Compression, Filtering, and Signal-to-Noise Ratio in a Pulse-Modulated System

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Some aspects of the transmission of Gaussian pulses in a frequency-division multiplex system have been calculated. It is assumed that each channel transmitter includes an amplifier, which may be nonlinear, with input and output filters, and that each channel receiver includes a linear amplifier with a thermal noise source at its input plus input and output filters. The influence that (a) transmitting amplifier compression, (b) maximally flat channel-dropping filters of first or second order, and (c) distribution of filtering in the system have on the signal-to-noise ratio, time crosstalk, and frequency crosstalk is calculated.

I. INTRODUCTION

A proposed long distance waveguide system[1] will transmit a large number of wideband PCM channels via a single waveguide by means of a frequency multiplexing arrangement. In previous papers[2,3] an idealized frequency multiplex system was analyzed. These papers considered a system in which each channel signal, consisting of on-off carrier pulses, arrives at a detector together with three unwanted signals: (a) time crosstalk, due to leading and trailing edges of neighboring pulses; (b) frequency crosstalk, which is the interference from neighboring channels; and (c) thermal noise* generated essentially in the first amplifier of the receiver. Since these unwanted signals cause errors in the reading of the pulses, the filtering characteristics are closely related to the probability of errors. It has been shown that for any given probability of error it is possible to design the filters in such a way that a very desirable result is achieved—namely, the simultaneous minimization of time spacing between successive pulses, of frequency separation between neighboring channels, and of signal-to-noise ratio.

* Throughout the paper, when we talk about "noise" we mean thermal noise.

These conclusions were derived by analyzing a linear system consisting of a transmitting filter and a receiving filter. Such a scheme does not give quantitative answers to three practical questions:

i. Because RF power is expensive it is desirable to drive the transmitting amplifiers as hard as possible, and consequently they must operate in the nonlinear portion of their characteristic. What is the influence of nonlinearity on the system's performance?

ii. Filters between transmitting and receiving amplifiers reduce the signal without changing the noise. Consequently, in order to increase signal-to-noise ratio it would be desirable to have no filter between those amplifiers. But then the frequency crosstalk would be prohibitively high. How much filtering before and after each amplifier makes a reasonable compromise?

iii. All of the immediately available channel-dropping filters that connect transmitters and receivers with the transmission media[4,5,6] are approximately Butterworth filters of first or second order. Pulses belonging to one channel will lose part of their power in passing through neighboring channel-dropping filters. How is the frequency spacing of channels affected by this loss?

The object of this paper is to answer those questions. More specifically, we want to find out the influences that compression, distribution of filtering with respect to the amplifiers, and the use of the available dropping filters have on time crosstalk, frequency crosstalk, and signal-to-noise ratio. The results extend those already derived previously.[2,3]

In Section II the system to be analyzed is described and the fundamental formulas are introduced. The influences of neighboring channel-dropping filters and nonlinearity of amplifiers are described in Sections III and IV. Signal-to-noise ratio is evaluated in Section V, and some design examples are given in Section VI. Section VII contains a résumé of results and conclusions.

II. SYSTEM DESCRIPTION AND ANALYSIS

The system to be analyzed, Fig. 1, is ideally phase-equalized. The effect of imperfect equalization can be derived[7] as a perturbation of the results obtained in this paper.

Table I shows that the transfer characteristic of each transmitter, excluding the amplifier is Gaussian and that that of each receiver is given by the sum of two slightly displaced Gaussian functions, which approximates the characteristic of a maximally flat Butterworth filter of third order and is easy to handle mathematically. Quantitative reasons for the selection of these filters can be found in a previous paper.[3]

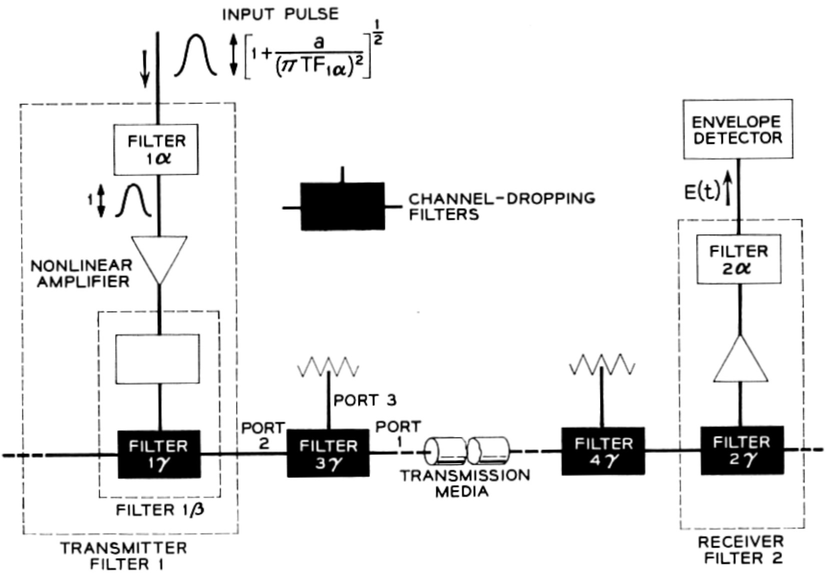


Fig. 1 — Block diagram of system (see definitions of filters in Table I).

Qualitative reasons are that these filters make a good compromise for low time crosstalk and frequency crosstalk. In effect, in order to minimize the tails of the output pulses and consequently the time crosstalk, it would be best to have both filters Gaussian. On the other hand, in order to minimize frequency crosstalk, it would be best to have both filters maximally flat (Butterworth) of high order. A reasonable compromise is achieved by making one of them Gaussian and the other maximally flat of high order. A transfer characteristic relatively easy to achieve without too much midband attenuation is that of a maximally flat filter of third order, which we adopt.

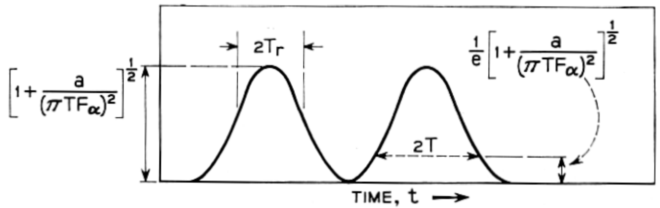
If the transmitter and receiver are centered at the same frequency, the transmission through them measures the insertion loss through a channel. If they are centered at different frequencies, the transmission through them measures the frequency crosstalk.

Each transmitter includes three filters and a nonlinear amplifier. The filter 1α preceding the amplifier and the combination of the two following it, 1β, are Gaussian. Each receiver consists of a linear amplifier between two filters 2α and 2γ whose combined transfer characteristic is approximately maximally flat of third order. This scheme allows one to evaluate the effects of the amount of filtering preceding and following each amplifier.

Table I—Definitions of Symbols

Filters in Fig. 1		Transfer Characteristics		3 db Band-width	Center Frequency	Notes
1	Transmitter, excluding nonlinear amplifier	Gaussian	$y_1 = \exp\{-a[(\lvert f\rvert - f_1)/F_1]^2\}$	$2F_1$	f_1	$a = 0.346$
1α	Filter before nonlinear amplifier	Gaussian	$y_\alpha = \exp\{-a[(\lvert f\rvert - f_1)/F_\alpha]^2\}$	$2F_\alpha$	f_1	$F_1^{-2} = F_\alpha^{-2} + F_\beta^{-2}$
1β	Filter after nonlinear amplifier	Gaussian	$y_\beta = \exp\{-a[(\lvert f\rvert - f_1)/F_\beta]^2\}$	$2F_\beta$	f_1	$F_1^{-2} = F_\alpha^{-2} + F_\beta^{-2}$
1γ	Channel-dropping filter at transmitting end	Maximally flat (qth order)	$y_{1\gamma} = \left[1 + \left(\frac{\lvert f\rvert - f_1}{F_1}\right)^{2q}\right]^{-\frac{1}{2}}$	$2F_{1\gamma}$	f_1	
2	Receiver	~Maximally flat (3rd order)	$y_2 = \exp\{-b[(\lvert f\rvert - f_1)/F_2]^2\}.\cosh n \frac{\lvert f\rvert - f_1}{F_2}$	$2F_2$	f_2	$b = 1.3$ $n = 1.61$
2γ	Channel-dropping filter at receiving end	Maximally flat (rth order)	$y_{2\gamma} = \left[1 + \left(\frac{\lvert f\rvert - f_2}{F_{2\gamma}}\right)^{2r}\right]^{-\frac{1}{2}}$	$2F_{2\gamma}$	f_2	
2α	Noise filter		$y_{2\alpha} = y_2/y_{2\gamma}$	—	—	
3γ	Neighboring channel-dropping filter at transmitting end	Maximally flat (qth order)	$y_{3\gamma} = \left[1 + \left(\frac{F_{1\gamma}}{\lvert f\rvert - f_3}\right)^{2q}\right]^{-\frac{1}{2}}$	$2F_{1\gamma}$	f_3	
4γ	Neighboring channel-dropping filter at receiving end	Maximally flat (rth order)	$y_{4\gamma} = \left[1 + \left(\frac{F_{2\gamma}}{\lvert f\rvert - f_4}\right)^{2r}\right]^{-\frac{1}{2}}$	$2F_{2\gamma}$	f_4	

Note — In Fig. 1, if $f_1 = f_2$, the output is transmission through channel; if $f_1 \neq f_2$, the output is crosstalk.



Envelope input (on-off Gaussian pulses).

The channel-dropping filters 1γ, 2γ, 3γ, and 4γ connecting the transmission medium and transmitters to receivers are maximally flat of first or second order. No channel-dropping filters have been indicated in Fig. 1 to the left of the transmitter nor to the right of the receiver, because such filters will not influence the transfer characteristic between transmitter and receiver.

