# The Effect of Driving Electrode Shape on the Electrical Properties of Piezoelectric Crystals

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*In the present paper general formulas for the electrical admittance of a piezoelectric crystal, in terms of its resonant frequencies and static and motional capacitances, are derived and applied to the investigation of the effect of electrode shape on the spectrum of resonances and the capacitance ratio of the crystal. Particular attention is given in two cases of practical importance, namely, small piezoelectric coupling and thin crystal plates.*

## I. INTRODUCTION

Piezoelectric crystals are often used as circuit elements in filters and oscillators. Fig. 1 shows a typical admittance curve for such a crystal and Fig. 2 shows the corresponding equivalent circuit. At very low frequencies the crystal behaves like a capacitor, with a capacitance approximately equal to the static capacitance between the driving electrodes. Due to the piezoelectric effect, an applied alternating electric field causes the crystal to vibrate and, at certain natural frequencies of free vibration, it is driven into mechanical resonance by the applied voltage.

In the neighborhood of such natural frequencies, the admittance of the crystal is closely approximated by the simple equivalent circuit of Fig. 3. This is the equivalent circuit commonly used in the applications. It is a good approximation over a frequency range proportional to the spacing between the resonant frequency, at which the admittance is infinite (in the absence of dissipation), and the antiresonant frequency, at which the admittance vanishes.

For small electromechanical coupling this spacing is proportional to the capacitance ratio $C_n/C_0$. It is desirable to make this ratio as large as possible. Bechmann and Parsons[1] have shown how this may be done in various simple cases.

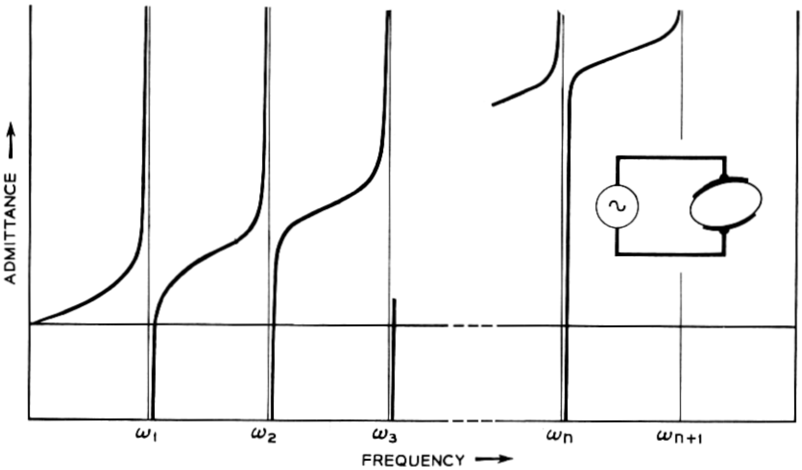


Fig. 1 — The equivalent electrical admittance of a piezoelectric crystal.

At high frequencies, where many resonances may occur within a narrow frequency range, the simple equivalent circuit of Fig. 3 may cease to be applicable. However, it is always possible, at least in principle, to find an electrode configuration which does not excite one or more of these resonances. A simple example is a symmetric electrode configuration, exciting only symmetric modes of free vibration. Vormer[2] has shown theoretically and experimentally what the appropriate electrode shape is for the longitudinal vibrations of a piezoelectric bar.

In the following, we consider both the question of capacitance ratio maximization and that of resonance suppression in some detail from the theoretical point of view. The investigation is divided into three parts.

In Sections II through VI we consider the general problem of steady vibrations of a piezoelectric body. The principal tools used in these sections are the piezoelectric analogs of various integral theorems of classical elasticity, found, for instance, in Love.[3] We prove, for example, that the material particle displacements corresponding to two different

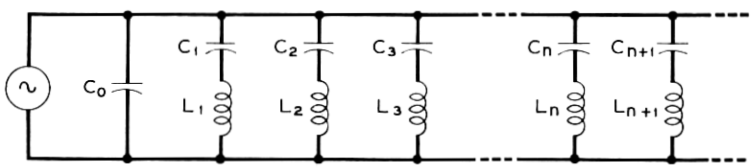


Fig. 2 — The equivalent electrical circuit of a piezoelectric crystal.

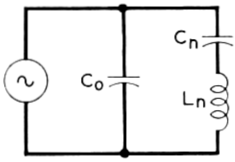


Fig. 3 — The equivalent electrical circuit near resonance.

modes of free vibration of a piezoelectric body are orthogonal just as they are for an ordinary elastic body.

The principal results of these sections are general expressions for the equivalent electrical admittance of any piezoelectric body and for the motional capacitances. The latter expression indicates how the electrode configuration must be chosen to suppress a given resonance.

Section VII is devoted to the case of small piezoelectric coupling, in which, to a first approximation, we may separate the mechanical and electrical problems completely, thus making it possible to obtain more explicit results than in the general case. A sample result states that the capacitance ratio $C_n/C_0$ is equal to the square of that portion of the input electrical energy which goes into exciting the $n$th mode, divided by the product of the electrostatic energy of the driving field and the strain energy of this mode of free vibration. As incidental by-products of our calculations, we obtain upper and lower bounds on the change in resonant frequency produced by a change in electrode configuration and an upper bound on the capacitance ratio.

Finally, in Sections VIII and IX, we make use of the simplifications possible when the piezoelectric body is a thin plate, in particular obtaining an explicit relation for the electrode configuration maximizing the capacitance ratio. As a simple although somewhat artificial example, we consider the low-frequency longitudinal vibrations of a piezoelectric bar (the example treated by Vormer[2]) in some detail, using this example to point out some of the advantages, as well as the difficulties, in the practical application of the foregoing techniques.

