Stochastic Processes with Balking in the Theory of Telephone Traffic*

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It is supposed that at a telephone exchange calls are arriving according to a recurrent process. If an incoming call finds exactly j lines busy then it either realizes a connection with probability p_j *or balks with probability* q_j $(p_j + q_j = 1)$. *The holding times are mutually independent random variables with common exponential distribution. In this paper the stochastic behavior of the fluctuation of the number of the busy lines is studied.*

I. INTRODUCTION

Many results in telephone traffic theory (and elsewhere) may be unified by the introduction of *balking*. A call is said to balk if for any reason it refuses service on arrival. A mathematical model for balking is constructed by assigning a probability to balking dependent only on the state of the system; if an incoming call finds exactly j lines busy, then it realizes a connection with probability p_j and balks with probability q_j $(p_j + q_j = 1)$. Thus if $p_j = 1$ $(j = 0, 1, \cdots)$ the system is one with an infinite number of lines and with no loss and no delay, the ideal for for any service, while if $p_j = 1$ $(j = 0, 1, \cdots, m - 1)$ and $p_j = 0$ $(j = m, m + 1, \cdots)$ the system is a loss system with m lines.

This balking model is examined here for recurrent input and exponential distribution of holding times. More specifically, the call arrival times are taken as the instants $\tau_1, \tau_2, \cdots, \tau_n, \cdots$, where the inter-arrival times $\theta_n = \tau_{n+1} - \tau_n$ $(n = 0, 1, \cdots; \tau_0 = 0)$ are identically distributed, mutually independent, positive random variables with distribution function

$$\mathbf{P}\{\theta_n \leqq x\} = F(x). \tag{1}$$

* Dedicated to the memory of my professor Charles Jordan (December 16, 1871–December 24, 1959)

The holding times are identically distributed, mutually independent random variables with distribution function

$$H(x) = \begin{cases} 1 - e^{-\mu x} & \text{if } x \geqq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{2}$$

The holding times are independent of the $\{\tau_n\}$ as well.

Let us denote by $\xi(t)$ the number of busy lines at the instant t. Define $\xi_n = \xi(\tau_n - 0)$; that is, ξ_n is the number of busy lines immediately before the arrival of the nth call. The system is said to be in state E_k at the instant t if $\xi(t) = k$. Let us denote by m the smallest nonnegative integer such that $p_m = 0$. If $p_j > 0$ $(j = 0, 1, 2, \cdots)$ then $m = \infty$.

In the present paper we shall give a method to determine the distribution of ξ_n for every n, the distribution of $\xi(t)$ for finite t values, and the limiting distributions of ξ_n and $\xi(t)$ as $n \to \infty$ and $t \to \infty$ respectively. Further, we shall determine the stochastic law of the transitions $E_k \to E_{k+1}$ $(k = 0, 1, 2, \cdots)$.

II. NOTATION

The Laplace-Stieltjes transform of the distribution function of the interarrival times will be denoted by

$$\varphi(s) = \mathbf{E}\{e^{-s\theta_n}\} = \int_0^\infty e^{-sx}\, dF(x),$$

which is convergent if $\Re(s) \geqq 0$. The expectation of the interarrival times will be denoted by

$$\alpha = \mathbf{E}\{\theta_n\} = \int_0^\infty x\, dF(x).$$

Let $\mathbf{P}\{\xi_n = k\} = P_k^{(n)}$ and $\mathbf{P}\{\xi(t) = k\} = P_k(t)$. Define

$$\Pi_k(s) = \int_0^\infty e^{-st} P_k(t)\, dt,$$

which is convergent if $\Re(s) > 0$. Let

$$\lim_{n\to\infty} P_k^{(n)} = P_k \qquad \text{and} \qquad \lim_{t\to\infty} P_k(t) = P_k^*,$$

provided that the limits exist.

Define

$$C_r = \prod_{i=1}^{r} \left(\frac{\varphi(i\mu)}{1 - \varphi(i\mu)} \right) \qquad (r = 0, 1, 2, \cdots), \tag{3}$$

where the empty product means 1; that is, $C_0 = 1$. We shall also use the abbreviation

$$\varphi_r = \varphi(r\mu) = \int_0^\infty e^{-r\mu x}\, dF(x) \qquad (r = 0, 1, 2, \cdots). \tag{4}$$

Denote by $M_k(t)$ the expected number of calls occurring in the time interval $(0,t]$ which find exactly k lines busy. The expected number of transitions $E_k \to E_{k+1}$ occurring in the time interval $(0,t]$ is clearly $p_k M_k(t)$. Denote by $N_k(t)$ the expected number of transitions $E_{k+1} \to E_k$ occurring in the time interval $(0,t]$.

Let $G_k(x)(k = 0, 1, 2, \cdots)$ be the distribution function of the time differences between successive transitions $E_{k-1} \to E_k$ and $E_k \to E_{k+1}$, while $R_k(x)(k = 0, 1, 2, \cdots)$ is the distribution function of the time differences between consecutive transitions $E_k \to E_{k+1}$. If $\xi(0) = 0$ then we say that a transition $E_{-1} \to E_0$ takes place at time $t = -0$. Write

$$\gamma_k(s) = \int_0^\infty e^{-sx}\, dG_k(x)$$

and

$$\rho_k(s) = \int_0^\infty e^{-sx}\, dR_k(x)$$

which are convergent if $\Re(s) \geqq 0$.

III. PREVIOUS RESULTS

3.1 *A. K. Erlang*

Erlang[1] has proved that, if $\{\tau_n\}$ forms a Poisson process of intensity λ—that is, $F(x) = 1 - e^{-\lambda x}$ for $x \geqq 0$—and further, $p_j = 1$ when $j < m$, $p_j = 0$ when $j \geqq m$, then

$$P_k^* = \frac{\dfrac{(\lambda/\mu)^k}{k!}}{\displaystyle\sum_{j=0}^{m} \frac{(\lambda/\mu)^j}{j!}} \qquad (k = 0, 1, \cdots, m). \tag{5}$$

In this case $P_k = P_k^*$ $(k = 0, 1, \cdots, m)$ also holds. This is the simplest loss system.

3.2 *Conny Palm*

Palm[2] has generalized the above result of Erlang for the case when $\{\tau_n\}$ forms a recurrent process and otherwise every assumption remains unchanged. Palm has proved that

$$P_m = \frac{1}{\sum_{r=0}^{m} \binom{m}{r} \frac{1}{C_r}}, \tag{6}$$

where C_r is defined by (3). In this case the complete limiting distributions $\{P_k\}$ and $\{P_k^*\}$ have been determined by Pollaczek,[3] Cohen,[4] and the author.[5,6] The transient behavior of the sequence $\{\xi_n\}$ was determined by Pollaczek[3] and Beneš,[7] and the transient behavior of the process $\{\xi(t)\}$ by Beneš[8] and by the author.[9]

3.3 *The Infinite Line Case*

The case when $p_j = 1$ $(j = 0, 1, 2, \cdots)$ has been investigated by the author,[10,11] who has proved that

$$P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} C_r \qquad (k = 0, 1, 2, \cdots) \tag{7}$$

and, if $F(x)$ is not a lattice distribution and if $\alpha < \infty$, then the limiting distribution $\{P_k^*\}$ exists and

$$\begin{aligned} P_k^* &= \frac{P_{k-1}}{k\alpha\mu} \qquad (k = 1, 2, \cdots), \\ P_0^* &= 1 - \frac{1}{\alpha\mu} \sum_{k=1}^{\infty} \frac{P_{k-1}}{k}. \end{aligned} \tag{8}$$

The transient behavior of the process $\{\xi(t)\}$ is also treated in Refs. 10 and 11.

