Asymptotic Behavior of General Queues with One Server

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The asymptotic behavior of the virtual delay in a single-server queue with order of arrival service is studied, under no restrictions on the stochastic nature of the arrival process and the service times, except for a weak stationarity condition. This behavior is shown to be governed by a functional equation closely analogous to the "fundamental equation" of branching processes, already used in special queueing models. A generalization of the Pollaczek-Khinchin delay formula is derived for the case in which delays do not build up.

I. INTRODUCTION

There is a queue in front of one server; waiting customers are served in order of arrival, and no defections from the queue occur. The delays incurred in such a system can be described by a stochastic process $\{W(t),t \geqq 0\}$, the virtual waiting time, defined as the time a customer would have to wait for service if he arrived at time t. Stochastic processes of this kind have been studied.[1,2]

In the present work we examine the asymptotic behavior of the probabilities $\Pr\{W(t) \leqq w\}$, as t becomes large, for a class of processes satisfying a weak condition of stationarity.

It is customary to define the process $W(\cdot)$ in terms of the arrival epoch t_k and the service time S_k of the kth arriving customer, for $k = 1, 2, \cdots$. We can, however, describe the service times and the arrival epochs simultaneously by a single function $K(\cdot)$, defined for $t \geqq 0$, left-continuous, nondecreasing, and constant between successive jumps. The locations of the jumps are the epochs of arrivals, and the magnitudes are the service times. With $K(0) = W(0)$ for convenience, $K(t)$ can be interpreted as the "work load" submitted to the server during $[0,t)$, and $W(t)$ as the work remaining to be done at t.

Formally, $W(\cdot)$ is defined in terms of $K(u)$ for $0 \leqq u < t$ by the equation

$$W(t) = K(t) - t + \int_0^t U(-W(u))\,du, \tag{1}$$

where $U(\cdot)$ is the unit step function, i.e., $U(x) = 1$ for $x \geqq 0$, and $U(x) = 0$ otherwise. Equation (1) may be interpreted as follows:

$$\begin{aligned}\text{work remaining at } t = {} & \text{load offered up to } t \\ & - \text{elapsed time} \\ & + \text{total time that server was idle} \\ & \quad \text{in } (0,t).\end{aligned} \tag{2}$$

The relationship of $W(\cdot)$ and $K(\cdot)$ has previously been described and illustrated.[1,2] A principal result of the latter paper (and the only one needed for reading the present one) was a formula for the distribution of $W(t)$ in terms of the functions

$$\Pr\{K(t) \leqq u\},$$

$$R(t,u,w) = \Pr\{K(t) - K(u) - t + u \leqq w \mid W(u) = 0\}.$$

The reference to $W(\cdot)$ in the condition is not circular because, as shown,[2] the events

$$\{W(u) = 0\}, \qquad \{\sup_{0<y<u} [K(u) - K(y) - u + y] \leqq 0\}$$

differ by at most a set of measure zero. The formula for distr$\{W(t)\}$ is

$$\begin{aligned}\Pr\{W(t) \leqq w\} = {} & \Pr\{K(t) - t \leqq w\} \\ & - \frac{\partial}{\partial w}\int_0^t R(t,u,w)\Pr\{W(u) = 0\}\,du, \qquad w > 0,\end{aligned} \tag{3}$$

or, in an integrated form,

$$\begin{aligned}\int_0^w \Pr\{K(t) \leqq u\}\,du = {} & \int_0^{t+w} \Pr\{K(t) \leqq u\}\,du \\ & - \int_0^t R(t,u,w)\Pr\{W(u) = 0\}\,du.\end{aligned} \tag{4}$$

For $-t \leqq w \leqq 0$, the chance $\Pr\{W(u) = 0\}$ that the server be idle at time u satisfies the Volterra equation

$$\begin{aligned}\int_0^{t+w} R(t,u,w)\Pr\{W(u) = 0\}\,du &= \int_0^{t+w} \Pr\{K(t) \leqq u\}\,du, \\ &= E\{\max[0, w - K(t) + t]\}.\end{aligned} \tag{5}$$

These results can be cast into a form more useful for the present endeavor by use of Laplace-Stieltjes transforms. Theorem 5 of the author's paper[2] implies that for $\mathrm{Re}(s) > 0$

$$E\{e^{-sW(t)}\} = E\{e^{-sK(t)+st}\} - s\int_0^t E\{e^{-s[K(t)-K(u)-t+u]} \mid W(u) = 0\} \Pr\{W(u) = 0\}\, du. \tag{6}$$

This formula for the Laplace-Stieltjes transform of the distribution of $W(t)$ can be obtained directly from (4) and (5) by taking the Laplace-Stieltjes transform with respect to w.

II. BASIC ASSUMPTIONS

Unless the input process $K(\cdot)$ has some asymptotic temporal uniformity, one cannot expect $W(t)$ to have a limiting distribution as $t \to \infty$. We shall assume that the increments of $K(\cdot)$ are *weakly stationary* in the sense[3] that the basic kernel $R(t,u,w)$ of (3) depends only on $(t - u)$ and w, i.e.,

$$\Pr\{K(t) - K(u) - t + u \leqq w \mid W(u) = 0\} = R(t - u,w). \tag{7}$$

The word "weak" refers to the dependence of the kernel only on the difference of its two temporal arguments.

The stationarity condition (7) turns (5) into an equation of *convolution* type. Our strategy will be to work with (5), find conditions under which $\Pr\{W(u) = 0\}$ converges as $u \to \infty$, and then use Abelian arguments on (4) and (6) to obtain the asymptotic behavior of

$$\Pr\{W(t) \leqq w\} \qquad \text{as } t \to \infty.$$

A similar approach has been used in two prior papers by the author.[3,4]

We shall only consider the case in which

$$W(0) = 0, \tag{8}$$

with probability one. This "start empty" condition is in a sense unavoidable in practice, and it leads to simple, elegant results. Equations (7) and (8) imply that for $t,y \geqq 0$

$$\begin{aligned} E\{e^{-sK(t)+st}\} &= E\{e^{-sK(t)+st} \mid W(0) = 0\} \\ &= E\{e^{-sK(t+y)+sK(y)+sy} \mid W(y) = 0\} \\ &= \int_0^\infty e^{-sw} R(t,dw). \end{aligned} \tag{9}$$

The condition $\{W(\cdot) = 0\}$ will therefore be omitted from now on in all probabilities and expectations.

