Transverse Electron Beam Waves in Varying Magnetic Fields

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The properties of electron cyclotron and synchronous waves in varying magnetic fields are discussed. Magnetic field variations in space and time are considered. The problem is treated by establishing the wave excitation from knowledge of the macroscopic beam motion. It is shown that the cyclotron wave is coupled to the synchronous wave and that both waves are always amplified in a changing field. Unless the charge density is an appreciable fraction of the full Brillouin value, however, the individual electron orbits will be amplified along with the waves causing beam expansion.

The phase velocity of the waves is shown to be approximately independent of space charge. In the case of a spatially varying field, one of the waves must be fast, carrying positive kinetic power, and the other slow, carrying negative kinetic power. The total kinetic power carried by the two waves is conserved. When the magnetic field varies in time, the kinetic power of the two waves is not conserved but the Manley-Rowe relation is satisfied. When the field varies at a rate greater than the signal frequency, both modes may carry positive kinetic power.

I. INTRODUCTION

The electron cyclotron wave has received a great deal of attention in the past year, since it forms the basis of a successful beam-type traveling-wave parametric amplifier. As was first demonstrated by Adler et al.,[1,2,3] the fast cyclotron wave is actively coupled to a fast cyclotron wave idler by the action of a high-frequency transverse electric quadrupole field. It was shown by Gordon[4,5] that space periodic quadrupole fields actively couple a fast and slow cyclotron wave.

The purpose of this paper is to demonstrate that cyclotron waves will also be amplified in varying magnetic fields. It will be shown that the idler mode associated with the cyclotron wave is a synchronous wave associated with the spatial configuration of the beam. Analogous to the quadrupole case, pumping in a time-varying magnetic field with the pump frequency greater than the signal frequency leads to active coupling of two fast modes. When the pump frequency is lower than the signal frequency one mode must be fast and the other slow. Pumping in a spatially varying magnetic field will be seen to correspond to the limiting case of zero pump frequency.

In Section II the cyclotron and synchronous waves will be described and some simple points of view established. Then the results of an earlier analysis,[6] in which a general solution for charged particle orbits in arbitrarily varying magnetic fields is given, will be utilized to determine the properties of coupled cyclotron and synchronous waves. The analysis is done in such a way as to include the possibility of pumping with time- and space-varying fields and succeeding sections will be devoted to each. Space-charge effects will be shown to be significant in preventing beam expansion.

II. CYCLOTRON AND SYNCHRONOUS WAVES

A wave description of the motion of an electron beam is appropriate only to the extent that it can describe the energy exchange between the beam and the electromagnetic field. In the case of transverse beam waves, the beam may be thought of as made up of thin discs with no longitudinal coupling. The appropriate wave variable is the center-of-mass of the disc. The validity of this model is justified by considering the significant coupling term between the electromagnetic field and the beam, namely the time average value of $\int \mathbf{J} \cdot \mathbf{E} d^3 r$, in which $\mathbf{J}$ is the electron conduction current and $\mathbf{E}$ is the RF electric field. In a given unit-volume cross section of the beam, the significant quantities which contribute to the time average value are

$$\mathbf{J} \cdot \mathbf{E} = -e \sum_{j=1}^{N} (\dot{\mathbf{r}}_j \cdot \mathbf{E}_r + v_j \mathbf{r}_j \cdot \mathrm{grad}_r E_z),$$

in which N is the electron density, v_j is the z-directed drift velocity and $\mathbf{r}_j$ is the transverse coordinate of the jth electron. It is assumed that the z-axis of the coordinate system is at a maximum of the transverse field and a zero of the longitudinal field. For a thin beam placed along the z-axis, $\mathbf{E}_r$ and $\mathrm{grad}_r E_z$ are constant over the cross section of the beam to within terms that are of order $(\beta r_j)^2 \ll 1$, in which β is the wave propagation constant. To the same degree of approximation, variations in the drift velocity resulting from longitudinal forces also may be neglected and v_j may be assumed to be equal to the unperturbed drift velocity.

Hence, the quantities

$$\mathbf{R} = N^{-1} \sum_{j=1}^{N} \mathbf{r}_j \qquad \text{and} \qquad \dot{\mathbf{R}} = N^{-1} \sum_{j=1}^{N} \dot{\mathbf{r}}_j ,$$

i.e., the position and velocity of the center-of-mass of the disc, are sufficient to describe the interaction. Since the external transverse forces on each electron in the cross section are identical and the internal forces between electrons cancel exactly, the motion of the disc is exactly the same as that of a single electron and is independent of the internal motions of the electrons within the disc or the beam density. Hence, we may conclude that the transverse waves are independent of the beam space charge.

In the uniform coupler fields the internal motion of the electrons with respect to the center-of-mass proceeds as if no external forces were applied. In pump fields the significant forces vary over the cross section of the beam. As a result the individual electron trajectories measured in the center-of-mass coordinate system may also be pumped. This can be avoided by making the beam sufficiently dense that the internal motions of the electrons within the beam are sufficiently different from the motion of the center-of-mass itself. This point will be discussed in detail later.

