A Nonlinear Integral Equation from the Theory of Servomechanisms

By V. E. BENEŠ

(Manuscript received March 28, 1961)

The equation $x(t) = s(t) - k*F(x)(t)$, where $s(\cdot)$ is a given signal, $F(\cdot)$ is a nonlinear function, $k(\cdot)$ is the response of a linear system, and $*$ denotes convolution, describes a general class of servomechanisms. Properties of a solution $x(\cdot)$ can be established by finding a fixed point in a specific set of a function space, using Schauder's theorem.

I. INTRODUCTION

A general class of nonlinear servomechanisms is described by the integral equation

$$x(t) = s(t) - \int_{-\infty}^{\infty} k(t-u)F(x(u))\,du, \qquad -\infty < t < \infty, \quad (1)$$

where $s(\cdot)$ is an input signal, $k(\cdot)$ is an impulse response function, and $F(\cdot)$ is a nonlinear function. The equation (1) represents the system diagram of Fig. 1, with $F(\cdot)$ as above, and with $K(\cdot)$ the transfer function corresponding to $k(\cdot)$. We assume that $F(\cdot)$ satisfies the uniform Lipschitz condition

$$| F(x) - F(y) | \leqq \beta | x - y |,$$

and that $F(0) = 0$.

A classical method for studying nonlinear servomechanisms like that of Fig. 1 is to specify exactly the nonlinear element $F(\cdot)$, to assume that the response $k(\cdot)$ is the Green's function of a differential operator of low order, and to use some sort of phase-plane analysis. This method has two theoretical disadvantages: it lacks generality, and, when applied, it tends to give more information than is needed; thus it provides detailed knowledge about a restricted class of cases.

In this paper we shall use a method that has the opposite characteristics: it provides a small amount of highly relevant information about a large class of cases. We shall exemplify the use of Schauder's fixed point theorem for studying solutions $x(\cdot)$ of (1) without specifying either $k(\cdot)$ or $F(\cdot)$ in detail. We establish definite properties of $x(\cdot)$ by finding a fixed point (corresponding to a solution of the equation) in a specific set of a function space. Since the function space and the set can be chosen in many ways, depending in part on what properties of $x(\cdot)$ are of interest, such a method can be used for a wide class of problems. The theory in the sequel is therefore restricted to sample results for the function space L_2 of square-integrable functions, and is to be regarded only as a particular example of the method described above.

II. FUNCTIONS OF FINITE ENERGY

In many situations it is desirable that the convolution term

$$\int_{-\infty}^{\infty} k(t - u)F(x(u))\, du$$

follow the input signal $s(\cdot)$. The error in this approximation is then $x(\cdot)$ itself. It is then reasonable to work in the space L_2 of real, square-integrable functions, i.e., functions of finite energy. Accordingly, we assume that $k(\cdot)$ and $s(\cdot)$ are in L_2, and we seek to bound the energy of a solution $x(\cdot)$ of (1).

Now the functions of L_2 cannot assume values appreciably different from zero on sets of arbitrarily large measure. Hence they may be viewed physically as pulses. By restricting $s(\cdot)$ and the solution $x(\cdot)$ to L_2 we are therefore studying the response of the system of Fig. 1 to certain pulses of finite energy. We shall be particularly interested in finding out how much of the energy of $x(\cdot)$ lies outside a given time interval.

The norm symbol $\| \cdot \|$ is used to denote the square root of the energy of a function. Thus for $x(\cdot)$ in L_2,

$$\| x \| = \left(\int_{-\infty}^{\infty} | x(t) |^2\, dt \right)^{\frac{1}{2}},$$

and a sequence of functions $\{x_n(\cdot), n \geqq 0\}$ is said to converge to a function $x(\cdot)$ in L_2-norm if $\| x - x_n \|$ approaches zero with increasing n.