Before commencing our discussion of these problems, a few general remarks may be appropriate. In order to apply the techniques presently proposed, considerable detailed information about the modes of free vibration is required. Until recently, such information was available only for a few special cases. However, at least for crystal plates, it appears that such information may be obtained by the approximate theoretical techniques developed by Mindlin and his co-workers* over the

* The output of Mindlin and his coworkers in this field is so extensive that a complete bibliography is impossible here. Refs. 4, 5, and 6 are most closely connected with our work.

last decade. Experimental methods using an electric probe which may be moved over the surface of the crystal, such as the methods developed by Van Dyke[7] and by Koga, Fukuyo, and Rhodes,[8] may also yield the desired information. Conversely, measurement of the electrical admittance of a given crystal for various electrode configurations and driving frequencies may yield considerable information about mode shapes.

## II. THE BASIC EQUATIONS

The equations governing the steady vibrations of a piezoelectric crystal at angular frequency $\omega$ may be written in the form

$$T_{ij,j} + \rho\omega^2 u_i = 0, \tag{1}$$

$$D_{i,i} = 0, \tag{2}$$

where $T_{ij}$ is the stress tensor, $\rho$ the mass density, $u_i$ the material particle displacement vector, and $D_i$ the electric displacement vector. Here and in the following, we use Cartesian tensor notation (see, for example, Jeffries[9]), in which commas denote differentiation with respect to the Cartesian coordinates $(x_1, x_2, x_3)$ and a repeated subscript indicates summation over all possible values of that subscript. Thus, for example, (2) is just the usual quasistatic electric field equation, stating that the divergence of the electric displacement vector vanishes. The symbols used for stress, strain, etc., are those used by Mason.[10]

In the case of a piezoelectric medium the stress $T_{ij}$ and electric displacement $D_i$ are given in terms of the strain $S_{ij}$ and electric field $E_i$ by the linear anisotropic constitutive relations

$$T_{ij} = c_{ijkm}{}^E S_{km} - e_{kij} E_k , \tag{3}$$

$$D_i = \epsilon_{ik}{}^S E_k + e_{ikm} S_{km} , \tag{4}$$

where $S_{ij}$ and $E_i$ are given in terms of the particle displacement $u_i$ and electrical potential $V$ by the relations

$$S_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \tag{5}$$

$$E_i = -V_{,i} . \tag{6}$$

The elastic constants $c_{ijkm}{}^E$, piezoelectric constants $e_{kij}$, and dielectric constants $\epsilon_{ij}{}^S$ satisfy the symmetry relations

$$c_{ijkm}{}^E = c_{jikm}{}^E = c_{ijmk}{}^E = c_{kmij}{}^E ,$$

$$e_{kij} = e_{kji} ,$$

$$\epsilon_{ij}{}^S = \epsilon_{ji}{}^S .$$

We shall also have occasion to use the alternate constitutive relations

$$T_{ij} = c_{ijkm}{}^{D} S_{km} - h_{kij} D_k , \tag{7}$$

$$E_i = \beta_{ik}{}^{S} D_k - h_{ikm} S_{km} , \tag{8}$$

$$S_{ij} = s_{ijkm}{}^{E} T_{km} + d_{kij} E_k , \tag{9}$$

$$D_i = \epsilon_{ik}{}^{T} E_k + d_{ikm} T_{km} , \tag{10}$$

where

$$\beta_{ik}{}^{S} \epsilon_{jk}{}^{S} = \delta_{ij} = \begin{cases} 1, & \text{for } i = j, \\ 0, & \text{for } i \neq j, \end{cases}$$

$$h_{ikm} = \beta_{ij}{}^{S} e_{jkm} ,$$

$$c_{ijkm}{}^{D} = c_{ijkm}{}^{E} + h_{pij} e_{pkm} ,$$

$$s_{ipkq}{}^{E} c_{jpmq}{}^{E} = \delta_{ij} \delta_{km} ,$$

$$d_{kij} = e_{kpq} s_{ijpq}{}^{E} ,$$

$$\epsilon_{ij}{}^{T} = \epsilon_{ij}{}^{S} + d_{ipq} e_{jpq} .$$

The magnitude of the piezoelectric effect is specified by the piezoelectric coupling coefficient $k$, where $k^2$ may be given by any one of the three dimensionless ratios, $e^2/c\epsilon$, $h^2/c\beta$, $d^2/s\epsilon$, with $e$, $c$, $\epsilon$, etc., being representative values of the corresponding material constants. For all real piezoelectric materials $k$ is small compared with unity.

## III. BOUNDARY CONDITIONS

The particle displacement $u_i$ and electric potential $V$, satisfying the preceding equations in a piezoelectric body $B$, are completely determined by the specification of certain conditions on its surface $S$ (see Fig. 4). We shall always assume that the body is supported in such a fashion that its surface is free of tractions, i.e.,

$$T_{ij} n_j = 0 \tag{11}$$

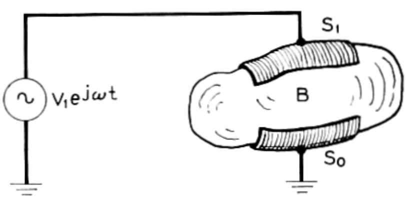


Fig. 4 — Piezoelectric body with driving electrodes.

on $S$, where $n_i$ is the outward normal vector. This assumption is not essential; we could consider just as well the case of a rigid (or compliant) support. It does, however, make the subsequent algebra somewhat simpler.

The body is assumed to be driven by an alternating voltage of constant amplitude $V_1$, applied between electrodes $S_0$ and $S_1$, plated on its surface. Thus

$$V = \begin{cases} V_1, & \text{on } S_1, \\ 0, & \text{on } S_0. \end{cases} \tag{12}$$

The remaining surface is assumed to be free of plating. On this portion of the surface, then, one has two conditions requiring that the potential and normal electric displacement be equal to the potential and normal electric displacement in the external field. We may take this external leakage field into account formally by simply requiring that the equations of the preceding section hold throughout all space (except on $S_0$ and $S_1$), with the stress, strain, and piezoelectric constants vanishing identically outside $B$ and the dielectric constant outside $B$ being that of free space. In the following we shall indicate that an integral is to be taken over all space by using $B_\infty$ in place of $B$. To complete the set of boundary conditions, we assume that $V$ vanishes at large distance from $B$. The external field is included here only for formal logical completeness; in almost all practical problems it is of negligible importance.