3.4 *The Case* $p_0 = 1$, $p_j = p$ $(j = 1, 2, \cdots)$, $q_j = q$ $(j = 1, 2, \cdots)$

This case, where $p + q = 1$, plays an important role in the theory of particle counters and has been investigated by the author,[12] who has found that

$$P_0 = \frac{p \sum_{r=0}^{\infty} (-p)^r C_r}{1 - q \sum_{r=0}^{\infty} (-p)^r C_r} \tag{9}$$

and

$$P_k = \frac{\sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} C_r}{1 - q \sum_{r=0}^{\infty} (-p)^r C_r} \qquad (k = 1, 2, \cdots). \tag{10}$$

If $F(x)$ is not a lattice distribution and if $\alpha < \infty$, then the limiting distribution $\{P_k^*\}$ exists and

$$\begin{aligned} P_{k+1}^* &= \frac{p_k P_k}{(k+1)\alpha\mu} \qquad (k = 0, 1, 2, \cdots), \\ P_0^* &= 1 - \frac{1}{\alpha\mu} \sum_{k=0}^{\infty} \frac{p_k P_k}{(k+1)}. \end{aligned} \tag{11}$$

The transient behavior of the process $\{\xi(t)\}$ is also treated in Ref. 12.

3.5 *The Distribution Function* $G_k(x)$

This function plays an important role in the investigation of overflow traffic. In the infinite line case, i.e., when $p_j = 1$ ($j = 0, 1, 2, \cdots$), Palm[2] has proved that $\gamma_k(s)(k = 0, 1, 2, \cdots)$ satisfies the following recurrence formula:

$$\gamma_k(s) = \frac{\gamma_{k-1}(s+\mu)}{1 - \gamma_{k-1}(s) + \gamma_{k-1}(s+\mu)} \qquad (k = 1, 2, \cdots), \tag{12}$$

where $\gamma_0(s) = \varphi(s)$. Palm has obtained $\gamma_k(s)$ explicitly when $\{\tau_n\}$ is a Poisson process; that is, $\varphi(s) = \lambda/(\lambda + s)$. Then

$$\gamma_k(s) = \frac{\sum_{j=0}^{k} \binom{k}{j} \frac{s(s+\mu)\cdots[s+(j-1)\mu]}{\lambda^j}}{\sum_{j=0}^{k+1} \binom{k+1}{j} \frac{s(s+\mu)\cdots[s+(j-1)\mu]}{\lambda^j}}. \tag{13}$$

The general solution of the recurrence formula (12) is

$$\gamma_k(s) = \frac{\sum_{r=0}^{k} \binom{k}{r} \prod_{i=0}^{r-1} \left[\frac{1-\varphi(s+i\mu)}{\varphi(s+i\mu)}\right]}{\sum_{r=0}^{k+1} \binom{k+1}{r} \prod_{i=0}^{r-1} \left[\frac{1-\varphi(s+i\mu)}{\varphi(s+i\mu)}\right]} \qquad (k = 0, 1, 2, \cdots), \tag{14}$$

where the empty product means 1. The formula (14) is proved in Refs. 10 and 11.

In the particular case $p_0 = 1$, $p_j = p$ ($j = 1, 2, \cdots$), $q_j =$

$q(j = 1, 2, \cdots)$, where $p + q = 1$, the Laplace-Stieltjes transform $\gamma_k(s)$ has been given explicitly in Ref. 12. We have

$$\gamma_k(s) = \frac{D_k(s)}{D_{k+1}(s)} \qquad (k = 0, 1, 2, \cdots), \tag{15}$$

where $D_0(s) \equiv 1$ and

$$\begin{aligned} D_k(s) = \Bigg\{ & p \sum_{r=0}^{k} \binom{k}{r} \prod_{i=0}^{r-1} \left[\frac{1 - \varphi(s + i\mu)}{p\varphi(s + i\mu)} \right] \\ & - \frac{q[1 - \varphi(s)]}{p\varphi(s)} \sum_{r=0}^{k} \binom{k}{r} \sum_{j=1}^{r-1} (-1)^j \prod_{i=j+1}^{r-1} \left[\frac{1 - \varphi(s + i\mu)}{p\varphi(s + i\mu)} \right] \Bigg\} \end{aligned} \tag{16}$$

if $k = 1, 2, \cdots$.

IV. THE TRANSIENT BEHAVIOR OF $\{\xi_n\}$

It is easy to see that the sequence of random variables $\{\xi_n\}$ forms a homogeneous Markov chain with transition probabilities

$$p_{jk} = \mathbf{P}\{\xi_{n+1} = k \mid \xi_n = j\} = \int_0^\infty \pi_{jk}(x)\, dF(x), \tag{17}$$

where

$$\pi_{jk}(x) = p_j \binom{j+1}{k} e^{-k\mu x}(1 - e^{-\mu x})^{j+1-k} + q_j \binom{j}{k} e^{-k\mu x}(1 - e^{-\mu x})^{j-k} \tag{18}$$

is the conditional transition probability given that the interarrival time $\theta_n = x$ (constant). For, if $\xi_n = j$ and $\theta_n = x$, then ξ_{n+1} has a Bernoulli distribution, either with parameters $j + 1$ and $e^{-\mu x}$ when the nth call realizes a connection, or with parameters j and $e^{-\mu x}$ when the nth call does not. The system is said to be in state E_k at the nth step if $\xi_n = k$.

Starting from the initial distribution $\{P_k^{(1)}\}$ the distributions $\{P_k^{(n)}\}$ can be determined successively by the following formulas:

$$P_k^{(n+1)} = \sum_{j=k-1}^{\infty} p_{jk} P_j^{(n)} \qquad (n = 1, 2, \cdots). \tag{19}$$

However, it turns out that in many cases it is more convenient to determine the binomial moments of $\{P_k^{(n)}\}$ first. By definition,

$$U_r^{(n)} = \mathbf{E}\left\{\binom{\xi_n}{r}\right\} = \sum_{k=r}^{\infty} \binom{k}{r} P_k^{(n)} \qquad (r = 0, 1, 2, \cdots) \tag{20}$$

is the rth binomial moment of $\{P_k^{(n)}\}$. If we suppose that $U_r^{(1)} < C_1^r/r!$ where C_1 is a constant, then it can be proved that every $U_r^{(n)}$ exists and $U_r^{(n)} < C^r/r!$ where C is a constant. Thus the distribution $\{P_k^{(n)}\}$ is uniquely determined by $\{U_r^{(n)}\}$. We obtain from (20) that

$$P_k^{(n)} = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} U_r^{(n)} \qquad (k = 0, 1, 2, \cdots). \tag{21}$$

This is the inversion formula of Jordan.[13]

It is convenient to use the related quantities

$$V_r^{(n)} = \mathbf{E}\left\{\binom{\xi_n}{r} p_{\xi_n}\right\} = \sum_{k=r}^{\infty} \binom{k}{r} p_k P_k^{(n)} \qquad (r = 0, 1, 2, \cdots), \tag{22}$$

whence by inversion

$$p_k P_k^{(n)} = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} V_r^{(n)}. \tag{23}$$

