On the Theory of Linear Multi-Loop Feedback Systems

By I. W. SANDBERG

(Manuscript received October 19, 1962)

It is well known that the concept of return difference plays a central role in the classical theory of linear feedback systems developed by Black, Nyquist, Blackman and Bode. This concept, which relates to the influence of a single algebraic system-constraint of the form $f_2 = \gamma f_1$ *where* f_1 *and* f_2 *respectively may be regarded as a controlling signal and a controlled signal, retains its prominence in the subsequent signal-flow graph theoretic extensions by Mason. It is particularly pertinent to the study of the stability of the system, its degree of immunity from parameter variations, and the determination of its transmission and driving-point properties.*

This paper reports on a generalization of Blackman's equation and on some generalizations of Bode's return difference theorems. Here attention is focused not on a single constraint of the form $f_2 = \gamma f_1$, *but instead on a set of constraining equations.*

I. INTRODUCTION

It is well known that the concept of return difference plays a central role in the classical theory of linear feedback systems developed by Black,[1] Nyquist,[2] Blackman[3] and Bode.[4] This concept, which relates to the influence of a single algebraic system-constraint of the form $f_2 = \gamma f_1$ where f_1 and f_2 respectively may be regarded as a controlling signal and a controlled signal, retains its prominence in the subsequent signal-flow graph theoretic extensions by Mason.[5] It is particularly pertinent to the study of the stability of the system, its degree of immunity from parameter variations, and the determination of its transmission and driving-point properties.

This paper reports on a generalization of Blackman's equation[3] and on some generalizations of Bode's return difference theorems.[4] Here attention is focused not on a single constraint of the form $f_2 = \gamma f_1$, but instead on a set of constraining equations. For this reason the results are applicable to systems which possess in the usual physical sense a multiplicity of feedback loops. However it will become clear that the results are not restricted to situations of this type.

In Section II we describe the basic system considered throughout the paper. Section III presents an explicit expression for $w(\mathbf{X})$, the transmission or driving-point function to be studied. We then introduce definitions, based on a simple topological characterization of the relation between the system input and output variables, of the loop-difference matrix, the null loop-difference matrix, and the complementary loop-difference matrix. The determinants of these matrices are of fundamental importance in the subsequent discussion. In Section VI we derive a generalization of Blackman's interesting equation. The material in Sections VII and VIII relates to generalizations of Bode's well-known return difference theorems. Some applications and illustrations of the theory are discussed in Section IX.

II. THE BASIC SYSTEM AND THE SET OF EQUATIONS $\mathfrak{F}$

We shall be concerned throughout with the transmission and driving-point properties of the structure shown in Fig. 1, an arbitrary linear time-invariant two-port network containing no independent sources.

The Laplace-transformed equilibrium equations which implicitly define the external properties of the two-port are of the form:

$$\sum_j (\alpha_{kj} e_j + \beta_{kj} i_j) = 0 \qquad (k = 1, 2, \cdots, K) \tag{1}$$

where e_j and i_j respectively are branch voltages and currents and the α_{kj} and β_{kj} are functions of the complex-frequency variable.

We wish to focus attention on the influence of a prescribed subset of the linear constraints implied by (1). This subset is assumed to be expressible as

$$f_k = \sum_{i=q+1}^{q+p} \gamma_{ki} f_i \qquad (k = 1, 2, \cdots, q) \tag{2}$$

in which each f_l is one of the e_j or one of the i_j. It is convenient to interpret these relations as corresponding to a set of q controlled (i.e.,

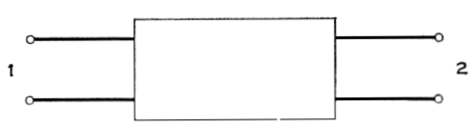

Fig. 1 — Two-port network.

dependent) sources, where the quantity on the left-hand side of each equation is a controlled-source output and each f_i on the right-hand side is a controlling variable. Thus we take the viewpoint that the two-port network contains a controlled-source subnetwork characterized by (2) even when the two-port does not contain devices such as vacuum tubes or transistors with which controlled-sources are ordinarily associated. We shall denote by $\mathfrak{F}$ the set of controlled-source constraints (2).

Let the components of a q-vector $\mathbf{\Psi}$ be the f_k $(k = 1, 2, \cdots, q)$ arranged in any one of the $q!$ possible orders, and let the components of a p-vector $\mathbf{\Phi}$ be the controlling variables f_i $(i = q + 1, q + 2, \cdots, q + p)$ arranged in any one of the $p!$ possible orders. Then (2) can be written as $\mathbf{\Psi} = \mathbf{X\Phi}$. We shall call the $q \times p$ matrix $\mathbf{X}$ the controlled-source matrix or the matrix of controlled-source coefficients.

III. EVALUATION OF THE TRANSFER AND DRIVING-POINT FUNCTIONS FOR THE TWO-PORT

Our primary concern here is the determination of the influence of $\mathfrak{F}$ (i.e., of the controlled-source subnetwork) on the transfer and driving-point functions for the two-port network.

Let $y_2 = w(\mathbf{X})y_1$ represent the relation to be studied in which y_2 and y_1 respectively are response and excitation functions (each function may be either a voltage or a current). If $w(\mathbf{X})$ is a driving-point immittance, y_2 and y_1 respectively are a current and a voltage or a voltage and a current at the same port.

Consider now an evaluation of $w(\mathbf{X})$. With y_1 and the components of $\mathbf{\Psi}$ treated as independent variables, we apply the superposition theorem to obtain†

$$y_2 = dy_1 + \mathbf{B\Psi} \tag{3}$$

where d and $\mathbf{B}$ are defined by the equation. In particular when $\mathbf{X} = \mathbf{0}$, $\mathbf{\Psi} = \mathbf{0}$, and $y_2 = dy_1$. That is, $d = w(\mathbf{0})$.

Similarly we can express $\mathbf{\Phi}$ as

$$\mathbf{\Phi} = \mathbf{A}y_1 + \mathbf{C\Psi} \tag{4}$$

where the matrices $\mathbf{A}$ and $\mathbf{C}$ are defined by the equation. After premultiplying both sides of (4) by $\mathbf{X}$ and using $\mathbf{\Psi} = \mathbf{X\Phi}$, we find that

$$\mathbf{\Psi} = [\mathbf{1}_q - \mathbf{XC}]^{-1}\mathbf{XA}y_1 \tag{5}$$

where $\mathbf{1}_q$ is the identity matrix of order q.

† We are assuming that y_2 is uniquely determined by y_1 and Ψ. See the relevant discussion in the next section.

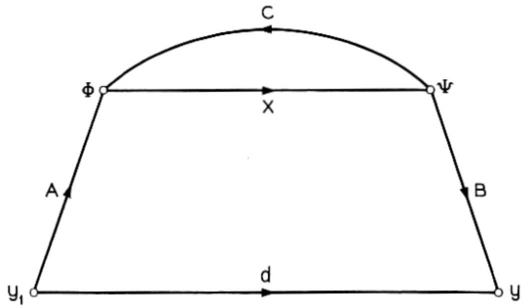

Fig. 2 — Basic flow-graph representation of the relation between y_1 and y_2.

From (3), (5), and $d = w(\mathbf{0})$

$$w(\mathbf{X}) = y_2(y_1)^{-1} = w(\mathbf{0}) + \mathbf{B}[\mathbf{1}q - \mathbf{XC}]^{-1}\mathbf{XA}. \tag{6}$$

IV. THE BASIC FLOW GRAPH

In order to express $w(\mathbf{X})$ in terms of quantities that are related to concepts of importance in the classical theory of linear feedback systems, we first represent the relation between y_1 and y_2 by the signal-flow graph[5] in Fig. 2. The matrix equation associated with "node-group" $\mathbf{\Phi}$ is (4), and so forth. This topological characterization of the influence of $\mathfrak{F}$ plays a central role in the subsequent discussion. We shall call **A**, **B**, **C**, and d "flow-matrices."

It is worth emphasizing at the outset that although the signal-flow graph exhibits feedback in the sense that the outputs of the controlled sources (i.e., the elements of $\mathbf{\Psi}$) influence the value of the controlling variables (i.e., the elements of $\mathbf{\Phi}$), it is not necessary that feedback exist in the network in the usual physical sense. For example, the physical system may comprise simply a set of driving-point impedances and the controlled sources may represent a certain subset of these one-port elements. Feedback arises in the topological characterization merely because of the form of the equations that we have chosen to write.

Before proceeding it is important to note that for some choices of $\mathfrak{F}$ it may not be possible to characterize the two-port by equations of the form (3) and (4) and hence by the flow graph in Fig. 2. The superposition theorem implies that when y_2 and $\mathbf{\Phi}$ are uniquely determined by y_1 and $\mathbf{\Psi}$, the relations are of the form (3) and (4); it does not imply that such relations exist. The one-port network shown in Fig. 3 is one

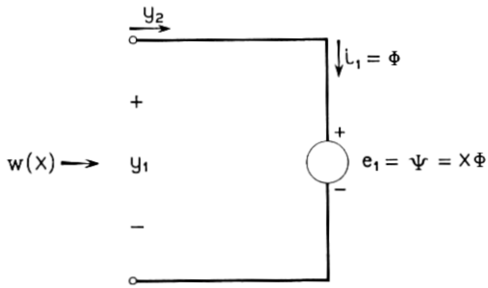

Fig. 3 — Single controlled-source one-port.

of the simplest structures which illustrate the difficulty. For this network, with $w(\mathbf{X})$ the well-defined driving-point admittance, all four flow matrices **A**, **B**, **C**, and d fail to exist.

There are much more sophisticated situations of this general type. Nevertheless, almost all feedback networks of interest do not exhibit this degeneracy. Furthermore when the difficulty does occur, it is generally possible to consider the limiting form of a network for which the degeneracy is not present. For example, if the one-port in Fig. 3 is modified by adding a series resistor, the flow matrices **A**, **B**, **C**, and d become well-defined and the expression obtained for the driving-point admittance reduces to $(\mathbf{X})^{-1}$ as the value of the series resistor approaches zero.

With this motivation we state the

Assumption: It is assumed throughout that the relation between y_1 and y_2 can be represented by a signal-flow graph of the type shown in Fig. 2.

Consider now the definition of three matrices that are related to the flow graph.

4.1 *The Loop-Difference Matrix for the Branch* $\mathbf{X}$: $\mathbf{F}_{\mathfrak{F}}(\mathbf{X}_1)$

The loop-difference matrix for the branch **X** is defined as follows. Introduce an additional node-group **P** in the flow-graph of Fig. 2 by replacing **X** with a cascade of any two branches $\mathbf{X}_1$ and $\mathbf{X}_2$ such that $\mathbf{X}_1\mathbf{X}_2 = \mathbf{X}$. Let the orders of $\mathbf{X}_1$ and $\mathbf{X}_2$ respectively be $q \times m$ and $m \times p$. Next, break the feedback loop by splitting **P** into a source node-group **P′** and a sink node-group **P″** to obtain the graph in Fig. 4.

With y_1 set equal to zero, suppose that an arbitrary signal vector $\mathbf{S}_m$, of order m, is applied to **P′**. The resulting signals at node-groups

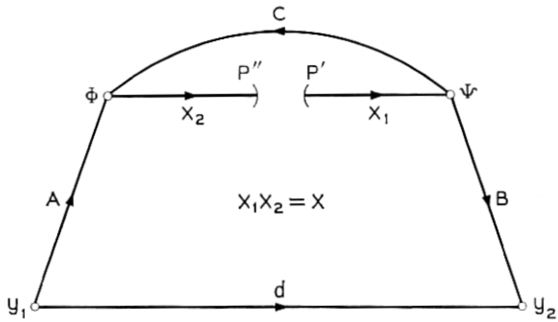

Fig. 4 — Signal-flow graph for defining the loop-difference matrix and the null loop-difference matrix.