## IV. INTEGRAL RELATIONS FOR PIEZOELECTRIC BODIES

Suppose that $u_i$ and $V$ are any (suitably continuous) vector and scalar functions, defined on $B + S$. Note that, by our previous convention, $V$ is actually defined throughout all space. We shall assume that $V$ is continuous across $S$ and zero at infinity. In general, of course, $u_i$ and $V$ will not satisfy the equations of the preceding section. However, by introducing suitable volume and surface forces, $F_i$, $T_i$, and volume and surface charges, $q_B$, $q_S$, we may construct a boundary value problem satisfied by these functions.

First we calculate strains and electric field components, $S_{ij}$, $E_i$, from (5) and (6) and stresses and electric displacement components from the constitutive relations, (3) and (4). The required distributions of volume force and charge are then given by the relations

$$\rho F_i = -(T_{ij,j} + \rho\omega^2 u_i),$$

$$q_B = D_{i,i},$$

where $\omega$ is an arbitrary positive constant. Similarly, the distributions of surface force and charge on $S$ are

$$T_i = T_{ij}n_j ,$$

$$q_S = [D_i]n_i ,$$

where $n_i$ is the outward normal to $S$ and $[D_i]$ is the jump in the electric displacement vector across $S$, given by

$$[D_i] = (D_i)_{\text{ext}} - (D_i)_B .$$

Now suppose we have any two pairs of such functions $(u_i', V')$, $(u_j'', V'')$ and calculate the corresponding stresses, electric displacements, etc. Then the divergence theorem yields

$$\int_B u_i' T_{ij,j}'' \, dB = \int_S u_i' T_{ij} n_j \, dS - \int_B S_{ij}' T_{ij}'' \, dB, \tag{13}$$

$$\int_{B_\infty} V' D_{i,i}'' \, dB = -\int_S V'[D_i'']n_i \, dS + \int_{B_\infty} E_i' D_i'' \, dB, \tag{14}$$

or, in terms of the equivalent forces and charges,

$$\int_B \rho u_i'(F_i'' + \omega''^2 u_i'') \, dB = -\int_S u_i' T_i'' \, dS + \int_B S_{ij}' T_{ij}'' \, dB \tag{15}$$

$$\int_{B_\infty} V' q_B'' \, dB = -\int_S V' q_S'' \, dS + \int_{B_\infty} E_i' D_i'' \, dB. \tag{16}$$

For example, for the forced vibrations previously considered, with $(u_i', V') = (u_i'', V'') = (u_i, V)$, $\omega'' = \omega$, $F_i'' = T_i'' = q_B'' = 0$, addition and rearrangement of (15) and (16) yields

$$\int_{B_\infty} (c_{ijkm}{}^E S_{ij} S_{km} + \epsilon_{ik}{}^S E_i E_k - \rho\omega^2 u_i u_i) \, dB = V_1 Q_1 , \tag{17}$$

where $V_1$ is constant and the total charge on $S_1$ is

$$Q_1 = \int_{S_1} q_S \, dS = \int_{S_1} [D_i]n_i \, dS. \tag{18}$$

Note that, by our previous convention, $c_{ijkm}{}^E$ and $u_i$ vanish outside $B$. Equation (17) essentially states that the sum of the potential energy, made up of the strain energy and electrostatic energy, and the kinetic energy is equal to the energy passing into $B$ through $S_1$ due to the applied voltage $V_1$.

For free vibrations, with $(u_i, V,) = (u_i^n, V^n, \omega_n)$, (17) yields the

Rayleigh quotient for the natural frequency $\omega_n$, i.e.,

$$\omega_n^2 = \frac{\int_{B_\infty} (c_{ijkm}^E S_{ij}^n S_{km}^n + \epsilon_{ik}^S E_i^n E_k^n) \, dB}{\int_B \rho u_i^n u_i^n \, dB}. \tag{19}$$

This relation suggests that the effect of piezoelectricity is to increase the natural frequencies over the values they would have in an ordinary elastic body with the elastic constants $c_{ijkm}^E$. In Section VII we shall prove that this is the case for small piezoelectric coupling coefficient $k$.

If we subtract (16) from (15) and rearrange terms, we obtain

$$\int_{B_\infty} [\rho u_i'(F_i'' + \omega''^2 u_i^{2''}) - V' q_B''] \, dB + \int_S (u_i' T_i'' - V' q_S'') \, dS = \int_{B_\infty} (S_{ij}' T_{ij}'' - E_i' D_i'') \, dB.$$

This equation still holds when we interchange primed and double-primed quantities. Furthermore

$$S_{ij}'' T_{ij}' - E_i'' D_i' = S_{ij}' T_{ij}'' - E_i' D_i'',$$

and thus

$$\begin{aligned} &\int_{B_\infty} [\rho u_i'(F_i'' + \omega''^2 u_i'') - V' q_B''] \, dB + \int_S (u_i' T_i'' - V' q_S'') \, dS \\ &= \int_{B_\infty} [\rho u_i''(F_i' + \omega'^2 u_i') - V'' q_B'] \, dB + \int_S (u_i'' T_i' - V'' q_S') \, dS. \end{aligned} \tag{20}$$

This is the analytical form of the so-called reciprocal theorem (see Ref. 3, Chapter VII, p. 174) for a piezoelectric body. In words it may be stated as follows:

*The difference between the mechanical and electrical work done by the forces (including kinetic reactions) and charges of the first set, acting over the displacements and potential produced by the second set, is equal to the difference between the mechanical and electrical work done by the forces and charges of the second set, acting over the displacements and potential produced by the first.*

