On Overflow Processes of Trunk Groups with Poisson Inputs and Exponential Service Times

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Recurrence formulas for the Laplace transforms and the moments of the interoverflow distributions are obtained under the assumptions that the traffic offered is random (Poissonian) and that the service times are independent of each other and have a common negative exponential law. Under the same assumptions, it is also shown that the distribution of the nonbusy period of a group of c trunks is identical to the interoverflow distribution of a group of c − 1 trunks and that the distribution of the number of consecutive successful calls is essentially a mixture of geometric distributions. Processes obtained by superposing two or more overflow processes from independent trunk groups are not of the renewal type because interoverflow intervals are no longer independent. It is shown here that the correlation between two consecutive interoverflow intervals of a composite overflow process is always positive.

I. INTRODUCTION

When telephone networks are engineered, the loads offered to the alternate routes cannot usually be assumed to be random (Poissonian) and, in these cases, it is of considerable practical interest to determine the characteristics of the traffic overflowing the trunk groups. In this paper we shall be mainly concerned with the interoverflow distribution, the term used here to designate the distribution of the time intervals separating consecutive epochs at which calls find all trunks busy (overflow). We shall first show that the distribution of the nonbusy periods of a group of c trunks is identical to the interoverflow distribution of a group of $c - 1$ trunks and then obtain new recurrence formulas for the Laplace transforms of the interoverflow distribution of a single trunk group under the assumptions that: (a) the load submitted to the group is random; (b) the service times are independent of each other and are all distributed according to the same negative exponential law; and (c) the requests which are placed when all the trunks are busy are either canceled or sent via some alternate route. As we shall see, these formulas are much simpler than the expressions obtained by C. Palm (cf Ref. 1, pp. 25–26, and Ref. 2, pp. 36–40) and are well suited to the computation of the moments. Then, under the same three assumptions, we shall also obtain the generating function of the probability distribution of the number of consecutive successful calls or, in other words, of the number of calls which are placed during a time interval whose end points coincide with two consecutive overflows. From the form of this generating function we can then infer that the distribution of the number of consecutive successful calls is essentially a mixture of geometric distributions.

In the remaining part of the paper we consider processes obtained by superposing (pooling) overflow or, more generally, renewal processes. In particular, it is shown that for processes obtained by superposition of two or more overflow processes, the covariance between the lengths of two intervals determined by three consecutive overflows is always positive.

II. RECURRENCE FORMULAS FOR THE MOMENTS OF THE INTEROVERFLOW DISTRIBUTION.

Consider a group of c trunks and assume that:

i. the calls are placed at random (Poisson input);

ii. the service times are independent of each other and of the state of the system and are distributed according to the negative exponential law with mean 1; and

iii. calls arriving when all trunks are busy do not wait but are either canceled or overflow to some other route (loss system).

Under these assumptions, the epochs at which overflows occur constitute a renewal process. We note also that if at some instant t the c trunks are busy, then the distribution of the time that elapses between t and the first overflow occurring after t is the same as the distribution of the intervals separating successive overflows. The cumulative distribution, $F_c(\cdot)$, of these intervals will be called the interoverflow distribution.

Before proceeding, we recall the following definitions of the busy and nonbusy periods: a busy period is a time interval during which all the trunks are continuously busy, and a nonbusy period is a time interval separating two consecutive busy periods. Under the present assumptions, the distribution of the busy periods is the negative exponential law with mean $1/c$.

Now, let $\gamma_c(\cdot)$ be the Laplace transform of the derivative of $F_c(\cdot)$. Then, as shown in Ref. 1, pp. 25–26, and Ref. 2, pp. 36–40,

$$\gamma_c(s) = D_c(s)/D_{c+1}(s) \tag{1}$$

with

$$D_n(s) = 1 + \sum_{j=1}^{n} \binom{n}{j} a^{-j} s(s+1) \cdots (s+j-1)$$

and a the demand rate. (Note that $D_n(s) = c_n(-s,a)$ with $c_n(\cdot,a)$ the Poisson-Charlier polynomial of degree n and parameter a.)