Now we shall prove

Theorem 1. We have $U_0^{(n)} = 1$ $(n = 1, 2, \cdots)$ *and*

$$U_r^{(n+1)} = \varphi_r(U_r^{(n)} + V_{r-1}^{(n)}) \qquad (n = 1, 2, \cdots; \quad r = 1, 2, \cdots), \tag{24}$$

where $\varphi_r = \varphi(r\mu)$. *Further*

$$V_r^{(n)} = \sum_{j=r}^{\infty} \binom{j}{r} (\Delta^{j-r} p_r) U_j^{(n)} \qquad (r = 0, 1, 2, \cdots), \tag{25}$$

where

$$\Delta^{j-r} p_r = \sum_{\nu=0}^{j-r} (-1)^{\nu} \binom{j-r}{\nu} p_{j-\nu}\,. \tag{26}$$

Proof. First of all we note that the rth binomial moment of the Bernoulli distribution $\{Q_k\}$ with parameters n and p, that is, that of

$$Q_k = \binom{n}{k} p^k (1-p)^{n-k} \qquad (k = 0, 1, \cdots, n),$$

is given by

$$B_r = \sum_{k=r}^{n} \binom{k}{r} Q_k = \binom{n}{r} p^r \qquad (r = 0, 1, \cdots, n). \tag{27}$$

Using (27), we get by (18) that

$$\mathbf{E}\left\{\binom{\xi_{n+1}}{r} \mid \xi_n = j, \theta_n = x\right\} = p_j \binom{j+1}{r} e^{-r\mu x} + q_j \binom{j}{r} e^{-r\mu x},$$

whence

$$\begin{aligned} \mathbf{E}\left\{\binom{\xi_{n+1}}{r} \mid \xi_n = j\right\} &= \varphi_r\left[p_j\binom{j+1}{r} + q_j\binom{j}{r}\right] \\ &= \varphi_r\left[\binom{j}{r} + p_j\binom{j}{r-1}\right]. \end{aligned} \tag{28}$$

If we multiply both sides of (28) by $P_j^{(n)}$ and add them for every j, then we get (24). We obtain (25) if we put (21) into (22). This completes the proof of the theorem.

Starting from $U_r^{(1)} (r = 1, 2, \cdots)$ the binomial moments $U_r^{(n)} (n = 2, 3, \cdots)$ can be obtained recursively by (24) and (25). If, specifically, $\xi(0) = i$ and $\tau_1 = x$ then ξ_1 has a Bernoulli distribution with parameters i and $e^{-\mu x}$ and thus, for $\xi(0) = i$,

$$U_r^{(1)} = \mathbf{E}\left\{\binom{\xi_1}{r}\right\} = \binom{i}{r}\varphi_r \qquad (r = 0, 1, 2, \cdots). \tag{29}$$

Remark 1. If we introduce the generating functions

$$U_r(w) = \sum_{n=1}^{\infty} U_r^{(n)} w^n \tag{30}$$

and

$$V_r(w) = \sum_{n=1}^{\infty} V_r^{(n)} w^n \tag{31}$$

and suppose that $\xi(0) = i$, then by (24) and (29) we get that

$$U_r(w) = \frac{w\varphi_r}{1 - w\varphi_r}\left[\binom{i}{r} + V_{r-1}(w)\right] \qquad (r = 1, 2, \cdots), \tag{32}$$

and evidently

$$U_0(w) = \frac{w}{1 - w}. \tag{33}$$

Note also that (21) implies that

$$\sum_{n=1}^{\infty} P_k^{(n)} w^n = \sum_{r=k}^{\infty} (-1)^{r-k}\binom{r}{k} U_r(w). \tag{34}$$

Example 1. In the infinite line case, i.e., when $p_j = 1 (j = 0, 1, 2, \cdots)$, $V_r^{(n)} = U_r^{(n)}$ and $V_r(w) = U_r(w)$ for $r = 0, 1, 2, \cdots$. If we suppose that $\xi(0) = i$, then by (32) we get

$$U_r(w) = \frac{w\varphi_r}{1 - w\varphi_r}\left[\binom{i}{r} + U_{r-1}(w)\right] \qquad (r = 1, 2, \cdots) \tag{35}$$

and $U_0(w) = w/(1 - w)$. The solution of these equations is given by

$$U_r(w) = \left\{\prod_{j=0}^{r}\left(\frac{w\varphi_j}{1 - w\varphi_j}\right)\right\}\left\{\sum_{j=0}^{r}\binom{i}{j}\prod_{\nu=0}^{j-1}\left(\frac{1 - w\varphi_\nu}{w\varphi_\nu}\right)\right\} \qquad (r = 0, 1, 2, \cdots),$$

where the empty product means 1. The distribution $\{P_k^{(n)}\}$ is determined by (34).

Example 2. For a loss system with m lines, i.e., when $p_j = 1$ $(j < m)$ and $p_j = 0$ $(j \geqq m)$, in the case $\xi(0) = i \leqq m$ we have

$$V_r^{(n)} = U_r^{(n)} - \binom{m}{r} U_m^{(n)} \qquad (r = 0, 1, 2, \cdots, m - 1)$$

and

$$V_r^{(n)} = U_r^{(n)} = 0 \qquad (r = m, m + 1, \cdots).$$

Thus,

$$V_r(w) = U_r(w) - \binom{m}{r} U_m(w) \qquad (r = 0, 1, 2, \cdots, m - 1)$$

and

$$V_r(w) = U_r(w) = 0 \qquad (r = m, m + 1, \cdots).$$

By (32) we get

$$U_r(w) = \frac{w\varphi_r}{1 - w\varphi_r}\left[\binom{i}{r} + U_{r-1}(w) - \binom{m}{r - 1} U_m(w)\right] \qquad (r = 1, 2, \cdots, m) \tag{36}$$

and $U_0(w) = w/(1 - w)$. The solution of these equations for $r = 0, 1, 2, \cdots, m$ is given by

$$U_r(w) = \frac{\Gamma_r(w)}{\sum_{j=0}^{m}\binom{m}{j}\frac{1}{\Gamma_j(w)}}\left\{\left[\sum_{j=r}^{m}\binom{m}{j}\frac{1}{\Gamma_j(w)}\right]\left[\sum_{j=0}^{r}\binom{i}{j}\frac{1}{\Gamma_{j-1}(w)}\right] - \left[\sum_{j=0}^{r-1}\binom{m}{j}\frac{1}{\Gamma_j(w)}\right]\left[\sum_{j=r+1}^{m}\binom{i}{j}\frac{1}{\Gamma_{j-1}(w)}\right]\right\} \tag{37}$$

where

$$\Gamma_r(w) = \prod_{i=0}^{r}\left(\frac{w\varphi_i}{1 - w\varphi_i}\right), \qquad (r = 0, 1, 2, \cdots)$$

and $\Gamma_{-1}(w) \equiv 1$. Finally, $\{P_k^{(n)}\}$ can be obtained by (34).

V. THE LIMITING DISTRIBUTION $\{P_k\}$

In the Markov chain $\{\xi_n\}$ the states $E_0, E_1, \cdots, E_m$ form an irreducible closed set, while $E_m, E_{m+1}, \cdots$ are transient states. If either $m = \infty$ or $m < \infty$, but we restrict ourselves to the states $E_0, E_1, \cdots, E_m$, then the Markov chain $\{\xi_n\}$ is irreducible. The Markov chain $\{\xi_n\}$ is always aperiodic. Accordingly

$$\lim_{n \to \infty} P_k^{(n)} = P_k \qquad (k = 0, 1, 2, \cdots)$$

always exists and is independent of the initial distribution. There are two possibilities: either every $P_k = 0$ $(k = 0, 1, 2, \cdots)$ or $\{P_k\}$ is a probability distribution. (In the second case $P_k > 0$ if $k \leqq m$ and $P_k = 0$ if $k > m$.) In the second case $\{P_k\}$ is the unique stationary distribution of the Markov chain $\{\xi_n\}$ and conversely if there exists a stationary distribution then it is unique and agrees with the limiting distribution $\{P_k\}$.