## V. EXPANSION IN NORMAL MODES

We now return to the problem of forced vibrations, described in Sections I and II. The equations and boundary conditions of those sections completely determine a unique particle displacement $u_i$ and potential $V$, except at certain natural frequencies $\omega_n (n = 1, 2, \cdots)$ where a non-trivial solution $(u_i^n, V^n)$ of the free vibration problem (with $V_1 = 0$) exists. If we denote the solution of the static problem (with $\omega = 0$) by $(u_i^s, V^s)$, we may satisfy all of the preceding equations and boundary conditions, except (1), by setting

$$u_i = u_i^s + \sum_{n=1}^{\infty} a_n u_i^n, \tag{21}$$

$$V = V^s + \sum_{n=1}^{\infty} a_n V^n, \tag{22}$$

where the $a_n$'s are to be determined to satisfy (1). Since

$$T_{ij,j}^s = T_{ij,j}^n + \rho \omega_n^2 u_i^n = 0,$$

in this case (1) is satisfied if

$$\rho \sum_{n=1}^{\infty} (\omega_n^2 - \omega^2) a_n u_i^n = \rho \omega^2 u_i^s. \tag{23}$$

Now suppose we set $(u_i', V') = (u_i^m, V^m)$, $(u_i'', V'') = (u_i^n, V^n)$, with $\omega_m \neq \omega_n$, in (20). We obtain

$$\rho(\omega_m^2 - \omega_n^2) \int_B u_i^m u_i^n \, dB = 0, \tag{24}$$

so that displacements corresponding to two different natural frequencies are orthogonal, just as they are in ordinary elasticity. Thus, multiplying (23) by $u_i^k$ and integrating over $B$, we obtain

$$\rho(\omega_k^2 - \omega^2) a_k \int_B u_i^k u_i^k \, dB = \rho \omega^2 \int_B u_i^s u_i^k \, dB.$$

Again applying (20), this time with $(u_i', V') = (u_i^k, V^k)$, $(u_i'', V'') = (u_i^s, V^s)$, we obtain

$$\rho \omega_k^2 \int_B u_i^s u_i^k \, dB = \int_{S_e} V^s [D_i^k] n_i \, dS = V_1 \int_{S_1} [D_i^k] n_i \, dS.$$

Thus

$$a_k = \frac{\omega^2 V_1 \int_{S_1} [D_i^k] n_i \, dS}{\rho \omega_k^2 (\omega_k^2 - \omega^2)}. \tag{25}$$

For future reference, we also note the identities

$$\rho\omega_n^2 \int_B u_i^n u_i^n \, dB = \int_B S_{ij}^n T_{ij}^n \, dB, \tag{26}$$

$$\int_{S_1} V_1[D_i^n] n_i \, dS = \int_B e_{kij}(S_{ij}^s E_k^n - S_{ij}^n E_k^s) \, dB, \tag{27}$$

obtained by direct application of the divergence theorem.

## VI. THE EQUIVALENT ELECTRICAL ADMITTANCE

Using the apparatus developed in the previous sections, it is a simple matter to obtain a general expression for the admittance of the crystal body. The admittance $Y(\omega)$ is the ratio of the total input current $I_1$ into the crystal to the voltage $V_1$ across its terminals. The current is the rate of increase of the total charge $Q_1$ on the electrode $S_1$, where

$$Q_1 = \int_{S_1} [D_i] n_i \, dS,$$

so that

$$Y(\omega) = I_1/V_1 = j\omega Q_1/V_1 = j\omega V_1^{-1} \int_{S_1} [D_i] n_i \, dS.$$

Substituting

$$[D_i] = [D_i^s] + \sum_{n=1}^{\infty} a_n [D_i^n],$$

we obtain

$$Y(\omega) = j\omega \left( C_s + \sum_{n=1}^{\infty} \frac{\omega^2 C_n}{\omega_n^2 - \omega^2} \right), \tag{28}$$

where the static capacitance $C_s$ is given by

$$C_s = V_1^{-1} \int_{S_1} [D_i^s] n_i \, dS, \tag{29}$$

and the motional capacitances $C_n$ by

$$C_n = \frac{\left( \int_{S_1} [D_i^n] n_i \, dS \right)^2}{\rho\omega_n^2 \int_B u_i^n u_i^n \, dB}. \tag{30}$$

If we set

$$C_0 = C_s - \sum_{n=1}^{\infty} C_n,$$

(28) may also be written in the form

$$Y(\omega) = j\omega\left(C_0 + \sum_{n=1}^{\infty} \frac{C_n}{1 - \omega^2/\omega_n^2}\right). \tag{31}$$

Fig. 2 shows the corresponding equivalent circuit with

$$L_n = 1/\omega_n^2 C_n, \qquad n = 1, 2, \cdots,$$

and Fig. 1 shows the corresponding behavior of the admittance with frequency. Since $C_n \ll C_s$ for real piezoelectric materials, the admittance is very nearly equal to $j\omega C_s$, except in the vicinity of a resonant frequency, where the admittance is that of a simple series-resonant circuit, shunted by the capacitance $C_0$. The corresponding antiresonant frequency $\omega_n'$ is given approximately by

$$\frac{\omega_n'^2}{\omega_n^2} = \frac{1 + C_n}{C_0},$$

or, since $C_n/C_0 \ll 1$,

$$\frac{\Delta\omega_n}{\omega_n} = \frac{\omega_n' - \omega_n}{\omega_n} = \frac{C_n}{2C_0}. \tag{32}$$

All of these considerations hold when the resonant frequencies are not too closely spaced, specifically when $(\omega_{n+1} - \omega_n)/\omega_n$ is large compared with $C_n/C_0$. At high frequencies, where many resonances may occur in a narrow frequency band, the region of applicability of the above simple single-resonance circuit may be very small, and the simple relation for the antiresonant frequency $\omega_n'$ given by (32) may not be an adequate approximation. In order to circumvent this difficulty, we may choose the shape of the electrodes $S_1$ and $S_0$ so that resonances in the vicinity of a desired resonance in this frequency range are not excited, i.e., so that the corresponding $C_n$'s vanish, or

$$\int_{S_1} [D_i^n] n_i \, dS = 0. \tag{33}$$

This is a necessary and sufficient condition that the $n$th resonance not be excited by the given plating shape.

