right] \right\} N_e N_n \,. \tag{32}$$

Terms proportional to $\cos \omega t$ have been neglected in (32) since, if the time average over one period of the oscillation is taken, no contribution from these terms would result.

$\Delta N_e'$ is the number of emitted photons if we ask only for the probability of photon emission and take the corresponding values for κ and σ and the upper signs. $\Delta N_a'$ is the number of absorbed photons. In order to obtain the net number of actually emitted photons, the difference $\Delta N' = \Delta N_e' - \Delta N_a'$ has to be taken.

$\Delta N_{\substack{e\\a}}'$ of equation (32) implies also that we are interested only in those photons which are emitted while the electron is scattered into a certain direction of space given by $\boldsymbol{\kappa}$ or $\boldsymbol{\sigma}$ into the solid angle $d\Omega$. In order to get all the photons, we have to calculate

$$\Delta N = \int (\Delta N_e' - \Delta N_a')\, d\Omega. \tag{33}$$

It is interesting to note that the last two terms in (32), stemming from the two-photon process, give only a negligible contribution to (33) and can be neglected.

We obtain†

$$\frac{\Delta N}{N} = \frac{8Z^2 e^6 N_e N_n}{m^3 v^4 f^2} \ln \frac{2}{\sqrt{\epsilon^2 + \eta^2}} \left(1 - \frac{e^2 N v^2}{\pi h f^3}\right) \tag{34}$$

with $\eta = \gamma/k \ll 1$ and $\epsilon = hf/mv^2 \ll 1$. Equation (34) holds as long as

$$\frac{e^2 N v^2}{\pi h f^3} \ll 1. \tag{34a}$$

Equation (34) is identical to equation (32) of Ref. 1 if $\eta = 0$ in the limit $N \to 0$. [$\ln(2/\epsilon) - 1$ has been approximated by omitting the 1.] However, we now have obtained an expression for the photon creation

† All equations in this paper are written in electrostatic c.g.s. units.

rate $\Delta N/N$, which depends on the energy density hfN of the radiation field. For increasingly larger N, the creation rate $\Delta N/N$ becomes decreasingly smaller. We therefore have the possibility of computing the available power.

V. ESTIMATE OF THE AVAILABLE POWER

The power P_r radiated from a cavity is given by

$$P_r = 2\pi h f^2 N V \left(\frac{1}{Q_L} - \frac{1}{Q_i}\right). \tag{35}$$

The energy density hfN in the cavity will build up to the point where the number of created photons ΔN equals the number of photons which are radiated from the cavity and absorbed by its walls. We obtain for the oscillation condition

$$\frac{\Delta N}{N} = \frac{2\pi f V}{Q_L}. \tag{36}$$

Substituting (34) into (36), solving for N, and substituting into (35), we obtain

$$P_r = \frac{2\pi^2 h^2 f^5 V}{e^2 v^2}\left(\frac{1}{Q_L} - \frac{1}{Q_i}\right)\left[1 - \frac{\pi m^3 v^4 f^3 V}{4e^2 Z^2 Q_L N_n N_e \ln \dfrac{2}{\sqrt{\epsilon^2 + \eta^2}}}\right] \tag{37}$$

(c.g.s. units are being used).

Because of (34a), this equation holds if the expression in the brackets is much smaller than 1 (but larger than 0).

As soon as the product $N_n N_e$ is large enough to make $P_r \geqq 0$, the cavity oscillates. The oscillation condition is, therefore

$$N_e N_n \geqq \frac{\pi m^3 v^4 f^3 V}{4e^2 Z^2 Q_L \ln \dfrac{2}{\sqrt{\epsilon^2 + \eta^2}}}. \tag{37a}$$

It is understood that the power carried by the incident electron beam must be substantially larger than the power calculated from (37).

The most surprising fact about (37) is its dependence on the fifth power of the frequency. This means that at low frequencies the power obtainable from stimulated emission of bremsstrahlung is small.