A Gaussian pulse modulating a carrier is the input to the system. The amplitude has been selected in such a way that the amplitude of the pulse reaching the nonlinear transmitter's amplifier is unity. The envelope of the output pulse reaching the detector has been calculated under the following assumptions:

i. Of all the neighboring channels, only those with center frequencies immediately adjacent to the carrier influence the output pulse.

ii. The transfer characteristics of the channel-dropping filters of the neighboring channels can be approximated by the first three terms of their power series expansions around the carrier frequency.

The envelope $E(t)$ of the pulse reaching the detector has been calculated in (79) of Appendix A:

$$E(t) = \left| \sum_{N=0}^{\infty} \left[I_N - \frac{i}{2\pi\varphi_1} \frac{\partial I_N}{\partial t} + \frac{1}{(2\pi\varphi_2)^2} \frac{\partial^2 I_N}{\partial t^2} \right] \right|. \tag{1}$$

Before detailing the exact meaning of each letter we will give a rough functional interpretation of this expression: φ_1 and φ_2 have to do with the presence of neighboring channel-dropping filters; if these filters had no influence φ_1 and φ_2 would be infinity and the envelope $E(t)$ would be reduced to

$$E(t) = \left| \sum_{N=0}^{\infty} I_N \right|. \tag{2}$$

The infinite number of terms is due to the nonlinearity of the transmitting amplifier, since in general the output is given by a summation of powers of the input. Those powers are an odd number $2N + 1$, and, if the term with $N = 0$ is the only one different from zero, the amplifier is linear; if only the terms with $N = 0$ and $N = 1$ are different from zero, the amplifier has compression of third order; if only the terms with $N = 0$, $N = 1$, and $N = 2$ are different from zero, the amplifier has compression of third and fifth order; and so forth.

Now we define the symbols in (1):

$$I_N = A_N \frac{T_N}{T} \sqrt{\frac{B_N}{(1+S)(1+R)}}\, e^{(n\mu A/2)^2 B_N - 4\rho^2 b(1 - b\mu^2 A^2 B_N) - B_N (t/T)^2} \cdot \cosh\left[2n\rho(1 - b\mu^2 A^2 B_N) + in\mu A B_N \frac{t}{T} \right], \tag{3}$$

$$A_N = 2^{-2N} \binom{2N+1}{N} a_{2N+1}, \tag{4}$$

$$T_N = T \sqrt{\frac{1 + \dfrac{a}{(\pi T F \alpha)^2}}{2N+1}}, \tag{5}$$

$$B_N = \frac{1}{1 + A^2(a + b\mu^2) - 2N \left(\dfrac{T_N}{T}\right)^2}, \tag{6}$$

$$\varphi_1 = \frac{f_1 - f_3}{\dfrac{qs}{1+S} \pm \dfrac{rR}{1+R}}, \tag{7}$$

$$\varphi_2 = \frac{f_1 - f_4}{\left[\dfrac{q(2q+1)S}{2(1+S)} + \dfrac{r(2r+1)R}{2(1+R)} \mp \dfrac{qrSR}{(1+S)(1+R)} - \dfrac{3}{2}\dfrac{(qS)^2}{(1+S)} - \dfrac{3}{2}\dfrac{(rR)^2}{(1+R)}\right]^{\frac{1}{2}}}, \tag{8}$$

$$R = \left(\frac{F_{2\gamma}}{f_1 - f_4}\right)^{2r}, \tag{9}$$

$$S = \left(\frac{F_{1\gamma}}{f_1 - f_3}\right)^{2q}; \tag{10}$$

$a = 0.346$ is a parameter normalizing the bandwidth of a Gaussian filter,

$b = 1.3$ and $n = 1.61$ are parameters that normalize the bandwidth of an approximately maximally flat filter of third order,

f_1 is the center frequency of the transmitter and the carrier frequency,

f_2 is the center frequency of the receiver,

f_3 and f_4 are the center frequencies of neighboring channels,

$2F_1$ is the transmitting bandwidth measured at half power assuming a linear amplifier,

$2F_2$ is the receiving bandwidth measured at half power assuming a linear amplifier,

$2F_{1\alpha}$ is the half-power bandwidth of the Gaussian filter preceding the transmitting amplifier,

$2F_{1\beta}$ is the half-power bandwidth of the Gaussian filter following the transmitting amplifier,

$2F_{1\gamma}$ is the half-power bandwidth of the transmitting Butterworth channel-dropping filter,

$2F_{2\gamma}$ is the half-power bandwidth of the receiving Butterworth channel-dropping filter,

q is the order of the Butterworth channel-dropping filter at the transmitting end,

r is the order of the Butterworth channel-dropping filter at the receiving end,

$2T$ is the width of the input Gaussian pulse measured 8.686 db down,

$$\mu = \frac{F_1}{F_2} \tag{11}$$

is the ratio of transmitting to receiving bandwidths,

$$\rho = \frac{|f_1 - f_2|}{2F_2} \tag{12}$$

is the ratio of channel spacing to receiving bandwidth,

and

$$A = \frac{1}{\pi T F_1} \tag{13}$$

a_{2N+1} are the odd coefficients in the power series

$$v_0 = \sum_0^\infty a_\nu \, v_i^\nu \tag{14}$$

that relates the input and output amplitudes v_i and v_0 of the nonlinear transmitting amplifier.

Where double signs are indicated, the upper one is to be used if $f_3 = 2f_1 - f_4$ (neighboring channels at opposite side of carrier f_1), and the lower one if $f_3 = f_4$ (neighboring channels at the same side of carrier f_1).

If the transmitting amplifier is linear ($a_\nu = 0$ for $\nu \neq 1$) and there is no interference from neighboring channels ($f_3 = f_4 = \infty$), expression (1) measures the transmission through a channel (if $f_1 = f_2$) or the frequency crosstalk (if $f_1 \neq f_2$), as calculated previously.[3]

III. THE INFLUENCE OF NEIGHBORING CHANNEL-DROPPING FILTERS ON SIGNAL-TO-NOISE RATIO AND CROSSTALK

Assume:

(a) Linear transmitting amplifier. Then

$$A_N = 0 \qquad \text{for } N \neq 0, \tag{15}$$

and only the first term of the summation (1) remains.

(b) Transmitter and receiver centered at the same frequency,

$$f_1 = f_2 . \tag{16}$$

Call $E_1(t)$ the output envelope (1) when conditions (15) and (16) are satisfied, and normalize this output to the value $E_0(0)$ that measures the peak of the output ($t = 0$) that would occur if the channels were infinitely spaced, ($f_3 = f_4 = \infty$). Then

$$\frac{E_1(t)}{E_0(0)} = \left\{ \frac{\left\{ \left[1 + \frac{1}{(2\pi\varphi_2)^2} \frac{\partial^2}{\partial t^2} \right] e^{-(t/T)^2 B_0} \cos \frac{\mu A B_0 n t}{T} \right\}^2}{(1 + S)(1 + R)} + \frac{\left[\frac{1}{2\pi\varphi_1} \frac{\partial}{\partial t} e^{-(t/T)^2 B_0} \cos \frac{\mu A B_0 n t}{T} \right]^2}{(1 + S)(1 + R)} \right\}^{\frac{1}{2}} . \tag{17}$$

For $t = 0$,

$$\frac{E_1(0)}{E_0(0)} = \frac{1 - \frac{2B_0}{(2\pi\varphi_2 T)^2} \left[1 - \frac{1}{2} (\mu A n)^2 B_0\right]}{\sqrt{(1 + S)(1 + R)}} . \tag{18}$$

The insertion loss in db due to the proximity of neighboring channel dropping filters is measured by

$$20 \log \frac{E_0(0)}{E_1(0)} .$$

This function has been plotted for maximally flat channel dropping filters of first order ($q = r = 1$) in Fig. 2 and for maximally flat filters of second order ($q = r = 2$) in Fig. 3. The abscissas

$$\rho = \frac{| f_1 - f_3 |}{2F_2} = \frac{| f_1 - f_4 |}{2F_2}$$

are proportional to channel spacing. The rows correspond to different ratios $F_{2\gamma}/F_2$ between the bandwidth of the channel-dropping receiving filter and the bandwidth of the receiver. The different columns correspond to different ratios $F_{1\gamma}/F_1$ between the bandwidth of the channel dropping transmitting filter and bandwidth of the transmitter. The solid lines correspond to the ratio between the transmitting and the receiving bandwidths $F_1/F_2 = 1.3$, the dashed lines to the ratio $F_1/F_2 = 1$; the upper solid and dashed lines in each set of curves correspond to $\pi T F_1 = 0.5$ and the lower solid and dashed lines correspond to $\pi T F_1 = 1$. We have chosen not to include as another variable the selection of signs in the expression for φ_2, (8), because its effect is small. The two possible