In order to make use of this condition, the surface charge $[D_i^n]n_i$ must be determined, either theoretically, perhaps by using an approximate theory of the type applied so successfully by Mindlin and his coworkers,[4,5,6] or experimentally using a point electrical probe as developed by Van Dyke[7] or Koga, Fukuyo, and Rhodes.[8] In the case of high-frequency vibrations, one would expect that the resulting electrode shape would be quite complex. Clearly the choice of electrode shape suppressing a finite number of resonances is not unique. For these and other even more pressing reasons, it is not clear whether the use of the condition is of any particular practical importance.

Besides affecting the value of $C_n$, the choice of electrode shape also affects the magnitude of the static capacitance $C_s$ (or $C_0$). Since $C_0$ shunts the series-resonant elements, it is desirable to make its value as small as possible, i.e., to make the capacitance ratio $C_n/C_0$ as large as possible at the desired resonance. Finally, because of piezoelectric coupling, the values of the resonant frequencies themselves depend on the electrode shape, although only to second order in the piezoelectric coupling coefficient $k$. In order to obtain more concrete information concerning these effects, in the next section we consider the case of small piezoelectric coupling.

## VII. SMALL PIEZOELECTRIC COUPLING

In this section we shall obtain expressions for the static capacitance, the capacitance ratios, and the shift in natural frequencies due to a change in electrode shape, valid for small piezoelectric coupling coefficient $k$. With an eye to the applications, these expressions should be the most convenient forms for application to the most common practical case, namely, a crystal plate.

First of all, it is clear that, to first order in $k$, the static capacitance $C_s$ is simply the ordinary electrostatic capacitance in the absence of any piezoelectric effect. Thus, to first order, the static potential $V^s$ satisfies the equations

$$D_i^s = \epsilon_{ik}^s E_k^s = -\epsilon_{ik}^s V_{,k}^s,$$

$$D_{i,i}^s = 0,$$

in $B$, and

$$V^s = \begin{cases} V_1, & \text{on } S_1, \\ 0, & \text{on } S_0. \end{cases}$$

(In this instance, one should recall our previously agreed upon convention concerning the treatment of the external region.) The static capacitance is then given by

$$C_s = V_1^{-1} \int_{S_1} [D_i^s] n_i \, dS = V_1^{-2} \int_{B_\infty} \epsilon_{ij}^s E_i E_j^s \, dB. \tag{34}$$

Next we consider the solution of the free vibration problem for small coupling. To first order in $k$ the particle displacement $u_i^n$ and natural frequency $\omega_n$ are those of the purely elastic problem, and the potential $V^n$ and electric displacement $D_i^n$ are of first order in $k$. Finally, the change in natural frequency is of second order in $k$. Thus we set

$$u_i = u_i^0 + u_i'',$$
$$V = V',$$
$$\omega^2 = \omega_0^2(1 + \nu''),$$

where the quantities with index "0" are of zero order in $k$, those with single primes first order, and those with double primes second order in $k$. We have also temporarily dropped the index "$n$" in order to reduce the number of indices present. The governing equations and boundary conditions then become

$$T_{ij,j}^0 + \rho\omega_0^2 u_i^0 = 0, \qquad \text{in } B,$$
$$T_{ij}^0 n_j = 0, \qquad \text{on } S,$$
$$D_{i,i}' = 0, \qquad \text{in } B_\infty,$$
$$V' = 0, \qquad \text{on } S_e = S_0 + S_1,$$
$$T_{ij,j}'' + \rho\omega_0^2(u_i'' + \nu'' u_i^0) = 0, \qquad \text{in } B,$$
$$T_{ij}'' n_j = 0, \qquad \text{on } S,$$

where the $T_{ij}$'s and $D_i$'s remain to be defined.

The definitions of these quantities, and thus $u_i^0$, $\omega_0$, $V'$, $u_i''$, and $\nu''$ depend on the choice of constitutive relations used. For example, if we use (3) and (4) for the stresses and electric displacements in terms of the strains and electric fields, we have

$$T_{ij}^0 = c_{ijkm}^E S_{km}^0,$$
$$D_i' = \epsilon_{ik}^S E_k' + e_{ikm} S_{km}^0,$$
$$T_{ij}'' = c_{ijkm}^E S_{km}'' - e_{kij} E_k',$$

whereas if we use (7) and (8) we have

$$T_{ij}^0 = c_{ijkm}^D S_{km}^0,$$
$$E_i' = \beta_{ik}^S D_k' - h_{ikm} S_{km}^0,$$
$$T_{ij}'' = c_{ijkm}^D S_{km}'' - h_{kij} D_k'.$$

Use of either of these sets of relations should give the same results (to the order considered) for the potential $V'$, the electric displacement $D_i'$, and the natural frequency $\omega^2 = \omega_0^2(1 + \nu'')$. We may apply the reciprocal theorem, (20), to obtain a simple expression for $\nu''$ in either case. We set $(u_i', V') = (u_i^0, 0)$, $(u_i'', V'') = (u_i'', V')$ to obtain the expressions

$$\nu_E = \int_{B_\infty} \epsilon_{ij}^S E_i' E_j' \, dB / \rho \omega_E^2 \int_B u_i^0 u_i^0 \, dB, \tag{35}$$

$$\nu_D = -\int_{B_\infty} \beta_{ij}^S D_i' D_j' \, dB / \rho \omega_D^2 \int_B u_i^0 u_i^0 \, dB, \tag{36}$$

where $\omega_E$ and $\omega_D$ are the natural frequencies found using the elastic constants $c_{ijkm}^E$ and $c_{ijkm}^D$, respectively, and $\nu_E$ and $\nu_D$ are the corresponding frequency shifts due to the piezoelectric effect. Since, to second order,

$$\omega^2 = \omega_E^2(1 + \nu_E) = \omega_D^2(1 + \nu_D)$$

and $\nu_E \geqq 0$, $\nu_D \leqq 0$, we must have

$$\omega_E \leqq \omega \leqq \omega_D, \tag{37}$$

with equality only if $\nu_E$ or $\nu_D$ vanish. According to (35) and (36), this can only happen if $E_i'$ or $D_i'$ vanishes identically, i.e., if the divergence of the vector $e_{ikm} S_{km}^0$ vanishes in the former case, or if the curl of the vector $h_{ikm} S_{km}^0$ vanishes in the latter case. Thus, in general, neither of these bounds will be attained. We can, however, prove that the lower bound $\omega_E$ is most nearly attained by a completely plated crystal and the upper bound $\omega_D$ by a completely unplated crystal. Furthermore, making the electrodes larger always decreases the resonant frequencies. These results, which are not of central importance in our present considerations, may be obtained by the application of Schwarz's inequality and Green's identity to (35).