The following functions occur frequently in the discussion and merit abbreviations:

$$E\{\max[0,t - K(t)]\} = F(t),$$
$$\Pr\{K(t) \leqq t\} = R(t,0) = R(t),$$
$$\Pr\{W(t) = 0\} = P(t).$$

In this notation, the Volterra equation (5) for $w = 0$ can be restated as

$$F(t) = \int_0^t R(t - u)P(u)\, du. \tag{10}$$

The Laplace-Stieltjes transform of a function is denoted by the corresponding *lower-case* Greek letter; e.g.,

$$\varphi(s) = \int_0^\infty e^{-st} dF(t), \qquad \text{Re}(s) > 0.$$

The ordinary Laplace transform is denoted by the corresponding *capital* Greek letter; e.g.,

$$\Pi(\tau) = \int_0^\infty e^{-\tau t} P(t)\, dt.$$

From (1) defining $W(\cdot)$ it is evident that the mean delay is

$$E\{W(t)\} = E\{K(t)\} - t + \int_0^t \Pr\{W(u) = 0\}\, du.$$

Also, we have

$$\begin{aligned} E\{K(t)\} &= \int_0^\infty \Pr\{K(t) > u\}\, du \\ &= \int_0^t [1 - \Pr\{K(t) \leqq u\}]\, du + \int_t^\infty \Pr\{K(t) > u\}\, du \\ &= t - F(t) + \int_t^\infty \Pr\{K(t) > u\}\, du, \end{aligned}$$

since

$$E\{\max(0, K(t) - t\} = \int_0^t \Pr\{K(t) \leqq u\}\, du.$$

In addition to the stationarity condition (7) we use the following assumptions (since some theorems do not depend on all the assumptions to be listed, the relevant hypotheses are repeated when a result is stated):

i. $\limsup_{t\to\infty} t^{-1}E\{K(t)\} < \infty$;

ii. $\liminf_{t\to\infty} t^{-1} \log \Pr\{K(t) = 0\} > -\infty$;

iii. There exists a neighborhood N of the positive real axis and a $T > 0$ such that $E\{e^{-sK(t)}\} \neq 0$ for $s \in N$ and $t > T$.

Hypothesis iii will only be needed when the system is recurrent-null; i.e., the service-factor is unity.

III. SUMMARY

Basic assumptions and preliminary lemmata occupy Sections II and IV, respectively, the principal result being the existence of a solution $s(\tau)$ of the functional equation

$$\tau - s(\tau) = a(s(\tau)), \qquad \tau \geqq 0,$$

where, for real $s \geqq 0$,

$$a(s) = \limsup_{t\to\infty} t^{-1} \log E\{e^{-sK(t)} \mid W(0) = 0\}.$$

This functional equation is closely analogous to the "fundamental equation" of branching processes. In Section V it is shown that the Laplace transform of $\Pr\{W(\cdot) = 0\}$ is just $1/s(\tau)$, whence it follows (Section VI) that the asymptotic value of $\Pr\{W(t) = 0\}$ as $t \to \infty$ (if any) can be determined from the limit of $a(s)/s$ as $s \to 0$.

The existence of a limiting value $\Pr\{W(\infty) = 0\}$ is investigated in Sections VII and VIII, by Mercerian and Tauberian methods respectively, applied to (5), for $w = 0$. Section IX contains limit theorems for $\Pr\{W(\cdot) \leqq w\}$, $w > 0$; these are obtained readily by Abelian methods from (4) or (6), once the convergence of $\Pr\{W(t) = 0\}$ as $t \to \infty$ has been ascertained.

IV. PRELIMINARY RESULTS

Let $s \geqq 0$, and define $a(s)$ to be the abscissa of convergence of the Laplace transform*

$$\Psi(\tau,s) = \int_0^\infty e^{-\tau t} E\{e^{-sK(t)}\}\, dt.$$

Obviously $a(0) = 0$, $a(s) \leqq 0$ for $s \geqq 0$, and $a(\cdot)$ is monotone decreasing in $s \geqq 0$. It is not obvious that $a(s) \neq -\infty$, at present.

* We assume throughout that $E\{e^{-sK(t)}\}$ is a measurable function of t for s-values under discussion.

Lemma 1: The abscissa $a(s)$ of convergence of the integral

$$\Psi(\tau,s) = \int_0^\infty e^{-\tau t} E\{e^{-sK(t)}\}\, dt \tag{11}$$

is given by the formula

$$a(s) = \limsup_{t\to\infty} t^{-1} \log E\{e^{-sK(t)}\}, \qquad s \geqq 0.$$

Proof: Let $s > 0$ be fixed and set $E\{e^{-sK(t)}\} = \psi(t)$. The function $\psi(\cdot)$ is monotone decreasing, and satisfies $0 \leqq \psi \leqq 1$. Hence, by Theorem 2.4d of Widder,[5] the abscissa of convergence of the integral

$$\int_0^\infty e^{-\tau t}\, d\psi(t) \tag{12}$$

is

$$\limsup_{t\to\infty} t^{-1} \log |\psi(t)| \leqq 0.$$

However,

$$\int_{0-}^{T} e^{-\tau t}\, d\psi(t) = e^{-\tau T}\psi(T) + \tau \int_0^T e^{-\tau t}\psi(t)\, dt,$$

so that (11) and (12) have the same abscissa of convergence.

Lemma 2: For $s,h > 0$ and $u \geqq 0$

$$E\{e^{-sK(u)}\}E\{e^{-hK(u)}\} \leqq E\{e^{-(s+h)K(u)}\}.$$

Proof: Let $K(u) = \xi$. Since x^{s+h} is convex in x^s and in x^h, it follows from Jensen's theorem that

$$0 \leqq E\{e^{-s\xi}\} \leqq E^{s/(s+h)}\{e^{-(s+h)\xi}\},$$
$$0 \leqq E\{e^{-h\xi}\} \leqq E^{h/(s+h)}\{e^{-(s+h)\xi}\}.$$

Multiplying these inequalities together gives the result.

Lemma 3: If

$$\limsup_{t\to\infty} E\{K(u)/u\} < \beta,$$

then, given $\epsilon > 0$, there exists h so small and t so large that

$$E\{e^{-hK(u)}\} \geqq e^{-\epsilon u}$$

for all $u > t$.