In the particular case $p_j = 1$ $(j = 0, 1, 2, \cdots)$ the limiting distribution always exists, as has been proved in Ref. 10. In this special case

$$P_0 = \sum_{r=0}^{\infty} (-1)^r C_r > 0.$$

If we consider an arbitrary sequence $\{p_j\}$ then evidently

$$P_0 \geqq \sum_{r=0}^{\infty} (-1)^r C_r > 0,$$

whence it follows that $\{\xi_n\}$ belongs to the second class; that is, $\{P_k\}$ is a probability distribution.

The stationary distribution $\{P_k\}$ is uniquely determined by the following system of linear equations:

$$P_k = \sum_{j=k-1}^{\infty} p_{jk} P_j \tag{38}$$

and

$$\sum_{k=1}^{\infty} P_k = 1. \tag{39}$$

Since in this case $P_k^{(n)} = P_k$ for every n, we get (38) by (19). Now let us introduce the binomial moments

$$U_r = \sum_{k=r}^{\infty} \binom{k}{r} P_k \qquad (r = 0, 1, 2, \cdots) \tag{40}$$

and define

$$V_r = \sum_{k=r}^{\infty} \binom{k}{r} p_k P_k . \tag{41}$$

By inversion we get, from (40),

$$P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} U_r \qquad (k = 0, 1, 2, \cdots) \tag{42}$$

and similarly, from (41),

$$p_k P_k = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} V_r . \tag{43}$$

The binomial moments $U_r (r = 0, 1, 2, \cdots)$ can be obtained by the following

Theorem 2. We have $U_0 = 1$ *and*

$$U_r = \frac{\varphi_r}{1 - \varphi_r} V_{r-1} \qquad (r = 1, 2, \cdots), \tag{44}$$

where $\varphi_r = \varphi(r\mu)$. *Further,*

$$V_r = \sum_{j=r}^{\infty} \binom{j}{r} (\Delta^{j-r} p_r) U_j \qquad (r = 0, 1, 2, \cdots), \tag{45}$$

where

$$\Delta^{j-r} p_r = \sum_{\nu=0}^{j-r} (-1)^{\nu} \binom{j-r}{\nu} p_{j-\nu} . \tag{46}$$

Proof. This theorem immediately follows from Theorem 1 if we put $U_r^{(n)} = U_r$, $V_r^{(n)} = V_r$ in (24) and (25).

Remark 2. In many cases there is a simple relation between the generating functions

$$U(z) = \sum_{k=0}^{\infty} P_k z^k \tag{47}$$

and

$$V(z) = \sum_{k=0}^{\infty} p_k P_k z^k \tag{48}$$

when U_r $(r = 0, 1, \cdots)$ can easily be obtained by (44). For,

$$U_r = \frac{1}{r!} \left[\frac{d^r U(z)}{dz^r} \right]_{z=1} \qquad (r = 0, 1, 2, \cdots) \tag{49}$$

and

$$V_r = \frac{1}{r!}\left[\frac{d^r V(z)}{dz^r}\right]_{z=1} \qquad (r = 0, 1, 2, \cdots). \tag{50}$$

Theorem 3. The binomial moments U_r $(r = 0, 1, 2, \cdots)$ *satisfy the following system of linear equations:*

$$\sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \left(p_k U_r - \frac{1 - \varphi_{r+1}}{\varphi_{r+1}} U_{r+1} \right) = 0 \qquad (r = 0, 1, 2, \cdots) \tag{51}$$

and

$$U_{r+1} = \frac{\varphi_{r+1}}{1 - \varphi_{r+1}} \sum_{j=r}^{\infty} \binom{j}{r} (\Delta^{j-r} p_r) U_j \qquad (r = 0, 1, 2, \cdots), \tag{52}$$

where $\Delta^{j-r} p_r$ *is defined by* (46).

Proof. If we put (42) into (43) and use the relation (44) then we get (51). If we eliminate V_r from (44) and (45) then we get (52).

Remark 3. If $p_m = 0$ then $U_r = 0$ for $r > m$, and in this case, starting from U_m, the unknowns $U_{m-1}, U_{m-2}, \cdots, U_0$ can be obtained successively either by (51) or by (52) and finally $U_0 = 1$ determines U_m. If the higher differences of p_r vanish, then (52) can be used successfully for the determination of the binomial moments U_r.

Example 3. If $p_j = 1$ $(j = 0, 1, 2, \cdots)$ then $V_r = U_r$ $(r = 0, 1, 2, \cdots)$ and, by (44),

$$U_r = \frac{\varphi_r}{1 - \varphi_r} U_{r-1} \qquad (r = 1, 2, \cdots),$$

whence

$$U_r = \prod_{j=1}^{r} \left(\frac{\varphi_j}{1 - \varphi_j} \right) \qquad (r = 1, 2, \cdots) \tag{53}$$

and $U_0 = 1$. The distribution $\{P_k\}$ is given by (42).

Example 4. Let $p_j = 1$ if $j < m$ and $p_j = 0$ if $j \geqq m$. Then

$$V_r = U_r - \binom{m}{r} U_m \qquad (r = 0, 1, \cdots, m)$$

and

$$V_r = U_r = 0 \qquad (r = m + 1, m + 2, \cdots).$$

By (44)

$$U_r = \frac{\varphi_r}{1-\varphi_r}\left[U_{r-1} - \binom{m}{r-1} U_m\right] \qquad (r = 1, 2, \cdots, m),$$

and the solution of this equation is

$$U_r = C_r \frac{\sum_{j=r}^{m} \binom{m}{j} \frac{1}{C_j}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_j}} \qquad (r = 0, 1, \cdots, m), \tag{54}$$

where C_r is defined by (3). $U_r = 0$ if $r > m$. Finally, $\{P_k\}$ is given by (42).

Example 5. Let $p_0 = 1$ and $p_j = p$ $(j = 1, 2, \cdots)$, $q_j = q$ $(j = 1, 2, \cdots)$, where $p + q = 1$. Then

$$V_r = pU_r \qquad (r = 1, 2, \cdots),$$

$$V_0 = pU_0 + qP_0 = 1 - q\,(U_1 - U_2 + U_3 - \cdots).$$

Putting V_r into (44) we get

$$U_r = \frac{p\varphi_r}{1-\varphi_r} U_{r-1} \qquad (r = 1, 2, \cdots)$$

and

$$U_r = \frac{\varphi_1}{1-\varphi_1}[1 - q(U_1 - U_2 + U_3 - \cdots)].$$

The solution of this system of linear equations is

$$U_r = \frac{p^r C_r}{1 - q\sum_{j=0}^{\infty}(-p)^j C_j} \qquad (r = 0, 1, 2, \cdots), \tag{55}$$

where C_r is defined by (3). Finally, $\{P_k\}$ is given by (42).

Example 6. Let $p_0 = 1$, $p_j = p$, and $q_j = q$ if $j = 1, 2, \cdots, m - 1$, where $p + q = 1$, and $p_j = 0$ if $j > m$. Then

$$V_0 = p + qP_0 - pP_m$$

$$= p + q[U_0 - U_1 + U_2 - \cdots + (-1)^m U_m] - pU_m,$$

$$V_r = pU_r - p\binom{m}{r} U_m \qquad (r = 1, 2, \cdots, m),$$

$$V_r = U_r = 0 \qquad (r = m + 1, m + 2, \cdots).$$

Now $U_0 = 1$ and, by (44),

$$U_r = \frac{p^r C_r \sum_{j=r}^{m} \binom{m}{j} \frac{1}{C_j p^j}}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_j p^j} - q \sum_{j=0}^{m} (-1)^j C_j p^j \sum_{i=j}^{m} \binom{m}{i} \frac{1}{C_i p^i}} \qquad (r = 1, 2, \cdots, m). \tag{56}$$

The distribution $\{P_k\}$ is given by (42).