III. HYPOTHESES AND PRELIMINARY RESULTS

If $x(\cdot)$ is a function of L_2, we let

$$Tx(\omega) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{i\omega t} x(t)\, dt$$

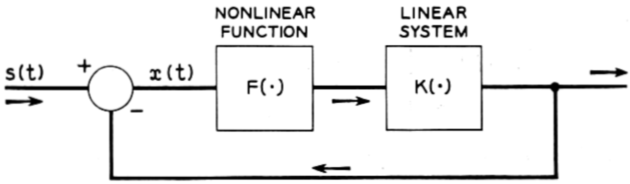

Fig. 1 — System diagram of servomechanism.

denote its Fourier transform; for $k(\cdot) \in L_2$, we reserve the special notation

$$Tk(\omega) = K(\omega).$$

The operator H on L_2 is defined by the condition

$$Hx(t) = \int_{-\infty}^{\infty} k(t-u)F(x(u))\,du.$$

Lemma 1: If $K(\cdot)$ is bounded in ω, then H is a continuous transformation of L_2 into itself.

Proof: For $x(\cdot) \in L_2$ the Lipschitz condition on $F(\cdot)$ yields $\| F(x) \| \leqq \beta \| x \|$, so that $F(x(\cdot)) \in L_2$. It is a known result that the convolution of two L_2 functions belongs to L_2. Hence $Hx \in L_2$. Also, by the Parseval relations,

$$\begin{aligned} \| Hx - Hy \|^2 &= \int_{-\infty}^{\infty} | K(\omega) |^2 \, | TF(x) - TF(y) |^2 \, d\omega \\ &\leqq \sup_{\omega} | K(\omega) |^2 \, \| F(x) - F(y) \|^2 \\ &\leqq \beta \sup_{\omega} | K(\omega) |^2 \, \| x - y \|^2, \end{aligned}$$

which shows that H is continuous.

Now let $w(\cdot)$ be a given non-negative function of L_2, and let S be the set of all $x(\cdot)$ in L_2 such that

$$| x(t) | \leqq w(t), \qquad \text{almost everywhere.} \tag{2}$$

Lemma 2: S is closed and convex.

Proof: Let $x_n(\cdot) \in S$ be a sequence of functions approaching $x(\cdot)$ in L_2. Then for $\epsilon > 0$ and $\mu(\cdot) =$ Lebesgue measure,

$$\begin{aligned} \| x - x_n \|^2 &\geqq \int_{|x_n - x| > \epsilon} | x_n - x |^2 \, dt \\ &\geqq \epsilon^2 \mu\{t : | x_n(t) - x(t) | > \epsilon\}. \end{aligned}$$

However,

$$| x(t) | \leqq | x(t) - x_n(t) | + | x_n(t) |$$
$$\leqq | x(t) - x_n(t) | + w(t).$$

Hence $| x(t) | - w(t) > \epsilon$ implies $| x(t) - x_n(t) | > \epsilon$ and

$$\mu\{t: | x(t) - x_n(t) | > \epsilon\} \geqq \mu\{t: | x(t) | - w(t) > \epsilon\}.$$

Letting n approach infinity on the left, we find that

$$\{t: | x(t) | - w(t) > \epsilon\}$$

has measure zero for each $\epsilon > 0$. Hence almost everywhere

$$| x(t) | \leqq w(t),$$

and so S is closed. The convexity of S is obvious.

We denote by B the subset of functions $x(\cdot)$ of L_2 which are "band-limited" to the frequency interval $(-\Omega,\Omega)$, i.e., representable as

$$x(t) = (2\pi)^{-\frac{1}{2}} \int_{-\Omega}^{\Omega} e^{i\omega t} Tx(\omega)\, d\omega.$$

The physical interpretation of membership in B is of course that the sinusoidal oscillations into which a function is decomposed by the Fourier transform are restricted in frequency to the interval $(-\Omega,\Omega)$; i.e., $Tx(\omega) = 0$ for $| \omega | > \Omega$.

The input signal $s(\cdot)$, and the response $k(\cdot)$ will be assumed to belong to B. If we define the operator J on L_2 by

$$Jx(t) = s(t) - Hx(t),$$

then the range of J is a subset of B. It follows that any solution of (1), i.e., any fixed point of J, will belong to B as long as $s(\cdot)$ and $k(\cdot)$ do so. Such a "band-limiting" restriction is natural physically, because of the known attenuation at high frequencies characteristic of physical circuits, and it will have an important mathematical role in finding fixed points of J. In particular, we note that $JS \subset B$.