Table I illustrates the situation for a practical example. We assume that we use bare nuclei with charge $Z = 1$. Correspondingly, we have to take $\eta = 0$. Furthermore, we assume that the expression in the bracket of (37) has the value of 0.1 and that $Q_L = 10^4$, $Q_i \gg Q_L$, $v = 2 \times 10^8$

TABLE I

f	P_r	N_eN_n	N_n/V
10 gc	10^{-10} w	10^{33} cm^{-2} sec^{-1}	10^{15} cm^{-3}
100 gc	10^{-5} w	10^{37} cm^{-2} sec^{-1}	10^{18} cm^{-3}
1000 gc	1 w	10^{40} cm^{-2} sec^{-1}	10^{21} cm^{-3}

cm/sec corresponding to an accelerating potential of 10 volts. The cavity volume is taken as $V = 10$ cm^3.

The values for N_eN_n given in Table I are the products of nuclei and electron density current necessary to satisfy the oscillation condition (37a). N_n/V in the last column of Table I is the density of nuclei which results from N_eN_n with $V = 10$ cm^3 if N_e is chosen to equal a current density of 0.1 amp/cm^2.

We see that the available power is very low at $f = 10$ gc and reaches interesting values for $f = 1000$ gc. However, the required density of nuclei (or ions) also increases very rapidly with increasing frequency. Equation (37), which was used to compute the values for the available power in Table I, is an approximation. It does not hold for $N_eN_n \to \infty$. However, it seems reasonable to believe that the results of Table I are correct to the order of magnitude. We can not hope to obtain one milliwatt of power if our theory predicts 10^{-10} w.

VI. USE OF NEUTRAL ATOMS AS SCATTERING CENTERS

We derived (37) under the assumption that the electrons are scattered by a potential

$$V = -Ze^2 \frac{e^{-\gamma r}}{r}.$$

According to the Thomas-Fermi statistical model of the electron[6]

$$\eta = \frac{\gamma}{k} = \frac{me^2Z^{\frac{1}{3}}}{\hbar^2 k} = \frac{e^2Z^{\frac{1}{3}}}{\hbar v}. \tag{38}$$

Equations (37a) and (38) allow us to calculate the required minimum number of atoms necessary to achieve selfsustained oscillations. It has to be remembered, however, that we have completely neglected the interaction of the incident electron with the bound electrons in the scattering atom. The only way the bound electrons enter the picture is by effectively shielding the Coulomb potential of the nucleus. In reality, the incident electron will excite the bound electrons into higher energy states and will also ionize some of the atoms. It might be that the process of ionization is accompanied by either stimulated emission or absorption

of radiation, just as the process of scattering is accompanied by a net stimulated emission of radiation if the incident electrons fly more or less parallel to the electric vector of the stimulating radiation field.[1]

For all practical cases we will have

$$\epsilon \ll \eta$$

and therefore we obtain as the oscillation condition from (37a)

$$N_e N_n \geqq \frac{\pi m^3 v^4 f^3 V}{4Z^2 e^6 Q_L \ln\left(\frac{2\hbar v}{e^2 Z^{\frac{1}{3}}}\right)}. \tag{39}$$

Table II lists the product $N_e N_n$ of (39), the atom density for an electron current of 100 ma/cm^2 and the velocity v as well as the corresponding accelerating potential U as a function of frequency for three different values of Z. The velocity is chosen so that $\eta = 0.2$ [equation (38)]. Z is equivalent to the order number of the atom in the periodic table of elements. We have again assumed $Q_L = 10^4$ and $V = 10$ cm^3.

VII. DISCUSSION OF THE FEASIBILITY OF A BREMSSTRAHLUNG OSCILLATOR

The examples given in the last section show the problems involved in building an oscillator utilizing stimulated emission of bremsstrahlung. At low frequencies, where it is easy to satisfy the oscillation condition, the available power is very low. The available power reaches useful proportions in the region of 100 gc and becomes abundant at frequencies higher than 1000 gc. However, the required number of scattering nuclei or atoms becomes very high.

There is a possibility of using periodic arrangements of scattering centers rather than the randomly distributed atoms or nuclei considered here. The author has considered scattering by a string of nuclei (or atoms) which are arranged on a straight line parallel to both the direc-