Example 7. If, in particular, $F(x) = 1 - e^{-\lambda x}$ for $x \geqq 0$, then $\varphi(s) = \lambda/(\lambda + s)$ and $\varphi_r = \lambda/(\lambda + r\mu)(r = 0, 1, 2, \cdots)$. In this case by (24) we have

$$r\mu U_r = \lambda V_{r-1} \qquad (r = 1, 2, \cdots),$$

whence

$$\mu U'(z) = \lambda V(z).$$

Forming the coefficient of z^{k-1} we obtain that

$$\mu k P_k = \lambda p_{k-1} P_{k-1} \qquad (k = 1, 2, \cdots), \tag{57}$$

whence

$$P_k = P_0 \frac{\left(\frac{\lambda}{\mu}\right)^k}{k!} p_0 p_1 \cdots p_k \qquad (k = 0, 1, 2, \cdots),$$

and P_0 is determined by the requirement that

$$\sum_{k=0}^{\infty} P_k = 1.$$

VI. THE TRANSIENT BEHAVIOR OF $\{\xi(t)\}$

In this section we suppose that $\xi(0) = i$ always. Denote by $M_j(t)$ the expectation of the number of calls occurring in the time interval $(0,t]$ which find exactly j lines busy. Let

$$\mu_j(s) = \int_0^\infty e^{-st} \, dM_j(t), \tag{58}$$

which is convergent if $\Re(s) > 0$. Now we shall prove the following

Lemma 1. Define

$$\Phi_r(s) = \sum_{j=r}^{\infty} \binom{j}{r} \mu_j(s) \qquad (r = 0, 1, 2, \cdots) \tag{59}$$

and

$$\Psi_r(s) = \sum_{j=r}^{\infty} \binom{j}{r} p_j \mu_j(s) \qquad (r = 0, 1, 2, \cdots), \tag{60}$$

which are convergent if $\Re(s) > 0$. *Then*

$$\Phi_0(s) = \frac{\varphi(s)}{1 - \varphi(s)} \tag{61}$$

and if $\xi(0) = i$ *then*

$$\Phi_r(s) = \frac{\varphi(s + r\mu)}{1 - \varphi(s + r\mu)} \left[\binom{i}{r} + \Psi_{r-1}(s) \right]. \tag{62}$$

Proof. Since evidently

$$M_j(t) = \sum_{n=1}^{\infty} \mathbf{P}\{\tau_n \leqq t, \quad \xi_n = j\}, \tag{63}$$

we have

$$\Phi_r(s) = \sum_{j=r}^{\infty} \binom{j}{r} \mu_j(s) = \sum_{n=1}^{\infty} \mathbf{E}\left\{ e^{-s\tau_n} \binom{\xi_n}{r} \right\} \tag{64}$$

and similarly

$$\Psi_r(s) = \sum_{j=r}^{\infty} \binom{j}{r} p_j \mu_j(s) = \sum_{n=1}^{\infty} \mathbf{E}\left\{ e^{-s\tau_n} \binom{\xi_n}{r} p_{\xi_n} \right\}. \tag{65}$$

Now we shall prove that

$$\mathbf{E}\left\{ e^{-s\tau_{n+1}} \binom{\xi_{n+1}}{r} \mid \xi_n = j, \quad \theta_n = x, \quad \tau_n = y \right\} = \left[p_j \binom{j+1}{r} + q_j \binom{j}{r} \right] e^{-r\mu x} e^{-s(x+y)}.$$

This follows from the fact that under the given condition ξ_{n+1} has a Bernoulli distribution either with parameters $j + 1$ and $e^{-\mu x}$ when the nth call gives rise to a connection, or with parameters j and $e^{-\mu x}$ when the nth call does not. Unconditionally we get

$$\mathbf{E}\left\{ e^{-s\tau_{n+1}} \binom{\xi_{n+1}}{r} \right\} = \varphi(s + r\mu) \left[\mathbf{E}\left\{ e^{-s\tau_n} \binom{\xi_n}{r} \right\} + \mathbf{E}\left\{ e^{-s\tau_n} \binom{\xi_n}{r-1} p_{\xi_n} \right\} \right]. \tag{66}$$

If $\xi(0) = i$ then

$$\mathbf{E}\left\{e^{-s\tau_1}\binom{\xi_1}{r}\right\} = \binom{i}{r}\varphi(s + r\mu). \tag{67}$$

If we add (66) for $n = 1, 2, \cdots$ and (67) then we get

$$\Phi_r(s) = \varphi(s + r\mu)\left[\binom{i}{r} + \Phi_r(s) + \Psi_{r-1}(s)\right] \qquad (r = 0, 1, 2, \cdots), \tag{68}$$

where $\Psi_{-1}(s) \equiv 0$. Thus we get (61) and (62). In many cases use of Lemma 1 determines $\Phi_r(s)$ $(r = 0, 1, 2, \cdots)$ explicitly.

Remark 4. From (59) we obtain by inversion

$$\mu_k(s) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \Phi_r(s). \tag{69}$$

The functions $\mu_k(s)(k = 0, 1, 2, \cdots)$ can be determined also by the following system of linear equations:

$$\sum_{k=r}^{\infty} \binom{k}{r} \mu_k(s) = \frac{\varphi(s + r\mu)}{1 - \varphi(s + r\mu)} \left[\binom{i}{r} + \sum_{k=r-1}^{\infty} \binom{k}{r - 1} p_k \mu_k(s)\right], \tag{70}$$

which we get if we put (59) and (60) into (62).

If we know $\Phi_r(s)$ $(r = 0, 1, 2, \cdots)$ then $P_k(t)$ can be determined by the following

Theorem 4. The Laplace transform $\Pi_k(s)$ *is given by*

$$\Pi_k(s) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \beta_r(s), \tag{71}$$

where

$$\beta_r(s) = \frac{[1 - \varphi(s + r\mu)]\Phi_r(s)}{\varphi(s + r\mu)(s + r\mu)} \qquad (r = 0, 1, 2, \cdots). \tag{72}$$

Proof. Let the rth binomial moment of $\{P_k(t)\}$ be defined by

$$B_r(t) = \mathbf{E}\left\{\binom{\xi(t)}{r}\right\} = \sum_{k=r}^{\infty} \binom{k}{r} P_k(t) \qquad (r = 0, 1, 2, \cdots). \tag{73}$$

By using the results of Ref. 10 we can see that $B_r(t) \leqq C^r/r!$ for every $t \geqq 0$, where C is a constant. Thus the probability distribution $\{P_k(t)\}$ is uniquely determined by its binomial moments. From (73) we get by inversion

$$P_k(t) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} B_r(t). \tag{74}$$

If

$$\beta_r(s) = \int_0^\infty e^{-st} B_r(t)\, dt$$

and we form the Laplace transform of (74), we get (71). Now let us determine $\beta_r(s)$ $(r = 0, 1, 2, \cdots)$.