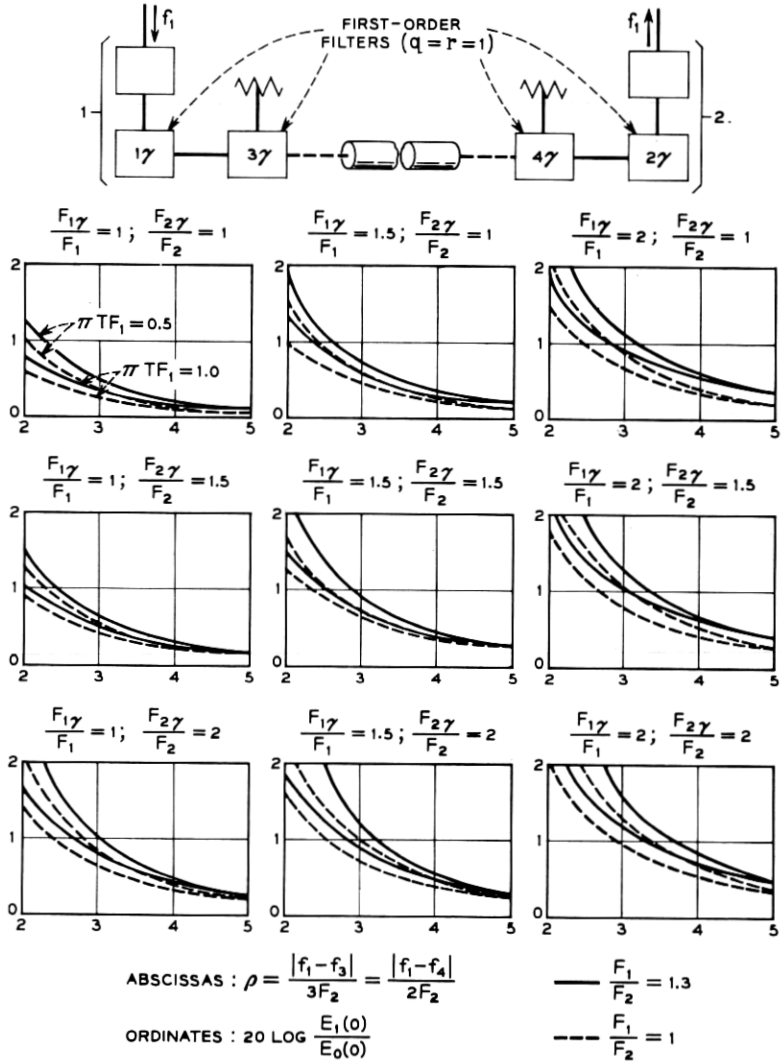


Fig. 2 — Insertion loss due to first-order neighboring channels.

choices of signs correspond to frequencies f_3 and f_4 of the neighboring channels on the same or different sides of f_1. The first case introduces less insertion loss, but for the ranges of values selected in Figs. 2 and 3, and for small abscissas, the difference in ordinates for the two cases is smaller than 0.5 db. As a compromise, Figs. 2 and 3 have been calculated by averaging the results.

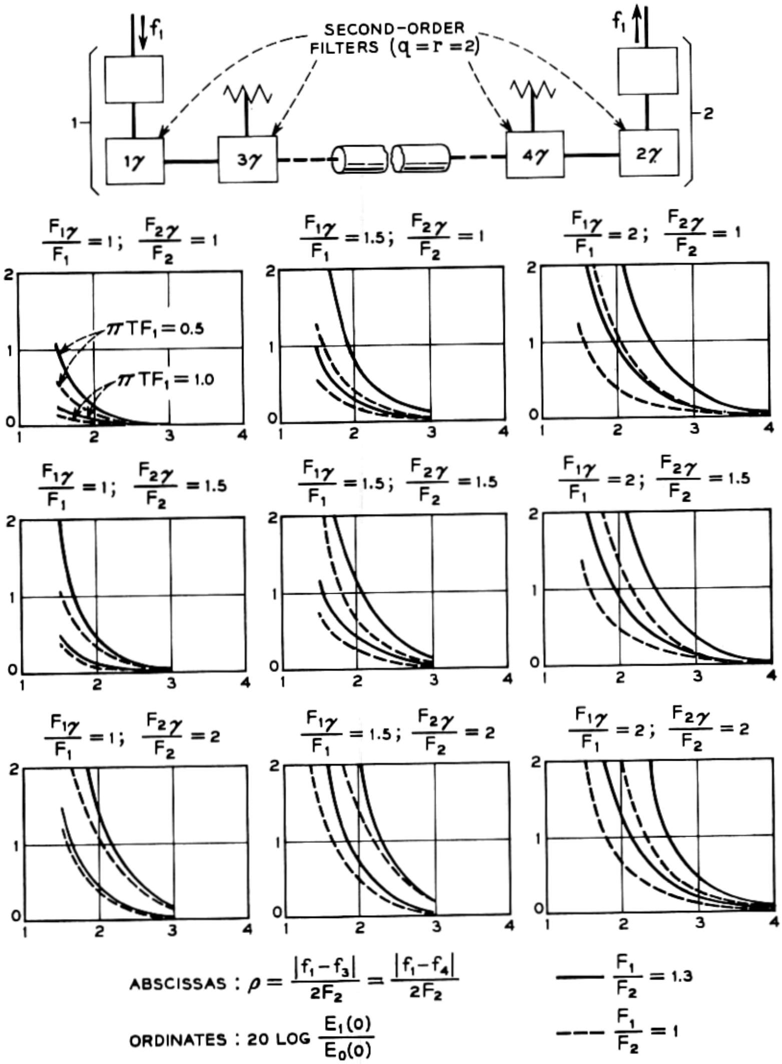


Fig. 3 — Loss due to second-order neighboring channels.

Let us concentrate on Fig. 2, ($q = r = 1$; that is, channel-dropping filters of first order). What parameters yield a system with low insertion loss due to neighboring channels? For a given abscissa (normalized channel spacing ρ), the smallest ordinate occurs in the lowest line in the upper left set of curves. This line is characterized by

$$F_{1\gamma} = F_{2\gamma} = F_1 = F_2 \,; \qquad \pi T F_1 = 1.$$

As could be expected, the system should have

(a) long input pulses,

(b) wide and equal bandwidths in transmitters and receivers,

(c) narrow and equal channel-dropping filters at transmitting and receiving ends.

In order to compare Figs. 2 and 3 we compute an example:
For

$$\frac{F_1}{F} = 1, \quad \frac{F_{1\gamma}}{F_1} = \frac{F_{2\gamma}}{F_2} = 1.5, \quad \pi T F_1 = 1, \quad \text{and} \quad \rho = \frac{(f_1 - f_3)}{2F_2} = 2,$$

the insertion loss due to neighboring channels is 1.25 db if $q = r = 1$ in Fig. 2 (channel-dropping filter of first order) and 0.25 db if $q = r = 2$ in Fig. 3 (channel-dropping filter of second order). Second-order filters make the insertion loss due to neighboring channels 1 db better than that of first-order filters. But this does not necessarily mean that the total insertion loss will be 1 db better in a practical case. There are two reasons for this statement:

i. A second-order maximally flat filter, has more cavities and consequently more heat loss than does one of first order. We don't know how much the difference is, but assuming it is 0.5 db, the increase of heat loss in the two channel-dropping filters is 1 db, and consequently the advantage cited previously is only apparent. This point becomes more important for smaller tolerable insertion loss due to neighboring channels (large ρ).

ii. Consider the transmitter's total Gaussian transfer characteristic

$$y_1 = e^{-a[(|f|-f_1)/F_1]^2} \tag{19}$$

and the transfer characteristic of its qth order channel-dropping filter

$$y_{1\gamma} = \frac{1}{\sqrt{1 + \left(\frac{|f| - f_1}{F_{1\gamma}}\right)^{2q}}}. \tag{20}$$

The transfer characteristic of the rest of the filter should be $y_1/y_{1\gamma}$. There are values of the frequency f for which this function is larger than unity, and therefore it is impossible to realize a passive filter with the desired transfer characteristic. The difficulty is solved by allowing the transfer characteristic of the transmitting filter to be not (19) but

$$\bar{y}_1 = \frac{e^{-a[(|f|-f_1)/F_1]^2}}{K}, \tag{21}$$

where K is a constant greater than unity and equal to the maximum value that $y_1/y_{1\gamma}$ can achieve; K expressed in db is an added insertion loss.

The previous reasoning applies also to the receiver, but, since we are finally going to be interested in signal-to-noise ratios, the value of K applied to the receiver drops out.

In order to have an idea of the values that K may achieve in Gaussian transmitters we have calculated some examples:

	$\frac{F_{1\gamma}}{F_1}$	$20 \log K$
First-order maximally flat channel-dropping filter ($q = r = 1$):	1	0.3 db
	1.5	< 0.1 db
	2	< 0.1 db
Second-order maximally flat channel-dropping filter ($q = r = 2$):	1	< 0.1 db
	1.5	< 0.1 db
	2	< 0.1 db

In general, the broader the channel dropping filter and the flatter the transfer characteristic of the transmitter or receiver, the smaller the added insertion loss 20 log K. This added insertion loss can be neglected except when $F_{1\gamma} = F_1$ and $q = r = 1$.

Now we turn to the increase of time and frequency crosstalk that is introduced by the presence of neighboring channel dropping filters. It is shown in Appendix B that, for systems in which the increase of insertion loss due to neighboring channel-dropping filters is not very large, (around 1 db), the increase of both time and frequency crosstalk is negligible.