We have now obtained expressions for the static capacitance and the shift in resonant frequency caused by the piezoelectric effect. We now derive a simple and symmetric expression for the capacitance ratio $C_n/C_s$, using (26) and (27). We have

$$\int_{S_1} V_1[D_i^n] n_i \, dS = \int_B e_{kij}(S_{ij}^s E_k^n - S_{ij}^n E_k^s) \, dB = -\int_B e_{kij} S_{ij}^n E_k^s \, dB,$$

neglecting terms of higher order in $k$. Also, from (26),

$$\rho \omega_n^2 \int_B u_i^n u_i^n \, dB = \int_B S_{ij}^n T_{ij}^n \, dB,$$

which, in the present case, is either the strain energy in terms of $c_{ijkm}^E$ or in terms of $c_{ijkm}^D$, depending on our choice of constitutive relations.

To first order in $k$ the two expressions are equal, of course. Finally, using (34), we obtain

$$\frac{C_n}{C_s} = \frac{\left(\int_B e_{kij} E_k{}^s S_{ij}{}^n \, dB\right)^2}{\left(\int_B \epsilon_{ij}{}^S E_i{}^s E_j{}^s \, dB\right)\left(\int_B c_{ijkm}{}^E S_{ij}{}^n S_{km}{}^n \, dB\right)}. \tag{38}$$

Direct calculation, again using Schwarz's inequality, leads to the upper bound

$$\frac{C_n}{C_s} \leqq \frac{\omega_D{}^2 - \omega_E{}^2}{\omega_E{}^2}. \tag{39}$$

Again we will not dwell on this incidental result, only remarking that it provides a general upper bound on a quantity which we usually wish to make as large as possible. An alternate form of (38), in terms of stresses, is

$$\frac{C_n}{C_s} = \frac{\left(\int_B d_{kij} E_k{}^s T_{ij}{}^n \, dB\right)^2}{\left(\int_B \epsilon_{ij}{}^T E_i{}^s E_j{}^s \, dB\right)\left(\int_B s_{ijkm}{}^E T_{ij}{}^n T_{km}{}^n \, dB\right)}. \tag{40}$$

## VIII. CRYSTAL PLATES

In this section we consider a thin piezoelectric crystal plate, driven by symmetrically placed electrodes on its face ($x_2 = 0$, $x_2 = h$), as sketched schematically in Fig. 5. In this case, if the thickness $h$ is small compared

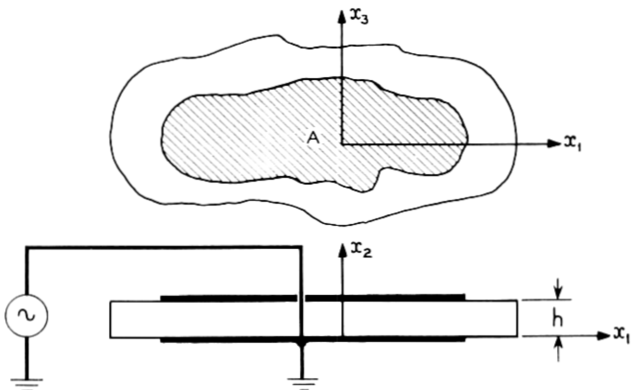


Fig. 5 — The crystal plate.

with the diameter of the electrodes, the static capacitance is given approximately by

$$C_s = \epsilon_{22}{}^S A/h, \tag{41}$$

where $A$ is the electrode area. To the same approximation, the static field is given by

$$E_1{}^s = E_3{}^s = 0,$$

$$E_2{}^s = \begin{cases} -V_1/h, & \text{for } (x_1, x_3) \text{ in } A, \\ 0, & \text{for } (x_1, x_3) \text{ outside } A. \end{cases}$$

(We now neglect the external field completely.) Thus we have

$$\int_B e_{kij} E_k{}^s S_{ij}{}^n \, dB = -V_1 \iint_A e_{2ij} \bar{S}_{ij}{}^n(x_1, x_3) \, dx_1 \, dx_3,$$

where

$$\bar{S}_{ij}{}^n(x_1, x_3) = h^{-1} \int_0^h S_{ij}{}^n(x_1, x_2, x_3) \, dx_2,$$

or, in terms of average stresses,

$$\int_B d_{kij} E_k{}^s T_{ij}{}^n \, dB = -V_1 \iint_A d_{2ij} \bar{T}_{ij}{}^n(x_1, x_3) \, dx_1 \, dx_3,$$

where

$$\bar{T}_{ij}{}^n(x_1, x_3) = h^{-1} \int_0^h T_{ij}{}^n(x_1, x_2, x_3) \, dx_2 .$$

With

$$\int_B \epsilon_{ij}{}^S E_i{}^s E_j{}^s \, dB = V_1{}^2 C_s = \epsilon_{22}{}^E A V_1{}^2/h,$$

the capacitance ratio is given by

$$\frac{C_n}{C_s} = \left(\frac{h}{\epsilon_{22}{}^S A}\right) \left[\iint_A p_n(x_1, x_3) \, dx_1 \, dx_3\right]^2, \tag{42}$$

where

$$p_n = \frac{e_{2ij} \bar{S}_{ij}{}^n}{\left(\int_B c_{ijkm}{}^E S_{ij}{}^n S_{km}{}^n \, dB\right)^{\frac{1}{2}}}, \tag{43}$$

in terms of strains, and

$$p_n = \frac{d_{2ij}\bar{T}_{ij}^{\ n}}{\left(\int_B s_{ijkm}{}^E T_{ij}{}^n T_{km}{}^n \, dB\right)^{\frac{1}{2}}}, \tag{44}$$

in terms of stresses. Note that the integrals in the denominators of these two expressions for $p_n$ are simply normalization factors. We retain them only to keep the dimensions straight.