Proof: Choose h so that $h\beta < \epsilon$, and t so that

$$E\{K(u)\} < \beta u$$

for all $u > t$. Then, by Jensen's theorem, for $u > t$,

$$E\{e^{-hK(u)}\} \geqq e^{-hE\{K(u)\}} \geqq e^{-\epsilon u}.$$

Lemma 4: If

$$\limsup_{t\to\infty} E\{K(u)/u\} < \infty,$$

then for given $\epsilon > 0$ and any $s > 0$, there exist h and t so that

$$t^{-1}\log E\{e^{-sK(u)}\} - t^{-1}\log E\{e^{-(s+h)K(u)}\} < \epsilon$$

for all $u > t$.

Proof: Lemmata 2 and 3.

Lemma 5: $a(\cdot)$ is a continuous function of s, $s \geqq 0$.

Proof: Set $t^{-1}\log E\{e^{-sK(t)}\} = g_s(t)$. Let $\epsilon > 0$ be given, and find, by Lemma 4, an h and a t so that $u > t$ implies

$$0 \leqq g_s(u) - g_{s+h}(u) \leqq \frac{\epsilon}{2}. \tag{13}$$

There exists a sequence $u_n \to \infty$ along which

$$a(s) - \frac{\epsilon}{4} \leqq g_s(u_n),$$

$$g_{s+h}(u_n) \leqq a(s+h) + \frac{\epsilon}{4},$$

for all n sufficiently large; this is because

$$a(s) = \limsup_{t\to\infty} g_s(t).$$

For such n, then, since $a(\cdot)$ is monotone decreasing and (13) holds,

$$0 \leqq a(s) - a(s+h) \leqq g_s(u_n) - g_{s+h}(u_n) + \frac{\epsilon}{2},$$

$$0 \leqq a(s) - a(s+h) \leqq \epsilon,$$

so that $a(\cdot)$ is right-continuous. A mirror image of this argument proves $a(\cdot)$ left-continuous.

Lemma 6: $a(\cdot)$ is convex in $s \geqq 0$.

Proof: For each $t > 0$, $E\{e^{-sK(t)}\}$ is a completely monotonic function of s, and so (Widder,[5] p. 167) is logarithmically convex; indeed, in $s \geqq 0$ we have

$$\frac{1}{t}\frac{\partial^2}{\partial s^2}\log E\{e^{-sK(t)}\} = \frac{\partial^2}{\partial s^2} g_t(s) \geqq 0.$$

This implies that for $s,h \geqq 0$

$$\tfrac{1}{2}g_t(s) + \tfrac{1}{2}g_t(s + h) \geqq g_t(s + \tfrac{1}{2}h).$$

Taking the lim sup as $t \to \infty$, we find that $a(\cdot)$ is convex.

Lemma 7: For $s,h > 0$,

$$\frac{a(s)}{s} \leqq \frac{a(s+h)}{s+h}, \tag{14}$$

and

$$\rho = -\lim_{s \to 0} \frac{a(s)}{s} > 0$$

exists as a finite or infinite limit.

Proof: If ξ is a positive variate then

$$E^{1/r}\{\xi^r\}$$

is a nondecreasing function of r. (See Loève,[6] p. 156.) Choosing $\xi = e^{-K(t)}$, we have

$$E^{1/s}\{e^{-sK(t)}\} \leqq E^{1/(s+h)}\{e^{-(s+h)K(t)}\},$$

$$s^{-1}g_t(s) \leqq (s+h)^{-1}g_t(s+h).$$

But

$$a(s) = \limsup_{t \to \infty} g_t(s),$$

so (14) is true.

Theorem 1: If

$$\liminf_{t \to \infty} t^{-1} \log \Pr\{K(t) = 0\} > -\infty \tag{16}$$

then for each $\tau \geqq 0$ there is at least one number $s(\tau)$ such that

$$\tau - s(\tau) = a(s(\tau)),$$

and

$$\begin{cases} \tau - s \geqq a(s) & \text{for } s \leqq s(\tau). \\ \tau - s \leqq a(s) & \text{for } s \geqq s(\tau) \end{cases} \tag{17}$$

Proof: The hypothesis implies that there is a constant $c > 0$ such that

$$\Pr\{K(t) \leqq 0\} \geqq e^{-ct}$$

for all sufficiently large t. However, this implies that

$$E\{e^{-sK(t)}\} \geqq e^{-ct}$$

for all t large enough, and so $a(s) \geqq -c$ for all $s \geqq 0$. Since $a(\cdot)$ is continuous it must intersect the line $\tau - s$ at least once, and the relations (17) follow from the convexity of $a(\cdot)$.

This section ends with some preliminary analytical results for $F(\cdot)$ and $P(\cdot)$.

Lemma 8: If $R(\cdot)$ is of bounded variation in every finite interval, then

$$F'(t) = d/dt\, F(t)$$

exists almost everywhere.

Proof: The hypothesis implies that the transform

$$\rho(s) = \int_{0-}^{\infty} e^{-st}\, dR(t)$$

exists in $\mathrm{Re}(s) > 0$. From (10) we find

$$\varphi(s) = \rho(s) \int_0^{\infty} e^{-st} P(t)\, dt = \rho(s)\Pi(s),$$

but $\Pi(\cdot)$ is the Laplace-Stieltjes transform of an absolutely continuous function; hence $\varphi(\cdot)$ is also.

In order to use various Tauberian theorems it will usually be necessary to impose "Tauberian" conditions on the oscillations of $P(\cdot)$. We recall (Widder,[5] p. 209) that a function $f(\cdot)$ is *slowly oscillating* if

$$\lim_{\substack{y-x\to 0\\ y\to\infty}} f(y) - f(x) = 0,$$

and $f(\cdot)$ is *slowly decreasing* if

$$\liminf_{y-x\to 0} f(y) - f(x) \geqq 0$$

as $x \to \infty$, $y = y(x) > x$.

Lemma 9: If F' is continuous uniformly in $t \geqq 0$, and $R(\cdot)$ is of bounded total variation, with the form

$$R = U + H_1 + H_2$$

where U is the unit step at zero, H_1 is absolutely continuous, and

$$\int |\, dH_2 \,| = \lambda < 1,$$

then $P(\cdot)$ is slowly oscillating.