The roots $r_1, r_2, \cdots, r_{c+1}$ of $D_{c+1}(\cdot)$ are all negative and distinct so that:

$$F_c(t) = \sum_{i=1}^{c+1} k_i[1 - \exp(-s_i t)], \qquad r_i = -s_i, \qquad i = 1, \cdots, c+1$$

where the coefficients k_i, $i = 1, \cdots, c+1$ are given by the relations:

$$k_i s_i \prod_{j \neq i} (s_j - s_i) = a^{c+1} D_c(-s_i), \qquad i = 1, 2, \cdots, c+1.$$

We shall now derive the distribution of the nonbusy period. To this end, let us consider a busy period starting at time 0, say, and let T be the epoch at which the next following overflow occurs. We distinguish now between two cases:

1. At least one call is placed during the busy period under consideration. In this instance, the conditional density function, $f_1(\cdot)$, of T has the following expression:

$$f_1(t) = (c+a) \exp[-(c+a)t].$$

2. No call is placed during the busy period under consideration. In this case, the interval T is made up of three independent subintervals, namely, the busy period, the nonbusy period that follows it, and the period that elapses from the time at which all the trunks are made busy again to the occurrence of the first overflow. Since the distribution of this last subinterval is also $F_c(\cdot)$, the conditional density function, $f_2(\cdot)$, of T is then:

$$f_2(t) = (c+a) \int_0^t \int_0^v \exp[-(c+a)u] h_c(v-u) F_c'(t-v) du \cdot dv$$

where $h_c(\cdot)$ is the density function of the nonbusy period and $F_c'(\cdot)$ is the derivative of $F_c(\cdot)$.

Cases (1) and (2) occur with probabilities respectively equal to

$a/(c + a)$ and $c/(c + a)$ and we have therefore:

$$F_c'(t) = a \exp[-(c + a)t] + c \int_0^t \int_0^v \cdot \exp[-(c + a)u] h_c(v - u) F_c'(t - v)\, du \cdot dv. \tag{2}$$

Taking the Laplace transform on both sides of (2) yields:

$$\gamma_c(s) = \frac{a}{(c + a + s)} + \frac{c}{(c + a + s)} \cdot \gamma_c(s) \cdot \varphi_c(s) \tag{3}$$

with $\varphi_c(\cdot)$ the transform of $h_c(\cdot)$. Hence, solving for $\varphi_c(\cdot)$, we find

$$\begin{aligned} \varphi_c(s) &= (c + a + s)/c - (a/c)[\gamma_c(s)]^{-1} \\ &= \frac{(c + a + s)D_c(s) - aD_{c+1}(s)}{cD_c(s)}. \end{aligned}$$

Since (cf Ref. 2, p. 38 and p. 83)

$$aD_{c+1}(s) = (c + a + s)D_c(s) - cD_{c-1}(s)$$

it follows that:

$$\varphi_c(s) = \frac{D_{c-1}(s)}{D_c(s)}$$

which is the same as (1) with c replaced by $c - 1$. This shows that the nonbusy period distribution of a group of c trunks is identical to the interoverflow distribution of a group of $c - 1$ trunks with the same demand rate and average service time. (Note also that, under the present circumstances, the nonbusy period distribution remains unchanged if the calls finding all trunks occupied are allowed to wait.)

We have, therefore:

$$\varphi_c(s) = \gamma_{c-1}(s)$$

and (3) can be rewritten as follows:

$$\gamma_c(s) = a[c + a + s - c\gamma_{c-1}(s)]^{-1}. \tag{4}$$

It is interesting to note that this recurrence formula is considerably simpler than the one obtained by Palm,[1] namely,

$$\gamma_c(s) = \gamma_{c-1}(s + 1)[1 - \gamma_{c-1}(s) + \gamma_{c-1}(s + 1)]^{-1}. \tag{5}$$

The latter, however[4], is valid for arbitrary recurrent input while (4) was obtained under the stronger assumption that the input is Poissonian. As one may expect, (4) can easily be obtained directly from (5). Indeed, the recurrences (cf Ref. 2, p. 38 and p. 83)

$$aD_{c+1}(s) = (c + a + s)D_c(s) - cD_{c-1}(s)$$

and

$$D_{c+1}(s) = D_c(s) + [D_1(s) - 1]D_c(s + 1)$$

imply that

$$\begin{aligned}\gamma_{c-1}{}^{-1}(s+1) &= \frac{D_c(s+1)}{D_{c-1}(s+1)} = \frac{D_{c+1}(s) - D_c(s)}{D_c(s) - D_{c-1}(s)} \\ &= \frac{1}{a}\,\frac{(c+a+s)D_c(s) - cD_{c-1}(s)}{D_c(s) - D_{c-1}(s)} - \frac{D_c(s)}{D_c(s) - D_{c-1}(s)} \\ &= \frac{1}{a}\,\frac{(c+a+s) - c\gamma_{c-1}(s)}{1 - \gamma_{c-1}(s)} - \frac{1}{1 - \gamma_{c-1}(s)}\,.\end{aligned}$$