To obtain a bound on the amount of energy that a solution $x(\cdot)$ has outside a given interval, we shall suppose that the non-negative function $w(\cdot)$ of L_2, used in the definition of S, satisfies the integral inequality

$$| s(t) | + \beta \int_{-\infty}^{\infty} | k(t - u) | \, w(u)\, du \leqq w(t). \tag{3}$$

This inequality may be thought of as defining an associated *linear* problem; it will be used to ensure that $Jx(\cdot)$ belongs to S if $x(\cdot)$ does. The nonlinear function $F(\cdot)$ enters formula (3) only via its Lipschitz constant (of order 1) β.

Lemma 3: If (3) holds, and $F(0) = 0$, then $JS \subset S$.

Proof: Let $x(\cdot)$ belong to S. Then

$$\begin{aligned} |Jx(t)| &\leqq |s(t)| + \int_{-\infty}^{\infty} |k(t-u)|\,|F(x(u))|\,du \\ &\leqq |s(t)| + \beta \int_{-\infty}^{\infty} |k(t-u)|\,w(u)\,du \\ &\leqq w(t). \end{aligned}$$

Our preliminaries are completed by

Lemma 4: $S \cap B$ is compact in L_2.

Proof: Let $E = \|w\|^2$. The functions of $S \cap B$ are (uniformly) equicontinuous with modulus

$$\left(\frac{2\Omega E}{\pi}\right)^{\frac{1}{2}} \left(1 - \frac{\sin \Omega\epsilon}{\Omega\epsilon}\right)^{\frac{1}{2}}.$$

This follows from the inequalities:

$$\begin{aligned} |x(t+\epsilon) - x(t)| &\leqq (2\pi)^{-\frac{1}{2}} \int_{-\Omega}^{\Omega} |e^{i\omega\epsilon} - 1|\, Tx(\omega)\, d\omega \\ &\leqq (2\pi)^{-\frac{1}{2}} \left(\int_{-\Omega}^{\Omega} |e^{i\omega\epsilon} - 1|^2\, d\omega\right)^{\frac{1}{2}} \|x\| \\ &\leqq \left(\frac{2E}{\pi} \int_{-\Omega}^{\Omega} (1 - \cos \omega\epsilon)\, d\omega\right)^{\frac{1}{2}} \\ &\leqq \left(\frac{2\Omega E}{\pi}\right)^{\frac{1}{2}} \left(1 - \frac{\sin \Omega\epsilon}{\Omega\epsilon}\right)^{\frac{1}{2}}, \end{aligned}$$

the last bound on the right being independent of t and $x(\cdot)$. Also, the inequalities

$$\begin{aligned} |x(t)| &\leqq (2\pi)^{-\frac{1}{2}} \int_{-\Omega}^{\Omega} |Tx(\omega)|\, d\omega \\ &\leqq (2\pi)^{-\frac{1}{2}} (2\Omega)^{\frac{1}{2}} \|x\| \\ &\leqq \left(\frac{\Omega E}{\pi}\right)^{\frac{1}{2}}, \end{aligned}$$

show that the functions of $S \cap B$ are uniformly bounded.

Since both S and B are closed sets, it suffices (to prove Lemma 4) to show that $S \cap B$ is sequentially compact. Let $x_n(\cdot) \in S \cap B$ be an arbitrary sequence of functions. The $x_n(\cdot)$ are uniformly bounded and uniformly equicontinuous. By a standard diagonal argument using the σ-compactness of the real line, we can select a subsequence $x_m(\cdot)$ which converges to a function $x(\cdot)$ uniformly on any compact set. We have

$$| x(t) | \leqq | x(t) - x_m(t) | + | x_m(t) |,$$

$$\int_{-t}^{t} | x(u) |^2 \, du \leqq \int_{-t}^{t} | x(u) - x_m(u) |^2 \, du + 2 \int_{-t}^{t} | x_m(u) | \, | x(u) - x_m(u) | \, du + \int_{-t}^{t} | x_m(u) |^2 \, du.$$