Proof:* Since F' exists we may differentiate (10) to find that

* The procedure of this proof is patterned after that of Karlin.[7]

$$F'(t) = \int_{0-}^{t} P(t-u)\,dR(u).$$

This is equivalent to the renewal equation

$$P(t) = F'(t) + \int_{0-}^{t} P(t-u)\,d_u\{U - R\},$$

or, iterating n times,

$$P(t) = \int_{0-}^{t} P(t-u)\,d_u\{L + M\} + \int_{0-}^{t} F'(t-u)\,dN(u),$$

where L is absolutely continuous,

$$\int |dM| < \lambda^n, \text{ and } \int |dN| < \infty.$$

Then

$$\begin{aligned} |P(t+\epsilon) - P(t)| \leqq \int_{-\infty}^{\infty} |L'(t+\epsilon) - L'(t)|\,dt + \lambda^n \\ + \int_{t}^{t+\epsilon} |F'(t+\epsilon-u)|\,|dN(u)| \\ + \int_{0}^{\infty} |F'(t+\epsilon-u) - F'(t-u)|\,|dN(u)|. \end{aligned}$$

The first term goes to zero with $\epsilon \to 0$, by a known result of Lebesgue (Wiener,[8] p. 14). The second term vanishes as n increases. The third term approaches zero as $t \to \infty$, since F' is bounded. The fourth term goes to zero with $\epsilon \to 0$ by the uniform continuity of F'.* Hence

$$\lim_{\substack{\epsilon \to 0 \\ t \to \infty}} |P(t+\epsilon) - P(t)| = 0.$$

The hypothesis that F' is continuous can be replaced by the condition that $F'(t)$ approach a limit as $t \to \infty$. Also the uniform continuity of F' could be replaced by a weaker but more complicated asymptotic condition.

V. THE TRANSFORM $\Pi(\cdot)$ OF $\Pr\{W(\cdot) = 0\}$

Theorem 2: Let s be real and positive, and let

$$\int_{0}^{\infty} e^{-\tau t} P(t)\,dt = \Pi(\tau);$$

* It is assumed that $F'(t) = 0$ for $t < 0$.

then (16) implies

$$s\Pi(s + a(s)) \equiv 1, \qquad s > 0. \tag{18}$$

Proof: Equation (6) implies that

$$\int_0^\infty e^{-\tau t} E\{e^{-sW(t)}\}\, dt = \Psi(\tau,s)[1 - s\Pi(\tau)].$$

The left-hand side has no singularities in $\mathrm{Re}(\tau) > 0$. But by Widder[5] (p. 58, Theorem 5b), $\Psi(\tau,s)$ is singular at $\tau = s + a(s)$, and so (18) holds.

Theorem 3: If $\tau > 0$, and (16) holds, then

$$\Pi(\tau) = \frac{1}{s(\tau)},$$

where $s(\tau)$ satisfies (and is determined uniquely by) the equation

$$\tau - s(\tau) = a(s(\tau)). \tag{19}$$

Proof: For given τ, there exists at least one $s(\tau) > 0$ satisfying (19). By Theorem 2, any such $s(\tau)$ has the property

$$\begin{aligned}\frac{1}{s(\tau)} &= \Pi(s(\tau) + a(s(\tau))),\\ &= \Pi(\tau),\end{aligned}$$

because of (18). Since $\Pi(\tau)$ is a strictly monotone function of real τ, being of the form

$$\Pi(\tau) = \int_0^\infty e^{-\tau t} P(t)\, dt, \qquad P(\cdot) \geqq 0,$$

the solution $s(\tau)$ of (19) is unique, and strictly monotone increasing in τ.

When $-a(s)/s \to 1$ as $s \to 0$, we need some additional properties of $a(\cdot)$. For each t, there is a neighborhood N_t of the positive real axis in which

$$E\{e^{-sK(t)}\}$$

has no zeros. However, we shall need the condition iii of Section II that there exist one neighborhood N of the positive real axis in which

$$E\{e^{-sK(t)}\} \neq 0$$

for all t sufficiently large, say $t > T$. Define, for $z \in N$,

$$g_z(t) = t^{-1} \log |E\{e^{-zK(t)}\}|.$$

Then $g_{(\cdot)}(t)$ is harmonic in N for $t > T$, and

$$\begin{aligned} | g_z(t) | &= | t^{-1} \log | E\{e^{-zK(t)}\} || \\ &\leqq t^{-1} | \log \Pr\{K(t) = 0\} | < \infty, \end{aligned}$$

so that $| g_z(t) | \leqq c$ for $z \in N$, where c is the constant of Theorem 1.

Theorem 4: If

$$\liminf_{t \to \infty} t^{-1} \log \Pr\{K(t) = 0\} > -\infty \tag{16}$$

and N is a neighborhood of the positive real axis such that

$$E\{e^{-zK(t)}\} \neq 0, \qquad z \in N$$

for all sufficiently large t, then $a(\cdot)$ can be extended to be harmonic in N.

Proof: For $t > T$ define a class V_t of functions on N as follows: V_t is the smallest class containing all $g_{(\cdot)}(t')$ for $t' > t$, and closed under the operations

(a) $$v_1, v_2 \rightarrow \max(v_1, v_2),$$

(b) $$v \rightarrow v_{\mathrm{mod}} = \begin{cases} v & \text{outside } \Delta \\ Pv & \text{inside } \Delta \end{cases}$$

where Δ is a disc with $\bar{\Delta} \subset N$, and P is the Poisson integral operator on the disc Δ. These operations preserve the property of being bounded by the constant c of Theorem 1. Now let

$$u_t(z) = \sup_{v \in V_t} v(z).$$

By a standard argument (Ahlfors,[9] p. 197), the function $u_t(\cdot)$ is harmonic in N, and clearly

$$u_t(z) \geqq \sup_{t_1 > t} g_z(t_1).$$

We prove the reverse inequality:

$$u_t(z) \leqq \sup_{t_1 > t} g_z(t_1).$$

It is enough to show that the operations (a) and (b) preserve this property. For (a) this is obvious; for (b), we observe that

$$Pv(z) = \frac{1}{2\pi} \int_0^{2\pi} v(z + re^{i\theta})\, d\theta \leqq \sup_{t_1 > t} g_z(t_1).$$

Hence for real z

$$a(z) = \inf_t \sup_{t_1 > t} g_z(t_1),$$
$$= \lim_{t \to \infty} u_t(z).$$

The functions $u_t(\cdot)$ are harmonic and monotone decreasing in t. Hence by Harnack's principle they either tend to $-\infty$ or to a harmonic function. The first alternative is ruled out by the inequality $| g_z(t) | \leqq c$. Hence $a(\cdot)$ is harmonic in N.