$\mathbf{\Psi}$, $\mathbf{\Phi}$, and $\mathbf{P}''$ respectively are $\mathbf{X}_1\mathbf{S}_m$, $\mathbf{C}\mathbf{X}_1\mathbf{S}_m$, and $\mathbf{X}_2\mathbf{C}\mathbf{X}_1\mathbf{S}_m$. The vector signal difference at $\mathbf{P}',\mathbf{P}''$ (i.e., the signal at $\mathbf{P}'$ less that at $\mathbf{P}''$) is $[\mathbf{1}_m - \mathbf{X}_2\mathbf{C}\mathbf{X}_1]\mathbf{S}_m$. We shall call $[\mathbf{1}_m - \mathbf{X}_2\mathbf{C}\mathbf{X}_1]$ the loop-difference matrix for the branch $\mathbf{X}$ and denote it by $\mathbf{F}_{\mathfrak{F}}(\mathbf{X}_1)$, which indicates explicitly its dependence on $\mathbf{X}_1$. When $\mathbf{X}_1$, $\mathbf{X}_2$, and $\mathbf{C}$ are scalars, $\mathbf{F}_{\mathfrak{F}}(\mathbf{X}_1)$ becomes independent of $\mathbf{X}_1$ and reduces to Mason's flow-graph definition of the loop-difference for the branch $\mathbf{X}$.

4.2 *The Null Loop-Difference Matrix for the Branch* X: $\hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$

This matrix is evaluated under the condition that y_1 is adjusted so that $y_2 = 0$. Specifically, if an arbitrary signal vector $\mathbf{S}_m$ is applied to $\mathbf{P}'$ in Fig. 4, the signal reaching y_2 by way of the branch $\mathbf{B}$ is $\mathbf{B}\mathbf{X}_1\mathbf{S}_m$ so that, if y_2 is to be zero, y_1 must be $-d^{-1}\mathbf{B}\mathbf{X}_1\mathbf{S}_m$. The total signal arriving at $\mathbf{\Phi}$ is $-d^{-1}\mathbf{A}\mathbf{B}\mathbf{X}_1\mathbf{S}_m + \mathbf{C}\mathbf{X}_1\mathbf{S}_m$, and hence the signal at $\mathbf{P}''$ is $[\mathbf{X}_2\mathbf{C}\mathbf{X}_1 - d^{-1}\mathbf{X}_2\mathbf{A}\mathbf{B}\mathbf{X}_1]\mathbf{S}_m$. Thus the vector signal difference at $\mathbf{P}',\mathbf{P}''$, under the condition that y_1 is adjusted so that $y_2 = 0$, is $[\mathbf{1}_m - \mathbf{X}_2\mathbf{C}\mathbf{X}_1 + d^{-1}\mathbf{X}_2\mathbf{A}\mathbf{B}\mathbf{X}_1]\mathbf{S}_m$. We call $[\mathbf{1}_m - \mathbf{X}_2\mathbf{C}\mathbf{X}_1 + d^{-1}\mathbf{X}_2\mathbf{A}\mathbf{B}\mathbf{X}_1]$ the null loop-difference matrix for the branch $\mathbf{X}$ and denote it by $\hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$. When $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{X}_1$, and $\mathbf{X}_2$ are all scalars, $\hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$ reduces to Truxal's definition[6] of the null loop-difference for the branch $\mathbf{X}$, and is independent of the choice of $\mathbf{X}_1$. It is convenient to write: $\hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1) = [\mathbf{1}_m - \mathbf{X}_2\hat{\mathbf{C}}\mathbf{X}_1]$, where $\hat{\mathbf{C}} = \mathbf{C} - d^{-1}\mathbf{A}\mathbf{B}$.

4.3 *The Complementary Loop-Difference Matrix for the Branch* X: $\bar{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$

The complementary loop-difference matrix, denoted by $\bar{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$, is defined by the requirement that when an arbitrary $\mathbf{S}_m$ is applied to

$\mathbf{P}'$, $\bar{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)\mathbf{S}_m$ is the vector signal difference at $\mathbf{P}',\mathbf{P}''$ under the condition that y_1 is adjusted so that $(y_1 + y_2) = 0$. This matrix is therefore obtained from the expression for the null loop-difference matrix by setting $d = 1$. That is, $\bar{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1) = [\mathbf{1}_m - \mathbf{X}_2\bar{\mathbf{C}}\mathbf{X}_1]$ where $\bar{\mathbf{C}} = \mathbf{C} - \mathbf{AB}$. The complementary loop-difference matrix is of utility when $d = 0$. In such cases the null loop-difference matrix is not defined.

It is evident that the concepts of loop-difference matrix, null loop-difference matrix, and complementary loop-difference matrix need not be restricted to the branch $\mathbf{X}$ in Fig. 2; they relate without ambiguity to any branch in a signal-flow graph which possesses a single input node and a single output node.

4.4 *Circuit Interpretations of* $\mathbf{C}$, $\hat{\mathbf{C}}$, *and* $\bar{\mathbf{C}}$

It is important to note that the matrices $\mathbf{C}$, $\hat{\mathbf{C}}$, and $\bar{\mathbf{C}}$ possess very explicit circuit interpretations. When the elements of $\mathbf{\Psi}$ (i.e., the outputs of the controlled sources) are treated as independent variables, the p-vector of voltages and currents $\mathbf{\Phi}$ is equal to $\mathbf{C\Psi}$, $\hat{\mathbf{C}}\mathbf{\Psi}$, and $\bar{\mathbf{C}}\mathbf{\Psi}$ respectively when y_1 is set equal to zero, y_1 is adjusted so that $y_2 = 0$, and y_1 is adjusted so that $(y_1 + y_2) = 0$. The evaluation of these matrices from the circuit is simplified considerably by the fact that the controlled sources are treated as independent sources.

V. THE DETERMINANTS det $\mathbf{F}_{\mathfrak{F}}(\mathbf{X}_1)$, det $\hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$, AND det $\bar{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$

Our primary interest in the matrices $\mathbf{F}_{\mathfrak{F}}(\mathbf{X}_1)$, $\hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$, and $\bar{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$ is with regard to their determinants. Here we wish to establish two elementary properties of these determinants which add to an understanding of the character of the expression for $w(\mathbf{X})$ presented in the next section. Further properties of these determinants are considered subsequent to the derivation of the expression for $w(\mathbf{X})$.

Although the matrices $\mathbf{F}_{\mathfrak{F}}(\mathbf{X}_1)$ and $\hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$ generally depend upon the choice of $\mathbf{X}_1$, it is true that

Lemma I:

$$det\, \mathbf{F}_{\mathfrak{F}}(\mathbf{X}_1) = det\, \mathbf{F}_{\mathfrak{F}}(\mathbf{1}_q)$$

$$det\, \hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1) = det\, \hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{1}_q).$$

The lemma is proved in Appendix A. It implies, of course, that the determinants are independent of the choice of $\mathbf{X}_1$. This property is evidently shared by det $\bar{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$ since $\bar{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$ can be obtained formally from $\hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$ by setting $d = 1$.

Lemma I is often of assistance in evaluating the determinants. For example, if the normal rank of $\mathbf{X}$ is unity, as is the case when $\mathfrak{F}$ contains a single equation, $\mathbf{X}$ can be written as $\mathbf{X}_1\mathbf{X}_2$ where $\mathbf{X}_1$ and $\mathbf{X}_2$ respectively are $q \times 1$ and $1 \times p$ matrices. In such instances the determinants of the loop-difference matrix, null loop-difference matrix, and complementary loop-difference matrix can be expressed respectively as simply $1 - \mathbf{X}_2\mathbf{C}\mathbf{X}_1$, $1 - \mathbf{X}_2\hat{\mathbf{C}}\mathbf{X}_1$, and $1 - \mathbf{X}_2\bar{\mathbf{C}}\mathbf{X}_1$. Similar simplifications can of course be exploited if $\mathbf{C}$, $\hat{\mathbf{C}}$, or $\bar{\mathbf{C}}$ is of unit normal rank.

With the result stated in Lemma I as motivation, we shall throughout the remainder of the paper denote det $\mathbf{F}_{\mathfrak{F}}(\mathbf{X}_1)$, det $\hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$, and det $\bar{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$ respectively by det $\mathbf{F}_{\mathfrak{F}}$, det $\hat{\mathbf{F}}_{\mathfrak{F}}$, and det $\bar{\mathbf{F}}_{\mathfrak{F}}$.

Recall that the components of $\mathbf{\Psi}$ and $\mathbf{\Phi}$ respectively are the controlled variables and controlling variables arranged in any definite orders. Thus the matrices $\mathbf{X}$, $\mathbf{F}_{\mathfrak{F}}(\mathbf{X}_1)$, and $\hat{\mathbf{F}}_{\mathfrak{F}}(\mathbf{X}_1)$ are not uniquely defined by the details of the two-port structure and the set of controlled-source constraints $\mathfrak{F}$. Nevertheless,

Lemma II: det $\mathbf{F}_{\mathfrak{F}}$ *and det* $\hat{\mathbf{F}}_{\mathfrak{F}}$ *are invariant with respect to the ordering of the components of* $\mathbf{\Psi}$ *and* $\mathbf{\Phi}$.

In other words, the determinants *are* uniquely defined by the details of the two-port and $\mathfrak{F}$. The proof of this important result is straightforward: Let $\mathbf{\Phi}'$ and $\mathbf{\Psi}'$ respectively denote vectors obtained from $\mathbf{\Phi}$ and $\mathbf{\Psi}$ by any reordering of the components. Then $\mathbf{\Phi}' = \mathbf{U}\mathbf{\Phi}$ and $\mathbf{\Psi}' = \mathbf{V}\mathbf{\Psi}$, where $\mathbf{U}$ and $\mathbf{V}$ are nonsingular matrices. In terms of $\mathbf{\Phi}'$ and $\mathbf{\Psi}'$ the controlled-source constraints read $\mathbf{\Psi}' = \mathbf{X}'\mathbf{\Phi}'$ where $\mathbf{X}' = \mathbf{V}\mathbf{X}\mathbf{U}^{-1}$, and the equations corresponding to (3) and (4) are

$$y_2 = dy_1 + \mathbf{B}'\mathbf{\Psi}' = dy_1 + \mathbf{B}\mathbf{V}^{-1}\mathbf{\Psi}' \tag{7}$$

$$\mathbf{\Phi}' = \mathbf{A}'y_1 + \mathbf{C}'\mathbf{\Psi}' = \mathbf{U}\mathbf{A}y_1 + \mathbf{U}\mathbf{C}\mathbf{V}^{-1}\mathbf{\Psi}'. \tag{8}$$

Finally, note that for all $d \neq 0$

$$\begin{aligned}\det [\mathbf{1}_q - \mathbf{X}'(\mathbf{C}' - d^{-1}\mathbf{A}'\mathbf{B}')] &= \det [\mathbf{1}_q - \mathbf{V}\mathbf{X}\mathbf{U}^{-1}(\mathbf{U}\mathbf{C}\mathbf{V}^{-1} - d^{-1}\mathbf{U}\mathbf{A}\mathbf{B}\mathbf{V}^{-1})] \\ &= \det [\mathbf{1}_q - \mathbf{X}(\mathbf{C} - d^{-1}\mathbf{A}\mathbf{B})].\end{aligned}$$

VI. GENERALIZATION OF BLACKMAN'S EQUATION

At this point we are in a position to state and prove the following generalization of Blackman's classical result.