Equation (4) then follows by substituting this expression in

$$\frac{1}{\gamma_c(s)} = 1 + \frac{1 - \gamma_{c-1}(s)}{\gamma_{c-1}(s+1)}$$

which is Palm's recurrence in the reciprocal form.

Equation (4) can be used to obtain recurrences for the moments of the interoverflow distribution. Indeed, writing $\mu_n(c)$ for the nth moment of $F_c(\cdot)$, we find upon taking the nth derivative on both sides of (4):

$$\begin{aligned}\frac{d^n}{ds^n}\gamma_c(s)\bigg|_{s=0} &= (-1)^n\mu_n(c) = a\,\frac{d^n}{ds^n}\,[c + a + s - c\gamma_{c-1}(s)]_{s=0}^{-1} \\ &= Y_n(fg_1, \cdots, fg_n)\end{aligned}$$

where $Y_n(fg_1, \cdots, fg_n)$ is a multivariable Bell polynomial (cf Ref. 3, p. 34–35 and p. 49) with:

$$\begin{aligned}f_k &= (-1)^k k!a^{-k} \\ g_1 &= 1 + c\mu_1(c-1) \\ g_k &= (-1)^{k+1}c\mu_k(c-1), \qquad k > 1.\end{aligned}$$

Since

$$\mu_n(0) = n!a^{-n}, \qquad n = 0, 1, 2, \cdots$$

these relations can be used to compute the moments of the interoverflow distribution recurrently. In particular, we have:

$$a\mu_1(c) = 1 + c\mu_1(c-1) = E_{1,c}{}^{-1}(a)$$

where $E_{1,c}(a)$, which is known as the first Erlang loss function, is the probability that a call is placed when all trunks are busy and is therefore cleared from the system. For $n = 2, 3$ and 4 we have the following recurrences:

$$\begin{aligned} a\mu_2(c) &= 2a\mu_1^2(c) + c\mu_2(c-1) \\ a\mu_3(c) &= 6a\mu_1^3(c) + 6c\mu_1(c)\mu_2(c-1) + c\mu_3(c-1) \\ a^2\mu_4(c) &= 24a^2\mu_1^4(c) + 36ac\,\mu_1^2(c)\mu_2(c-1) + 6c^2\mu_2^2(c-1) \\ &\qquad + 8\,ac\,\mu_1(c)\mu_3(c-1) + ac\,\mu_4(c-1). \end{aligned}$$

Finally we note that repeated use of the first of these relations yields the following expression for the second moment:

$$\mu_2(c) = 2\sum_{n=0}^{c} \frac{(c)_n}{a^n}\,\mu_1^2(c-n)$$

with $(c)_0 = 1$ and $(c)_n = c(c-1)\cdots(c-n+1)$.

III. DISTRIBUTION OF THE NUMBER OF CONSECUTIVE SUCCESSFUL CALLS

In this section we consider a group of c indexed trunks with calls always assigned to the free trunk having the lowest index. The calls which find the first m trunks busy will be referred to as m-overflows, and those which are placed when at least one of the first m trunks is free will be said to be m-successful. (c-successful calls are simply said to be successful.)

We shall designate by $F_m(\cdot)$ the cumulative distribution of the time interval separating two consecutive m-overflows and by $P_m(t,n)$ the probability that exactly $n - 1$ calls are m-successful during a time interval of length t whose end points coincide with two consecutive epochs at which m-overflows occur.