Theorem 5: Under the conditions of Theorem 4, let

$$\rho = \lim_{s \to 0} - \frac{a(s)}{s}.$$

If $\rho > 1$, there exists a largest root $\zeta > 0$ of the equation $\zeta = a(\zeta)$, and $s(\tau) \to \zeta$ as $\tau \to 0$. If $\rho \leqq 1$, then $s(\tau) \to 0$ as $\tau \to 0$.

Proof: $s(\cdot)$ is monotone, nonnegative, and nonincreasing, so $s(0+) \geqq 0$ exists. If $\rho > 1$, then $a(x) < -x$ for some $x > 0$. However, for $\tau > 0$,

$$a(s(\tau)) = \tau - s(\tau) > -s(\tau),$$

so $s(0+) > x > 0$. Since $s(\cdot)$ is continuous we may let $\tau \to 0$ to find $\zeta = a(\zeta) = s(0+) > 0$.

Next, if $\rho < 1$, suppose that $s(0+) > 0$. Then

$$\frac{a(s(0+))}{s(0+)} = -1,$$

which is impossible, since

$$\frac{a(s)}{s} \to -\rho > -1 \qquad \text{as } s \to 0,$$

in a monotone decreasing manner.

The case $\rho = 1$ requires a special argument. If $s(0+) > 0$, then again

$$\frac{a(s(0+))}{s(0+)} = -1$$

and we must have

$$\frac{a(s)}{s} \equiv 1, \qquad 0 \leqq s \leqq s(0+).$$

However, since $a(\cdot)$ is harmonic in a neighborhood N of the positive real axis, this would imply, by the Cauchy-Riemann relations, that $a(s) = s$ throughout N; but $a(s) \geqq -c$ for real s. Hence $s(0+) = 0$.

The hypothesis about the existence of the neighborhood N is needed only for the case $\rho = 1$.

Theorem 6: Suppose that $P(\infty) = \lim P(t)$ as $t \to \infty$ exists. Then

$$P(\infty) = \begin{cases} 0 & \text{if } \rho \geqq 1, \\ 1 - \rho & \text{if } \rho < 1. \end{cases}$$

Proof: Since $\tau - s(\tau) = a(s(\tau))$, we find that

$$\tau\Pi(\tau) = \frac{\tau}{s(\tau)} = 1 - \frac{a(s(\tau))}{s(\tau)} .$$

Then if $\rho \leqq 1$, the result follows from the previous theorem and a standard Abelian theorem for the Laplace transform. If $\rho > 1$ then $P(\cdot) \in L$, and if $P(\infty)$ exists it must be zero.

VI. ASYMPTOTIC RELATIONSHIPS

We now prove some lemmata that exhibit some of the basic asymptotic relationships between ρ, $a(\cdot)$, $F(\cdot)$, $R(\cdot)$, etc.

Lemma 10:

$$1 - \rho = 1 + a'(0) \geqq \limsup_{t \to \infty} 1 - E\{K(t)/t\}.$$

Proof: By Jensen's theorem

$$E\{e^{-sK(t)}\} \geqq e^{-sE\{K(t)\}},$$

$$t^{-1} \log E\{e^{-sK(t)}\} \geqq -sE\{K(t)/t\}.$$

Definition:

$$Q(t) = \int_t^\infty \Pr\{K(t) > u\}\, du = E\{\max[0, K(t) - t]\}$$

Lemma 11: If $Q(t) = o(t)$ as $t \to \infty$, then

i. $\rho \leqq 1$;

ii. If $d/dt\, E\{K(t)\}$ also exists and approaches a limit as $t \to \infty$, and if $Q(\cdot)$ can be differentiated in the usual way, then $R(t) \to 1$ as $t \to \infty$.

Proof:

$$0 \leqq t^{-1}F(t) \leqq 1$$

and

$$t^{-1}F(t) = 1 - E\{K(t)/t\} + Q(t)/t,$$

so that

$$\liminf_{t\to\infty} t^{-1}F(t) = \liminf_{t\to\infty} 1 - E\{K(t)/t\}.$$

Then i follows by the previous lemma. To prove ii note that

$$F'(t) = 1 - \frac{d}{dt} E\{K(t)\} + 1 - R(t) + \int_t^\infty \frac{\partial}{\partial t} \Pr\{K(t) > u\}\, du.$$

The last term is positive, so

$$R(t) \geqq 1 + 1 - \frac{d}{dt} E\{K(t)\} - F'(t),$$

$$R(t) \to 1.$$

Lemma 12: If $R(t) \to R(\infty) > 0$ as $t \to \infty$, then $\rho \leqq 1$.
Proof:

$$E\{e^{-sK(t)+st}\} \geqq E\{e^{s \min[0, t-K(t)]}\},$$
$$\geqq \Pr\{K(t) \leqq t\}.$$

So

$$E\{e^{-sK(t)}\} \geqq R(t)e^{-st}.$$

Hence also

$$1 + \frac{a(s)}{s} \geqq \frac{1}{s} \limsup_{t\to\infty} t^{-1} \log R(t).$$

If $R(\infty) > 0$, the lim sup is zero, so that

$$1 + \frac{a(s)}{s} \geqq 0, \qquad \rho \leqq 1.$$

Lemma 13: If

$$\limsup_{t\to\infty} t^{-1} \log R(t) = 0,$$

then $\rho \leqq 1$. If this lim sup is less than 0, then $R(\cdot)$ is integrable and $R(\infty) = 0$. Hence also $P(\infty) = 0$ and $\rho \geqq 1$.
Lemma 14: If

$$\liminf_{t\to\infty} Q(t) = 0,$$

then $\rho \leqq 1$.
Proof:

$$E\{\min [0,t - K(t)]\} = E\{t - K(t)\} - F(t),$$

so

$$\begin{aligned} 1 + \frac{a(s)}{s} &\geqq \limsup_{t\to\infty} \left[1 - \frac{E\{K(t)\}}{t} - F(t)\right], \\ &\geqq \limsup_{t\to\infty} [-Q(t)], \\ &\geqq -\liminf_{t\to\infty} Q(t). \end{aligned}$$

VII. CONVERGENCE OF $\Pr\{W(t) = 0\}$: MERCERIAN METHODS

A Mercerian theorem (see Pitt,[10] p. 94) is one which, for example, enables us to study the asymptotic behavior of $P(\cdot)$ directly from that of the convolution

$$F(t) = \int_0^t P(t - u)R(u)\, du,$$

i.e., that of $F(\cdot)$, without the intervention of "Tauberian" hypotheses on the oscillations of $P(\cdot)$. Tauberian methods usually require $P(\cdot)$ to be a slowly decreasing function; such methods are considered in Section VIII.