Theorem I: If $w(\mathbf{0}) \neq 0$,

$$w(\mathbf{X}) = w(\mathbf{0}) \frac{det\ \hat{\mathbf{F}}_{\mathfrak{F}}}{det\ \mathbf{F}_{\mathfrak{F}}}.$$

6.1 *Proof*

Consider equation (6), which for convenience is repeated below:

$$w(\mathbf{X}) = y_2(y_1)^{-1} = w(\mathbf{0}) + \mathbf{B}[\mathbf{1}_q - \mathbf{XC}]^{-1}\mathbf{XA}. \tag{9}$$

Recall that adj $(\mathbf{M})$, the adjoint of an arbitrary square matrix $\mathbf{M}$ of order q, is defined by

$$\mathbf{M}\,\mathrm{adj}\,(\mathbf{M}) = \mathrm{adj}\,(\mathbf{M})\mathbf{M} = \mathbf{1}_q \det \mathbf{M}.$$

Thus $w(\mathbf{X})$ can be expressed as

$$\begin{aligned} w(\mathbf{X}) &= w(\mathbf{0}) + \mathbf{B}\,\mathrm{adj}\,(\mathbf{1}_q - \mathbf{XC})\mathbf{XA}[\det\,(\mathbf{1}_q - \mathbf{XC})]^{-1} \\ &= w(\mathbf{0})\left[\frac{\det\,[\mathbf{1}_q - \mathbf{XC}] + w(\mathbf{0})^{-1}\mathbf{B}\,\mathrm{adj}\,(\mathbf{1}_q - \mathbf{XC})\mathbf{XA}}{\det\,[\mathbf{1}_q - \mathbf{XC}]}\right]. \end{aligned}$$

Since $\det \mathbf{F}_{\mathfrak{F}} = \det\,[\mathbf{1}_q - \mathbf{XC}]$, we must prove that $\det\,[\mathbf{1}_q - \mathbf{XC}] + w(\mathbf{0})^{-1}\mathbf{B}\,\mathrm{adj}\,(\mathbf{1}_q - \mathbf{XC})\mathbf{XA} = \det \hat{\mathbf{F}}_{\mathfrak{F}}$ or, more explicitly, that

$$\begin{aligned} \det\,[\mathbf{1}_q - \mathbf{XC}] &+ w(\mathbf{0})^{-1}\mathbf{B}\,\mathrm{adj}\,(\mathbf{1}_q - \mathbf{XC})\mathbf{XA} \\ &= \det\,[\mathbf{1}_q - \mathbf{X}(\mathbf{C} - w(\mathbf{0})^{-1}\mathbf{AB})]. \end{aligned} \tag{10}$$

We first prove the following result which will be used also in a later section.

Lemma III: Let $\mathbf{G}$ *and* $\mathbf{H}$ *respectively be nonsingular matrices of orders* n *and* m, *and let* $\mathbf{I}$ *and* $\mathbf{J}$ *respectively denote matrices of orders* $n \times m$ *and* $m \times n$. *Then,*

$$det[\mathbf{G} + \mathbf{IH}^{-1}\mathbf{J}]det\,\mathbf{H} = det[\mathbf{H} + \mathbf{JG}^{-1}\mathbf{I}]det\,\mathbf{G}.$$

The proof follows from the lemma established in Appendix A which states that $\det\,[\mathbf{1}_n + \mathbf{DE}] = \det\,[\mathbf{1}_m + \mathbf{ED}]$ for arbitrary matrices $\mathbf{D}$ and $\mathbf{E}$ respectively of orders $n \times m$ and $m \times n$. Taking $\mathbf{D} = \mathbf{G}^{-1}\mathbf{I}$ and $\mathbf{E} = \mathbf{H}^{-1}\mathbf{J}$, we have: $\det\,[\mathbf{1}_n + \mathbf{G}^{-1}\mathbf{IH}^{-1}\mathbf{J}] \det \mathbf{H} \det \mathbf{G} = \det\,[\mathbf{1}_m + \mathbf{H}^{-1}\mathbf{JG}^{-1}\mathbf{I}] \det \mathbf{G} \det \mathbf{H}$. Moreover, $\det\,[\mathbf{1}_n + \mathbf{G}^{-1}\mathbf{IH}^{-1}\mathbf{J}] \det \mathbf{G} = \det\,[\mathbf{G} + \mathbf{IH}^{-1}\mathbf{J}]$, and $\det\,[\mathbf{1}_m + \mathbf{H}^{-1}\mathbf{JG}^{-1}\mathbf{I}] \det \mathbf{H} = \det\,[\mathbf{H} + \mathbf{JG}^{-1}\mathbf{I}]$.

The identity (10) is a direct consequence of the following corollary of Lemma III.

Lemma IV: Let $\mathbf{G}$, $\mathbf{I}$, *and* $\mathbf{J}$ *respectively denote matrices of orders* $n \times n$, $n \times 1$, *and* $1 \times n$. *Then,*

$$det[\mathbf{G} + \mathbf{IJ}] = det\,\mathbf{G} + \mathbf{J}\,adj(\mathbf{G})\mathbf{I}.$$

To establish this result, consider Lemma III with $\mathbf{H} = 1$ and $m = 1$. Clearly when $\mathbf{G}$ is nonsingular, $\det\,[\mathbf{G} + \mathbf{IJ}] = (1 + \mathbf{JG}^{-1}\mathbf{I}) \det \mathbf{G} =$

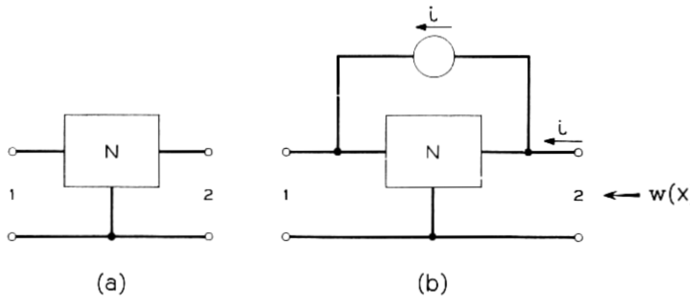

Fig. 5 — (a) Network with the open-circuit transfer impedance from port 1 to port 2 equal to $w(\mathbf{X})$; (b) modified network with driving-point impedance $w(\mathbf{X})$.

det $\mathbf{G} + \mathbf{J}$ adj $(\mathbf{G})\mathbf{I}$. A continuity argument of the type used in Appendix A shows that Lemma IV is valid also when $\mathbf{G}$ is singular.

Expression (10) can be obtained from Lemma IV by taking $\mathbf{G} = (\mathbf{1}_q - \mathbf{XC})$, $\mathbf{I} = \mathbf{XA}$, and $\mathbf{J} = w(\mathbf{0})^{-1}\mathbf{B}$. This completes the proof of Theorem I.

6.2 *Remarks Relating to Theorem I*

When $w(\mathbf{X})$ is a driving-point impedance function, the null loop-difference matrix is evaluated under the condition that the current at the terminal pair at which the impedance is defined is adjusted so that the voltage at the terminal pair is zero. Thus, in this case, the null loop-difference matrix is equal to the loop-difference matrix evaluated from the network when the terminal pair is shorted. Similarly when $w(\mathbf{X})$ is a driving-point admittance function, the null loop-difference matrix is equal to the loop-difference matrix evaluated from the network when the terminal pair is open-circuited.

The conceptual and computational simplifications mentioned in the last paragraph are not valid if $w(\mathbf{X})$ is a transfer function.† However, by introducing an additional controlled source, the evaluation of the open-circuit transfer impedance or short-circuit transfer admittance of any three-terminal network can be reduced to the evaluation of a driving-point function. This reduction is illustrated in Fig. 5 for the transfer impedance case.

† Of course we assume here that $w(\mathbf{X})$ is not both a transfer function and a driving-point function associated with the two-port in Fig. 1.

6.3 *Relation of Theorem I to Earlier Work*

The expression given in Theorem I reduces to Blackman's classical result[3] when it is applied to the driving-point immittance case in which $\mathfrak{F}$ characterizes a single unilateral amplifier, and the equation is expressed in terms of physically-defined open-circuit and short-circuit return differences.

A particularly succinct derivation of an equation of this form (for the single amplifier case) was subsequently presented by Bode[4] who exploited the properties of the determinant of the network immittance matrix. In addition Bode introduced the useful concept of return difference with respect to a two-terminal element. Signal-flow graph interpretations of Blackman's expression were later considered by Mason[5] and by Truxal.[6] Our proof of Theorem I is a generalization of Truxal's work.

Exploiting a suggestion by Bode,[4] Mulligan[7] has stated an entirely different generalization† of Blackman's equation for cases in which a multiplicity of nonreciprocal elements are to be considered. His result is an explicit expression obtained by repeated application of Blackman's original result.

6.4 *Generalization of Bode's Relation between Feedback and Impedance*

Theorem II: Let det $\mathbf{F}'_{\mathfrak{F}}$ *denote the determinant of the loop-difference matrix for the set of controlled-source constraints* $\mathfrak{F}$, *evaluated under the condition that* $y_1 = \eta^{-1}y_2$. *Let* $w(\mathbf{X}) - w(\mathbf{0})$ *not vanish identically in* $\mathbf{X}$. *Then det* $\mathbf{F}'_{\mathfrak{F}}$ *is a linear-fractional function of* η *that vanishes in* η *if and only if* $\eta = w(\mathbf{X})$.

The proof is based on Lemma IV: From the signal-flow graph obtained from Fig. 2 by adding a branch from y_2 to y_1 with transmission η^{-1}, it is clear that

$$\det \mathbf{F}'_{\mathfrak{F}} = \det \{\mathbf{1}_q - \mathbf{X}[\mathbf{C} + \mathbf{AB}(\eta - d)^{-1}]\}.$$

Using Lemma IV,

$$\det \mathbf{F}'_{\mathfrak{F}} = \det [\mathbf{1}_q - \mathbf{XC}] - \mathbf{B} \operatorname{adj} (\mathbf{1}_q - \mathbf{XC})\mathbf{XA}(\eta - d)^{-1}$$

Thus the equation $\det \mathbf{F}'_{\mathfrak{F}} = 0$ implies that $\eta = w(\mathbf{X})$, assuming that

† Subsequent to preparing this paper and soliciting comments from colleagues, it came to the writer's attention that a result similar to Theorem I is contained in the recently published book by Yutze Chow and Etienne Cassignol: Linear Signal-Flow Graphs and Applications, John Wiley and Sons, New York, 1962. They consider the situation that corresponds here to the special case in which $\mathfrak{F}$ is a set of equations of the form $f_k = \gamma_k f_{k+q}$ $(k = 1, 2, \ldots, q)$. Their proof is considerably different from ours.

$\mathbf{B}$ adj $(\mathbf{1}_q - \mathbf{XC})\mathbf{XA} = w(\mathbf{X}) - w(\mathbf{0})$ does not vanish identically in $\mathbf{X}$.

Of course when $w(\mathbf{X})$ is a driving-point impedance function, the condition $y_1 = \eta^{-1}y_2$ corresponds to adding an impedance $-\eta$ in series with the one-port.[4]

6.5 *The Expression for $w(\mathbf{X})$ when $w(\mathbf{0}) = 0$*

Theorem I does not apply when $w(\mathbf{0}) = 0$. In fact, in all such instances the null loop-difference matrix does not exist. The following corollary which involves the complementary loop-difference matrix is of assistance in these cases.

Corollary I: If $w(\mathbf{0}) = 0$,

$$w(\mathbf{X}) = \frac{\det \bar{\mathbf{F}}_{\mathfrak{F}} - \det \mathbf{F}_{\mathfrak{F}}}{\det \mathbf{F}_{\mathfrak{F}}}.$$

This result is readily established: Here $d = 0$ in the flow graph of Fig. 2. Suppose that the branch d is replaced with one having unit transmission. Then the ratio of y_2 to y_1 for the resulting graph is $w(\mathbf{X}) + 1$. However, using Theorem I and the definition of $\bar{\mathbf{F}}_{\mathfrak{F}}$, $w(\mathbf{X}) + 1 = \det \bar{\mathbf{F}}_{\mathfrak{F}}[\det \mathbf{F}_{\mathfrak{F}}]^{-1}$.