For each fixed t, the first two terms on the right of the last inequality approach zero as m becomes large, and the third term is at most $\| w \|^2 = E$ uniformly in t. Hence $\| x \|^2 \leqq E$ and $x(\cdot) \in L_2$. Using Minkowski's inequality, we find

$$\begin{aligned} \| x - x_m \| &\leqq \left(\int_{|u|>t} | x - x_m |^2 \, du \right)^{\frac{1}{2}} + \left(\int_{|u| \leqq t} | x - x_m |^2 \, du \right)^{\frac{1}{2}} \\ &\leqq \left(\int_{|u|>t} | x(u) |^2 \, du \right)^{\frac{1}{2}} + \left(\int_{|u|>t} | x_m(u) |^2 \, du \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{|u| \leqq t} | x - x_m |^2 \, du \right)^{\frac{1}{2}}. \end{aligned}$$

The first two terms on the right can be made arbitrarily small by a large enough choice of t, uniformly in m; for t fixed, the third term goes to zero as $m \to \infty$. Hence the $x_m(\cdot)$ converge to $x(\cdot)$ in L_2, which proves Lemma 4.

IV. PRINCIPAL RESULTS FOR GENERAL $F(\cdot)$

Theorem 1: Let $s(\cdot)$ and $k(\cdot)$ belong to B, with $K(\cdot)$ bounded, let $F(0) = 0$, and let the integral inequality (3) obtain. Then there exists a solution $x(\cdot)$ of (1) in the set $S \cap B$, with the properties

$$| x(t) | \leqq w(t) \qquad (\text{and so } \| x \| \leqq \| w \|),$$

$$| x(t) | \leqq \left(\frac{\Omega}{\pi} \right)^{\frac{1}{2}} \| x \|,$$

$$x(t) = (2\pi)^{-\frac{1}{2}} \int_{-\Omega}^{\Omega} e^{i\omega t} Tx(\omega) \, d\omega.$$

Proof: J is a continuous mapping of the closed convex set S into its compact subset $S \cap B$. By the "strong form" of Schauder's theorem,[1,2] there exists a point $x(\cdot)$ in $S \cap B$ such that $x = Jx$. The properties listed above are immediate consequences of belonging to $S \cap B$.

The following slight modification of Theorem 1 involves no new principle:

Extension: If, in addition to the hypotheses of Theorem 1,

$$\beta \sup_{\omega} | K(\omega) | < 1, \tag{4}$$

then to the conclusion of Theorem 1 can be added

$$\| x \| \leqq \frac{\| s \|}{1 - \beta \sup_{\omega} | K(\omega) |}.$$

Proof: Let a denote the bound on the right of the last inequality. Then the intersection Q of S with the closed ball of radius a is closed and convex. With condition (4), and $x(\cdot) \in Q$, the inequalities

$$\begin{aligned} \| Jx \| &\leqq \| s \| + \left(\int_{-\Omega}^{\Omega} | K(\omega) |^2 \, | TF(x) |^2 \, d\omega \right)^{\frac{1}{2}} \\ &\leqq \| s \| + \sup_{\omega} | K(\omega) | \, \| F(x) \| \\ &\leqq \| s \| + \beta \sup_{\omega} | K(\omega) | \, \| x \| \end{aligned}$$

show that $JQ \subset Q \cap B$. Since the topology is Hausdorff, $Q \cap B$ is a closed subset of the compact set $S \cap B$, so it is compact. The result follows from Schauder's theorem.

V. PRELIMINARIES FOR $F(\cdot)$ NEARLY LINEAR AT THE ORIGIN

It is clear that stronger assumptions concerning the nonlinear function $F(\cdot)$ are necessary if we are to obtain results that make the energy of $x(\cdot)$ less than that of $s(\cdot)$. A particularly important case is one in which

$$F(t) = t + o(t), \qquad \text{as } t \to 0;$$

that is, $F(\cdot)$ is linear near the origin.