In what follows the norm symbol (when used) refers to the total variation of a function of bounded total variation. The subscripts "dis" and "sing" (applied to a function symbol) denote the discontinuous and singular components, respectively.

Theorem 7: If $R(\cdot)$ is of bounded variation in any finite interval, and for some $\sigma \geqq 0$,

$$\inf_{\tau} | \rho(\sigma - i\tau) | > 0, \tag{20}$$

$$\| (R_\sigma)_{\text{sing}} \| < \inf_{\tau} | \rho_{\text{dis}}(\sigma - i\tau) |, \tag{21}$$

then $P(\cdot)$ can be represented by the inversion formula

$$\int_{-\infty}^t F'(t - u)e^{\sigma u}\, dG_\sigma(u), \qquad \int_{-\infty}^\infty | dG_\sigma | < \infty,$$

provided that

$$R_\sigma(t) = \int_{-\infty}^t e^{-\sigma u}\, dR(u) \tag{22}$$

is of bounded *total* variation.

Proof: The hypotheses (20) through (22), together with Theorem 1 of Wiener and Pitt,[11] imply that there exists a function $G_\sigma(\cdot)$ of bounded total variation whose Fourier-Stieltjes transform is

$$[\rho(\sigma - i\tau)]^{-1}, \qquad \sigma \text{ fixed.}$$

We then solve the equation

$$e^{-\sigma t}F(t) = \int_0^t e^{-\sigma(t-u)}P(t-u)e^{-\sigma u}R(u)\,du$$

by Laplace transforms, obtaining

$$\int_0^\infty e^{-(p+\sigma)t}P(t)\,dt = \frac{\varphi(p+\sigma)}{\rho(p+\sigma)}.$$

However, by Lemma 8, F' exists and has Laplace transform $\varphi(\cdot)$. Therefore

$$e^{-\sigma t}P(t) = \int_{-\infty}^t F'(t-u)e^{-\sigma(t-u)}\,dG_\sigma(u),$$

and division by $e^{-\sigma t}$ completes the proof.

This result provides another way of proving that $P(\cdot)$ is slowly oscillating, as follows:

Lemma 15: If the hypotheses of Theorem 7 are true with $\sigma = 0$, and if $F'(\cdot)$ is uniformly continuous and bounded, then $P(\cdot)$ is slowly oscillating.

Proof: Let b be a bound on F'. Then

$$\begin{aligned} |P(t+\epsilon) - P(t)| &\leqq \int_{-\infty}^\infty |F'(t+\epsilon-u) - F'(t-u)|\,|dG_0(u)| \\ &\leqq 2b\int_{|u|>T} |dG_0(u)| + \int_{-T}^T |F'(t+\epsilon-u) \\ &\qquad - F'(t-u)|\,|dG_0(u)|. \end{aligned}$$

Choose first T large, then ϵ small, using the uniform continuity of F' to let $t \to \infty$.

Theorem 8: If the hypotheses of Theorem 7 are true with $\sigma = 0$, if F' is bounded below, and for some constant A

$$\varphi(s) \sim \frac{A}{s} \qquad \text{as } s \to 0+, \tag{23}$$

then

$$P(t) \underset{(C,1)}{\longrightarrow} \frac{A}{R(\infty)} \qquad \text{as } t \to \infty.$$

Proof: The hypotheses imply that $P(\cdot)$ can be written as

$$P(u) = \int_{-\infty}^{u} F'(u - y)\, dG_0(y)$$

with $G_0(\cdot)$ of bounded total variation; then also

$$t^{-1} \int_0^t P(u)\, du = \int_0^t \frac{F(t - u)}{t - u} \frac{t - u}{t}\, dG_0(u).$$

There exists a constant $b > 0$ such that

$$F(t) + bt$$

is nondecreasing in $t \geqq 0$; hence by a known Tauberian result (Widder,[5] p. 197, Theorem 4.6), condition (23) implies that

$$\frac{F(t)}{t} \to A \geqq 0 \qquad \text{as } t \to \infty,$$

the convergence being bounded, since $0 \leqq F(t) \leqq t$. It follows that

$$t^{-1} \int_0^t P(u)\, du \to A[G_0(+\infty) - G_0(-\infty)]$$

$$\to \frac{A}{R(\infty)}.$$

The condition (20) of Theorem 7 guarantees that $R(\infty) > 0$.

Theorem 9: If the hypotheses of Theorem 7 are true with $\sigma = 0$, and $F' \to F'(\infty)$ boundedly as $t \to \infty$, then

$$P(t) \to \frac{F'(\infty)}{R(\infty)} \qquad \text{as } t \to \infty.$$

Proof: The hypotheses imply that

$$P(t) = \int_{-\infty}^{t} F'(t - u)\, dG_0(u),$$

where $G_0(\cdot)$ is of bounded total variation. The result then follows from a known Abelian lemma, e.g., Lemma 1 of Smith.[12]

Theorem 10: If the function $R(\cdot) - U(\cdot)$ is absolutely continuous, if $F'(\infty)$ exists, and if $\rho(s)$ has no zeros in $\text{Re}(s) \geqq 0$, then

$$P(t) \to \frac{F'(\infty)}{R(\infty)} \qquad \text{as } t \to \infty.$$

Proof: Set $[R(\cdot) - U(\cdot)]' = k(\cdot)$, so that (10) implies (by differentiation) that

$$F'(t) = \int_{0-}^{\infty} P(t-u)\, dR(u) = P(t) + \int_0^t P(t-u)\, d_u(R-U)$$
$$= P(t) + \int_0^t P(t-u)k(u)\, du.$$

Then, as $t \to \infty$,

$$P(t) + \int_0^t P(t-u)k(u)\, du \to F'(\infty).$$

By Theorem XVII of Paley and Wiener,[13] this, together with the conditions that $P(\cdot)$ be bounded and $\rho(s) \neq 0$ in $\mathrm{Re}(s) \geqq 0$, implies that, as $t \to \infty$,

$$P(t) \to \frac{F'(\infty)}{1 + \int_0^{\infty} k(u)\, du} = \frac{F'(\infty)}{R(\infty)},$$

since for $t > 0$

$$R(t) = 1 + \int_0^t k(u)\, du.$$

By a theorem of Pitt[10] (p. 115), the restriction that $\rho(s) \neq 0$ for $\mathrm{Re}(s) \geqq 0$ can be weakened to $\rho(i\tau) \neq 0$ for real τ.