IV. INFLUENCE OF THE NONLINEARITY OF THE TRANSMITTING AMPLIFIER ON SIGNAL-TO-NOISE RATIO AND CROSSTALK

Let us assume that, in Fig. 1, (a) the transmitting amplifier is driven so hard that the input signal is not linearly amplified, and (b) the presence of neighboring channel dropping filters can be neglected; this means that $f_3 = f_4 = \infty$ and, from (7) and (8),

$$\varphi_1 = \varphi_2 = \infty . \tag{22}$$

Call $E_2(t)$ the output envelope (1) when (22) is satisfied and normalize this output to the value $E_0(0)$ that measures the peak of the output ($t = 0$) that would occur if there were no compression ($A_N = 0$ for $N \neq$

0), and with the transmitter and receiver centered at the same frequency, $f_1 = f_2$. Then,

$$\frac{E_2(t)}{E_0(0)} = \left| \sum_{N=0}^{\infty} \frac{A_N}{A_0} \frac{T_N}{T_0} \sqrt{\frac{B_N}{B_0}}\, e^{(n\mu A/2)^2 (B_N - B_0) - 4\rho^2 b(1 - b\mu^2 A^2 B_N) - B_N (t/T)^2} \cdot \cosh\left[2n\rho(1 - b\mu^2 A^2 B_N) + in\mu A B_N \frac{t}{T}\right] \right|. \tag{23}$$

As explained in Appendix A, this expression has a simple physical interpretation. The Gaussian pulse of unit amplitude and width

$$2T \sqrt{1 + \frac{a}{(\pi T F_\alpha)^2}}$$

entering the nonlinear amplifier Fig. 1 produces an output that is a summation of Gaussian pulses each characterized by the integer N. The larger N is, the smaller are the amplitude and width of the corresponding pulse. Because of the linearity of the circuit following the amplifier, the envelope of the normalized output of the system (23) is the envelope of the sum of the transients produced by the Gaussian pulses.

For

$$t = \rho = 0, \tag{24}$$

(23) measures the normalized maximum output intensity through a channel ($\rho = 0$) which, expressed in db, we call output compression:

$$20 \log \frac{E_3(0)}{E_0(0)} = 20 \log_{10}\left[1 + \sum_{N=1}^{\infty} \frac{A_N}{A_0} \frac{T_N}{T_0} \sqrt{\frac{B_N}{B_0}}\, e^{(n\mu A/2)^2 (B_N - B_0)}\right]. \tag{25}$$

Naturally, if there were no compression (linear amplifier), A_N would be zero for all N and the output compression would be zero db.

The amplifier compression characterizes the nonlinear amplifier. It measures in db the ratio between the amplitude of a sine wave input and the amplitude of the output without harmonics. It is derived from (25), (5), (6), and (13), making $T = \infty$:

$$20 \log \frac{E_4(0)}{E_0(0)} = 20 \log\left(1 + \sum_1^{\infty} \frac{A_N}{A_0}\right). \tag{26}$$

Assuming the summations in (25) and (26) to be small compared to unity, it is possible to expand the logarithms in series and to retain the first term; then the ratio between output compression and amplifier compression is

$$\frac{\log \dfrac{E_3(0)}{E_0(0)}}{\log \dfrac{E_4(0)}{E_0(0)}} = \frac{\displaystyle\sum_1^\infty \frac{A_N}{A_0} \sqrt{\frac{B_N}{(2N+1)B_0}}\, e^{(n\mu A/2)^2(B_N - B_0)}}{\displaystyle\sum_1^\infty \frac{A_N}{A_0}}. \tag{27}$$

Furthermore, if the amplifier has only compression of order $2N + 1$, the previous expression becomes

$$\frac{\log \dfrac{E_3(0)}{E_0(0)}}{\log \dfrac{E_4(0)}{E_0(0)}} = \sqrt{\frac{B_N}{(2N+1)B_0}}\, e^{(n\mu A/2)^2(B_N - B_0)}. \tag{28}$$

This function has been plotted in Fig. 4. In abscissas we carry both $(F_1/F_{1\beta})^2$, the square of the ratio between the total transmitting bandwidth and the bandwidth of the filter after the amplifier, and $F_{1\alpha}/F_{1\beta}$, the ratio between bandwidth of the filters preceding and following the

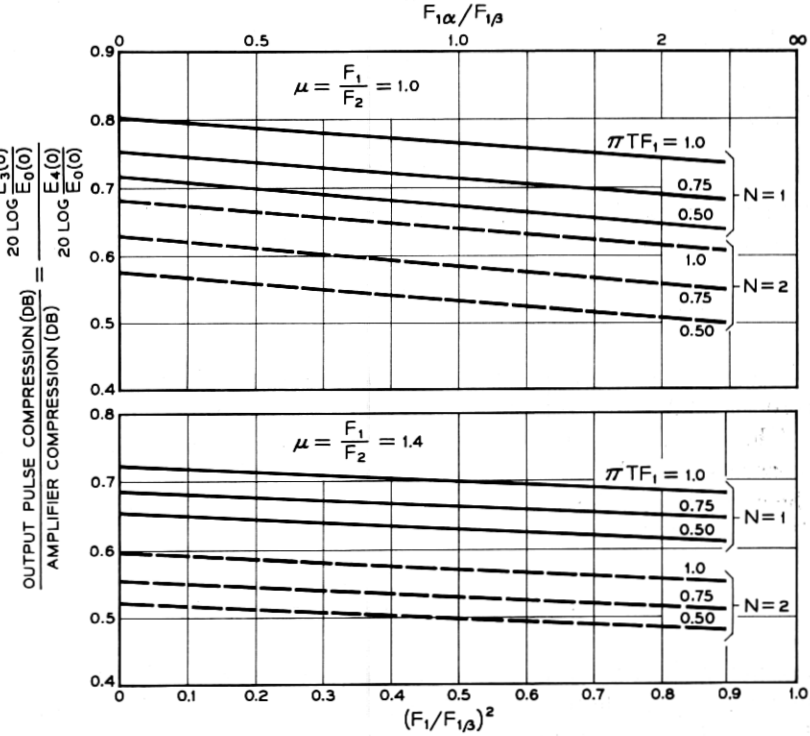


Fig. 4 — Compression of output pulse due to nonlinearity of amplifier.

amplifier in the transmitting end. The two functions are related by the following expression:

$$\frac{F_{1\alpha}}{F_{1\beta}} = \left[\left(\frac{F_{1\alpha}}{F_1}\right)^2 - 1\right]^{\frac{1}{2}}.$$

The upper and lower sets of curves correspond to different ratios between transmitting and receiving bandwidths $\mu = F_1/F_2$. The solid lines correspond to amplifier compression of third order ($2N + 1 = 3$) and the dashed lines to amplifier compression of fifth order ($2N + 1 = 5$). The parameter πTF_1 is proportional to input pulse width $2T$, and transmitting bandwidth $2F_1$.

Let us use these curves for an example. Suppose that the amplifier compression is -2 db. This means that the amplifier output is 2 db below what it would be if the amplifier were linear. Let us assume further that the amplifier's compression is of third order ($N = 1$) and that

$$\mu = \frac{F_1}{F_2} = 1; \quad \frac{F_{1\alpha}}{F_{1\beta}} = 1; \quad \pi TF_1 = 1.$$

For these parameters the upper curve in Fig. 4 yields

$$\frac{\text{output compression}}{\text{amplifier compression}} = 0.763,$$

and, since the amplifier compression is -2 db, the system output compression is -1.53 db. This means that the amplifier compresses more than the system of which it is a part.

The output compression can be reduced by decreasing πTF_1 and by increasing F_1/F_2, $F_{1\alpha}/F_{1\beta}$, and N. In other words, for given input pulse width T, the output compression is reduced by placing as much filtering as possible after the amplifier and by using an amplifier that works like a limiter (large N). These conclusions were to be anticipated if we recall that the compression of order $2N + 1$ introduces a spurious Gaussian pulse in the output of the amplifier. Its effect on the output can be reduced not only by filtering as much as possible but also by making that spurious pulse very narrow (large N). Using the example elaborated previously and making $N = 2$, the output compression is reduced from -1.53 db to -1.4 db.

Because of the compression of the top of the output pulse, time and frequency crosstalks are increased by

$$20 \log \frac{E_3(0)}{E_0(0)} \text{ db},$$

and the signal-to-noise ratio is reduced by the same amount with respect to the values that would be obtained if the amplifier were linear.

The crosstalks are different from the linear case, not only because of the compression of the top of the output pulse but also because the tails of the pulses are changed, thus modifying the time crosstalk (extra time crosstalk), and because the narrow spurious pulses have a wide spectrum that change the frequency crosstalk (extra frequency crosstalk). It is shown in Appendix C that in computing time crosstalk it is slightly conservative not to take into account the extra time crosstalk, and in computing frequency crosstalk it is conservative to ignore both the output compression and the extra frequency crosstalk.

V. SIGNAL-TO-NOISE RATIO

We want to calculate the maximum signal-to-noise ratio in the system shown in Fig. 1. Assume (a) the transmitter and receiver tuned at the same frequency $f_1 = f_2$, (b) the nonlinearity of the transmitting amplifier of order $2N + 1$, and (c) small insertion losses (about 2 db) due to compression and proximity of neighboring channel-dropping filters.

Then, except for a constant measuring the insertion loss of the transmission media, the signal-to-noise ratio in db is derived from (1) and (100) of Appendix D:

$$20 \log \frac{E(0)}{\sigma} = 20 \log \left[\frac{E(0)}{\sigma}\right]_0 + 20 \log \frac{E_1(0)}{E_0(0)} + 20 \log \frac{E_3(0)}{E_0(0)}, \tag{29}$$

in which the first term,

$$20 \log \left[\frac{E(0)}{\sigma}\right]_0 =$$

$$20 \log \left(\frac{2}{\pi}\right)^{\frac{1}{4}} \frac{\sqrt{\left[1 + \frac{a}{(\pi T F_{1\alpha})^2}\right] B_0 e^{(n\mu A/2)^2 B_0}}}{\left\{W_0 F_2 \left[1 + \frac{1}{(-2)^r} \left(\frac{F_2}{F_{2\gamma}}\right)^{2r} \frac{\partial^r}{\partial b^r}\right] \frac{1 + e^{n^2/2b}}{\sqrt{b}}\right\}^{\frac{1}{2}}}, \tag{30}$$

measures in db the reference signal-to-noise ratio in the system — that is, the one that would be obtained if there were no compression and no neighboring channels. The other terms are already-known corrections. The second term measures the insertion loss due to neighboring channels (18) and is plotted in Figs. 2 and 3; the third term measures the insertion loss due to compression (25), and can be derived from Fig. 4.