Taking the average service time as unity, we have then the following recurrence:

$$\begin{aligned} P_c(t,n)\,dF_c(t) &= e^{-t}P_{c-1}(t,n)\,dF_{c-1}(t) \\ &+ \sum_{k=1}^{n-1}\int_0^t (1-e^{-u})P_c(t-u,n-k)P_{c-1}(u,k)\,dF_c(t-u)\,dF_{c-1}(u). \end{aligned} \tag{6}$$

Indeed, let us assume that a c-overflow occurs at time 0. Then the event "the first c-overflow after time 0 occurs in the interval $(t,t + \Delta t)$ and there are $n - 1$ successful calls during $(0,t)$" can be split as follows:

i. The first c-overflow after time 0 occurs during $(t,t + \Delta t)$; the call being served by the cth trunk at time 0 does not terminate before t; and the number of $(c-1)$-successful calls is equal to $n-1$. The probability of this event is equal to

$$e^{-t}P_{c-1}(t,n)\Delta F_{c-1}(t) + o(\Delta t).$$

ii. The first $(c-1)$-overflow after time 0 occurs during $(u,u+\Delta u)$, $u < t$; the first c-overflow after time 0 occurs during $(t,t+\Delta t)$; the call being served by the cth trunk at time 0 terminates before u; and the number of $(c-1)$-successful calls during $(0,u)$ is equal to $k-1$, while the number of c-successful calls during (u,t) is equal to $n-k-1$, $k = 1, \cdots, n-1$. The probability of this event is, in first approximation:

$$(1-e^{-u})P_c(t-u,n-k)P_{c-1}(u,k)\Delta F_c(t-u)\Delta F_{c-1}(u).$$

Equation (6) is then obtained by summing up these probabilities and then passing to the limit $(\Delta u \to 0, \Delta t \to 0)$ and integrating with respect to u $(0 \leqq u \leqq t)$.

Now write:

$$\lambda_c(x,w) = \sum_{n=1}^{\infty} x^n \int_0^{\infty} e^{-wt}P_c(t,n)\, dF_c(t).$$

Then (6) implies:

$$\lambda_c(x,w) = \frac{\lambda_{c-1}(x,w+1)}{1-\lambda_{c-1}(x,w)+\lambda_{c-1}(x,w+1)} \tag{7}$$

which is of the same form as Palm's recurrence (5). (Note that while (5) holds for arbitrary recurrent input, (7) is valid only when the input is Poissonian.)

Clearly, $\lambda_c(x,w)$ can be written as a ratio†:

$$\lambda_c(x,w) = \frac{D_c(x,w)}{D_{c+1}(x,w)}. \tag{8}$$

Substituting this expression in (7), we find that

$$\frac{D_{c+1}(x,w)-D_c(x,w)}{D_c(x,w+1)} = \frac{D_c(x,w)-D_{c-1}(x,w)}{D_{c-1}(x,w+1)} = \cdots = \frac{D_1(x,w)-D_0(x,w)}{D_0(x,w+1)}. \tag{9}$$

† The method used here to solve (7) is formally identical to the one used in Ref. 2 to solve Palm's recurrence (5).

Setting $D_0(x,w) \equiv 1$, which is not a restriction of the generality, we obtain, using (9):

$$D_{r+1}(x,w) = D_r(x,w) + [D_1(x,w) - 1]D_r(x,w + 1), \qquad r \geqq 1. \quad (10)$$

We note that:

$$\lambda_0(x,w) = \int_0^\infty e^{-wt}x\, dF_0(t) = ax \int_0^\infty \exp[-t(a + w)]\, dt = \frac{ax}{w + a}$$

so that: $D_1(x,w) = (w + a)/ax$.

Solving now (10) recurrently, we find:

$$D_m(x,w) = 1 + \sum_{j=1}^{m} \binom{m}{j} \left(\frac{1}{ax}\right)^j \prod_{k=1}^{j} [a(1 - x) + w + (k - 1)].$$

From this, it follows that the probability generating function, $H_c(\cdot)$, of the number of successful calls between two consecutive c-overflows, is

$$H_c(x) = \frac{\lambda_c(x,0)}{x} = \frac{1}{x} \frac{D_c(x,0)}{D_{c+1}(x,0)}$$

with

$$D_m(x,0) = 1 + \sum_{j=1}^{m} \binom{m}{j} \left(\frac{1}{ax}\right)^j \prod_{k=1}^{j} [a(1 - x) + (k - 1)].$$