VII. THE MATRICES $\mathbf{F}_{\mathfrak{F}\mathfrak{F}_0}$ AND $\hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0}$, AND THE NOTION OF A "RESIDUAL SET OF EQUATIONS" OBTAINED FROM $\mathfrak{F}$

In this section we generalize the definitions of the loop-difference matrix and the null loop-difference matrix and then introduce the notion of a "residual set of equations" obtained from $\mathfrak{F}$, in order to both facilitate and provide a firm general basis for the subsequent derivation of some fundamental properties of det $\mathbf{F}_{\mathfrak{F}}$ and det $\hat{\mathbf{F}}_{\mathfrak{F}}$. The complementary loop-difference matrix need not be considered separately since it is merely a special case of the null loop-difference matrix. The material presented in this and the next section is in many respects a generalization of Bode's classical theory relating to return difference with respect to a single element.[4]

7.1 *The Matrices $\mathbf{F}_{\mathfrak{F}\mathfrak{F}_0}$ and $\hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0}$*

The relation between y_1 and the quantities $\mathbf{\Phi}$, $\mathbf{\Psi}$, and y_2 clearly remains unchanged if the signal-flow graph shown in Fig. 2 is replaced with the graph in Fig. 6, where $\mathbf{X}_0$ is an arbitrary $q \times p$ matrix. Let the loop-difference matrix and null loop-difference matrix for the branch $(\mathbf{X} - \mathbf{X}_0)$ in Fig. 6 be defined in the same manner as for $\mathbf{X}$ in Fig. 2,

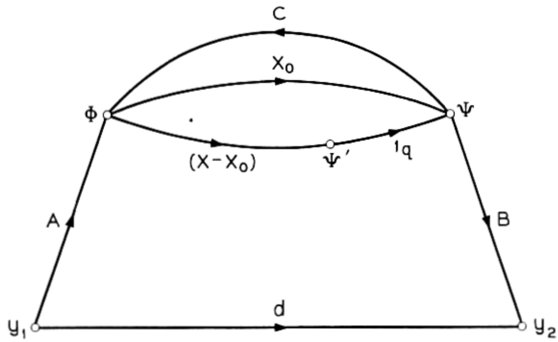

Fig. 6 — Signal-flow graph for the definition of $\mathbf{F}_{\mathfrak{F}\mathfrak{F}_0}$ and $\hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0}$.

and let $\mathfrak{F}_0$ denote the "reference set" of equations obtained from (2) when the controlled-source matrix $\mathbf{X}$ is set equal to $\mathbf{X}_0$. We introduce

Definition I: The matrices $\mathbf{F}_{\mathfrak{F}\mathfrak{F}_0}$ *and* $\hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0}$ *respectively denote the loop-difference matrix and the null loop-difference matrix for the branch* $(\mathbf{X} - \mathbf{X}_0)$ *in Fig.* 6. *When* $\mathbf{X}_0 = \mathbf{0}$, *we write:* $\mathbf{F}_{\mathfrak{F}\mathfrak{F}_0} = \mathbf{F}_{\mathfrak{F}}$, $\hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0} = \hat{\mathbf{F}}_{\mathfrak{F}}$. *The determinants det* $\mathbf{F}_{\mathfrak{F}\mathfrak{F}_0}$ *and det* $\hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0}$ *respectively are referred to as the determinant of the loop-difference matrix and the determinant of the null loop-difference matrix for the set of controlled-source constraints* $\mathfrak{F}$, *with respect to the reference set* $\mathfrak{F}_0$.

The branch $(\mathbf{X} - \mathbf{X}_0)$ in Fig. 6, which corresponds to the matrix relation $\mathbf{\Psi}' = (\mathbf{X} - \mathbf{X}_0)\mathbf{\Phi}$, may be interpreted as characterizing a set of controlled sources which focuses attention on the departure of the elements of $\mathbf{X}$ from those of $\mathbf{X}_0$ or, equivalently, on the departure of $\mathfrak{F}$ from $\mathfrak{F}_0$.

We wish to prove

Theorem III: Let det $\mathbf{F}_{\mathfrak{F}} |_{\mathfrak{F}=\mathfrak{F}_0}$ *and det* $\hat{\mathbf{F}}_{\mathfrak{F}} |_{\mathfrak{F}=\mathfrak{F}_0}$ *respectively denote the determinants obtained from det* $\mathbf{F}_{\mathfrak{F}}$ *and det* $\hat{\mathbf{F}}_{\mathfrak{F}}$ *by replacing each element of* $\mathbf{X}$ *with the corresponding element of* $\mathbf{X}_0$. *Then,*

$$det\ \mathbf{F}_{\mathfrak{F}\mathfrak{F}_0} = \frac{det\ \mathbf{F}_{\mathfrak{F}}}{det\ \mathbf{F}_{\mathfrak{F}} |_{\mathfrak{F}=\mathfrak{F}_0}}$$

$$det\ \hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0} = \frac{det\ \hat{\mathbf{F}}_{\mathfrak{F}}}{det\ \hat{\mathbf{F}}_{\mathfrak{F}} |_{\mathfrak{F}=\mathfrak{F}_0}}.$$

Node-group $\mathbf{\Psi}'$ in Fig. 6 can be eliminated to obtain the flow graph in Fig. 7. For example, the branch transmission from $\mathbf{\Psi}'$ to $\mathbf{\Phi}$ in Fig. 7

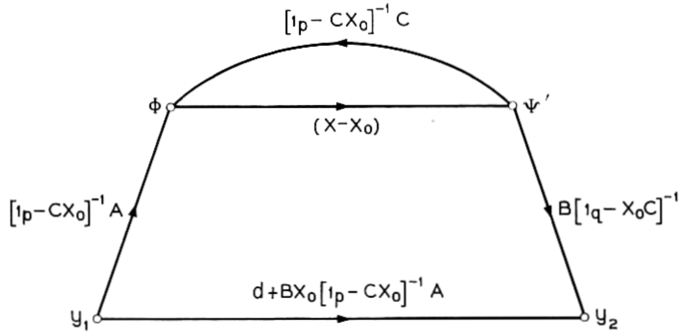

Fig. 7 — Flow-graph obtained from Fig. 6 by eliminating node-group $\mathbf{\Psi}$.

is the transmission of the subgraph in Fig. 8. From Fig. 7 it is clear that

$$\det \mathbf{F}_{\mathfrak{F}\mathfrak{F}_0} = \det [\mathbf{1}_p - (\mathbf{1}_p - \mathbf{CX}_0)^{-1}\mathbf{C}(\mathbf{X} - \mathbf{X}_0)]. \tag{11}$$

Thus,

$$\det \mathbf{F}_{\mathfrak{F}\mathfrak{F}_0} = \det [(\mathbf{1}_p - \mathbf{CX}_0)^{-1}] \det [\mathbf{1}_p - \mathbf{CX}]$$
$$= \frac{\det \mathbf{F}_{\mathfrak{F}}}{\det \mathbf{F}_{\mathfrak{F}} |_{\mathfrak{F}=\mathfrak{F}_0}} . \tag{12}$$

Now consider $\det \hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0}$. Since

$$w(\mathbf{X}) = w(\mathbf{0}) \frac{\det \hat{\mathbf{F}}_{\mathfrak{F}}}{\det \mathbf{F}_{\mathfrak{F}}} = w(\mathbf{X}_0) \frac{\det \hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0}}{\det \mathbf{F}_{\mathfrak{F}\mathfrak{F}_0}} \tag{13}$$

$$= [d + \mathbf{BX}_0(\mathbf{1}_p - \mathbf{CX}_0)^{-1}\mathbf{A}] \frac{\det \hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0} \det \mathbf{F}_{\mathfrak{F}} |_{\mathfrak{F}=\mathfrak{F}_0}}{\det \mathbf{F}_{\mathfrak{F}}} \tag{14}$$

where (12) has been used to obtain the last expression in (14), we have

$$\det \hat{\mathbf{F}}_{\mathfrak{F}} = [1 + d^{-1}\mathbf{BX}_0(\mathbf{1}_p - \mathbf{CX}_0)^{-1}\mathbf{A}] \det \hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0} \det \mathbf{F}_{\mathfrak{F}} |_{\mathfrak{F}=\mathfrak{F}_0} \tag{15}$$

$$= \{\det [\mathbf{1}_p - \mathbf{CX}_0] + d^{-1}\mathbf{BX}_0 \operatorname{adj} (\mathbf{1}_p - \mathbf{CX}_0)\mathbf{A}\} \det \hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0} . \tag{16}$$

Although it is assumed in (13) that $\det \mathbf{F}_{\mathfrak{F}}$ and $\det \mathbf{F}_{\mathfrak{F}\mathfrak{F}_0}$ do not vanish

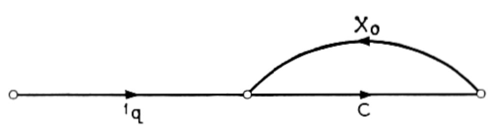

Fig. 8 — Subgraph for evaluation of transmission from $\mathbf{\Psi}'$ to $\mathbf{\Phi}$ in Fig. 7.

identically in the complex-frequency variable, it can readily be shown with a continuity argument that the validity of (16) does not depend upon these assumptions. According to Lemma IV, the expression within the branches in (16) is equal to det $[\mathbf{1}_p - \hat{\mathbf{C}}\mathbf{X}_0]$. Therefore

$$\det \hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0} = \frac{\det \hat{\mathbf{F}}_{\mathfrak{F}}}{\det \hat{\mathbf{F}}_{\mathfrak{F}} \big|_{\mathfrak{F}=\mathfrak{F}_0}}$$

7.2 *The "residual set of equations"*

Recall that the set $\mathfrak{F}$ is the collection of q equations:

$$f_k = \sum_{i=q+1}^{q+p} \gamma_{ki} f_i \qquad (k = 1, 2, \cdots, q). \tag{17}$$

Let $\mathfrak{R}$ denote an arbitrary subset of $\mathfrak{R}_0$, the set of all ordered pairs of integers (j, i) such that $1 \leqq j \leqq q$, $q + 1 \leqq i \leqq q + p$. Let $\sum_{\mathfrak{R}}$ denote a sum over all integers i such that $(k,i)\varepsilon\mathfrak{R}$ [i.e., such that (k,i) is an element of $\mathfrak{R}$], and denote by $\mathfrak{R}^*$ the complement of $\mathfrak{R}$ with respect to $\mathfrak{R}_0$. Then it is certainly true that

$$f_k = f_k' + \sum_{\mathfrak{R}^*} \gamma_{ki} f_i \qquad (k = 1, 2, \cdots, q)$$

where

$$f_k' = \sum_{\mathfrak{R}} \gamma_{ki} f_i \qquad (k = 1, 2, \cdots, q). \tag{18}$$

In accordance with the controlled-source interpretation of (17), the f_k' are the contributions to the controlled-source outputs associated with the subset of coefficients $\{\gamma_{ki} \mid (k,i)\varepsilon\mathfrak{R}\}$. Let $\mathfrak{F}\cdot\mathfrak{R}$ denote the set of r equations ($r \leqq q$) obtained from (18) by omitting all equations of the form $f_k' = 0$. The set $\mathfrak{F}\cdot\mathfrak{R}$ is referred to as a residual set of equations obtained from $\mathfrak{F}$.