In the earlier paper,[6] it is shown that the position of a charged particle in a plane transverse to the applied magnetic field is conveniently described in terms of a complex vector quantity $\mathbf{r} = x + iy = \mathbf{r}_g + \mathbf{r}_b$, in which x and y are the rectangular coordinates of the charged particle in the transverse plane. The quantity $\mathbf{r}_g$ represents the instantaneous center of curvature of the charged particle orbit, called the guiding center, and $\mathbf{r}_b$, called the radius vector, denotes the instantaneous radius of curvature of the trajectory and the position of the particle with respect to the guiding center (see Fig. 1). In a uniform magnetic field,

$$\mathbf{r} = \mathbf{r}_{g0} + \mathbf{r}_{b0} e^{i\omega_c (t - t_0)}, \tag{1}$$

in which $\mathbf{r}_{g0}$ and $\mathbf{r}_{b0}$ are the appropriate values at $t = t_0$ and ω_c is the cyclotron frequency. The motion of the beam disc is precisely the same. The four independent quantities, x, $\dot{x}$, y and $\dot{y}$, of the beam disc center-of-mass make possible four transverse waves on the electron beam. These are the fast and slow cyclotron and synchronous waves. It will be seen that the radius vector and guiding center of the beam disc are the appropriate quantities to express the cyclotron and synchronous wave excitation of the beam.

The cyclotron wave is excited by a circularly polarized transverse

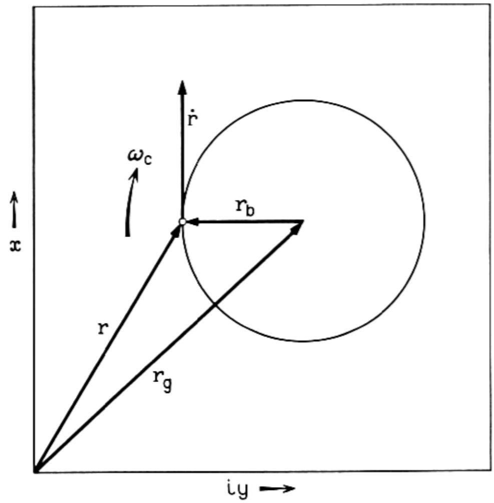

Fig. 1 — Particle coordinates for motion in a magnetic field directed along the z-axis.

electric field. The electric vector may rotate in the same or the opposite direction to the natural cyclotron motion of the disc. If the exciting electric field has a signal component at a frequency ω_s then we may say that ω_s is positive if the electric field rotates with the disc and is negative if the electric field rotates in the opposite direction (see Fig. 2). The initial phase of the field-excited rotation of the disc depends on the phase of the field at the time of entry. As a result, a cyclotron wave excitation at a signal frequency ω_s can be described in the following way: Each disc rotates about the axis with an angular velocity ω_c . An observer, standing at a fixed plane, will note that the *phase angle of rotation* of successive discs as they pass through the plane *will change at a rate* ω_s because of the spatial twist of the beam. If ω_s is positive, the excitation is a fast cyclotron wave. The slow wave is associated with a negative value of ω_s . If a cyclotron wave excitation exists, the phase angle of rotation of any disc which passes some plane, say $z = 0$, at a time $t = t_0$ can always be written as $\omega_s t_0$. The radius vector of the rotational motion can then be written

$$\mathbf{R}_{b0} = R_{b0} e^{i\omega_s t_0}, \tag{2}$$

in which R_{b0} is a real number representing the radius of the disc orbit. Thus the radius vector of any disc along the beam, with arbitrary t_0 , can be written for a uniform static field as

$$\mathbf{R}_b = \mathbf{R}_{b0} e^{i\omega_c (t-t_0)} = R_{b0} e^{i\omega_s (t-z/v_p)}, \tag{3}$$

in which we have used the relation, $t - t_0 = z/v$, v being the electron drift velocity. Clearly this is a wave motion for which $v_p = \omega_s v/(\omega_s - \omega_c)$ is the phase velocity. Note that when ω_s is positive, v_p is either greater than v or negative, and the wave is said to be "fast." When ω_s is negative, v_p is always less than v and the wave is said to be "slow."

In Fig. 2 the cyclotron wave is assumed to be in synchronism with a circularly polarized electromagnetic field of phase velocity u. The Doppler-shifted frequency in the frame of reference moving with the beam is $\omega_s(1 - v/u)$. Equating $v_p = u$ yields $\omega_c = \omega_s(1 - v/u)$; hence, synchronism implies that cyclotron resonance occurs at the Doppler-shifted frequency and the rotating disc remains in phase with the rotating uniform electric field, $E_\perp$.

The longitudinal forces on the rotating disc arise from the uniform transverse magnetic field, $H_\perp$, and longitudinal electric field, $E_\parallel$, which increases with distance from the axis. The forces excited by these fields are proportional to the rotational velocity and radius respectively. Since the electromagnetic fields in free space are related by Maxwell's equations, and the transverse velocity and radius are related by the cyclotron frequency, the changes in rotational and drift energy of each disc are uniquely related independent of the structure propagating the electromagnetic wave.