These relations can be expressed in a simpler manner. Indeed (cf Ref. 2, p. 83):

$$axD_{m+1}(x,0) = (m + a)D_m(x,0) - mD_{m-1}(x,0)$$

and, with the notation $N_m(x) = (ax)^m D_m(x,0)$, we have, therefore:

$$H_c(x) = a \frac{N_c(x)}{N_{c+1}(x)}$$

where:

$$N_0(x) \equiv 1, \qquad N_1(x) \equiv a$$

and

$$N_{m+1}(x) = (m + a)N_m(x) - axm\, N_{m-1}(x). \quad (11)$$

The polynomials $N_m(\cdot)$ have the following properties which are immediate consequences of (11):

i. $N_m(x) > 0$ for $x \leqq 0$, $m = 0, 1, 2, \cdots$.

ii. the degree $\nu = \nu(m)$ of $N_m(\cdot)$ is the integral part of $m/2$.

iii. The coefficients of $x^0, x^1, \cdots, x^\nu$ in $N_m(x)$ are alternately positive and negative. Taking (*ii*) into account, it then follows, as x tends to ∞, that $N_m(x)$ tends to ∞ if $\nu(m)$ is even and to $-\infty$ if $\nu(m)$ is odd.

Repeated application of (11) yields:

$$\begin{aligned}(m + a - 2)N_{m+1}(x) + \{ax[m(m + a - 2) + (m + a)(m - 1)] \\ - (m + a)(m + a - 1)(m + a - 2)\}N_{m-1}(x) \qquad (12) \\ + (m + a)(m - 1)(m - 2)(ax)^2 N_{m-3}(x) = 0.\end{aligned}$$

Using (12) and the properties (*i*)–(*iii*) of the polynomials $N_m(\cdot)$, it is easily shown by consideration of the signs that $N_m(\cdot)$ has $\nu(m)$ distinct roots which are positive and separated by the $\nu(m) - 1$ roots of $N_{m-2}(\cdot)$, $m = 4, 5, \cdots$. Let now $x_1 < x_2 < \cdots < x_{\nu(c+1)}$ be the roots of $N_{c+1}(\cdot)$. Then the generating function $H_c(\cdot)$ is of the form:

$$H_c(x) = \gamma_0 + \sum_{i=1}^{\nu(c+1)} \gamma_i \left(1 - \frac{x}{x_i}\right)^{-1}$$

where $\gamma_0 = 0$ if c is odd and $\gamma_0 > 0$ if c is even, and where the constants γ_i, $i = 1, \cdots, \nu(c + 1)$ are strictly positive.

Consequently,† the distribution of the number of successful calls is a mixture of $\nu(m)$ distinct geometric distributions when c is odd; when c is even, the distribution of the number of successful calls is a mixture of $\nu(m) + 1$ distributions, one of the latter being the distribution with probability mass 1 at the origin and the remaining ones being distinct geometric distributions.

Finally, we note that the recurrences

$$axD_{m+1}(x,w) = (m + w + a)D_m(x,w) - mD_{m-1}(x,w)$$

and

$$D_{m+1}(x,w) = D_m(x,w) + [D_1(x,w) - 1]D_m(x,w + 1)$$

allow us to write (7) in the simpler form

$$\lambda_c(x,w) = ax[c + a + w - c\lambda_{c-1}(x,w)].^{-1}$$

Hence, we also have

$$H_c(x) = a[c + a - cxH_{c-1}(x)]^{-1}$$

which may be used to compute the moments of the distribution of the number of consecutive successful calls recurrently.

† Since $H_c(\cdot)$ is analytic for $|x| < 1$, it follows that the roots $x_1, x_2, \cdots, x_{\nu(c+1)}$ are all larger than 1.

IV. COVARIANCE BETWEEN INTEROVERFLOW INTERVALS

Under the assumptions made here, the overflow process of a single trunk group is of the renewal type. The processes obtained by superposing two or more such processes (called here composite overflow processes) do not, however, have this property because successive interoverflow intervals are no longer independent. We shall now prove that the correlation between two consecutive interoverflow intervals of a composite overflow process is always positive.