If $\xi(0) = i$, then

$$\begin{aligned} B_r(t) &= \binom{i}{r} e^{-r\mu t}[1 - F(t)] \\ &+ \sum_{j=0}^{\infty} \left[p_j \binom{j+1}{r} + q_j \binom{j}{r} \right] \int_0^t e^{-r\mu(t-u)}[1 - F(t-u)]\, dM_j(u), \end{aligned} \tag{75}$$

where $M_j(t)$ is defined by (63). For, if there is no call in the time interval $(0,t]$ then $\xi(t)$ has a Bernoulli distribution with parameters i and $e^{-\mu t}$. If the last call in the time interval $(0,t]$ occurs at the instant u and in that instant the number of busy lines is j, then $\xi(t)$ has a Bernoulli distribution, either with parameters $j + 1$ and $e^{-\mu(t-\mu)}$ when this call gives rise to a connection or with parameters j and $e^{-\mu(t-\mu)}$ when this call does not. If we also take into consideration that the last call occurring in the time interval $(0,t]$ may be the 1st, 2nd, $\cdots$, nth, $\cdots$ one, then we get (75). Forming the Laplace transform of (74) we get

$$\begin{aligned} \beta_r(s) &= \frac{1 - \varphi(s + r\mu)}{s + r\mu} \left\{ \binom{i}{r} \right. \\ &\left. + \sum_{j=0}^{\infty} \left[p_j \binom{j+1}{r} + q_j \binom{j}{r} \right] \mu_j(s) \right\} \end{aligned} \tag{76}$$

where $\mu_j(s)$ is defined by (58). By using the notations (59) and (60) we can write also that

$$\beta_r(s) = \frac{1 - \varphi(s + r\mu)}{s + r\mu} \left\{ \binom{i}{r} + \Phi_r(s) + \Psi_{r-1}(s) \right\}. \tag{77}$$

Taking into consideration the relation (68) we obtain finally

$$\beta_r(s) = \frac{1 - \varphi(s + r\mu)}{(s + r\mu)} \frac{\Phi_r(s)}{\varphi(s + r\mu)}, \tag{78}$$

which was to be proved.

Example 8. Define

$$C_r(s) = \prod_{i=0}^{r} \left(\frac{\varphi(s + i\mu)}{1 - \varphi(s + i\mu)} \right) \qquad (r = 0, 1, 2, \cdots) \tag{79}$$

and

$$C_{-1}(s) \equiv 1.$$

If $p_j = 1$ $(j = 0, 1, 2, \cdots)$ and $\xi(0) = i$, then $\Psi_r(s) = \Phi_r(s)$ $(r = 0, 1, 2, \cdots)$ and, by (62), we get

$$\Phi_r(s) = \frac{\varphi(s + r\mu)}{[1 - \varphi(s + r\mu)]}\left[\binom{i}{r} + \Phi_{r-1}(s)\right] \qquad (r = 0, 1, \cdots), \quad (80)$$

where $\Phi_{-1}(s) = 0$. The solution of this recurrence formula is

$$\Phi_r(s) = C_r(s) \sum_{j=0}^{r} \binom{i}{j} \frac{1}{C_{j-1}(s)}, \qquad (81)$$

where $C_r(s)$ is defined by (79).

Example 9. If $p_j = 1$ when $j < m$ and $p_j = 0$ when $j \geqq m$ and $\xi(0) = i \leqq m$, then

$$\Psi_r(s) = \Phi_r(s) - \binom{m}{r} \Phi_m(s) \qquad (r = 0, 1, \cdots, m)$$

and

$$\Psi_r(s) = \Phi_r(s) = 0 \qquad (r = m + 1, m + 2, \cdots).$$

By (62)

$$\Phi_r(s) = \frac{\varphi(s + r\mu)}{[1 - \varphi(s + r\mu)]}\left[\binom{i}{r} + \Phi_{r-1}(s) - \binom{m}{r - 1}\Phi_m(s)\right] \quad (82)$$

for $r = 1, 2, \cdots, m$. The solution of this equation is

$$\Phi_r(s) = \frac{C_r(s)}{\sum_{j=0}^{m} \binom{m}{j} \frac{1}{C_j(s)}} \left\{\left[\sum_{j=r}^{m} \binom{m}{j} \frac{1}{C_j(s)}\right]\left[\sum_{j=0}^{r} \binom{i}{j} \frac{1}{C_{j-1}(s)}\right] - \left[\sum_{j=0}^{r-1} \binom{m}{j} \frac{1}{C_j(s)}\right]\left[\sum_{j=r+1}^{m} \binom{i}{j} \frac{1}{C_{j-1}(s)}\right]\right\} \quad (83)$$

where $C_j(s)$ is defined by (79).

VII. THE LIMITING DISTRIBUTION $\{P_k^*\}$

Now we shall prove

Theorem 5. If $F(x)$ is not a lattice distribution and its mean α is finite, then the limiting distribution

$$\lim_{t \to \infty} P_k(t) = P_k^* \qquad (k = 0, 1, \cdots)$$

exists and is independent of the initial distribution. We have

$$P_{k+1}^* = \frac{p_k P_k}{(k+1)\alpha\mu} \qquad (k = 0, 1, 2, \cdots) \tag{84}$$

and

$$P_0^* = 1 - \frac{1}{\alpha\mu} \sum_{k=0}^{\infty} \frac{p_k P_k}{k+1}, \tag{85}$$

where $\{P_k\}$ *is defined by* (38).

Proof. By the theory of Markov chains we can conclude that

$$\lim_{t\to\infty} \frac{M_k(t)}{t} = \frac{P_k}{\alpha}. \tag{86}$$

Furthermore, it is clear that the difference of the number of transitions $E_k \to E_{k+1}$ and $E_{k+1} \to E_k$ occurring in the time interval $(0,t]$ is at most 1. Accordingly, if we denote by $N_k(t)$ the expectation of the number of transitions $E_{k+1} \to E_k$ occurring in the time interval $(0,t]$, then

$$| p_k M_k(t) - N_k(t) | \leqq 1 \tag{87}$$

for all $t \geqq 0$. Further,

$$N_k(t) = (k+1)\mu \int_0^t P_{k+1}(u)\, du, \tag{88}$$

for, if we consider the process $\{\xi(t)\}$ only at those instants when there is state E_{k+1}, then the transitions $E_{k+1} \to E_k$ form a Poisson process of density $(k+1)\mu$. Thus, by (86), (87), and (88),

$$\lim_{t\to\infty} \frac{(k+1)\mu}{t} \int_0^t P_{k+1}(u)\, du = \lim_{t\to\infty} \frac{N_k(t)}{t} = \lim_{t\to\infty} \frac{p_k M_k(t)}{t} = \frac{p_k P_k}{\alpha};$$

that is,

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t P_{k+1}(u)\, du = \frac{p_k P_k}{(k+1)\alpha\mu} \qquad (k = 0, 1, 2, \cdots). \tag{89}$$

If we prove that the limiting distribution

$$\lim_{t\to\infty} P_k(t) = P_k^* \qquad (k = 0, 1, 2, \cdots)$$

exists, then it follows by (89) that

$$P_{k+1}^* = \frac{p_k P_k}{(k+1)\alpha\mu} \qquad (k = 0, 1, 2, \cdots), \tag{90}$$

and so

$$P_0{}^* = 1 - \sum_{k=0}^{\infty} P_{k+1}{}^* = 1 - \frac{1}{\alpha\mu} \sum_{k=0}^{\infty} \frac{p_k P_k}{(k+1)}. \qquad (91)$$