Let $F(\cdot)$ have the form [where $n(k)$ are integers, $n(1) = 1$]

$$F(t) = \sum_{k=1}^{\infty} t^{n(k)} (-1)^{k-1} f_k , \tag{5}$$

with $f_k > 0, f_1 = 1, n(k+1) > n(k)$, the series converging for $| t | < \rho$, where

$$\rho^{-1} = \limsup_{k \to \infty} | f_k |^{1/k}.$$

Suppose also that there is a number a, $0 < a < \rho$, such that

$$a^{n(k+1)-n(k)} < \frac{f_k}{f_{k+1}}, \qquad \text{for } k \geqq 1. \tag{6}$$

Then $| t | < a$ implies that $F(t)$ has the sign of t and

$$| F(t) - t | \leqq f_2 | t |^{n(2)}, \tag{7}$$

for then $F(\cdot)$ is represented by a power series of alternating sign whose terms are monotone in magnitude.

Since we are comparing (1) to a linearized version of (1) obtained by setting $F(t) = t$, we shall need the solution of the resulting linearized equation: this is a function $y(\cdot)$ defined by its Fourier transform

$$Ty(\omega) = \frac{Ts(\omega)}{1 + K(\omega)}.$$

Similarly, the closed-loop transfer function of the linearized loop is the Fourier transform

$$Tz(\omega) = \frac{K(\omega)}{1 + K(\omega)}$$

of a function $z(\cdot)$. These definitions will be justified in the theorem to be proved.

By dint of our stronger assumptions on $F(\cdot)$, we can use a different integral inequality from (3). We assume instead that there exists a real non-negative function $v(\cdot) \in L_2$ such that

$$| y(t) | + f_2 \int_{-\infty}^{\infty} v^{n(2)}(t - u) \, | z(u) | \, du \leqq v(t). \tag{8}$$

With this inequality playing the role of (3), the method used to prove Theorem 1 can be applied almost without modification.

However, since the integral inequality (8) is nonlinear in $v(\cdot)$, we shall digress a little and give a sufficient condition for its validity. One way to do this is to find a non-negative $v(\cdot) \in L_2$ that satisfies (8) *with equality*, i.e., is a solution of the nonlinear equation

$$\begin{aligned} v(t) &= | y(t) | + f_2 \int_{-\infty}^{\infty} v^{n(2)}(t - u) \, | z(u) | \, du \\ &= Mv(t). \end{aligned} \tag{9}$$

We shall show how the classical contraction principle for complete metric spaces can be used to find a solution $v(\cdot)$ of (9), i.e., a fixed point of M. Such a result is exemplified by

Lemma 5: If for

$$b = f_2 \int_{-\infty}^{\infty} | z(u) | \, du < \infty, \qquad \alpha = \left(\frac{1}{bn(2)}\right)^{1/[n(2)-1]}$$

we have for some $\delta > 0$

$$\sup_u | y(u) | < \alpha \left(1 - \frac{1}{n(2)}\right) - \delta, \tag{10}$$

then the map M is contracting on the closed set Y of $x(\cdot) \in L_2$ such that

$$x(\cdot) \geqq 0$$

$$\operatorname{ess\,sup}_u x(u) \leqq \alpha - \delta.$$

Proof: Consider the equation for $a > 0$,

$$\sup_u | y(u) | + ba^{n(2)} = a. \tag{11}$$

The left-hand side has unity slope at the point $a = \alpha$, and the inequality (10) implies that at this point the left-hand side is less than the right. Hence (11) has two roots in $a > 0$, and, for $x \in Y$,

$$\begin{aligned}\sup_t | Mx(t) | &\leqq \sup_u | y(u) | + f_2 \int_{-\infty}^{\infty} x^{n(2)}(t-u) \, | z(u) | \, du \\ &\leqq \sup_u | y(u) | + b\alpha^{n(2)} \\ &\leqq \alpha - \delta.\end{aligned}$$