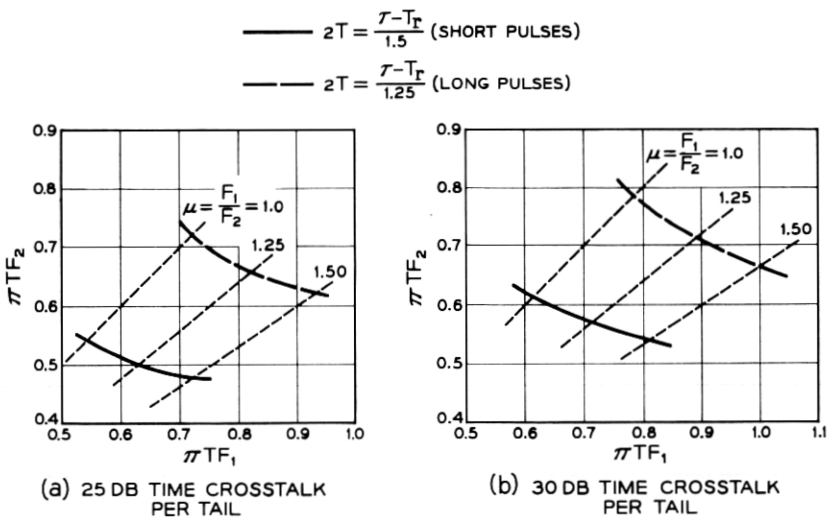


Fig. 5 — Relation between tolerable crosstalk filtering and pulses.

Before plotting the first term we are going to establish a functional relationship among several parameters. This relationship answers the following question: Given the tolerable time crosstalk at the detector of a system in which the input is a train of Gaussian pulses $2T$ wide and τ seconds apart and the sampling time is $2T_r$, what are the possible combinations of bandwidths $2F_1$ and $2F_2$, of the Gaussian transmitter and maximally flat receiver respectively, that satisfy those demands? The answers have been given in Figs. 4 and 7 of Ref. 3. We reproduce them in a convenient way in Figs. 5(a) and 5(b). They correspond to 25 and 30 db of time crosstalk per tail. The parameter for the solid curves is

$$2T = \frac{\tau - T_r}{1.5} \quad \text{(short pulses)}$$

and for the dashed curves it is

$$2T = \frac{\tau - T_r}{1.25} \quad \text{(long pulses).}$$

Straight lines of constant ratios $\mu = F_1/F_2$ have been added for convenience. The curves have been limited to the region between the straight lines $\mu = 1$ and $\mu = 1.5$ because, as will be seen later, that is the region where the signal-to-noise ratios pass through desirable maxima.

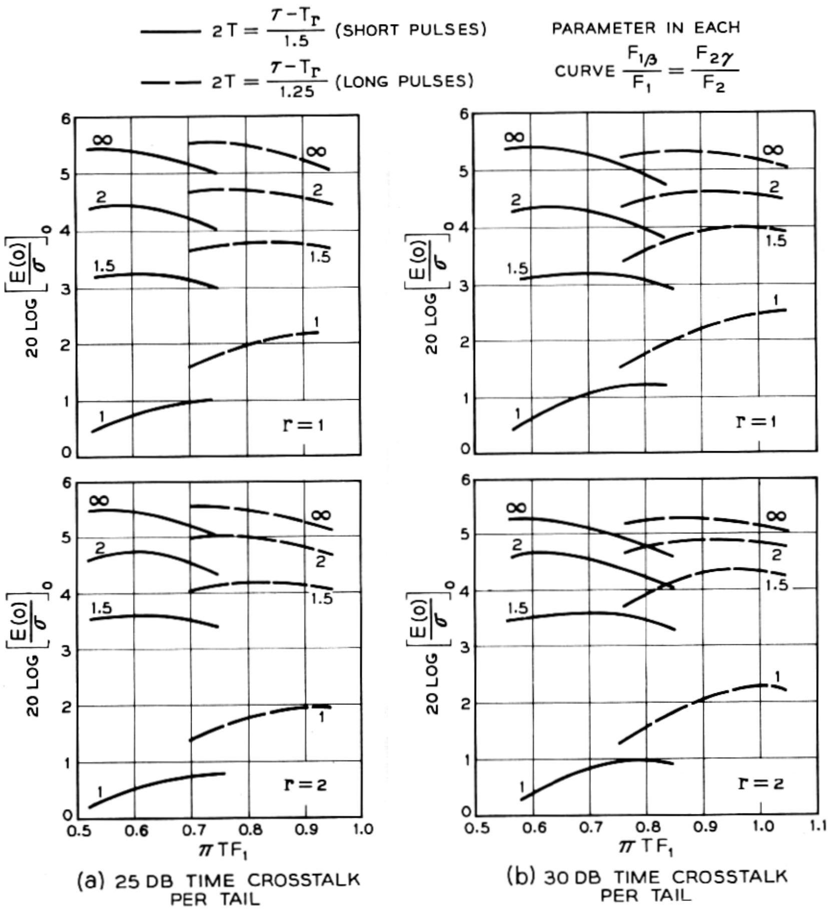


Fig. 6 — Reference signal-to-noise ratio in db.

The values of F_1, F_2, and T from the curves in Fig. 5 have been substituted in (30), and this reference signal-to-noise ratio expressed in db has been plotted with arbitrary reference in Fig. 6. The first and second rows of curves correspond to receiving channel-dropping filters of order $r = 1$ and $r = 2$ respectively. The first and second columns of curves correspond to 25 and 30 db of time crosstalk per tail. The solid lines correspond to shorter input pulses than do the dashed ones, and the parameters on each curve are different ratios between the bandwidths $2F_{1\beta}$ and $2F_{2\gamma}$ of the filters between amplifiers and transmission mediums and the bandwidths $2F_1$ and $2F_2$ of transmitter and receiver.

Let us examine how sensitive the signal-to-noise ratio is to each parameter.

Any curve in the first row ($r = 1$) and the corresponding one in the second ($r = 2$) differ only by a few tenths of a db, and therefore the signal-to-noise ratio is fairly insensitive to the order r of the receiving channel dropping filter.

Corresponding curves in the first and second columns (different time crosstalk) have similar ordinates. This means that crosstalk can be reduced at the expense of bandwidth without changing signal-to-noise level. Signal-to-noise level is improved by changing from short input pulses (solid lines) to long input pulses (dashed lines), but the advantage is paid for in channel bandwidth.

Finally, we find that signal-to-noise ratio is very sensitive to the parameter

$$\frac{F_{1\beta}}{F_1} = \frac{F_{2\gamma}}{F_2},$$

which is essentially proportional to the bandwidth of the filter connecting transmitting and receiving amplifiers. The signal-to-noise ratio grows by widening this bandwidth, but it cannot be widened indefinitely because it increases the insertion loss due to neighboring channel-dropping filters, Figs. 2 and 3. Nevertheless, curves corresponding to

$$\frac{F_{1\beta}}{F_1} = \frac{F_{2\gamma}}{F_2} = \infty$$

have been included in Fig. 6 to show the maximum achievable signal-to-noise ratio.

VI. DESIGN EXAMPLES

Given

(a) $1/\tau = 160$ mc, pulse repetition rate;

(b) $T = T_r$, equal sampling time and input pulse width;

(c) 25 db time crosstalk per tail;

(d) 2 db, third-order amplifier compression;

(e) $q = r = 1$, channel-dropping filters of first order;

calculate $2T$, $2F_1$, $2F_2$, $2F_{1\alpha}$, $2F_{1\beta}$, $2F_{1\gamma}$, $2F_{2\gamma}$, $|f_1 - f_3|$. In order to have low insertion loss due to neighboring channels and large signal-to-noise ratio we select

$$\mu = \frac{F_1}{F_2} = 1, \tag{31}$$

$$F_{1\gamma} = F_{1\beta} = F_{2\gamma}. \tag{32}$$

From (a), (b), (31), and Fig. 5(a), we obtain for short input pulses

$$\pi T F_1 = 0.54, \tag{33}$$

$$2T = 3.125 \text{ millimicroseconds}, \tag{34}$$

$$2F_1 = 2F_2 = 220 \text{ mc}. \tag{35}$$

From (c), (e), (33), and Fig. 6, the reference signal-to-noise ratio results:

$$20 \log \left[\frac{E(0)}{\sigma}\right]_0 = \begin{cases} 4.4 \text{ db} & \text{for } \dfrac{F_{1\beta}}{F_1} = \dfrac{F_{1\gamma}}{F_1} = \dfrac{F_{2\gamma}}{F_2} = 2 \\ 3.2 \text{ db} & \text{for } \dfrac{F_{1\gamma}}{F_1} = 1.5 \end{cases}. \tag{36}$$

From (d), (31), (33), and Fig. 4, we calculate the insertion loss due to compression and consequent increase in time crosstalk per tail:

$$20 \log \frac{E_0(0)}{E_3(0)} = \begin{cases} 0.7 \times 2 = 1.4 \text{ db} & \text{for } \dfrac{F_{1\gamma}}{F_1} = 2 \\ 0.68 \times 2 = 1.36 \text{ db} & \text{for } \dfrac{F_{1\gamma}}{F_1} = 1.5 \end{cases}. \tag{37}$$

From (e), (31), (32), and (33), the insertion loss due to neighboring channels is given approximately in the upper dotted curves in the center and lower right set of curves in Fig. 2. The abscissa which determines the frequency occupancy $f_1 - f_3$, should be selected as small as possible. We will elaborate two examples:

Example 1.