To prove the existence of the limiting distribution we need the following auxiliary theorem: If $F(x)$ is not a lattice distribution, then

$$\lim_{t\to\infty} \frac{p_k M_k(t+h) - p_k M_k(t)}{h} \qquad (92)$$

exists for every $h > 0$ and is independent of h and the initial state. This is a consequence of a theorem of Blackwell.[14] For the time differences between successive transitions $E_k \to E_{k+1}$ are identically distributed, independent, positive random variables, and, if $F(x)$ is not a lattice distribution, then these random variables have no lattice distribution either. If (92) exists, then it follows that

$$\lim_{t\to\infty} \frac{M_k(t+h) - M_k(t)}{h} = \lim_{t\to\infty} \frac{M_k(t)}{t} = \frac{P_k}{\alpha} \qquad (k = 0, 1, 2, \cdots). \quad (93)$$

Now, by the theorem of total probability, we can write

$$P_k(t) = \binom{i}{k} e^{-k\mu x} (1 - e^{-\mu x})^{i-k} [1 - F(t)] + \sum_{j=k-1}^{\infty} \int_0^t \pi_{jk}(t-u)[1 - F(t-u)]\, dM_j(u), \qquad (94)$$

where $\pi_{jk}(t)$ is defined by (18) and it is supposed that $\xi(0) = i$. The event $\xi(t) = k$ may occur in several mutually exclusive ways: there is no call in the time interval $(0,t]$ and, with the exception of k, all the i connections terminate by t; or the last call in the time interval $(0,t]$ is the nth $(n = 1, 2, \cdots)$ one and it finds state E_j $(j = k - 1, k, \cdots)$. If $\tau_n = u$ $(0 < u \leqq t)$, then during the time interval $(u,t]$ no new call arrives [the probability of which is $1 - F(t-u)$] and with the exception of k connections every connection terminates by t [the probability of which is $\pi_{jk}(t-u)$].

Applying Blackwell's theorem to (94) and using $\alpha < \infty$, it follows that

$$\lim_{t\to\infty} P_k(t) = P_k{}^* \qquad (k = 0, 1, \cdots)$$

exists and

$$P_k{}^* = \sum_{j=k-1}^{\infty} p_{jk}{}^* P_j, \qquad (95)$$

where

$$p_{jk}{}^* = \frac{1}{\alpha} \int_0^\infty \pi_{jk}(x)[1 - F(x)]\,dx. \tag{96}$$

It is easy to see from (95) that $\{P_k{}^*\}$ is a probability distribution.

VIII. THE DETERMINATION OF $\gamma_k(s)$

Define

$$\gamma_k(s) = \int_0^\infty e^{-sx}\,dG_k(x) = \frac{D_k(s)}{D_{k+1}(s)}, \tag{97}$$

where $D_0(s) = 1$. We are going to determine $D_r(s)(r = 1, 2, \cdots)$.

Write $D_r(s)$ in the following form:

$$D_r(s) = \sum_{j=0}^{r} \binom{r}{j} \Delta^j D_0(s), \tag{98}$$

where $\Delta^j D_0(s)$ is the jth difference of $D_r(s)$ at $r = 0$; that is,

$$\Delta^j D_0(s) = \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} D_i(s). \tag{99}$$

Then $D_r(s)$ is uniquely determined by its differences.

Now we shall prove

Theorem 6. Starting from $D_0(s) = \Delta^0 D_0(s) = 1$, *the functions* $D_r(s)$ $(r = 0, 1, 2, \cdots)$ *and the differences* $\Delta^j D_0(s)$ $(j = 0, 1, 2, \cdots)$ *can be obtained successively by the recurrence formulas*

$$\sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} D_j(s) = \varphi(s + j\mu) \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} [p_j D_{j+1}(s) + q_j D_j(s)] \tag{100}$$

and

$$\Delta^j D_0(s) = \frac{\varphi(s + j\mu)}{1 - \varphi(s + j\mu)} \sum_{i=0}^{j} \binom{j}{i} (\Delta^{j-i} p_i) \Delta^{i+1} D_0(s) \tag{101}$$

respectively. Here

$$\Delta^{j-i} p_i = \sum_{\nu=0}^{j-i} (-1)^\nu \binom{j-i}{\nu} p_{j-\nu}. \tag{102}$$

Proof. By the theorem of total probability we can write for $r = 0, 1, 2, \cdots$ that

$$G_r(x) = \int_0^x \sum_{j=0}^{r} \binom{r}{j} e^{-j\mu y} (1 - e^{-\mu y})^{r-j} \cdot [p_j G_{j+1}(x - y) * \cdots * G_r(x - y) + q_j G_j(x - y) * \cdots * G_r(x - y)] \, dF(y), \tag{103}$$

where the empty convolution product is equal to 1. Let us consider the instant of a transition $E_{r-1} \to E_r$ and measure time from this instant. Then $G_r(x)$ is the probability that the next transition $E_r \to E_{r+1}$ occurs in the time interval $(0,x]$. This event may occur in the following mutually exclusive ways: the first call in the time interval $(0,x]$ arrives at the instant $y(0 < y \leqq x)$, it finds state E_j $(j = 0, 1, \cdots, r)$, the probability of which is

$$\binom{r}{j} e^{-j\mu y}(1 - e^{-\mu y})^{r-j},$$

and, in the time interval $(y,x]$, a transition $E_r \to E_{r+1}$ occurs, the probability of which is

$$p_j G_{j+1}(x - y) * \cdots * G_r(x - y) + q_j G_j(x - y) * \cdots * G_r(x - y).$$

Introduce the notation

$$q_{r,j}(s) = \binom{r}{j} \int_0^\infty e^{-sx} e^{-j\mu x}(1 - e^{-\mu x})^{r-j} \, dF(x) \tag{104}$$

and form the Laplace-Stieltjes transform of (103); then

$$\gamma_r(s) = \sum_{j=0}^{r} q_{rj}(s) \left[p_j \prod_{i=j+1}^{r} \gamma_i(s) + q_j \prod_{i=j}^{r} \gamma_i(s) \right] \qquad (r = 0, 1, 2, \cdots),$$

where the empty product is 1. Now using (97) we find

$$D_r(s) = \sum_{j=0}^{r} q_{rj}(s)[p_j D_{j+1}(s) + q_j D_j(s)] \qquad (r = 0, 1, 2, \cdots). \tag{105}$$

This is already a recurrence formula for the determination of $D_r(s)$ $(r = 0, 1, 2, \cdots)$, but the coefficients can be simplified further.

If we form

$$\Delta^j D_0(s) = \sum_{l=0}^{j} (-1)^{j-l} \binom{j}{l} D_l(s),$$

where $D_l(s)$ is replaced by (105), and take into consideration that

$$\sum_{l=i}^{j} (-1)^{j-l} \binom{j}{l} q_{l,i}(s) = (-1)^{j-i} \binom{j}{i} \varphi(s + j\mu), \tag{106}$$

then we obtain

$$\Delta^j D_0(s) = \varphi(s + j\mu) \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} [p_i D_{i+1}(s) + q_i D_i(s)]. \tag{107}$$

Now, comparing (99) and (107), we obtain (100).

On the other hand, by (107) it follows that

$$\Delta^j D_0(s) = \varphi(s + j\mu)\Delta^j D_0(s) + \varphi(s + j\mu) \sum_{i=0}^{j} (-1)^{j-i} \binom{j}{i} p_i \Delta D_i(s),$$

whence

$$\Delta^j D_0(s) = \frac{\varphi(s + j\mu)}{1 - \varphi(s + j\mu)} \Delta^j [p_0 \Delta D_0(s)] \tag{108}$$

and here

$$\Delta^j [p_0 \Delta D_0(s)] = \sum_{i=0}^{j} \binom{j}{i} (\Delta^{j-i} p_i) \Delta^{i+1} D_0(s), \tag{109}$$

where

$$\Delta^{j-i} p_i = \sum_{\nu=0}^{j-i} (-1)^{\nu} \binom{j-i}{\nu} p_{j-\nu}. \tag{110}$$

This proves (101).