Thus $MY \subset Y$. To show that Y is a closed set we recall that convergence in L_2 implies convergence in measure. Let $x_n \in Y$ converge to x in L_2; then, as $n \to \infty$,

$$\mu\{t: | x_n(t) - x(t) | \geqq \epsilon\} \to 0$$

for each $\epsilon > 0$. However, almost everywhere we have

$$-| x(t) - x_n(t) | \leqq x(t) \leqq | x_n(t) - x(t) | + \alpha - \delta,$$

and so

$$\left.\begin{matrix}\mu\{t: x(t) < -\epsilon\} \\ \mu\{t: x(t) \geqq \alpha - \delta + \epsilon\}\end{matrix}\right\} \leqq \mu\{t: | x_n(t) - x(t) | \geqq \epsilon\},$$

where $\mu(\cdot)$ denotes Lebesgue measure. Letting $\epsilon \to 0$, we find

$$\operatorname*{ess\,sup}_{u} x(t) \leqq \alpha - \delta,$$

$$\operatorname*{ess\,inf}_{u} x(t) \geqq 0.$$

To show that M is contracting on Y, let $x(\cdot)$ and $y(\cdot)$ be arbitrary functions in Y. Then

$$\| Mx - My \| \leqq f_2 \left(\int_{-\infty}^{\infty} | Tz |^2 \, | Tx^{n(2)} - Ty^{n(2)} |^2 \, d\omega \right)^{\frac{1}{2}}$$

$$\leqq f_2 \sup_{\omega} | Tz | \, \| x^{n(2)} - y^{n(2)} \| .$$

However, on Y

$$| x^{n(2)} - y^{n(2)} | \leqq | x - y | \, n(2)(\alpha - \delta)^{n(2)-1},$$

$$\| x^{n(2)} - y^{n(2)} \| \leqq \| x - y \| \, n(2)(\alpha - \delta)^{n(2)-1},$$

and so, since $z \in L_1$,

$$\| Mx - My \| \leqq f_2 \sup | Tz | \, n(2)(\alpha - \delta)^{n(2)-1} \, \| x - y \|$$

$$\leqq bn(2)(\alpha - \delta)^{n(2)-1} \, \| x - y \|.$$

But $bn(2)(\alpha - \delta)^{n(2)-1} < 1$, so M is contracting on Y.

Lemma 5 implies, by the contraction principle, that there exists a unique solution $v(\cdot)$ of (9) in the set Y, obtainable as the limit of successive approximations starting at any point of Y.

VI. PRINCIPAL RESULTS FOR $F(\cdot)$ NEARLY LINEAR AT THE ORIGIN

Let R be the set of functions $x(\cdot)$ of L_2 that satisfy the condition

$$| x(t) | \leqq v(t), \qquad \text{almost everywhere},$$

where $v(\cdot)$ is the function in the inequality (8). The argument of Lemma 2 shows that R is closed and convex, and that of Lemma 4 shows that $R \cap B$ is compact.

Theorem 2: If $a > 0$ and $F(\cdot)$ have the properties (5) and (6), and if $k(\cdot)$ and $s(\cdot)$ both belong to B, with $k(\cdot) \in L_1$ and $K(\omega) \neq -1$, and if (8) holds with

$$\| v \|^2 < \frac{a^2 \pi}{\Omega}, \tag{12}$$

then a solution $x(\cdot)$ of (1) exists in B with the properties

$$|x(t)| \leqq v(t),$$

$$|x(t)| \leqq \left(\frac{\Omega}{\pi}\right)^{\frac{1}{2}} \| x \|.$$

Proof: Since $K(\omega)$ is continuous, and tends to zero at ∞, it must be bounded away from -1; hence by the Wiener-Lévy theorem,[3,4]

$$[1 + K(\omega)]^{-1}$$

is the Fourier transform of an integrable function $g(\cdot)$, and so

$$Ts(\omega)[1 + K(\omega)]^{-1}$$

is the Fourier transform of a function $y(\cdot)$ of $L_2 \cap B$, and also

$$K(\omega)[1 + K(\omega)]^{-1}$$

is the Fourier transform of a function $z(\cdot)$ of $L_1 \cap L_2 \cap B$.