We select

$$\rho = \left|\frac{f_1 - f_3}{2F_2}\right| = 2. \tag{38}$$

The insertion loss due to neighboring channels is

$$20 \log \frac{E_0(0)}{E_1(0)} = \begin{cases} \sim 4 \text{ db} & \text{for } \dfrac{F_{1\gamma}}{F_1} = 2 \\ 1.7 \text{ db} & \text{for } \dfrac{F_{1\gamma}}{F_1} = 1.5 \end{cases}. \tag{39}$$

The total signal-to-noise ratio (29) derived from (36), (37), and (39) is

$$20 \log \frac{E(0)}{\sigma} = \begin{cases} 4.4 - 1.4 - 4 = -1 \text{ db} & \text{for } \dfrac{F_{1\gamma}}{F_1} = 2 \\ 3.2 - 1.36 - 1.7 = 0.14 \text{ db} & \text{for } \dfrac{F_{1\gamma}}{F_1} = 1.5 \end{cases}. \tag{40}$$

Because of the better signal-to-noise ratio we adopt

$$\frac{F_{1\gamma}}{F_1} = 1.5 \tag{41}$$

and deduce from (32), (34), (35), (38), and (41):

$$\begin{aligned} 2T &= 3.125 \text{ millimicroseconds}, \\ 2F_1 &= 2F_2 = 220 \text{ mc}, \\ 2F_1 &= 2F_{1\beta} = 2F_{2\gamma} = 330 \text{ mc}, \\ 2F_{1\alpha} &= \frac{2F_1}{\sqrt{1 - \left(\dfrac{F_1}{F_{1\gamma}}\right)^2}} = 295 \text{ mc}, \\ |f_1 - f_3| &= 440 \text{ mc}. \end{aligned} \tag{42}$$

Example 2.

We select

$$\rho = \frac{|f_1 - f_3|}{2F_2} = 3. \tag{43}$$

The insertion loss due to neighboring channels is (Fig. 2),

$$20 \log \frac{E_0(0)}{E_1(0)} = \begin{cases} 1.3 \text{ db} & \text{for } \dfrac{F_{1\gamma}}{F_1} = 2 \\ 0.7 \text{ db} & \text{for } \dfrac{F_{1\gamma}}{F_1} = 1.5 \end{cases}. \tag{44}$$

Then from (29), (36), (37), and (44), the total signal-to-noise ratio is

$$20 \log \frac{E(0)}{\sigma} = \begin{cases} 4.4 - 1.4 - 1.3 = 1.7 \text{ db} & \text{for } \dfrac{F_{1\gamma}}{F_1} = 2 \\ 3.2 - 1.36 - 0.7 = 1.14 \text{ db} & \text{for } \dfrac{F_{1\gamma}}{F_1} = 1.5 \end{cases}. \tag{45}$$

Selecting the best signal-to-noise ratio, we adopt

$$\frac{F_{1\gamma}}{F_1} = 2 \tag{46}$$

and deduce from (32), (34), (35), (43), and (46):

$$\begin{aligned}
2T &= 3.125 \text{ millimicroseconds},\\
2F_1 &= 2F_2 = 220 \text{ mc},\\
2F_{1\gamma} &= 2F_{1\beta} = 2F_{2\gamma} = 440 \text{ mc},\\
2F_{1\alpha} &= \frac{2F_1}{\sqrt{1 - \left(\frac{F_1}{F_{1\gamma}}\right)^2}} = 254 \text{ mc},\\
|f_1 - f_3| &= 660 \text{ mc}.
\end{aligned} \tag{47}$$

In the first example, the signal-to-noise ratio (40) and the channel spacing (42) are 0.14 db and 440 mc respectively. In the second example these values, (45) and (47), are 1.7 db and 660 mc. The first example provides better frequency occupancy and the second provides greater spacing between successive repeaters.

VII. RÉSUMÉ OF RESULTS AND CONCLUSIONS

The influence of (a) neighboring channel-dropping filters, (b) nonlinearity of transmitting amplifiers, and (c) filtering preceding and following the transmitting and receiving amplifiers on the signal-to-noise ratio, time crosstalk and frequency crosstalk in a frequency-division multiplex PCM system has been calculated.

Neighboring channel-dropping filters reduce the reference* signal-to-noise ratio by increasing the signal insertion loss, Figs. 2 and 3. If these filters are of first order, the channel spacing is determined by the tolerable insertion loss they introduce and not by the frequency crosstalk, as calculated in Ref. 3. If these filters are of order higher than one, the channel spacing may be determined by the tolerable insertion loss or by the tolerable frequency crosstalk, (Figs. 5 and 8 of Ref. 3), depending on which demands more channel spacing. Neighboring channel-dropping filters have no practical bearing on time crosstalk.

Nonlinearity of the transmitting amplifiers decreases the reference signal-to-thermal-noise ratio and increases time crosstalk by the same amount, Fig. 4, but it is conservative to ignore its effect on frequency crosstalk.

* Signal-to-noise ratio in the absence of neighboring channels and with linear amplifiers.

The tolerance to compression is increased by concentrating as much filtering as possible between the nonlinear transmitting amplifier and the detector.

Even for compression as large as 2 db the increase in frequency and time crosstalk is smaller than the amplifier compression. This means that the system can be readily designed to take advantage of the greater power available from an amplifier operating in the nonlinear region.

Finally, reducing the filtering between transmitting and receiving amplifiers increases the reference signal-to-noise ratio, Fig. 6, but it also increases the signal insertion loss due to the presence of neighboring channel-dropping filters, Figs. 2 and 3.

APPENDIX A

Transmission Through Nonlinear Amplifier and Filters

Given a system consisting of five filters and a nonlinear amplifier, Fig. 1, we want to calculate the envelope of the response to a certain input.

The transfer characteristics of the cascaded filters are:

$$y_{1\alpha} = e^{-a[(|f|-f_1)/F_{1\alpha})^2}, \tag{48}$$

$$y_{1\beta} = e^{-a[(|f|-f_1)/F_{1\beta}]^2}, \tag{49}$$

$$y_2 = e^{-b[(|f|-f_2)/F_2]^2} \cosh n \frac{|f| - f_2}{F_2}, \tag{50}$$

$$y_{3\gamma} = \frac{1}{\sqrt{1 + \left(\dfrac{F_{1\gamma}}{|f| - f_3}\right)^{2q}}}, \tag{51}$$

$$y_{4\gamma} = \frac{1}{\sqrt{1 + \left(\dfrac{F_{2\gamma}}{|f| - f_4}\right)^{2r}}}, \tag{52}$$

where

$$a = 0.346,$$

$$b = 1.3,$$

$$n = 1.61,$$

and f_1, f_2, f_3, f_4 are the center frequencies of the different filters; $2F_{1\alpha}$, $2F_{1\beta}$, $2F_2$, $2F_{1\gamma}$, $2F_{2\gamma}$ are their half-power bandwidths; and $y_{1\alpha}$ and $y_{1\beta}$ are Gaussian transfer characteristics and y_2 is the transfer characteristic of an approximately third-order maximally flat filter.[3]

The characteristic $y_{3\gamma}$ is the transfer function between two ports of a three-port channel-dropping filter[4,5,6] proposed for use in a long distance waveguide communication system.[1] This filter 3γ (see Fig. 1) has port 1 matched at all frequencies, and the transfer function between ports 1 and 3 is that of a phase-equalized maximally flat filter of order q, bandwidth $2F_{1\gamma}$, and center frequency f_3. Because of conservation of energy it follows that the phase equalized transfer function between ports 1 and 2 is (51). $y_{4\gamma}$ is similar to $y_{3\gamma}$.

Between the first two filters 1α and 1β in Fig. 1 there is a nonlinear amplifier. Its infinitely wide band output voltage $v_0(t)$ as a function of the input $v_i(t)$ is

$$v_0(t) = \sum_0^\infty a_\nu [v_i(t)]^\nu. \tag{53}$$

The input to the system,

$$i(t) = \sqrt{1 + \frac{a}{(\pi T F_\alpha)^2}}\, e^{-(t/T)^2} \cos 2\pi f_1 t, \tag{54}$$

is a Gaussian pulse $2T$ seconds wide, at 8.686 db down, that modulates a carrier f_1. The amplitude

$$\sqrt{1 + \frac{a}{(\pi T F_\alpha)^2}}$$

has been selected to make the maximum amplitude of the signal $v_i(t)$ reaching the amplifier unity.

In order to evaluate the system's output we start calculating the amplifier's output $v_0(t)$. This is achieved in (53) by expressing $v_i(t)$ in terms of the input $i(t)$ to the system

$$v_0(t) = \sum_0^\infty a_\nu \left[\int_{-\infty}^{\infty} g(f) y_{1\alpha} e^{i2\pi f t}\, df \right]^\nu, \tag{55}$$

in which

$$g(f) = \sqrt{\pi}\, \frac{T}{2} \sqrt{1 + \frac{a}{(\pi T F\alpha)^2}}\, \left(e^{-[\pi T(f-f_1)]^2} + e^{-[\pi T(f+f_1)]^2}\right) \tag{56}$$

is the Fourier transform of $i(t)$ of (54).