Example 10. In the infinite line case, i.e., when $p_j = 1$ ($j = 0, 1, 2, \cdots$), (101) has the following simple form:

$$\Delta^{j+1} D_0(s) = \frac{1 - \varphi(s + j\mu)}{\varphi(s + j\mu)} \Delta^j D_0(s) \qquad (j = 0, 1, 2, \cdots), \tag{111}$$

whence

$$\Delta^j D_0(s) = \prod_{i=0}^{j-1} \left[\frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)} \right] \tag{112}$$

and

$$D_r(s) = \sum_{j=0}^{r} \binom{r}{j} \prod_{i=0}^{j-1} \left(\frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)} \right). \tag{113}$$

Example 11. If $p_0 = 1$ and $p_j = p$ $(j = 1, 2, \cdots)$, then (101) reduces to the following difference equation:

$$\Delta^{j+1}D_0(s) - \frac{1 - \varphi(s + j\mu)}{\varphi(s + j\mu)} \Delta^j D_0(s) + (-1)^j \frac{q[1 - \varphi(s)]}{p\varphi(s)} = 0 \qquad (j = 0, 1. 2, \cdots). \tag{114}$$

A simple calculation shows that the solution of (114) is

$$\Delta^j D_0(s) = \left\{ p \prod_{i=0}^{j-1} \left[\frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)} \right] - \frac{q[1 - \varphi(s)]}{p\varphi(s)} \sum_{\nu=1}^{j-1} (-1)^\nu \prod_{i=\nu+1}^{j-1} \left[\frac{1 - \varphi(s + i\mu)}{p\varphi(s + i\mu)} \right] \right\}, \tag{115}$$

and finally,

$$D_r(s) = \sum_{j=0}^{r} \binom{r}{j} \Delta^j D_0(s). \tag{116}$$

Theorem 7. Suppose that $\xi(0) = 0$ *and under this condition denote by* $M_k(t)$ *the expectation of the number of calls arriving in the time interval* $(0,t]$ *which find exactly* k *lines busy. Let*

$$\mu_k(s) = \int_0^\infty e^{-st}\, dM_k(t). \tag{117}$$

Then

$$\rho_k(s) = 1 - \frac{1}{p_k D_{k+1}(s) \mu_k(s)}, \tag{118}$$

where $D_{k+1}(s)$ *is given by Theorem 6 and* $\mu_k(s)$ *is given by*

$$\mu_k(s) = \sum_{r=k}^{\infty} (-1)^{r-k} \binom{r}{k} \Phi_r(s), \tag{119}$$

where $\Phi_r(s)$ *can be obtained by Lemma 1.*

Proof. The expected number of transitions $E_k \to E_{k+1}$ occurring in the time interval $(0,t]$ is evidently $p_k M_k(t)$. The time differences between consecutive transitions $E_k \to E_{k+1}$ are identically distributed, independent random variables with distribution function $R_k(x)$. By using renewal theory we can write that

$$p_k M_k(t) = G_0(t) * G_1(t) * \cdots * G_k(t) * [I(t) + R_k(t) + R_k(t) * R_k(t) + \cdots], \tag{120}$$

where $I(t) = 1$ if $t \geqq 0$ and $I(t) = 0$ if $t < 0$. Forming the Laplace-

Stieltjes transform of (121), we obtain

$$p_k\mu_k(s) = \frac{\gamma_0(s)\gamma_1(s)\cdots\gamma_k(s)}{1 - \rho_k(s)} = \frac{1}{D_{k+1}(s)[1 - \rho_k(s)]}, \qquad (121)$$

whence (118) follows.

Since we know the distribution functions $G_k(x)$ and $R_k(x)$ ($k = 0, 1, 2, \cdots$), the distribution of the number of transitions $E_k \to E_{k+1}$ occurring in the time interval $(0,t]$ can be obtained easily.

IX. THE OVERFLOW TRAFFIC

Suppose that $p_j = 1$ ($j = 0, 1, 2, \cdots$) and that the telephone lines are numbered by 1, 2, 3, $\cdots$. Further suppose that an incoming call realizes a connection through the idle line that has the lowest serial number. Consider the group $(1, 2, \cdots, m)$. Denote by $\pi_m^{(n)}$ the probability that the nth call finds every line busy in the group $(1, 2, \cdots, m)$. The distances between successive calls which find every line busy in the group $(1, 2, \cdots, m)$ are evidently identically distributed, independent random variables with distribution function, say, $G_m(x)$.

Palm[2] proved that

$$\pi_m = \lim_{n\to\infty} \pi_m^{(n)} = \frac{1}{\sum_{r=0}^{m} \binom{m}{r} \frac{1}{C_r}}, \qquad (122)$$

where C_r is defined by (3). This is in agreement with (6). In this case it is easy to see that $\pi_m^{(n)} = P_m^{(n)}$, where the distribution $\{P_k^{(n)}\}$ is defined in Example 2 of Section IV.

In Refs. 10 and 11 it is shown that

$$\int_0^\infty e^{-sx}\, dG_m(x) = \frac{\sum_{r=0}^{m} \binom{m}{r} \prod_{i=0}^{r-1} \left[\frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)}\right]}{\sum_{r=0}^{m+1} \binom{m+1}{r} \prod_{i=0}^{r-1} \left[\frac{1 - \varphi(s + i\mu)}{\varphi(s + i\mu)}\right]}, \qquad (123)$$

where the empty product means 1. It is easy to see that $G_m(x)$ agrees with the corresponding $G_m(x)$ defined in Section VIII when $p_j = 1$ ($j = 0, 1, 2, \cdots$). Thus (123) can be obtained from (97) and (113).

Remark 5. Denote by Γ_m the expectation of the random variable which is the difference of call numbers of successive calls, both of which find all lines busy in the group $(1, 2, \cdots, m)$. Knowing Γ_m, we can write that

$$\pi_m = \lim_{n\to\infty} \pi_m^{(n)} = \frac{1}{\Gamma_m} \qquad (124)$$

and

$$\int_0^\infty x\, dG_m(x) = \alpha\Gamma_m. \tag{125}$$

In Ref. 6 it is shown that Γ_r $(r = 1, 2, \cdots)$ satisfies the following recurrence formula:

$$\begin{aligned}\Gamma_r = q_{r,0}(\Gamma_1 + \Gamma_2 + \cdots + \Gamma_r) + q_{r,1}(\Gamma_2 + \Gamma_3 + \cdots + \Gamma_r) \\ + \cdots + q_{r,r-2}(\Gamma_{r-1} + \Gamma_r) + q_{r,r-1}\Gamma_r + 1,\end{aligned} \tag{126}$$

where

$$q_{r,j} = \binom{r}{j} \int_0^\infty e^{-j\mu x}(1 - e^{-\mu x})^{r-j}\, dF(x) \qquad (j = 0, 1, \cdots, r). \tag{127}$$

The solution of (126) is given by

$$\Gamma_r = \sum_{j=0}^{r} \binom{r}{j} \prod_{i=1}^{j} \left(\frac{1 - \varphi_i}{\varphi_i}\right) \qquad (r = 1, 2, \cdots). \tag{128}$$

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