Let us consider n trunk groups $G_1, G_2, \cdots, G_n (n \geqq 2)$, of sizes $c_1, c_2, \cdots, c_n$, respectively, and let A_i be the (random) load offered to G_i. Let us now place ourselves at an overflow epoch and suppose that the overflow call in question comes from group G_1. Let also U and V be the two consecutive interoverflow intervals separated by the overflow under consideration. Then, by an argument similar to one used by Cox and Smith,[4] it follows that:

$$\Pr [U \geqq u, V \geqq v] = [1 - F_1 (u)] \cdot [1 - F_1 (v)] \prod_{i=2}^{n} \int_{u+v}^{\infty} \frac{1 - F_i(x)}{\mu_i} \, dx \tag{13}$$

where $F_i(\cdot)$ is the cumulative interoverflow distribution of G_i (considered by itself) and

$$\mu_i = \int_0^\infty x dF_i (x).$$

[Equation (13) implies that the process obtained by superposing continuous renewal processes is itself a renewal process if and only if all the distributions $F_i(\cdot)$ are negative-exponentials. An overflow process is therefore of the renewal type if and only if $n = 1$.]

Under the present assumptions we have:[2]

$$\begin{aligned} F_i(x) &= \sum_{j=1}^{c_i+1} a_{ij}[1 - \exp(-s_{ij}x)], \qquad x \geqq 0 \\ F_i(x) &= 0, \; x < 0 \end{aligned} \tag{14}$$

where $a_{ij} > 0$, $s_{ij} > 0$, $j = 1, \cdots, c_i + 1$, $i = 1, 2, \cdots, n$, and

$$\sum_{j=1}^{c_i+1} a_{ij} = 1$$

$$\sum_{j=1}^{c_i+1} \frac{a_{ij}}{s_{ij}} = \mu_i .$$

Upon substituting (14) into (13) we find that:

$$\begin{aligned}
\Pr [U \geqq u, V \geqq v] \\
&= \frac{1}{M_1}\left[\sum_{j=1}^{c_1+1} a_{1j} \exp(-s_{1j}u)\right]\left[\sum_{j=1}^{c_1+1} a_{1j} \exp(-s_{1j}v)\right] \\
&\quad \cdot \prod_{i=2}^{n} \int_{u+v}^{\infty} \left[\sum_{j=1}^{c_1+1} a_{ij} \exp(-s_{ij}x)\right] dx \\
&= \frac{1}{M_1}\left[\sum_{j=1}^{c_1+1} a_{1j} \exp(-s_{1j}u)\right]\left[\sum_{j=1}^{c_1+1} a_{1j} \exp(-s_{1j}v)\right] \qquad (15) \\
&\quad \cdot \prod_{i=2}^{n} \left[\sum_{j=1}^{c_i+1} \frac{a_{ij}}{s_{ij}} \exp[-s_{ij}(u+v)]\right] \\
&= \frac{1}{M_1}\left[\sum_{j=1}^{c_1+1} a_{1j} \exp(-s_{1j}u)\right]\left[\sum_{j=1}^{c_1+1} a_{1j} \exp(-s_{1j}v)\right] \\
&\cdot \sum{}^{*} \frac{a_{2j_2}a_{3j_3}\cdots a_{nj_n}}{s_{2j_2}s_{3j_3}\cdots s_{nj_n}} \exp[-(s_{2j_2}+s_{3j_3}+\cdots+s_{nj_n})(u+v)]
\end{aligned}$$

where the summation $\sum^*$ is to be extended over all $(n-1)$-tuples $(j_2, \cdots, j_n)$ arising when multiplying out

$$\prod_{i=2}^{n}\left[\sum_{j=1}^{c_i+1} \frac{a_{ij}}{s_{ij}}\right].$$

From here on we shall use the letter J as a generic symbol for any one of these $(n-1)$-tuples.