Substituting (48) and (56) in (55), performing the integration, and assuming

$$e^{-(\pi T F_1)^2} \ll 1, \tag{57}$$

one derives the explicit value of the amplifier's output:

$$v_0(t) = \sum_0^\infty a_\nu \cos^\nu 2\pi f_1 t \exp\left[- \frac{\nu}{1 + \dfrac{a}{(\pi T F_\alpha)^2}} \left(\frac{t}{T}\right)^2 \right]. \tag{58}$$

Since

$$\cos^\nu 2\pi f_1 t = \frac{1}{2^\nu} \sum_{r=0}^{\nu} \binom{\nu}{r} \cos (\nu - 2r) 2\pi f_1 t, \tag{59}$$

where

$$\binom{\nu}{r} = \frac{\nu!}{(\nu - r)!\, r!}, \tag{60}$$

(58) becomes

$$v_0(t) = \sum_{\nu=0}^{\infty} \sum_{r=0}^{\nu} \frac{a_\nu}{2^\nu} \binom{\nu}{r} \exp\left[- \frac{\nu}{1 + \dfrac{a}{(\pi T F_\alpha)^2}} \left(\frac{t}{T}\right)^2 \right] \cos (\nu - 2r) 2\pi f_1 t. \tag{61}$$

A further simplification: assume that only frequencies in the neighborhood of f_1 are allowed to pass through the filters following the amplifier. Physically, this means that all harmonics of f_1 are filtered out; mathematically, this is translated by keeping in (61) only the terms in which

$$\nu - 2r = \pm 1. \tag{62}$$

Consequently,

$$v_0(t) = \cos 2\pi f_1 t \sum_0^\infty A_N\, e^{-(t/T_N)^2}, \tag{63}$$

where

$$A_N = 2^{-2N} \binom{2N + 1}{N} a_{2N+1}, \tag{64}$$

$$T_N = T \sqrt{\frac{1 + \dfrac{a}{(\pi T F_\alpha)^2}}{2N + 1}}. \tag{65}$$

The output from the nonlinear amplifier $v_0(t)$ is, then, a summation of Gaussian pulses each of amplitude A_N and width $2T_N$ measured 8.686 db down.

Using $v_0(t)$ as input to the second filter 1β in Fig. 1, we calculate the Fourier transform of each term with the help of (56) and deduce the system's output:

$$e(t) = \frac{\sqrt{\pi}}{2} \sum_{0}^{\infty} A_N T_N \int_{-\infty}^{\infty} y_{1\beta} y_2 y_{3\gamma} y_{4\gamma} e^{-[\pi T_N(|f|-f_1)]^2 + i2\pi f t} \, df. \tag{66}$$

We perform a change of variable,

$$f = \varphi + f_1 , \tag{67}$$

and assume

$$e^{-(\pi T_N f_1)} \ll 1. \tag{68}$$

Then the envelope of the output (66) becomes

$$E(t) = \sqrt{\pi} \left| \sum_{0}^{\infty} A_N T_N \int_{-\infty}^{\infty} e^{-(\pi T_N \varphi)^2 + i2\pi\varphi t} y_{1\beta} y_2 y_{3\gamma} y_{4\gamma} \, d\varphi \right| , \tag{69}$$

where $y_{1\beta}$, y_2 , $y_{3\gamma}$, $y_{4\gamma}$ are deduced from (49), (50), (51), (52), and (67) to be

$$y_{1\beta} = e^{-a(\varphi/F_{1\beta})^2}, \tag{70}$$

$$y_2 = e^{-b(\varphi/F_2)^2} \cosh n \frac{\varphi + f_1 - f_2}{F_2} , \tag{71}$$

$$y_{3\gamma} = \frac{1}{\sqrt{1 + \left(\dfrac{F_{1\gamma}}{\varphi + f_1 - f_3}\right)^{2q}}} , \tag{72}$$

$$y_{4\gamma} = \frac{1}{\sqrt{1 + \left(\dfrac{F_{2\gamma}}{\varphi + f_1 - f_4}\right)^{2r}}} . \tag{73}$$

In order to solve the integral in (69), we simplify part of the integrand by noticing that, on account of the exponential functions, most of the contribution to the result comes from small values of φ, and consequently $y_{3\gamma}$, $y_{4\gamma}$ can be expanded in power series. We keep the first three terms

$$y_{3\gamma} y_{4\gamma} = \frac{1 + \dfrac{\varphi}{\varphi_1} - \left(\dfrac{\varphi}{\varphi_2}\right)^2}{\sqrt{(1 + S)(1 + R)}} , \tag{74}$$

where

$$S = \left(\frac{F_{1\gamma}}{f_1 - f_3}\right)^{2q} , \tag{75}$$

$$R = \left(\frac{F_{2\gamma}}{f_1 - f_4}\right)^{2r} , \tag{76}$$

$$\varphi_1 = \frac{f_1 - f_3}{\dfrac{qS}{1+S} \mp \dfrac{rR}{1+R}}, \tag{77}$$

$$\varphi_2 = \frac{f_1 - f_4}{\left[\dfrac{q(2q+1)S}{2(1+S)} + \dfrac{r(2r+1)R}{2(1+R)} \pm \dfrac{qrSR}{(1+S)(1+R)} - \dfrac{3}{2}\left(\dfrac{qS}{1+S}\right)^2 - \dfrac{3}{2}\left(\dfrac{rR}{1+R}\right)^2\right]^{\frac{1}{2}}}. \tag{78}$$

Where double signs are indicated, the upper one is to be used if $f_3 = 2f_1 - f_4$ (neighboring channels at opposite sides of carrier f_1), and the lower one if $f_3 = f_4$ (neighboring channels at the same side of carrier f_1).

Substituting (70), (71), and (74) in (69) and integrating, one obtains the output envelope

$$E(t) = \left| \sum_{N=0}^{\infty} \left[I_N - \frac{i}{2\pi\varphi_1} \frac{\partial I_N}{\partial t} + \frac{1}{(2\pi\varphi_2)^2} \frac{\partial^2 I_N}{\partial t^2} \right] \right|, \tag{79}$$

in which

$$I_N = A_N \frac{T_N}{T} \sqrt{\frac{B_N}{(1+S)(1+R)}}\, e^{(n\mu A^2/2)B_N - 4\rho^2 b(1 - b\mu^2 A^2 B_N) - (t/T)^2 B_N} \cdot \cosh\left[2n\rho(1 - b\mu^2 A^2 B_N) + in\mu A B_N \frac{t}{T}\right], \tag{80}$$

$$B_N = \frac{1}{1 + A^2(a + b\mu^2) - 2N\left(\dfrac{T_N}{T}\right)^2}, \tag{81}$$

$$\mu = \frac{F_1}{F_2}, \tag{82}$$

$$\rho = \frac{|f_1 - f_2|}{2F_2}. \tag{83}$$

APPENDIX B

Influence of Channel-Dropping Filters on Crosstalk

We want to compare the frequency and time crosstalks of the system in Fig. 1, assuming a linear amplifier with the already known[3] frequency and time crosstalk of the same system obtained assuming not only linear amplifier but also no neighboring channel-dropping filters.

The maximum frequency crosstalk between neighboring channels is deduced from (69), making

$A_N = 0$ for $N \neq 0$ (linear amplifier),

$f_1 \neq f_2$ (transmitter and receiver centered at neighboring frequencies),

$f_3, f_4 \neq f_1$ (f_3 and f_4 are the center frequencies of neighboring channel-dropping filters),

$t = 0$ (in order to maximize the integrand).

Then

$$E_1(0) = \sqrt{\pi} A_0 T_0 \int_{-\infty}^{\infty} e^{-(\pi T_0 \varphi)^2} y_{1\beta} y_2 y_{3\gamma} y_{4\gamma} \, d\varphi. \tag{84}$$

We eliminate the effect of neighboring channel-dropping filters by assuming f_3 and f_4 to be infinitely large frequencies. Then $y_{3\gamma}$ and $y_{4\gamma}$ given in (72) and (73) become unity and

$$E_0(0) = \sqrt{\pi} A_0 T_0 \int_{-\infty}^{\infty} e^{-(\pi T_0 \varphi)^2} y_{1\beta} y_2 \, d\varphi. \tag{85}$$

This result has been calculated elsewhere[3] and consequently is known. The reader can check that $E_1(0)$ is slightly smaller than $E_0(0)$, because the product $y_{3\gamma} y_{4\gamma}$ in (84) is slightly smaller than one for values of φ for which the rest of the always-positive integrand contributes substantially to the result. Therefore, it is slightly conservative to ignore the effect of the neighboring channel-dropping filters on frequency crosstalk.

In order to evaluate the influence of neighboring channel-dropping filters on time crosstalk, we must compare the maximum values of two transients during the sampling time. The first transient (17) is that of a pulse through a system with neighboring channel-dropping filters; the second transient is that of the same pulse through the same system except for the removal of the neighboring channel-dropping filters. The last transient is obtained from (17) making $f_3 = f_4 = \infty$,

$$\frac{E_0(t)}{E_0(0)} = e^{-(t/T)^2 B} \cos \frac{\mu A B n t}{T}, \tag{86}$$

and the time crosstalk that is derived from this expression is known.[3]

If the pulses must be closely spaced, the maximum time crosstalk must be fixed by the sloping part of the transients, and consequently the broadening of the pulse due to the presence of the neighboring channel-dropping filters indirectly measures the time crosstalk. For small values of t, (17) and (24) yield, after power series expansion,

$$\frac{E_1(t)}{E_0(0)} \cong 1 - (1 + \tfrac{1}{2} n^2 \mu^2 A^2 B) \left[1 - \frac{2B(1 + \frac{1}{2} n^2 \mu^2 A^2 B)}{(2\pi \varphi_1 T)^2} \right] B \frac{t^2}{T^2}, \tag{87}$$

$$\frac{E_G(t)}{E_0(0)} \cong 1 - (1 + \tfrac{1}{2}n^2\mu^2A^2B)B\frac{t^2}{T^2}. \tag{88}$$