Equation (42) gives the capacitance ratio in a form which is particularly easy to study for given $p_n$. We wish to know either how to choose $A$ to make $C_n/C_s$ zero, in order to suppress a given resonance, or how to choose $A$ to make $C_n/C_s$ as large as possible, to increase the useful bandwidth of the crystal.

Clearly the former problem has many, many solutions. If, for example, $p_n$ is both positive and negative over the plate area, as will be the case unless we are dealing with some sort of fundamental mode, we need only distribute the electrode area $A$ so that the integrals over the positive and negative portions are equal. The apparent difficulty when $p_n$ is everywhere positive (or negative) can be circumvented by a simple artifice. We obtained (42) by assuming at the outset that voltages of the same polarity were applied between all portions of the top and bottom electrodes. If instead we imagine $A$ to be divided into two parts, $A^+$ and $A^-$, with voltages of opposite polarities applied to these parts, then

$$\iint_A p_n \, dx_1 \, dx_3 = \iint_{A^+} p_n \, dx_1 \, dx_3 - \iint_{A^-} p_n \, dx_1 \, dx_3 ,$$

which clearly can be made to vanish by choosing $A^+$ and $A^-$ so that the corresponding integrals are equal.

The problem of maximizing the capacitance ratio is mostly visualized simply as a geometric problem. First of all, note that in this case we may replace $p_n$ by its absolute value in the integral in (42), for in regions where $p_n$ is negative we may assume that the polarity of the driving voltage has been reversed. This obviously will always increase the capacitance ratio over the value it has with unreversed polarity in such regions. Now we imagine the surface $x_2 = | p_n(x_1, x_3) |$ to be erected above the plane $x_2 = 0$. The integral over $| p_n |$ can then be interpreted as the volume between this surface and the base plane, cut out by a cylinder of cross section $A$. Denoting this volume by $V$, we then must choose $A$ to make $V^2/A$ as large as possible. This problem has a very simple solution, although one which is not particularly well adapted to calculation.

First of all, we observe that, for *fixed* area, to maximize the volume we should choose $A$ to be the region bounded by the appropriate level curve of $| p_n |$. Any other choice of $A$, giving the same area, must consist of this region less a subregion plus an external region having the same area. But the value of $| p_n |$ for points outside the level curve is smaller than for points inside, so that the integral can only be decreased by this alternate choice.

Let us denote the area bounded by the level curve $| p_n(x_1, x_3) | = p$ = a constant by $A_p$ and the corresponding volume by $V_p$. We must now choose $A_p$ so that $V_p^2/A_p$ is maximized, i.e., so that

$$\frac{d(V_p^2/A_p)}{dA_p} = 0.$$

Now the change in $V_p$ due to an increment $\Delta A_p$ in $A_p$ is $p\Delta A_p$, since the height of the surface above the base plane is constant around its boundary, and thus

$$\frac{d(V_p^2/A_p)}{dA_p} = \frac{V_p(2pA_p - V_p)}{A^2_p},$$

which vanishes when $pA_p = \frac{1}{2}V_p$, i.e., when the volumes above and below a plane through the level curve are equal. Explicitly, to maximize the capacity ratio we have the condition

$$pA_p = \frac{1}{2} \iint_{A_p} | p_n(x_1, x_3) | \, dx_1 \, dx_3, \tag{45}$$

where $A_p$ is the total area of the electrode on either plate face, bounded by the level curve $| p_n(x_1, x_3) | = p$, and $p_n$ is given by (43) or (44). We also assume that the electrode polarity is positive when $p_n$ is positive and negative when $p_n$ is negative. Since $A_p$ must of course be less than or equal to the plate area, the maximum capacitance ratio given by (45) may not actually be attained. If this is the case, a plate wholly covered with electrodes gives the largest possible value to the capacitance ratio. It may also happen that the maximizing area $A_p$ may be partly bounded by a level curve and partly bounded by the plate edge.

Bechmann and Parsons[1] have considered the excitation of piezoelectric bars and plates with partially applied electrodes and have obtained results similar to the above for the capacitance ratio, both theoretically and experimentally.

## IX. THE LONGITUDINAL VIBRATIONS OF A PIEZOELECTRIC BAR

To illustrate the foregoing results, we now consider a concrete example. To keep the analysis simple, we treat the example discussed by Vormer[2] of the low-frequency longitudinal vibrations of the piezoelectric bar sketched in Fig. 6. We assume that $h \ll a \ll L$ and restrict our discussion to vibrations of wavelength large compared with $a$. Then we may assume that all the stresses $T_{ij}$ are zero except $T_{11}$, which we take to be a function of $x_1$ only. Thus, for free vibrations,

$$T_{11} = T_{11}^{\,n}(x_1),$$

satisfying the equation of motion

$$T_{11,1}^{\,n} + \rho\omega_n^{\,2} u_1^{\,n} = 0,$$

and the stress-strain relation

$$S_{11}^{\,n} = u_{1,1}^{\,n} = s_{1111}^{\,E} T_{11}^{\,n}.$$

Eliminating $u_1^{\,n}$ between these two equations leads to the equation

$$T_{11,11}^{\,n} + (\omega_n/c)^2 T_{11}^{\,n} = 0,$$

where $c^2 = 1/\rho s_{1111}^{\,E}$. Together with the boundary conditions

$$T_{11}^{\,n}(0) = T_{11}^{\,n}(L) = 0,$$

this equation yields

$$T_{11}^{\,n}(x_1) = \sin n\pi x_1/L, \qquad \omega_n = n\pi c/L.$$

If we assume that the width of the electrode $A$ at $x = x_1/L$ is $af(x)$, as in Fig. 6, we find that the capacitance ratio is given by

$$\frac{C_n}{C_s}\, 2k^2 \left(\frac{A_0}{A}\right)\left[\int_0^1 f(x) \sin n\pi x \, dx\right]^2, \tag{46}$$