VIII. THEOREMS CONCERNING det $\mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}}$ AND det $\hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}}$

To each choice of $\mathfrak{R}$ there corresponds a signal-flow graph characterization of the relation between y_1 and y_2 of the type shown in Fig. 2 where the elements of the flow matrices analogous to **A**, **B**, **C**, and d are independent of those literal coefficients γ_{ki} for which $(k,i)\varepsilon\mathfrak{R}$, and where the branch analogous to **X** is associated with the set of equations $\mathfrak{F}\cdot\mathfrak{R}$. Accordingly, each choice of $\mathfrak{R}$ defines a set of controlled-source interpretable equations $\mathfrak{F}\cdot\mathfrak{R}$, a pair of determinants det $\mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}}$ and det $\hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}}$, and an initial state of the physical system [i.e., a system obtained by setting $\gamma_{ki} = 0$ for all (k,i) contained in $\mathfrak{R}$]. The primary purpose of this section is to derive some fundamental results that relate det $\mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_1}$ and det $\hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}_1}$ respectively to det $\mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_2}$ and det $\hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}_2}$, where $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are any two subsets of $\mathfrak{R}_0$. The set $\mathfrak{F}\cdot\mathfrak{R}_0 = \mathfrak{F}$ is of interest here only in that $\mathfrak{F}\cdot\mathfrak{R}_1$ and $\mathfrak{F}\cdot\mathfrak{R}_2$ are obtained from it in a prescribed manner.

Our first objective is to relate det $\mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}}$ and det $\hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}}$ respectively to det $\mathbf{F}_{\mathfrak{F}}$ and det $\hat{\mathbf{F}}_{\mathfrak{F}}$. Consider

Lemma V: Let $\mathbf{A}'$, $\mathbf{B}'$, $\mathbf{C}'$, *and* d' *respectively denote the flow matrices in Fig.* 7 *which correspond to* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, *and* d *in Fig.* 2. *Let* $\mathbf{\Phi}$, $\mathbf{\Psi}'$, $\mathbf{A}'$, $\mathbf{B}'$, *and* $\mathbf{C}'$ *be partitioned as follows:*

$$\mathbf{\Phi} = \begin{bmatrix} \mathbf{\Phi}_1 \\ \mathbf{\Phi}_2 \end{bmatrix} \begin{matrix} s \\ (p-s) \end{matrix}, \qquad \mathbf{\Psi}' = \begin{bmatrix} \mathbf{\Psi}_1' \\ \mathbf{\Psi}_2' \end{bmatrix} \begin{matrix} r \\ (q-r) \end{matrix}, \qquad \mathbf{A}' = \begin{bmatrix} \mathbf{A}_1' \\ \mathbf{A}_2' \end{bmatrix} \begin{matrix} s \\ (p-s) \end{matrix}$$

$$\mathbf{B}' = \underset{r \qquad (q-r)}{[\mathbf{B}_1' \quad \mathbf{B}_2']}, \qquad \mathbf{C}' = \underset{r \qquad (q-r)}{\begin{bmatrix} \mathbf{C}_{11} & \mathbf{C}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} \end{bmatrix}} \begin{matrix} s \\ (p-s) \end{matrix},$$

and let all of the nonzero elements of $(\mathbf{X} - \mathbf{X}_0)$ *be restricted to* $\mathbf{X}_{rs}$, *the* $r \times s$ *submatrix standing in the upper left-hand corner. Then det* $\mathbf{F}_{\mathfrak{F}\mathfrak{F}_0}$ *and det* $\hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0}$ *respectively are equal to the determinant of the loop-difference matrix and the determinant of the null loop-difference matrix for the branch* $\mathbf{X}_{rs}$ *in the flow graph of Fig.* 9.

The proof follows at once from the expressions for det $\mathbf{F}_{\mathfrak{F}\mathfrak{F}_0}$ and det $\hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0}$ in terms of d' and the submatrices of $\mathbf{A}'$, $\mathbf{B}'$ and $\mathbf{C}'$. The details are omitted.

Suppose now that the elements of $\mathbf{\Psi}$ and $\mathbf{\Phi}$ are chosen so that all of the coefficients $\gamma_{ki}[(k,i)\varepsilon\mathfrak{R}]$ are contained in the $r \times s$ submatrix in the

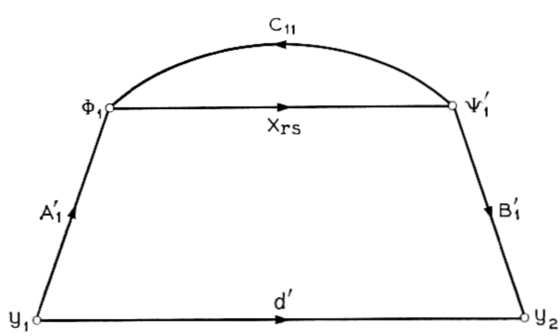

Fig. 9 — Flow-graph relevant to Lemma V.

upper left-hand corner of $\mathbf{X}$, where s is the smallest integer for which this is possible, and that $\mathbf{X}_0$ in Fig. 7 is obtained from $\mathbf{X}$ by replacing the $\gamma_{ki}[(k,i)\varepsilon\mathfrak{R}]$ by zeros. It then follows from Lemmas II and V that $\det \mathbf{F}_{\mathfrak{F}\mathfrak{F}_0} = \det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}}$ and $\det \hat{\mathbf{F}}_{\mathfrak{F}\mathfrak{F}_0} = \det \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}}$. Thus a direct application of Theorem III at this point proves

Theorem IV:

$$det\ \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}} = \frac{det\ \mathbf{F}_{\mathfrak{F}}}{det\ \mathbf{F}_{\mathfrak{F}}\Big|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}}}$$

$$det\ \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}} = \frac{det\ \hat{\mathbf{F}}_{\mathfrak{F}}}{det\ \hat{\mathbf{F}}_{\mathfrak{F}}\Big|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}}}.$$

Note that $\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}}$ and $\det \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}}$ are linear-fractional functions in each of the $\gamma_{ki}[(k,i)\varepsilon\mathfrak{R}^*]$. This fact is often of assistance in evaluating the determinants.

Now consider $\det \mathbf{F}_{\mathfrak{F}}\Big|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}}$ and $\det \hat{\mathbf{F}}_{\mathfrak{F}}\Big|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}}$.

Recall that these quantities are respectively the determinant of the loop-difference matrix and the determinant of the null loop-difference matrix for the branch $\mathbf{X}$ in Fig. 2 when $\gamma_{ki} = 0[(k,i)\varepsilon\mathfrak{R}]$. Consequently it follows from Lemma II, Lemma V, and the significance of $\mathfrak{F}\cdot\mathfrak{R}^*$ that:

$$\begin{aligned}\det \mathbf{F}_{\mathfrak{F}}\Big|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}} &= \det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}^*}\Big|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}}\\ &= \det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}^*}\Big\|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}}.\end{aligned} \tag{19}$$

$$\begin{aligned}\det \hat{\mathbf{F}}_{\mathfrak{F}}\Big|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}} &= \det \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}^*}\Big|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}}\\ &= \det \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}^*}\Big\|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}}.\end{aligned} \tag{20}$$

where $\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}^*}\Big\|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}}$ and $\det \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}^*}\Big\|_{\substack{\gamma_{ki}=0\\(k,i)\varepsilon\mathfrak{R}}}$ respectively

are equal to $\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}^*}$ and $\det \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}^*}$ evaluated from the flow graph or directly from the circuit under the condition that $\gamma_{ki} = 0$ for all (k,i) contained in $\mathfrak{R}$. Theorem IV and identities (19) and (20) imply

Theorem V: Let $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *denote two arbitrary subsets of* $\mathfrak{R}_0$ *. Then*

$$\frac{det\ \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_1}}{det\ \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_2}} = \frac{det\ \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_2^*}\left\| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_2\end{matrix}\right.}{det\ \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_1^*}\left\| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_1\end{matrix}\right.}$$

$$\frac{det\ \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}_1}}{det\ \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}_2}} = \frac{det\ \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}_2^*}\left\| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_2\end{matrix}\right.}{det\ \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}_1^*}\left\| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_1\end{matrix}\right.}.$$

We wish now to focus attention on the particular situation in which $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are disjoint. We shall prove

Theorem VI: Let $\mathfrak{R}_1$ *and* $\mathfrak{R}_2$ *be any two disjoint subsets of* $\mathfrak{R}_0$ *. Then*

$$\frac{det\ \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_1}}{det\ \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_2}} = \frac{det\ \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_1}\left\| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_2\end{matrix}\right.}{det\ \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_2}\left\| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_1\end{matrix}\right.}$$

$$\frac{det\ \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}_1}}{det\ \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}_2}} = \frac{det\ \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}_1}\left\| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_2\end{matrix}\right.}{det\ \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathfrak{R}_2}\left\| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_1\end{matrix}\right.}.$$

Consider Theorem V. Observe that here $\mathfrak{F}\cdot\mathfrak{R}_1$ and $\mathfrak{F}\cdot\mathfrak{R}_2$ respectively can be regarded as residual sets of equations obtained from $\mathfrak{F}\cdot\mathfrak{R}_2^*$ and $\mathfrak{F}\cdot\mathfrak{R}_1^*$. Formally, in accordance with the notation introduced earlier, $\mathfrak{F}\cdot\mathfrak{R}_1 = (\mathfrak{F}\cdot\mathfrak{R}_2^*)\cdot\mathfrak{R}_1$ and $\mathfrak{F}\cdot\mathfrak{R}_2 = (\mathfrak{F}\cdot\mathfrak{R}_1^*)\cdot\mathfrak{R}_2$. Using Theorem IV

$$\det \mathbf{F}_{\mathfrak{F}.\mathfrak{R}_1} = \frac{\det \mathbf{F}_{\mathfrak{F}.\mathfrak{R}_2^*}}{\det \mathbf{F}_{\mathfrak{F}.\mathfrak{R}_2^*}\left| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_1\end{matrix}\right.}.$$

Hence

$$\frac{\det \mathbf{F}_{\mathfrak{F}.\mathfrak{R}_2^*}\left| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_2\end{matrix}\right.}{\det \mathbf{F}_{\mathfrak{F}.\mathfrak{R}_2^*}\left| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_1 \cup \mathfrak{R}_2\end{matrix}\right.} = \det \mathbf{F}_{\mathfrak{F}.\mathfrak{R}_1}\left| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_2\end{matrix}\right. \tag{21}$$

$$= \det \mathbf{F}_{\mathfrak{F}.\mathfrak{R}_1}\left\| \begin{matrix}\gamma_{ki} = 0 \\ (k,i)\varepsilon\mathfrak{R}_2\end{matrix}\right..$$

Next let $\mathfrak{R}_3 = (\mathfrak{R}_1 \cup \mathfrak{R}_2)^*$ and note that $\mathfrak{F}\cdot\mathfrak{R}_3 = (\mathfrak{F}\cdot\mathfrak{R}_1^*)\cdot\mathfrak{R}_3 = (\mathfrak{F}\cdot\mathfrak{R}_2^*)\cdot\mathfrak{R}_3$. Again using Theorem IV

$$\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_3} = \frac{\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_1^*}}{\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_1^*}\left| \begin{matrix}\gamma_{ki} = 0\\ (k,i)\varepsilon\mathfrak{R}_3\end{matrix}\right.} = \frac{\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_2^*}}{\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_2^*}\left| \begin{matrix}\gamma_{ki} = 0\\ (k,i)\varepsilon\mathfrak{R}_3\end{matrix}\right.}.$$

Thus

$$\begin{aligned} \det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_3}\left| \begin{matrix}\gamma_{ki} = 0\\ (k,i)\varepsilon\mathfrak{R}_1 \cup \mathfrak{R}_2\end{matrix}\right. &= \det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_1^*}\left| \begin{matrix}\gamma_{ki} = 0\\ (k,i)\varepsilon\mathfrak{R}_1 \cup \mathfrak{R}_2\end{matrix}\right. \\ &= \det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_2^*}\left| \begin{matrix}\gamma_{ki} = 0\\ (k,i)\varepsilon\mathfrak{R}_1 \cup \mathfrak{R}_2 .\end{matrix}\right. \end{aligned} \tag{22}$$

Finally, from (21), (22), (19) and the fact that (21) remains valid when the subscripts 1 and 2 are interchanged, it follows directly that

$$\frac{\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_2^*}\left\| \begin{matrix}\gamma_{ki} = 0\\ (k,i)\varepsilon\mathfrak{R}_2\end{matrix}\right.}{\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_1^*}\left\| \begin{matrix}\gamma_{ki} = 0\\ (k,i)\varepsilon\mathfrak{R}_1\end{matrix}\right.} = \frac{\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_1}\left\| \begin{matrix}\gamma_{ki} = 0\\ (k,i)\varepsilon\mathfrak{R}_2\end{matrix}\right.}{\det \mathbf{F}_{\mathfrak{F}\cdot\mathfrak{R}_2}\left\| \begin{matrix}\gamma_{ki} = 0\\ (k,i)\varepsilon\mathfrak{R}_1\end{matrix}\right.}.$$

when $\mathfrak{R}_1$ and $\mathfrak{R}_2$ are disjoint. It is obvious at this point that a similar argument suffices to establish the second identity stated in the theorem.