Since expressions (87) and (88) have the same functional dependence, the transient of a system with neighboring channel-dropping filters is equivalent to that of a system without neighboring channel-dropping filters that operates with a wider input pulse

$$2T_1 = 2T\left[1 + B\frac{1 + \tfrac{1}{2}n^2\mu^2A^2B}{(2\pi\varphi_1T)^2}\right]. \tag{89}$$

It is possible to prevent any widening, $T_1 = T$, by making $\varphi_1 = \infty$. This is achieved in (6) by selecting the center frequencies f_3 and f_4 of the neighboring channels at opposite sides of the center frequency f_1 of the channel in which the time crosstalk is being studied. But even if φ_1 is finite, the relative pulse width increase $(T_1/T) - 1$ is negligibly small for typical cases. For example, if one selects the parameters of a system with 1 db insertion loss due to neighboring channel-dropping filters (center set of curves, Fig. 2):

$$\frac{F_{1\alpha}}{F_1} = \frac{F_{2\alpha}}{F_2} = 1.5,$$

$$\mu = \frac{F_1}{F_2} = 1,$$

$$q = r = 1,$$

$$\pi TF_1 = 0.5,$$

$$\frac{|f_1 - f_3|}{2F_2} = \frac{|f_1 - f_4|}{2F_2} = 2.6,$$

it follows from (89) that

$$\frac{T_1}{T} - 1 = 0.025.$$

APPENDIX C

Influence of Nonlinearity of the Transmitting Amplifier on Frequency and Time Crosstalk

The nonlinearity of the transmitting amplifier changes crosstalk by compressing the top of the output pulse and because the spurious pulses generated by the amplifier change the tail of the main output pulse (extra time crosstalk) and part of their spectra is fed into neighboring channels (extra frequency crosstalk).

The time crosstalk at the instant $t = \zeta T$ is by definition the ratio between the envelope of the channel output at that instant and the maximum output expressed in db. From (23), calling $E_3(t)$ the value acquired by $E_2(t)$ if $\rho = 0$ $(f_1 = f_2)$, it is

$$\begin{aligned} 20 \log \frac{E_3(\zeta T)}{E_3(0)} &= 20 \log e^{-B_0\zeta^2} \mid \cos n\mu A B_0 \zeta \mid \\ &- 20 \log \left[1 + \sum_1^\infty \frac{A_N}{A_0} \frac{T_N}{T_0} \sqrt{\frac{B_N}{B_0}}\, e^{(n\mu A/2)^2 (B_N - B_0)} \right] \\ &+ 20 \log \left| 1 + \sum_1^\infty \frac{A_N}{A_0} \frac{T_N}{T_0} \sqrt{\frac{B_N}{B_0}}\, e^{[(n\mu A/2)^2 - \zeta^2](B_N - B_0)} \cdot \frac{\cos n\mu A B_N \zeta}{\cos n\mu A B_0 \zeta} \right| . \end{aligned} \tag{90}$$

The first term is the main contribution to time crosstalk and has been computed elsewhere.[3] The second term is the contribution due to output compression, 20 log $[E_3(0)/E_0(0)]$, (25), and the third term is what we called extra time crosstalk in the text.

Assume:

(a) only one term of nonlinearity (only compression of third, fifth, etc. order),

(b) small output compression (2 db at most),

(c) such a pessimistic combination of parameters that

$$\cos n\mu A B_0 \zeta = \cos n\mu A B_N \zeta = 1. \tag{91}$$

Then (90) becomes

$$20 \log \frac{E_3(\zeta T)}{E_3(0)} \cong 20 \log e^{-B_0\zeta} - (1 - e^{-\zeta^2 (B_N - B_0)})\, 20 \log \frac{E_3(0)}{E_0(0)} . \tag{92}$$

For the range of values of the different parameters given in Fig. 4 and $\zeta > 2$,

$$e^{-\zeta^2 (B_N - B_0)} \ll 1, \tag{93}$$

and consequently it is a slightly conservative assumption to neglect the extra time crosstalk due to compression.

Now we turn to frequency crosstalk. The maximum frequency crosstalk is by definition the ratio between $E_3(0)$, the maximum of the envelope output in a neighboring channel, and $E_2(0)$, the maximum output in the channel. $E_2(0)$ is calculated from (23), making $t = 0$, and $E_3(0)$ is also derived from (23), making $t = \rho = 0$. The ratio expressed in db is

$$20 \log \frac{E_2(0)}{E_3(0)} = 20 \log e^{-4\rho^2 b(1-b\mu^2 A^2 B_0)} \cosh 2n\rho(1 - b\mu^2 A^2 B_0)$$
$$- 20 \log \left[1 + \sum_{N=1}^{\infty} \frac{A_N}{A_0} \frac{T_N}{T_0} \sqrt{\frac{B_N}{B_0}} e^{(n\mu A/2)^2 (B_N - B_0)}\right]$$
$$+ 20 \log \left[1 + \sum_{1}^{\infty} \frac{A_N}{A_0} \frac{T_N}{T_0} \sqrt{\frac{B_N}{B_0}} e^{(\mu A/2)^2 [n^2 + (4\rho b)^2](B_N - B_0)} \cdot \frac{\cosh 2n\rho(1 - b\mu^2 A^2 B_N)}{\cosh 2n\rho(1 - b\mu^2 A^2 B_0)}\right]. \tag{94}$$

The first term is the main contribution to frequency crosstalk and it has been computed elsewhere.[3] The second term is the contribution due to output compression 20 log $[E_3(0)/E_0(0)]$, (25), and the third term is the extra frequency crosstalk.

Assuming that the amplifier has only one term of nonlinearity and that the output compression is small, the second term of (94) is simplified to

$$20 \log \frac{E_3(0)}{E_0(0)} \cong 8.686 \frac{A_N}{A_0} \frac{T_N}{T_0} \sqrt{\frac{B_N}{B_0}} e^{(n\mu A/2)^2 (B_N - B_0)}, \tag{95}$$

and consequently (94) can be rewritten

$$20 \log \frac{E_2(0)}{E_3(0)} = 20 \log e^{-4\rho^2 b(1-b\mu^2 A^2 B_0)} \cosh 2n\rho(1 - b\mu^2 A^2 B_0)$$
$$- 20 \log \frac{E_3(0)}{E_0(0)} + 20 \log \left[1 + 2.3 \log \frac{E_3(0)}{E_0(0)} e^{(2\mu A \rho b)^2 (B_N - B_0)} \cdot \frac{\cosh 2n\rho(1 - b\mu^2 A^2 B_N)}{\cosh 2n\rho(1 - b\mu^2 A^2 B_0)}\right]. \tag{96}$$

If the channels must be crowded, ρ is small, but in that case the extra frequency crosstalk can be expanded in series and the second and third terms tend to cancel each other. In the limit, for $\rho = 0$, 20 log $[E_2(0)/E_3(0)] = 0$, as it should be, since by definition $E_2(0)$ and $E_3(0)$ coalesce to the same value.

On the other hand, for large values of ρ the extra frequency crosstalk (third term) becomes more important than the frequency crosstalk due to compression (second term), but then the total frequency crosstalk 20 log $[E_2(0)/E_3(0)]$ is negligibly small.

Consider, for instance, the following system, in which the parameters $\pi T F_1 = 0.5$, $F_1/F_2 = 1.4$, $F_{1\beta} = F_{2\alpha} = \infty$, $N = 2$, and 2 db of amplifier compression have been chosen to exaggerate the influence of nonlinearity on frequency crosstalk. Substituting these values in (96),

$$20 \log \frac{E_2(0)}{E_3(0)} = \begin{cases} -29.1 + 1.04 - 2.9 = -30.96 \text{ (in db)} & \text{for } \rho = 2, \\ -67.2 + 1.04 - 37.1 = -103.26 \text{ (in db)} & \text{for } \rho = 3. \end{cases} \tag{97}$$

The first term corresponds to frequency crosstalk for a linear amplifier, the second term is the frequency crosstalk correction due to output compression, and the third term is the extra frequency crosstalk. For small ρ ($\rho = 2$) it is conservative to ignore the second and third terms. For large ρ ($\rho = 3$) the total frequency crosstalk is negligibly small. Therefore, for all values of ρ and for all practical purposes the total frequency crosstalk can be calculated as if the amplifier were linear.

APPENDIX D

Noise Level at the Detector.

The distribution of the filtering before and after the first amplifier of the receiver, Fig. 1, is important because it determines the thermal noise power reaching the detector.

If the noise from the amplifier is white and the power density is W_0 watts per cycle, then the mean noise power received by the detector is

$$\sigma^2 = W_0 \int_{-\infty}^{\infty} | y_{2\alpha} |^2 \, df, \tag{98}$$

where

$$y_{2\alpha} = \frac{y_2}{y_{2\gamma}} = e^{-b[(|f|-f_2)/F_2]^2} \cosh n \frac{|f| - f_2}{F_2} \sqrt{1 + \left(\frac{|f| - f_2}{F_{2\gamma}}\right)^{2r}} \tag{99}$$

is the transfer characteristic of the filter connecting the amplifier and the detector. This characteristic has been found as the ratio between y_2, the transfer characteristic of the receiver (approximately third-order maximally flat), and $y_{2\gamma}$, the transfer characteristic of the receiving channel-dropping filter (maximally flat of order r).

Performing the integration (98), the average noise power received by the detector is

$$\sigma^2 = \sqrt{\frac{\pi}{2}}\, W_0 \left[1 + \frac{1}{(-2)^r} \left(\frac{F_2}{F_{2\gamma}} \right)^{2r} \frac{\partial^r}{\partial b^r} \right] \left(\frac{e^{n^2/2b} + 1}{\sqrt{b}} \right) F_2 . \quad (100)$$

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