TABLE II

f	Z	$N_e N_n$	N_n/V	v	U
10 gc	1	3×10^{37} cm^{-2} sec^{-1}	4.8×10^{19} cm^{-3}	1.1×10^9 sec	345 volts
100 gc		3×10^{40} cm^{-2} sec^{-1}	4.8×10^{22} cm^{-3}		
1000 gc		3×10^{43} cm^{-2} sec^{-1}	4.8×10^{25} cm^{-3}		
10 gc	10	6.5×10^{36} cm^{-2} sec^{-1}	1×10^{18} cm^{-3}	2.37×10^9 sec	1600 volts
100 gc		6.5×10^{39} cm^{-2} sec^{-1}	1×10^{21} cm^{-3}		
1000 gc		6.5×10^{42} cm^{-2} sec^{-1}	1×10^{24} cm^{-3}		
10 gc	50	2.2×10^{36} cm^{-2} sec^{-1}	3.5×10^{17} cm^{-3}	4.05×10^9 sec	4660 volts
100 gc		2.2×10^{39} cm^{-2} sec^{-1}	3.5×10^{20} cm^{-3}		
1000 gc		2.2×10^{42} cm^{-2} sec^{-1}	3.5×10^{23} cm^{-3}		

tion of the incident electrons and the electric field vector of the stimulating radiation field. The result was that, if the distance between successive atoms is $d = v/f$, the number N_n of scattering centers entering (34), or (39) in case of a random distribution, is replaced by N_n^2. Scattering by periodically arranged atoms thus greatly enhances the effect.

However, if the atoms are not aligned in one straight line but rather are in periodically arranged planes, N_n has to be replaced by nN_n, where N_n is the total number of atoms, while n is the number of planes. If 100 periodically arranged planes filled with scattering atoms were used, the number N_n quoted in Tables I and II would be reduced by a factor of 100, and the corresponding atoms would have to be distributed over these 100 planes. Using a frequency of 100 gc and atoms with $Z = 50$ would require a total number of 4×10^{19} atoms or 4×10^{17} atoms in each plane. If a plane of 10 cm^2 area is used, its thickness would have to be of the order of 1000 angstroms.

There is a limit to the number of planes which can be used in practice. This limit is set by the requirement that all electrons have to pass the distance between all these planes in the same time interval. The spread in electron velocities which can be tolerated decreases in inverse proportion to the number of planes used.

One could use electrons moving in conduction bands of solids rather than free electron beams and let them scatter from impurities in the crystal. Since the concentration of ionized impurities in semiconductors can be very high and since extremely high current densities can be applied, it appears that stimulated emission of bremsstrahlung should occur.

VIII. LIST OF SYMBOLS

$\mathbf{A} = (A_x, 0, 0)$	vector potential of the electromagnetic field
$c = 3 \times 10^{10}$ cm/sec	velocity of light
$\delta_{\kappa,k}$	Kronecker δ symbol equals 1 if $\kappa = k$ and is 0 otherwise
E	energy of the physical system composed of RF field and electron exclusive of the interaction energy
$e = 4.803 \times 10^{-10}$ dyn$^{\frac{1}{2}}$ cm	electron charge (or base of natural Logarithm)
$\epsilon = hf/mv^2$	
$\eta = \gamma/k$	
f	frequency of the stimulating RF field
Φ	state function of the physical system
γ	shielding factor of the potential of the neutral atom

$h = 2\pi\hbar = 6.624 \times 10^{-27}$ erg·sec	Planck's constant
H_0	Hamiltonian of the noninteracting system
H_i	interaction Hamiltonian
$i = \sqrt{-1}$	or sometimes used as subscript
$k = k_x$	propagation constant of the incident electron
$\kappa = \sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_z^2}$	magnitude of propagation constant of the scattered electron after the emission of one photon
L	length of the fictitious box used for box normalization
$m = 9.11 \times 10^{-28}$ gram	electron mass
n	number of photons in box of volume L^3
$N = n/L^3$	photon density
N_e	number of electrons per second per cm^2
N_n	total number of scattering nuclei or atoms
$\omega = 2\pi f$	
$p = mv = \hbar k$	momentum of the incident electron
$p = i\hbar(\partial/\partial x)$	momentum operator
q	absorption operator
q^*	emission operator
Q_L	loaded Q of the resonant cavity
Q_i	intrinsic Q of the resonant cavity
r	radius of polar coordinate system
ρ_F	number of electron states per unit energy interval
$\sigma = \sqrt{\sigma_x^2 + \sigma_y^2 + \sigma_z^2}$	magnitude of the electron propagation vector after the emission of two photons
S	scattering matrix
t	time
U	acceleration voltage of the electron
v	velocity of the incident electron
V	volume of the resonant cavity, also used to describe the shielded Coulomb potential
Z	number of elementary charges on scattering nucleus

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