It is worth stating explicitly the following direct corollary of Theorem VI.

Corollary II: Let N_1 and N_2 be two arbitrary disjoint subsets of $\{1, 2, \cdots, q\}$. Let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively denote the subsets of equations obtained from $\mathfrak{F}$ by including only those for which $k\varepsilon N_1$ and $k\varepsilon N_2$. Then

$$\frac{\det \mathbf{F}_{\mathfrak{F}_1}}{\det \mathbf{F}_{\mathfrak{F}_2}} = \frac{\det \mathbf{F}_{\mathfrak{F}_1}\left\| \begin{matrix}\gamma_{ki} = 0\\ k\varepsilon N_2\end{matrix}\right.}{\det \mathbf{F}_{\mathfrak{F}_2}\left\| \begin{matrix}\gamma_{ki} = 0\\ k\varepsilon N_1\end{matrix}\right.}$$

$$\frac{\det \hat{\mathbf{F}}_{\mathfrak{F}_1}}{\det \hat{\mathbf{F}}_{\mathfrak{F}_2}} = \frac{\det \hat{\mathbf{F}}_{\mathfrak{F}_1}\left\| \begin{matrix}\gamma_{ki} = 0\\ k\varepsilon N_2\end{matrix}\right.}{\det \hat{\mathbf{F}}_{\mathfrak{F}_2}\left\| \begin{matrix}\gamma_{ki} = 0\\ k\varepsilon N_1\end{matrix}\right.}.$$

The corollary is of utility in evaluating $\det \mathbf{F}_{\mathfrak{F}_2}$ and $\det \hat{\mathbf{F}}_{\mathfrak{F}_2}$ when $\det \mathbf{F}_{\mathfrak{F}_1}$ and $\det \hat{\mathbf{F}}_{\mathfrak{F}_1}$ are known, since

$$\det \mathbf{F}_{\mathfrak{F}_2}\left\| \begin{matrix}\gamma_{ki} = 0\\ k\varepsilon N_1\end{matrix}\right. \quad \text{and} \quad \det \hat{\mathbf{F}}_{\mathfrak{F}_2}\left\| \begin{matrix}\gamma_{ki} = 0\\ k\varepsilon N_1\end{matrix}\right.$$

are often considerably easier to evaluate than $\det \mathbf{F}_{\mathfrak{F}_2}$ and $\det \hat{\mathbf{F}}_{\mathfrak{F}_2}$. Of course similar remarks apply to Theorem VI. Frequently Theorem V also is useful in this respect. These results are generalizations of a theorem due to Bode.[4]

The following factorization theorem can be obtained by repeated applications of Theorem IV.

Theorem VII: Let $\{\mathcal{R}_1, \mathcal{R}_2, \cdots, \mathcal{R}_n\}$ *denote any collection of disjoint subsets of* $\mathcal{R}_0$ *such that* $\bigcup_{i=1}^{n} \mathcal{R}_i = \mathcal{R}_0$. Then

$$det\, \mathbf{F}_{\mathfrak{F}} = det\, \mathbf{F}_{\mathfrak{F}\cdot\mathcal{R}_1} \prod_{i=2}^{n} det\, \mathbf{F}_{\mathfrak{F}\cdot\mathcal{R}_i} \,\Bigg\|\, \begin{matrix} \gamma_{ki} = 0 \\ (k,i)\varepsilon\mathcal{R}_i' \end{matrix}$$

$$det\, \hat{\mathbf{F}}_{\mathfrak{F}} = det\, \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathcal{R}_1} \prod_{i=2}^{n} det\, \hat{\mathbf{F}}_{\mathfrak{F}\cdot\mathcal{R}_i} \,\Bigg\|\, \begin{matrix} \gamma_{ki} = 0 \\ (k,i)\varepsilon\mathcal{R}_i' \end{matrix}$$

where $\mathcal{R}_i' = \bigcup_{j=1}^{i-1} \mathcal{R}_j$.

Observe that det $\mathbf{F}_{\mathfrak{F}}$ is expressed as a product of ordinary loop-differences[4,5] when each $\mathcal{R}_i (i = 1, 2, \cdots, n)$ contains a single element.

IX. SOME SIMPLE APPLICATIONS OF THE THEORY

The first three examples relate to a specific vacuum-tube circuit that has been considered by Truxal.[6] He presents a detailed classical flow-graph analysis.

9.1 *An Application of Theorem I*

To illustrate the application of Theorem I we shall compute the driving-point impedance at port 2 of the circuit shown in Fig. 10 when port 1 is short-circuited. The pertinent linear incremental model is shown in Fig. 10. It is assumed that $\mu_1 = \mu_2 = 20$, $r_{p1} = r_{p2} = 10$, and $R_{L2} = R_{L1}R(R_{L1} + R)^{-1} = 200$.

Here we choose as the set $\mathfrak{F}$ the two equations

$$e_a = \mu_1 e_{g1}$$

$$e_b = \mu_2 e_{g2}$$

Let $\mathbf{\Psi} = [e_a, e_b]^t$ and $\mathbf{\Phi} = [e_{g1}, e_{g2}]^t$, where the superscript t indicates transposition. Hence $\mathbf{X} = \text{diag}\,[\mu_1, \mu_2]$. It is a trivial matter to show that

$$w(\mathbf{0}) = \frac{440R_k + 4200}{4.2R_k + 441}.$$

Recall that $\mathbf{C}$ is defined here by $\mathbf{\Phi} = \mathbf{C\Psi}$ where the elements of $\mathbf{\Psi}$ are treated as independent variables and port 2 is open-circuited. Similarly $\mathbf{\Phi} = \hat{\mathbf{C}}\mathbf{\Psi}$ when port 2 is short-circuited. An elementary analysis yields:

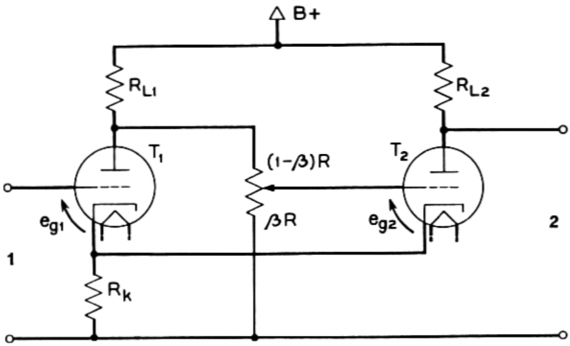

Fig. 10 — Circuit diagram for the example.

$$\mathbf{C} = \begin{bmatrix} \dfrac{-2.1R_k}{4.2R_k + 441} & \dfrac{-2.1R_k}{4.2R_k + 441} \\[2ex] \dfrac{-R_k(2\beta + 2.1) - 420\beta}{4.2R_k + 441} & \dfrac{R_k(2\beta - 2.1)}{4.2R_k + 441} \end{bmatrix} \tag{23}$$

$$\hat{\mathbf{C}} = \begin{bmatrix} \dfrac{-R_k}{22R_k + 210} & \dfrac{-21R_k}{22R_k + 210} \\[2ex] \dfrac{-R_k(20\beta + 1) - 200\beta}{22R_k + 210} & \dfrac{21R_k\left(\dfrac{20}{21}\beta - 1\right)}{22R_k + 210} \end{bmatrix}.$$

The determinants $\det \mathbf{F}_{\mathfrak{F}} = \det [\mathbf{1}_2 - \mathbf{CX}]$ and $\det \hat{\mathbf{F}}_{\mathfrak{F}} = \det [\mathbf{1}_2 - \hat{\mathbf{C}}\mathbf{X}]$ can be evaluated in a particularly simple manner by exploiting the fact that they are known at the outset to be linear-fractional functions in R_k. We find that

$$\det \mathbf{F}_{\mathfrak{F}} = \frac{(88.2 - 840\beta)R_k + 441}{4.2R_k + 441} \tag{24}$$

$$\det \hat{\mathbf{F}}_{\mathfrak{F}} = \frac{(462 - 8400\beta)R_k + 210}{22R_k + 210}$$

when $\mu_1 = \mu_2 = 20$. Therefore

$$w(\mathrm{diag}\,[20,20]) = w(\mathbf{0}) \frac{\det \hat{\mathbf{F}}_{\mathfrak{F}}}{\det \mathbf{F}_{\mathfrak{F}}}$$

$$= 200 \frac{(46.2 - 840\beta)R_k + 21}{(88.2 - 840\beta)R_k + 441}.$$

9.2 *An Application of Corollary I*

In this section let $w(\mathbf{X})$ be the (port 1 to port 2) open-circuit voltage transfer function for the circuit in Fig. 10, and let $\mathfrak{F}$, $\mathbf{\Psi}$, and $\mathbf{\Phi}$ be chosen as in the last section. Our objective here is to evaluate the transfer function in accordance with Corollary I which is obviously relevant.

Since $\mathbf{C}$ and det $\mathbf{F}_{\mathfrak{F}}$ have already been evaluated, consider the determination of the pertinent flow matrices $\mathbf{A}$ and $\mathbf{B}$. By inspection of the circuit, $\mathbf{A} = [1,0]^t$ and a simple calculation yields

$$\mathbf{B} = \left[\frac{2R_k}{4.2R_k + 441}, \qquad \frac{-2(R_k + 210)}{4.2R_k + 441}\right].$$

It follows that

$$\det \overline{\mathbf{F}}_{\mathfrak{F}} = \det [\mathbf{1}_2 - (\mathbf{C} - \mathbf{AB})\mathbf{X}]$$

$$= \det \begin{bmatrix} \dfrac{86.2R_k + 441}{4.2R_k + 441} & \dfrac{2R_k - 20(420)}{4.2R_k + 441} \\ \dfrac{R_k(40\beta + 42) + 8400\beta}{4.2R_k + 441} & \dfrac{(46.2 - 40\beta)R_k + 441}{4.2R_k + 441} \end{bmatrix}$$

$$= \frac{(928.2 - 840\beta)R_k + 441 + 160{,}000\beta}{4.2R_k + 441}. \tag{25}$$

Therefore from Corollary I, (24), and (25)

$$w(\text{diag}\,[20,20]) = \frac{840R_k + 160{,}000\beta}{(88.2 - 840\beta)R_k + 441}.$$

9.3 *An Application of Corollary II*

Consider the model in Fig. 11. Here let $\mathfrak{F}$ denote the set of equations

$$e_a = \mu_1 e_{g1}$$

$$e_b = \mu_2 e_{g2}$$

$$e_k = R_k i_k$$

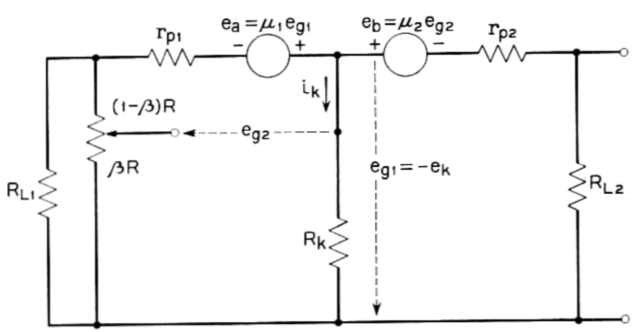

Fig. 11 — Linear model for the network in Fig. 10 when port 1 is short-circuited.

and let $\mathfrak{F}_1$ and $\mathfrak{F}_2$ respectively denote the subsets

$$\left.\begin{aligned} e_a &= \mu_1 e_{g1} \\ e_b &= \mu_2 e_{g2} \end{aligned}\right. , \qquad e_k = R_k i_k.$$

Truxal shows[6] that $\det \mathbf{F}_{\mathfrak{F}_2} = 1 - [-0.2 + (40/21)\beta]R_k$ when the network parameters have the values given in Section 9.1. We wish to determine the corresponding expression for $\det \mathbf{F}_{\mathfrak{F}_1}$ using Corollary II.