We now write (1) as

$$x(t) + \int_{-\infty}^{\infty} k(t-u)x(u)\,du = s(t) - \int_{-\infty}^{\infty} k(t-u)[F(x(u)) - x(u)]\,du.$$

Taking Fourier transforms gives

$$Tx = Ty - Tz\, T[F(x) - x].$$

We shall therefore consider the equivalent equation

$$x(t) = y(t) - \int_{-\infty}^{\infty} z(t-u)[F(x(u)) - x(u)]\,du,$$

$$= Gx(t).$$

This is of exactly the same form as (1); in particular, G is a continuous map. To apply Schauder's theorem it remains to verify that $GR \subset R$. For $x(\cdot) \subset R$,

$$|Gx(t)| \leqq |y(t)| + \int_{-\infty}^{\infty} |F(x(t-u)) - x(t-u)|\,|z(u)|\,du.$$

But $|x(\cdot)| \leqq a$, by (12); so (7) gives

$$|F(x(t-u)) - x(t-u)| \leqq f_2 |x(t-u)|^{n(2)},$$

$$\leqq f_2 |v^{n(2)}(t-u)|, \quad \text{almost everywhere.}$$

Hence (8) implies that $|Gx(t)| \leqq v(t)$.

The energy of the solution of the linearized equation with the input signal $s(\cdot)$ is

$$\left(\int_{-\infty}^{\infty} \left| \frac{Ts(\omega)}{1 + K(\omega)} \right|^2 d\omega \right)^{\frac{1}{2}} = \| y \| .$$

The gain of the closed linearized loop at the frequency ω is

$$\left| \frac{K(\omega)}{1 + K(\omega)} \right| . \tag{13}$$

It is reasonable to expect that, if the function $F(\cdot)$ is close to being linear, then the solution $x(\cdot)$ will have an energy close to that of the linear solution $y(\cdot)$, in the sense that, for some constant ξ that approaches unity as $F(\cdot)$ becomes linear, we have

$$\| x \| \leqq \xi \| y \| .$$

A precise form of this intuitive idea, depending on the linearized loop gain (13), is given in

Theorem 3: If, in addition to the hypotheses of Theorem 2, it is true that

$$c = a^{n(2)-2} f_2 \sup_{\omega} \left| \frac{K(\omega)}{1 + K(\omega)} \right| < 1 , \tag{14}$$

then to the conclusion of Theorem 2 may be added

$$\| x \| \leqq \min \left(\frac{\| y \|}{1 - c}, \quad \| v \| \right) .$$

Proof: The intersection V of R which has the closed ball of radius $\| y \| / (1 - c)$ is closed and convex. With condition (14), and $x(\cdot) \in V$, the inequalities

$$\begin{aligned} \| Gx \| &\leqq \| y \| + \left(\int_{-\infty}^{\infty} | Tz |^2 \, | T[F(x) - x] |^2 \, d\omega \right)^{\frac{1}{2}} \\ &\leqq \| y \| + \sup_{\omega} | Tz | \, \| F(x) - x \| \\ &\leqq \| y \| + c \| x \| \leqq \frac{\| y \|}{(1 - c)} \end{aligned}$$

show that $GV \subset V \cap B$. Also, $V \cap B$ is a closed subset of the compact set $R \cap B$. So the result follows from Schauder's theorem.

The condition (14) used in Theorem 3 relates the maximum gain of the linearized loop with the second nonzero coefficient f_2 in the expansion of

$F(t)$ around the origin, and with the power $n(2)$ associated with this coefficient.

VII. ACKNOWLEDGMENT

The author is indebted to H. Landau for reading an early draft and making helpful suggestions.

REFERENCES

1. Schauder, J., Der Fixpunktsatz in Funktionalraum, Studia Mathematica, **2,** 1930, pp. 171–181.
2. Bers, L., Introduction to Topology, Lecture XV, New York Univ., New York, 1954–5.
3. Pitt, H. R., *Tauberian Theorems*, Oxford Univ. Press, Oxford, 1958, p. 97.
4. Paley, R. E. A. C., and Wiener, N., *Fourier Transforms in the Complex Domain*, American Mathematical Society Colloquium Publications, XIX, New York, 1934, p. 63.