Integrating now (15) with respect to u and v yields:

$$\int_0^\infty \int \Pr[U \geqq u, V \geqq v]\, du\cdot dv = \frac{1}{M_1}\sum{}^{*} m_1(J)R_1^{\,2}(J)$$

with:

$$m_1(J) = \frac{a_{2j_2}\cdots a_{nj_n}}{s_{2j_2}\cdots s_{nj_n}}, \qquad J = (j_2, \cdots, j_n)$$

and

$$R_1(J) = \sum_{j=1}^{c_1+1} \frac{a_{1j}}{s_{1j} + (s_{2j_2}+\cdots+s_{nj_n})}, \qquad J = (j_2, \cdots, j_n)$$

We also note that:

$$\begin{aligned}
\mu_1 &= \int_0^\infty [1 - F_1(u)]\left[\prod_{i=2}^{n}\int_u^\infty \frac{1-F_i(x)}{\mu_i}\,dx_i\right] du \\
&= \frac{1}{M_1}\sum{}^{*} m_1(J)\cdot R_1(J).
\end{aligned}$$

When the overflow call separating the two intervals U and V is from group G_i, $i \neq 1$, we define M_i, $m_i(J)$ and $R_i(J)$ to have the meaning corresponding to M_1, $m_1(J)$ and $R_1(J)$, respectively. Let also P_i be the probability that the separating call is from group G_i. Then with μ denoting the average length between successive overflow calls and $\text{Cov}(U,V)$ standing for the covariance between U and V, we have:

$$\mu = \sum_{i=1}^{n} \frac{P_i}{M_i} \sum{}^* m_i(J) \cdot R_i(J)$$

and

$$\text{Cov}(U,V) = \sum_{i=1}^{n} \frac{P_i}{M_i} \sum{}^* m_i(J) R_i^2(J) - \mu^2.$$

We note now that:

$$\begin{aligned} \mu^2 &= \sum_{i=1}^{n} \frac{P_i}{M_i} \sum{}^* m_i(J) \left[\frac{R_i(J)}{\mu} - 1 \right]^2 \\ &= \sum_{i=1}^{n} \frac{P_i}{M_i} \sum{}^* m_i(J) R_i^2(J) - 2\mu^2 + \mu^2 \sum_{i=1}^{n} \frac{P_i}{M_i} \sum{}^* m_i(J). \end{aligned} \tag{16}$$

Therefore, since:

$$\sum{}^* m_i(J) = \prod_{\substack{k=i \\ k \neq i}}^{n} \int_0^\infty \sum_{j=1}^{c_k+1} [a_{kj} \exp(-s_{kj} x)]\, dx = M_i$$

we obtain, upon substituting this last expression in (16):

$$\text{Cov}(U,V) = \sum_{i=1}^{n} \frac{P_i}{M_i} \sum{}^* m_i(J) [R_i(J) - \mu]^2 \geqq 0. \tag{17}$$

In order to complete the proof, we have to show that the equality sign in (17) cannot hold when $n > 1$. This, however, is an immediate consequence of the fact that the s_{ij}'s occurring in $F_i(\cdot)$ are then all distinct.

V. SOME ADDITIONAL PROPERTIES

It is stated in Ref. 4 that the processes obtained by superposing n identical renewal processes tend usually to renewal processes with negative-exponential distributions as n tends to infinity. To get an idea about the speed of that convergence, we shall examine how fast the correlation between two consecutive intervals, U, V, determined by three consecutive arrival epochs tends to zero (arrival epoch = epoch at which a renewal occurs in any one of the n processes).

Let $F(\cdot)$ be the cumulative distribution function of the intervals separating pairs of consecutive renewal epochs from the same process. Under some mild restrictions, namely that $F(\cdot)$ and its first derivative are continuous in some closed interval $[0, \delta]$, $\delta > 0$, it can be shown that:

$$\lim_{n\to\infty} n^3 \operatorname{Cov}(U,V) = \gamma \tag{18}$$

where γ, a constant, is strictly positive if $F'(0) > 1$, strictly negative if $F'(0) < 1$, and zero if $F'(0) = 1$. [Here $F'(0)$ has to be understood as being the right-hand derivative of $F(\cdot)$ at 0.] Further:

$$\operatorname{Cov}(U,V) = O(n^{-4}), \qquad n \to \infty, \qquad \text{when } F'(0) = 1. \tag{19}$$

Let $\operatorname{Var}(X)$ stand for the variance of X and $\rho(X,Y)$ for the coefficient of correlation of X and Y.

We have here $\operatorname{Var}(U) = \operatorname{Var}(V)$ and

$$\lim_{n\to\infty} n^2 \operatorname{Var}(U) = \sigma$$

where σ is a strictly positive constant.