By inspection of Fig. 11

$$\det \mathbf{F}_{\mathfrak{F}_1} \|^{R_k=0} = 1, \qquad \det \mathbf{F}_{\mathfrak{F}_2} \|^{\mu_1=\mu_2=0} = 1 + \frac{1}{105} R_k .$$

Thus

$$\det \mathbf{F}_{\mathfrak{F}_1} = \frac{(88.2 - 840\beta)R_k + 441}{4.2R_k + 441}$$

which of course is identical to the right-hand side of (24).

9.4 *A Flow Graph Demonstration of Corollary II*

Corollary II can be demonstrated by considering the flow graph in Fig. 12. Let the determinants of the loop-difference matrices for the branches $\mathbf{Y}_1$ and $\mathbf{Y}_2$ respectively be denoted by $\det \mathbf{F}_{\mathfrak{F}_1}$ and $\det \mathbf{F}_{\mathfrak{F}_2}$. Straightforward evaluation shows that

$$\det \mathbf{F}_{\ 1} = \det \{\mathbf{1}_m - \mathbf{Y}_1[\mathbf{C}_1 + \mathbf{K}(\mathbf{1}_n - \mathbf{Y}_2\mathbf{C}_2)^{-1}\mathbf{Y}_2\mathbf{L}]\}$$

$$\det \mathbf{F}_{\mathfrak{F}_2} = \det \{\mathbf{1}_n - \mathbf{Y}_2[\mathbf{C}_2 + \mathbf{L}(\mathbf{1}_m - \mathbf{Y}_1\mathbf{C}_1)^{-1}\mathbf{Y}_1\mathbf{K}]\}$$

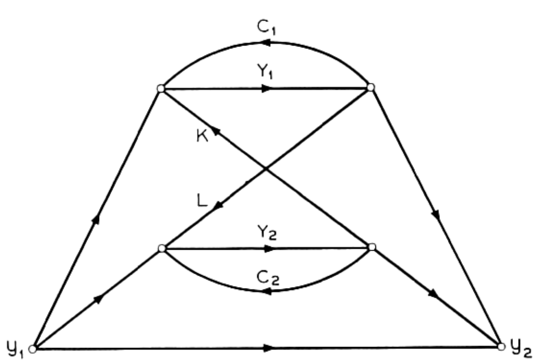

Fig. 12 — Flow-graph relating to the validity of Corollary II.

where n and m respectively are the number of rows of $\mathbf{Y}_2$ and the number of rows of $\mathbf{Y}_1$.

The corollary implies that

$$\frac{\det\{\mathbf{1}_m - \mathbf{Y}_1[\mathbf{C}_1 + \mathbf{K}(\mathbf{1}_n - \mathbf{Y}_2\mathbf{C}_2)^{-1}\mathbf{Y}_2\mathbf{L}]\}}{\det\{\mathbf{1}_n - \mathbf{Y}_2[\mathbf{C}_2 + \mathbf{L}(\mathbf{1}_m - \mathbf{Y}_1\mathbf{C}_1)^{-1}\mathbf{Y}_1\mathbf{K}]\}} = \frac{\det[\mathbf{1}_m - \mathbf{Y}_1\mathbf{C}_1]}{\det[\mathbf{1}_n - \mathbf{Y}_2\mathbf{C}_2]}$$

which can readily be verified independently with the aid of Lemma III.

9.5 *Foster's Results Via Theorem I*

As a final illustration of the theory we shall outline briefly an alternative proof of Foster's well-known results concerning the realizability of two-element-kind one-ports.

Let an arbitrary passive RC one-port containing only resistors and q parallel-RC combinations with capacitance and resistance values respectively c_k and $(\alpha c_k)^{-1}$ $(k = 1, 2, \cdots, q)$ be characterized by the driving-point admittance function $y(s + \alpha)$, where α is real and positive. Let $\mathfrak{F}$ denote the set of capacitor volt-ampere constraints: $i_k = sc_ke_k$ $(k = 1, 2, \cdots, q)$ and consider the evaluation of $y(s + \alpha)$ in accordance with Theorem I (some of the flow matrices in Fig. 2 may not exist if $\alpha = 0$).

From Theorem I and Lemma I, $y(s + \alpha)$ can be expressed as

$$y(s + \alpha) = w(\mathbf{0}) \frac{\det[\mathbf{1}_q - s\mathbf{C}_1\hat{\mathbf{C}}\mathbf{C}_1]}{\det[\mathbf{1}_q - s\mathbf{C}_1\mathbf{C}\mathbf{C}_1]} \tag{26}$$

where $\mathbf{C}_1 = \operatorname{diag}[(c_1)^{\frac{1}{2}}, (c_2)^{\frac{1}{2}}, \cdots, (c_q)^{\frac{1}{2}}]$. Observe that here $-\mathbf{C}$ and $-\hat{\mathbf{C}}$ are passive open-circuit resistance matrices. A moment's reflection shows that $\mathbf{A} = \mathbf{B}^t$. Hence $-\hat{\mathbf{C}} = -\mathbf{C} + \mathbf{D}$ where $\mathbf{D}$ is a nonnegative definite matrix of unit rank. With the aid of (26) and the following theorem it becomes a simple matter to show that

$$y(s) = g_0 + sc_\infty + \sum_{k=1}^{q'} \frac{sa_k}{s + b_k}$$

where $g_0, c_\infty \geqq 0$; $a_k, b_k \geqq 0$ $(k = 1, 2, \cdots, q')$.

Theorem VIII: Let $\mathbf{P}$ *and* $\mathbf{Q}$ *denote two nonnegative definite hermitian matrices of order* n; *let* $\mathbf{Q}$ *have unit rank. Then*

$$p_1 \leqq r_1 \leqq p_2 \leqq r_2 \cdots \leqq p_n \leqq r_n$$

where $\{p_1, p_2, \cdots, p_n\}$ *is the set of eigenvalues of* $\mathbf{P}$ *and* $\{r_1, r_2, \cdots, r_n\}$ *is the set of eigenvalues of* $(\mathbf{P} + \mathbf{Q})$. *Further, if* $p_j (j = 1, 2, \cdots, n)$ *is an eigenvalue of* $\mathbf{P}$ *of multiplicity* $m_j (m_j \geqq 1)$, *then* p_j *is an eigenvalue of* $(\mathbf{P} + \mathbf{Q})$ *of multiplicity at least* $(m_j - 1)$ *and at most* $(m_j + 1)$.

The discussion in Appendix B shows that Theorem VIII is representative of more general results that can be proved by remarkably simple arguments centering about Lemma IV. Eigenvalue inequalities of the type discussed in the Appendix are ordinarily deduced from the extremal properties of eigenvalues.[8]

APPENDIX A.

Proof of Lemma I

Since $\mathbf{X}_1\mathbf{X}_2 = \mathbf{X}$, Lemma I is a direct consequence of the following result.

Lemma A: If $\mathbf{D}$ *and* $\mathbf{E}$ *respectively are* $n \times m$ *and* $m \times n$ *matrices, det* $[\mathbf{1}_n + \mathbf{DE}] =$ *det* $[\mathbf{1}_m + \mathbf{ED}]$.

We prove first that the lemma is true when $\mathbf{D}$ and $\mathbf{E}$ are square matrices. Let $\tilde{\mathbf{D}}$ and $\tilde{\mathbf{E}}$ be $p \times p$ matrices. Then, if $\tilde{\mathbf{D}}$ is nonsingular,

$$\det [\mathbf{1}_p + \tilde{\mathbf{D}}\tilde{\mathbf{E}}] = \det [\tilde{\mathbf{D}}^{-1}(\mathbf{1}_p + \tilde{\mathbf{D}}\tilde{\mathbf{E}})\tilde{\mathbf{D}}] = \det [\mathbf{1}_p + \tilde{\mathbf{E}}\tilde{\mathbf{D}}]. \quad (27)$$

If $\tilde{\mathbf{D}}$ is singular, it has a zero characteristic root, and hence there exists a positive number σ_0 such that $\tilde{\mathbf{D}} + \sigma\mathbf{1}_p$ is nonsingular for all real σ satisfying $0 < |\sigma| < \sigma_0$. Thus when $0 < |\sigma| < \sigma_0$,

$$\det [\mathbf{1}_p + (\tilde{\mathbf{D}} + \sigma\mathbf{1}_p)\tilde{\mathbf{E}}] = \det [\mathbf{1}_p + \tilde{\mathbf{E}}(\tilde{\mathbf{D}} + \sigma\mathbf{1}_p)]. \quad (28)$$

Both sides of (28) are polynomials in σ of at most degree p. Furthermore these polynomials must be identical since they agree throughout the real interval $(0, \sigma_0)$. Therefore (28) is valid when $\sigma = 0$.

Consider now the cases in which $\mathbf{D}$ and $\mathbf{E}$ are not square. Let $p = m + n$,

$$\tilde{\mathbf{D}} = \overset{\quad m \quad n}{\begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}} \begin{matrix} n \\ m \end{matrix}, \qquad \tilde{\mathbf{E}} = \overset{\quad n \quad m}{\begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}} \begin{matrix} m \\ n \end{matrix};$$

and let the symbol $\dotplus$ denote a direct sum of matrices. Observe that $\det [\mathbf{1}_p + \tilde{\mathbf{D}}\tilde{\mathbf{E}}] = \det [(\mathbf{1}_n + \mathbf{DE}) \dotplus \mathbf{1}_m] = \det [\mathbf{1}_n + \mathbf{DE}]$, and that $\det [\mathbf{1}_p + \tilde{\mathbf{E}}\tilde{\mathbf{D}}] = \det [(\mathbf{1}_m + \mathbf{ED}) \dotplus \mathbf{1}_n] = \det [\mathbf{1}_m + \mathbf{ED}]$.

This proves the lemma.

APPENDIX B.

On the Eigenvalues of a Sum of Matrices: an Application of Lemma IV

In the following discussion $\mathbf{M}^*$ and $\rho(\mathbf{M})$ respectively denote the complex-conjugate transpose and the rank of an arbitrary matrix $\mathbf{M}$.