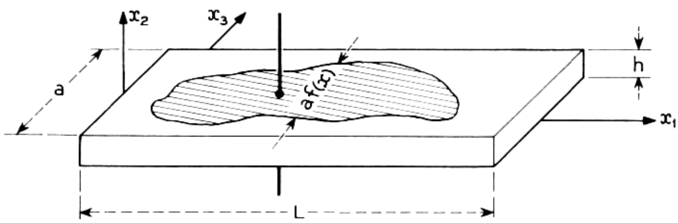


Fig. 6 — The crystal bar.

where $k$ is the piezoelectric coupling coefficient, here given by

$$k^2 = \frac{(d_{211})^2}{s_{1111}{}^E \epsilon_{22}{}^S},$$

$A_0 = aL$ is the bar-face area, and $A$ is the electrode area, given by

$$A = A_0 \int_0^1 | f(x) | \, dx.$$

We must assume that $0 \leqq | f(x) | \leqq 1$, and we may take into account portions of electrode of reversed polarity by assuming that $f(x)$ may be both positive and negative.

For the wholly covered bar ($f \equiv 1$), for example, (46) gives

$$\frac{C_n}{C_s} = \begin{cases} 4k^2/n^2, & \text{for } n = 1, 3, \cdots, \\ 0, & \text{for } n = 2, 4, \cdots. \end{cases}$$

(Clearly an even excitation, such as the above, cannot excite the odd modes.) This already provides an example of resonance suppression of a very trivial nature. In the present case, within the limitations of the theory, we can actually find an electrode shape which suppresses all resonances except one, for the $T_{11}{}^n$'s are orthogonal. Thus, if we set

$$f(x) = \sin m\pi x,$$

we find

$$\frac{C_n}{C_s} = \begin{cases} \pi k^2/4, & \text{for } n = m, \\ 0, & \text{otherwise.} \end{cases}$$

Of course, since the application of the present simple theory is restricted to low frequencies, we cannot expect the above to be valid for large $n$ and $m$. On the other hand, the low-frequency resonances are sufficiently widely spaced so that resonance suppression is of no particular practical importance in this case. Thus the present example must be taken as illustrative, rather than practical.

In the present case of one-dimensional vibrations, the level lines of $| p_n |$, considered in the previous section, are straight lines $x_1$ = a constant. Thus the electrode shape maximizing the capacitance ratio consists simply of a sequence of rectangular bands of suitable widths and polarities, extending across the width of the bar. To determine the optimum width, it suffices to consider the fundamental mode ($n = 1$). For convenience we shift the origin of coordinates to the center of the plate so that

$$T_{11}{}^1(x_1) = \cos (\pi x_1/L) = \cos \pi x,$$

and assume that the electrode band covers the interval $|x| \leqq x_0$. We must choose $x_0$ according to (45),

$$2a \int_0^{x_0} \cos \pi x \, dx = 2a(2x_0 \cos \pi x_0),$$

or

$$\tan \pi x_0 = 2\pi x_0 ,$$

which is satisfied by $x_0 = 0.371$, which is the value found also by Bechmann and Parsons[1] in this case, i.e., 74.2 per cent of the bar length covered by electrodes. This maximizes the capacitance ratio $C_1/C_s$ for the fundamental mode. The capacitance ratio for overtone modes may be maximized similarly, since the overtone modes may be imagined to be made up of a set of fundamental modes for $n$ bars, $1/n$ in length, set end to end.

For the fundamental mode, this maximum capacitance ratio is $0.909k^2$ compared with the values $0.811k^2$ and $0.786k^2$ for a completely covered bar and for a bar with sinusoidal plating which suppresses all overtones. Thus we may increase the capacitance ratio about 10 per cent over its value for the wholly covered bar by the above technique.

A difficulty which we do not encounter with the fundamental mode shows up when we consider the excitation of the first overtone ($n = 2$).

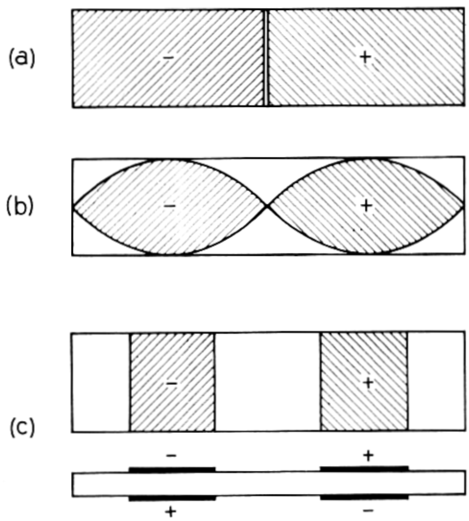


Fig. 7 — Various electrode configurations for excitation of first overtone in crystal bar.

In order to excite this mode, we must use electrodes having odd polarity about the middle of the plate, as shown in Fig. 7. For example, we may use complete (split) electrodes, as in Fig. 7(a), sinusoidal electrodes, as in Fig. 7(b), or band electrodes, as in Fig. 7(c). Especially in the first case, there will be a substantial contribution to the total static capacitance from the capacitance between adjoining electrodes of opposite polarity. This contribution has been completely neglected in our previous calculations. To make it small compared with the ordinary plate capacitance, proportional to electrode area, the spacing between adjoining electrodes of opposite polarity should be large compared with the plate thickness. This consideration makes the band electrodes [Fig. 7(c)], which maximize the capacitance ratio, particularly attractive.

The presence of a substantial electrical field component, parallel to the bar faces, also changes the form of the function $p_n$ and thus the motional capacitance. At high frequencies, with a complicated resonance spectrum, this field component may excite unwanted resonances. In the present case of low-frequency longitudinal vibrations, this effect is probably not important.

## X. ACKNOWLEDGMENT

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