We have, therefore:

$$\lim_{n\to\infty} n \cdot \rho(U,V) = k \tag{20}$$

where k, a constant, is strictly positive if $F'(0) > 1$, strictly negative if $F'(0) < 1$, and zero if $F'(0) = 1$. Further:

$$\rho(U,V) = O(n^{-2}), \quad n \to \infty, \quad \text{when } F'(0) = 1. \tag{21}$$

We proceed now with the proof of (18) and (19). Assuming that $EU = EV = 1$, we have:

$$\operatorname{Cov}(U,V) + n^{-2} = \int_0^\infty \int [1 - F(u)][1 - F(v)] \exp[(n-1)H(u+v)] \cdot du \cdot dv$$

with

$$H(x) = \log \int_x^\infty [1 - F(t)]\, dt.$$

Let now $u = (y + z)/2$, $v = (z - y)/2$. Then:

$$\operatorname{Cov}(U,V) + n^{-2} = \int_0^\infty \exp[(n-1)H(z)] \int_0^z \left[1 - F\left(\frac{z+y}{2}\right)\right]\left[1 - F\left(\frac{z-y}{2}\right)\right] \cdot dz \cdot dy.$$

We note, however, that:

$$x \log \int_{x^{-(\frac{1}{2}+\epsilon)}}^{\infty} [1 - F(t)]\, dt = - \left| 0(x^{\frac{1}{2}-\epsilon}) \right|, \quad 0 \leqq \epsilon < \tfrac{1}{2}, \quad x \to \infty$$

so that:

$$\mathrm{Cov}(U,V) + n^{-2} \sim \int_0^{(n-1)^{-(\frac{1}{2}+\epsilon)}} \exp\,[(n-1)H(z)]$$

$$\cdot \int_0^z \left[1 - F\left(\frac{z+y}{2}\right)\right]\left[1 - F\left(\frac{z-y}{2}\right)\right] \cdot dz \cdot dy$$

$$= \int_0^{(n-1)^{-(\frac{1}{2}+\epsilon)}} \exp\,[(n-1)H(z)]\,[z - F'(\theta z)z^2 + O(z^3)] \cdot dz$$

$$0 \leqq \theta \leqq 1, \qquad 0 \leqq \epsilon < \tfrac{1}{2}.$$

Upon expanding exp $(n - 1)H(z)$ and making the substitution $t = (n - 1)z$, we find:

$$\mathrm{Cov}(U,V) + n^{-2}$$

$$\sim \frac{1}{(n-1)^2} \int_0^{(n-1)^{\frac{1}{2}-\epsilon}} e^{-t} \left\{1 + \frac{F'(\theta' t/n-1) - 1}{2}\right.$$

$$\left. \cdot \frac{t^2}{n-1} + t^3 O(n^{-2})\right\} \left[t - F'(\theta t/n-1)\,\frac{t^2}{n-1} + t^3 O(n^{-2})\right] \cdot dt$$

where $0 \leqq \theta' \leqq 1$.

We can therefore conclude that:

$$\mathrm{Cov}(U,V) \sim \frac{1}{(n-1)^2} - \frac{2F'(0)}{(n-1)^3} + \frac{3[F'(0) - 1]}{(n-1)^3} - \frac{1}{n^2}$$

$$+ O(n^{-4}) = \frac{F'(0) - 1}{(n-1)^3} + O(n^{-4}).$$

This relation implies (18) and (19)

Finally, we shall state the following three properties:

i. If $F(\cdot)$ is the uniform distribution, then $\mathrm{Cov}(U,V) < 0$ for all values of $n(\geqq 2)$.

ii. For any given $n(\geqq 2)$, it is always possible to choose $F(\cdot)$ in such a way that the correlation between U and V vanishes while the variables U and V are not independent.

iii. Let $n = 2$ and assume that:

$$F(x) = 0, \qquad x < \alpha$$

$$F(x) = p, \qquad \alpha \leqq x < \beta$$

$$F(x) = 1, \qquad \alpha \geqq \beta$$

where $\alpha p + \beta(1 - p) = 1$.

Then $\mathrm{Cov}(U,V) = 0$ if and only if

$$6p\alpha(1 - \alpha)^2 + 8(p\alpha)^2(1 - \alpha) + 2(p\alpha)^2(p\alpha - 1) - 1 = 0$$

When this condition is satisfied, $\rho(U,V) = 0$ although U and V are dependent.

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