Our principal result is

Theorem A: Let $\{a_1, a_2, \cdots, a_n\}$ *denote the set of eigenvalues of the hermitian matrix* $\mathbf{A}$, *where the first* p *eigenvalues vanish if* $\mathbf{A}$ *is of nullity* p, *and*

$$-(a_{p+1})^{-1} \leqq -(a_{p+2})^{-1} \leqq \cdots \leqq -(a_n)^{-1}.$$

Let $\mathbf{B}$ *denote a nonnegative definite hermitian matrix of unit rank and order* n*; and let*

$$k = \lim_{s \to \infty} \frac{det\,[\mathbf{1}_n + s(\mathbf{A} + \mathbf{B})]}{det\,[\mathbf{1}_n + s\mathbf{A}]}.$$

Then $-\infty < k \leqq \infty$ *and the set of eigenvalues of* $(\mathbf{A} + \mathbf{B})$ *can be written as* $\{c_1, c_2, \cdots, c_n\}$ *where*

1. *if* $0 < k < \infty$, *the first* p *eigenvalues vanish and* $-(a_{p+1})^{-1} \leqq -(c_{p+1})^{-1} \leqq -(a_{p+2})^{-1} \leqq -(c_{p+2})^{-1} \leqq \cdots \leqq -(a_n)^{-1} \leqq -(c_n)^{-1}$
2. *if* $-\infty < k < 0$, *the first* p *eigenvalues vanish and* $-(c_{p+1})^{-1} \leqq -(a_{p+1})^{-1} \leqq -(c_{p+2})^{-1} \leqq -(a_{p+2})^{-1} \leqq \cdots \leqq -(c_n)^{-1} \leqq -(a_n)^{-1}$
3. *if* $k = 0$, *the first* $(p + 1)$ *eigenvalues vanish and* $-(a_{p+1})^{-1} \leqq -(c_{p+2})^{-1} \leqq -(a_{p+2})^{-1} \leqq -(c_{p+3})^{-1} \leqq \cdots \leqq -(c_n)^{-1} \leqq -(a_n)^{-1}$
4. *if* $k = \infty$, *the first* $(p - 1)$ *eigenvalues vanish and* $-(c_p)^{-1} \leqq -(a_{p+1})^{-1} \leqq -(c_{p+1})^{-1} \leqq -(a_{p+2})^{-1} \leqq \cdots \leqq -(a_n)^{-1} \leqq -(c_n)^{-1}$.

Proof:

The statements relating to the number of vanishing eigenvalues of $(\mathbf{A} + \mathbf{B})$ are a direct consequence of (i) $\rho(\mathbf{A}) = \rho(\mathbf{A} + \mathbf{B})$ when $0 < |k| < \infty$, (ii) $\rho(\mathbf{A}) = 1 + \rho(\mathbf{A} + \mathbf{B})$ when $k = 0$, and (iii) $\rho(\mathbf{A}) = \rho(\mathbf{A} + \mathbf{B}) - 1$ when $k = \infty$.

Consider now the real rational function in s:

$$y(s) = \frac{\det\,[\mathbf{1}_n + s(\mathbf{A} + \mathbf{B})]}{\det\,[\mathbf{1}_n + s\mathbf{A}]}.$$

It is well-known that there exist two unitary matrices $\mathbf{P}$ and $\mathbf{Q}$ such that $\mathbf{PAP}^{-1} = \text{diag}\,[a_1, a_2, \cdots, a_n]$ and $\mathbf{Q}(\mathbf{A} + \mathbf{B})\mathbf{Q}^{-1} = \text{diag}\,[c_1, c_2, \cdots, c_n]$ where the a_i and c_i are real. Hence

$$y(s) = \prod_{i=1}^{n} (1 + sc_i)(1 + sa_i)^{-1}.$$

Consider the evaluation of $y(s)$ in accordance with Lemma IV. Here $\mathbf{B}$ can be written as $\mathbf{DD}^*$ where $\mathbf{D}$ is an n-vector. Thus

$$\begin{aligned} y(s) &= 1 + s\mathbf{D}^*[\mathbf{1}_n + s\mathbf{A}]^{-1}\mathbf{D} \\ &= 1 + s\mathbf{D}^*\mathbf{P}^{-1}[\mathrm{diag}(1 + sa_1, 1 + sa_2, \cdots, 1 + sa_n)]^{-1}\mathbf{PD} \\ &= 1 + s\mathbf{H}^*\mathrm{diag}[(1 + sa_1)^{-1}, (1 + sa_2)^{-1}, \cdots, (1 + sa_n)^{-1}]\mathbf{H} \\ &= 1 + \sum_{i=1}^{n} \frac{s \mid h_i \mid^2}{1 + sa_i} \end{aligned} \tag{29}$$

where $\mathbf{H} = \mathbf{PD} = [h_1, h_2, \cdots, h_n]^t$. The theorem is obviously true when $\sum_{i=1}^{n} \mid h_i \mid^2 = 0$. Assume therefore that $\sum_{i=1}^{n} \mid h_i \mid^2 > 0$.

Observe that $y'(s) > 0$ for all real s such that $\mid y(s) \mid < \infty$, and that the poles of $y(s)$ are simple. Thus the interval on the real-axis of the s-plane between two adjacent poles contains one and only one zero of $y(s)$. Note that when $\lim_{s\to\infty} y(s) = k > 0$, the right-most critical point of $y(s)$ is a zero and that this critical point is a pole if $-\infty < k \leqq 0$. Similarly the left-most critical point is a zero if $k = \infty$ or $-\infty < k < 0$; it is a pole if $0 \leqq k < \infty$. The inequalities stated in the theorem follow directly from these observations. The equal signs make provision for possible coincident nonzero eigenvalues of $\mathbf{A}$ and $(\mathbf{A} + \mathbf{B})$. Note that (29) implies that $-\infty < k \leqq \infty$.

The following corollary of Theorem A appears to be useful in the study of linear dynamical systems.

Corollary A: Let $\tilde{\mathbf{A}}$ *and* $\tilde{\mathbf{B}}$ *denote two hermitian matrices of order n, with* $\tilde{\mathbf{A}}$ *nonnegative definite. Let* $\{\tilde{a}_1, \tilde{a}_2, \cdots, \tilde{a}_n\}$ *in which* $\tilde{a}_1 \leqq \tilde{a}_2 \leqq \cdots \leqq \tilde{a}_n$ *denote the set of eigenvalues of* $\tilde{\mathbf{A}}$*; and let* $\{\tilde{c}_1, \tilde{c}_2, \cdots, \tilde{c}_n\}$ *in which* $\tilde{c}_1 \leqq \tilde{c}_2 \leqq \cdots \leqq \tilde{c}_n$ *denote the set of eigenvalues of* $(\tilde{\mathbf{A}} + \tilde{\mathbf{B}})$. *Then*

$$\tilde{a}_{i-\beta} \leqq \tilde{c}_i \qquad [1 + \beta \leqq i \leqq n]$$

$$\tilde{c}_i \leqq \tilde{a}_{i+\alpha} \qquad [1 \leqq i \leqq (n - \alpha)]$$

where $\alpha = \frac{1}{2}(\tilde{r} + \tilde{s})$, $\beta = \frac{1}{2}(\tilde{r} - \tilde{s})$ *in which* $\tilde{r}$ *and* $\tilde{s}$ *respectively are the rank and signature of* $\tilde{\mathbf{B}}$.

Proof:

Consider Theorem A and assume that $\mathbf{A}$ is nonnegative definite. Then either $k > 0$ or $k = \infty$ [i.e., either $\rho(\mathbf{A} + \mathbf{B}) = \rho(\mathbf{A})$ or $\rho(\mathbf{A} + \mathbf{B}) = \rho(\mathbf{A}) + 1$]. Thus in either case, since the c_i and a_i are nonnegative here,

$$a_1 \leqq c_1 \leqq a_2 \leqq c_2 \cdots \leqq a_n \leqq c_n \tag{30}$$

We may express $\tilde{\mathbf{B}}$ as $\sum_{i=1}^{\alpha} \tilde{\mathbf{B}}_i - \sum_{i=\alpha+1}^{r} \tilde{\mathbf{B}}_i$ where the $\tilde{\mathbf{B}}_i$ are rank 1 nonnegative definite hermitian matrices. It is certainly true that $\tilde{c}_i$ does not exceed the corresponding eigenvalue of $\tilde{\mathbf{A}} + \sum_{i=1}^{\alpha} \tilde{\mathbf{B}}_i$ and is not less than the corresponding eigenvalue of $\tilde{\mathbf{A}} - \sum_{i=\alpha+1}^{r} \tilde{\mathbf{B}}_i$. Hence by an α-fold application of inequalities of the type (30) we obtain the upper bound on $\bar{c}_i$ stated in the corollary. If $\mathbf{B}$ in Theorem A were a nonpositive definite matrix the inequalities in (30) would be reversed, and hence a β-fold application of the inequalities suffices to establish the lower bound on $\bar{c}_i$.

Our final result relates to eigenvalue multiplicities.

Theorem B: Let $\bar{\mathbf{A}}$ *and* $\bar{\mathbf{B}}$ *be matrices of order* n. *Let* $\bar{\mathbf{A}}$ *be similar to a diagonal matrix and let the rank of* $\bar{\mathbf{B}}$ *be unity. Then if* $\bar{a}$ *is an eigenvalue of* $\bar{\mathbf{A}}$ *of multiplicity* $m(m \geqq 1)$, $\bar{a}$ *is an eigenvalue of* $(\bar{\mathbf{A}} + \bar{\mathbf{B}})$ *of multiplicity at least* $(m - 1)$. *Further, if in addition* $\bar{\mathbf{A}}$ *and* $\bar{\mathbf{B}}$ *are hermitian,* $\bar{a}$ *is an eigenvalue of* $(\bar{\mathbf{A}} + \bar{\mathbf{B}})$ *of multiplicity at most* $(m + 1)$.

Proof:

Consider

$$\bar{y}(s) = \frac{\det [\mathbf{1}_n + s(\bar{\mathbf{A}} + \bar{\mathbf{B}})]}{\det [\mathbf{1}_n + s\bar{\mathbf{A}}]}$$

Since $\bar{\mathbf{B}}$ is of unit rank and $\bar{\mathbf{A}}$ is similar to a diagonal matrix, the lemma can be used, as in the proof of Theorem A, to show that $\bar{y}(s)$ has only simple poles. This proves the first part of the theorem which is essentially equivalent to the statement that if the dimensionality of the null-space of $[\bar{a}\mathbf{1}_n - \bar{\mathbf{A}}]$ is m, then the dimensionality of the null-space of $[\bar{a}\mathbf{1}_n - (\bar{\mathbf{A}} + \bar{\mathbf{B}})]$ is not less than $(m - 1)$ [assuming that the Jordan form of $\bar{\mathbf{A}}$ is diagonal and $\bar{\mathbf{B}}$ is of unit rank]. It is not difficult to produce proofs of the result which are based on this interpretation.

If $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are hermitian, $\bar{y}'(s)$ is either positive for all real s such that $|\bar{y}(s)| < \infty$ or is negative for all such s (we assume that $\bar{y}(s) \neq 1$), since $\bar{\mathbf{B}}$ must be either nonnegative definite or nonpositive definite. Thus $\bar{y}(s)$ can have only simple zeros, from which the second part of the theorem follows at once.

REFERENCES

1. Black, H. S., U. S. Patent No. 2,102,671.
2. Nyquist, H., Regeneration Theory, B.S.T.J., **11,** January 1932, pp. 126–147.
3. Blackman, R. B., Effect of Feedback on Impedance, BSTJ, **22,** October 1943, pp. 268–277.
4. Bode, H. W., *Network Analysis and Feedback Amplifier Design,* New York, D. Van Nostrand, 1945.
5. Mason, S. J., Feedback Theory — Some Properties of Signal Flow Graphs, Proc. IRE, **41,** September 1953, pp. 1144–1156.
6. Truxal, J. G., *Control System Synthesis,* New York, McGraw-Hill, 1955.
7. Mulligan, J. H. Jr., Signal Transmission in Non-reciprocal Systems, MRI Symposium Proceedings, Polytechnic Institute of Brooklyn, Vol. **X,** 1960, pp. 125–153.
8. Riesz, F. and Sz-Nagy, B., *Functional Analysis,* New York, Frederick Ungar Publishing Co., 1955, pp. 237–239.