VIII. CONVERGENCE OF $\Pr\{W(t) = 0\}$: TAUBERIAN METHODS

The addition of "Tauberian" conditions on $P(\cdot)$,* such as the property of slow decrease, makes it possible to change or weaken the hypotheses on $R(\cdot)$ and $F(\cdot)$ necessary to ensure the convergence of $P(\cdot)$. The next result shows how the convergence of $F'(\cdot)$ in Theorem 9 can be relaxed.

Theorem 11: If

i. For some integrable function $k(\cdot)$ with a nonvanishing Fourier transform $K(\cdot)$, and some number A,

$$\int_0^{\infty} k(t-u)\, dF(u) \to A \int_{-\infty}^{\infty} k(u)\, du \qquad \text{as } t \to \infty;$$

ii. $P(\cdot)$ is slowly decreasing;

iii. $R(\cdot)$ is of bounded total variation with

* Such conditions were briefly studied in Section IV.

$$\rho(-i\tau) \neq 0,$$
$$\| R_{\text{sing}} \| < \inf_{\tau} | \rho_{\text{dis}}(-i\tau) |;$$

then

$$P(t) \to \frac{A}{R(\infty)} \qquad \text{as } t \to \infty.$$

Proof: The conditions on $\rho(\cdot)$ imply, by Theorem 1 of Wiener and Pitt,[11] that

$$\frac{K(\tau)}{\rho(-i\tau)}$$

is the nonvanishing Fourier transform of an integrable function $x(\cdot)$ such that

$$\int_0^\infty k(t-u)\, dF(u) = \int_0^\infty x(t-u)P(u)\, du.$$

The theorem to be proved then follows from Pitt's form of Wiener's fundamental Tauberian theorem (see Theorem 10a, p. 211 of Widder[5]).

Instead of imposing conditions on $F(\cdot)$ and $R(\cdot)$, one can place them on the Laplace-Stieltjes transforms $\varphi(s)$ and $\rho(s)$. We shall use the following Tauberian result:

Theorem 12: Let $\gamma(s)$ be the Laplace transform, convergent in $\text{Re}(s) > 0$, of a bounded function $C(\cdot) \geqq 0$, vanishing for negative argument, and slowly decreasing; i.e.,

$$\liminf_{\substack{\epsilon \to 0 \\ t \to \infty}} C(t+\epsilon) - C(t) \geqq 0.$$

Suppose that, for some function $g(\cdot)$, constant c, and $s = \sigma + i\tau$,

$$\lim_{\sigma \to 0+} \gamma(s) - s^{-1}c = g(\tau) \tag{24}$$

exists uniformly in every finite interval, $-a \leqq \tau \leqq +a$. Then

$$C(t) \to c \qquad \text{as } t \to \infty.$$

Proof: This is a variant of Ikehara's theorem, Widder[5] (p. 233). Define, for each $\lambda > 0$,

$$K_\lambda(x) \begin{cases} = 2\lambda(1 - |x/2\lambda|) & |x| \leqq 2\lambda, \\ = 0 & |x| > 2\lambda. \end{cases} \tag{25}$$

This is the Fourier transform of the function

$$k_\lambda(x) = 2\lambda(2\pi)^{-\frac{1}{2}}\left(\frac{\sin \lambda x}{\lambda x}\right)^2.$$

For $\epsilon > 0$, set

$$\begin{aligned} I_\lambda(x) &= (2\pi)^{-\frac{1}{2}} \int_0^\infty k_\lambda(x - u)[C(u) - c]e^{-\epsilon u}\, du \\ &= \frac{1}{2\pi}\int_0^\infty [C(u) - c]e^{-\epsilon u}\, du \int_{-2\lambda}^{2\lambda} K_\lambda(y)e^{-iy(x-u)}\, dy \\ &= \frac{1}{2\pi}\int_{-2\lambda}^{2\lambda} K_\lambda(y)e^{-iyx}\, dy \int_0^\infty [C(u) - c]e^{-t(\epsilon+iy)}\, dt \\ &= \frac{1}{2\pi}\int_{-2\lambda}^{2\lambda} K_\lambda(y)e^{-iyx}[\gamma(\epsilon + iy) - (\epsilon + iy)^{-1}c]\, dy. \end{aligned}$$

By (24) we may take the limit under the integral as $\epsilon \to 0$, to obtain

$$\begin{aligned} \lim_{\epsilon\to 0+} I_\lambda(x) &= (2\pi)^{-1}\int_{-2\lambda}^{2\lambda} K_\lambda(x)e^{-ixy}g(y)\, dy \\ &= (2\pi)^{-\frac{1}{2}}\int_0^\infty k_\lambda(x - u)[C(u) - c]\, du, \end{aligned}$$

the last identity following from Widder[5] (p. 183, Corollary 1c). The function $K_\lambda(\cdot)g(\cdot)$ belongs to L_1, and so, by the Riemann-Lebesgue lemma,

$$(2\pi)^{-\frac{1}{2}}\int_0^\infty k_\lambda(x - u)[C(u) - c]\, du \to 0 \qquad \text{as } x \to \infty.$$

Since $C(\cdot)$ is bounded and slowly decreasing, we conclude from Widder[5] (p. 209, Theorem 9), that

$$C(u) \to c \qquad \text{as } u \to \infty.$$

The theorem just proved implies directly the following consequence:

Corollary: Suppose that

i. For some number L and function $g(\cdot)$

$$\frac{\varphi(s)}{\rho(s)} - \frac{L}{s} \to g(\tau) \qquad \text{as } \sigma \to 0, \qquad (s = \sigma + i\tau) \tag{26}$$

uniformly in every finite interval;

ii. $P(\cdot)$ is slowly decreasing.

Then $P(t) \to L$ as $t \to \infty$.

Some simple conditions on $F(\cdot)$ and $R(\cdot)$ which ensure that hypothesis i of the corollary obtains are described in the next result:

Theorem 13: If

i. For some number L the function

$$F(t) - L \int_0^t R(u) \, du \tag{27}$$

(assumed to vanish for $t < 0$) is of bounded total variation;

ii. $R(\cdot)$ is of bounded total variation, and

$$\rho(-i\tau) \neq 0, \qquad \| R_{\text{sing}} \| < \inf_{\tau} | \rho_{\text{dis}}(-i\tau) |;$$

then there exists a $\mu(\cdot)$ such that

$$\frac{\varphi(s)}{\rho(s)} - \frac{L}{s} = \mu(s) \rightarrow \mu(\tau) \qquad \text{as } \sigma \rightarrow 0$$

uniformly in τ, and $\mu(\tau)$ is the Fourier-Stieltjes transform of a function of bounded total variation.

Proof: The Laplace-Stieltjes transform of (27) is

$$\varphi(s) - \frac{L}{s} \rho(s), \qquad \text{Re}(s) \geqq 0.$$

The condition ii ensures, by the Wiener-Pitt theorem, that $[\rho(-i\tau)]^{-1}$ is the Fourier-Stieltjes transform of a function of bounded total variation. It follows that $\mu(-i\tau)$ is also. But $\mu(\cdot)$ is the transform of $P(\cdot) - L$, and so extends analytically into $\text{Re}(s) > 0$ with the representation

$$\mu(s) = \int_0^\infty e^{-su} \, dG(u), \qquad \| G \| < \infty.$$

Then

$$| \mu(s) - \mu(\tau) | \leqq \int_0^\infty | 1 - e^{-\sigma u} | \, | dG(u) |$$

$$\leqq (1 - e^{-\sigma T}) \int_0^T | dG(u) | + \int_T^\infty | dG |.$$

Let $\epsilon > 0$ be given, and pick T so large that the second term on the right is less than $\epsilon/2$; then pick σ so small that the first term is also. This proves the result.

The condition i of Theorem 13 is satisfied, for example, if $F'(\infty)$ exists and both of

ii. for each $\tau \neq 0$, the function $E\{e^{i\tau[K(u)-u]}\}$ belongs to $L_1(0, \infty)$ and vanishes at ∞ (as a function of u);

iii. the function

$$\lambda(\tau) = \Pr\{W(\infty) = 0\} \cdot i\tau \int_0^\infty E\{e^{i\tau[K(u)-u]}\}\, du \tag{30}$$

is continuous at $\tau = 0$, with $\lambda(0) = 1$;
then there is a distribution function

$$L(\cdot) = \Pr\{W(\infty) \leqq \cdot\}$$

such that

$$\lim_{t\to\infty} E\{e^{i\tau W(t)}\} = \lambda(\tau), \tag{31}$$

$$\lim_{k\to\infty} \Pr\{W(t) \leqq w\} = L(w),$$

at continuity points of $L(\cdot)$, at least.

Proof: A direct application of Lemma 1 of Smith[12] to the real and imaginary parts of (6) with $s = -i\tau$, τ real, proves (31). The rest is a consequence of the standard continuity theorem for characteristic functions. Formula (30) is a generalization of the Pollaczek-Khinchin formula; this may be verified by letting $K(\cdot)$ be a compound Poisson process.

If $\Pr\{W(\infty) = 0\} = 0$, and condition ii of Theorem 16 is fulfilled, then a similar argument shows that

$$E\{e^{i\tau W(t)}\} \to 0 \qquad \text{as } t \to \infty$$

for all $\tau \neq 0$, so no limiting distribution exists.

Alternatively, one can apply similar Abelian methods directly to formula (3). For convenience we shall use, instead of (3), the w-integrated form ($w \geqq 0$)

$$\begin{aligned}\int_0^w \Pr\{W(t) \leqq u\}\, du = \int_0^{t+w} &\Pr\{K(t) \leqq u\}\, du \\ &- \int_0^t R(t-u,w) \Pr\{W(u) = 0\}\, du,\end{aligned} \tag{4}$$

in which $R(t,u,w)$ has been replaced by $R(t - u,w)$, in accordance with the stationarity condition (7).

Theorem 17: If

i. For each $w > 0$

$$\int_0^n [R(u,w) - R(u,0)]\, du = r(w) < \infty;$$

ii. $R(t) \to 1$ as $t \to \infty$;
iii. $P(t) = \Pr\{W(t) = 0\} \to \Pr\{W(\infty) = 0\}$ as $t \to \infty$;
then, for $w > 0$,

$$\lim_{t\to\infty} \int_0^w \Pr\{W(t) \leqq u\}\, du = w - r(w) \Pr\{W(\infty) = 0\}.$$

Proof: By subtracting the case $w = 0$ of (4) from the case $w > 0$, we find

$$\begin{aligned}\int_0^w \Pr\{W(t) \leqq u\}\, du &= \int_t^{t+w} \Pr\{K(t) \leqq u\}\, du \\ &\quad - \int_0^t [R(t-u,w) - R(t-u,0)]P(u)\, du.\end{aligned} \tag{32}$$

If $u \geqq t$, then

$$\Pr\{K(t) \leqq u\} \geqq \Pr\{K(t) \leqq t\} = R(t).$$

Hence, by ii,

$$\lim_{t\to\infty} \int_t^{t+w} \Pr\{K(t) \leqq u\}\, du = w.$$

By i. and Lemma 1 of Smith,[12] the last term of (32) approaches

$$r(w) \Pr\{W(\infty) = 0\}.$$

To clarify the meaning of the condition i of Theorem 17, we note that

$$\begin{aligned}R(t,w) - R(t,0) &= \Pr\{0 < K(t) - t \leqq w\}, \\ r(w) &= E\{\text{amount of time that } 0 < K(t) - t \leqq w\}.\end{aligned}$$

Other limit theorems can readily be obtained, e.g., for the nonintegrated equation (3), as soon as suitable differentiability conditions are imposed. Since no new principle is involved, we shall leave the matter here.

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