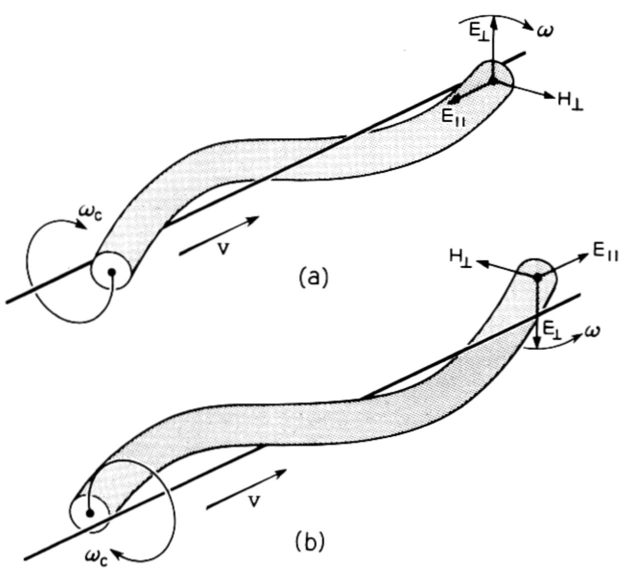

Fig. 2 — Beam motion for (a) fast and (b) slow cyclotron waves excited at a frequency ω.

Each disc experiences precisely the same forces and hence undergoes the same changes in rotational and drift energy. As a result, the total power carried by the beam changes. The power change is called the kinetic power of the beam wave and for the cyclotron wave it has two components:

$$
\begin{aligned}
P_{k\perp} &= \tfrac{1}{2}m(I_0/e)\omega_c^2 R_b^2, \\
P_{k\|} &= \tfrac{1}{2}m(I_0/e)\omega_c(\omega_s - \omega_c)R_b^2,
\end{aligned}
\tag{4}
$$

in which I_0 is the beam current. The total kinetic power $P_k = P_{k\perp} + P_{k\|}$ has the value[7,8,9]

$$P_k = \tfrac{1}{2}m(I_0/e)\omega_s\omega_c R_b^2. \tag{5}$$

The longitudinal kinetic power of the slow cyclotron wave is always negative ($\omega_s < 0$) and larger in magnitude than the transverse kinetic power. The longitudinal kinetic power of the fast cyclotron wave may be positive or negative depending upon whether the fast wave is a forward wave ($\omega_s > \omega_c$) or a backward wave ($\omega_s < \omega_c$). When it is negative its magnitude is always less than that of the transverse energy. When $\omega_s = \omega_c$, the longitudinal part of the kinetic power is zero since the phase velocity becomes infinite. Such a wave is excited by electromagnetic waves of infinite phase velocity which do not have a longitudinal component of electric field nor transverse magnetic field. In this limit $\omega_s\omega_c R_b^2$ becomes just the transverse velocity squared.

Now consider the average position or guiding center of the beam discs. Suppose that the guiding centers of the discs are correlated in such a way that we may write

$$
\begin{aligned}
\mathbf{R}_{g0} &= R_{g0}e^{-i\omega_s t_0} \\
&= R_{g0}e^{-i\omega_s(t-z/v)}
\end{aligned}
\tag{6}
$$

for the discs which pass the plane $z = 0$ at $t = t_0$. This represents a wave motion with phase velocity equal to v. It implies that the beam is shaped in the form of a helix which translates forward with a velocity v without rotation. To an observer standing at a fixed plane, the *phase angle of position* with respect to the axis of the helix of each successive disc, as it passes the plane, *changes at a rate* $-\omega_s$ even though the discs themselves may not be executing orbital motion. When ω_s is positive the discs form a right-handed or clockwise helix in space; when ω_s is negative the discs form a left-handed or counterclockwise helix, as shown in Fig. 3. The excitation associated with such a spatial configuration is known as a synchronous wave.

The synchronous wave is also excited by circularly polarized electromagnetic fields. Synchronism requires $v = u$, so that the Doppler-shifted frequency as observed by the drifting discs is zero and they remain in phase with the exciting field always. Hence the discs will drift radially outward with a velocity $E_\perp/B$, in which B is the DC magnetic flux density, in a direction depending upon the phase of the field $E_\perp$. When the field is removed, the discs stop drifting outward and there is no net change in the transverse energy. The change in drift energy comes from the longitudinal electric field. The kinetic power is given by[7,8,9]

$$P_k = \tfrac{1}{2}m(I_0/e)\omega_s\omega_c R_g^2. \qquad (7)$$

The synchronous wave which forms a left-handed helix carries negative kinetic power, while the wave which forms a right-handed helix carries positive kinetic power, as can be seen from Fig. 3. Consequently, a convention of positive frequencies referring to positive kinetic power waves and negative frequencies to negative kinetic power waves is pertinent to both cyclotron and synchronous waves. Henceforth, the terms "fast" and "slow" synchronous waves will be used in reference to their kinetic power, although the phase velocity of both waves is always synchronous with the drift velocity of the beam.

The kinetic power carried by the synchronous wave is proportional to

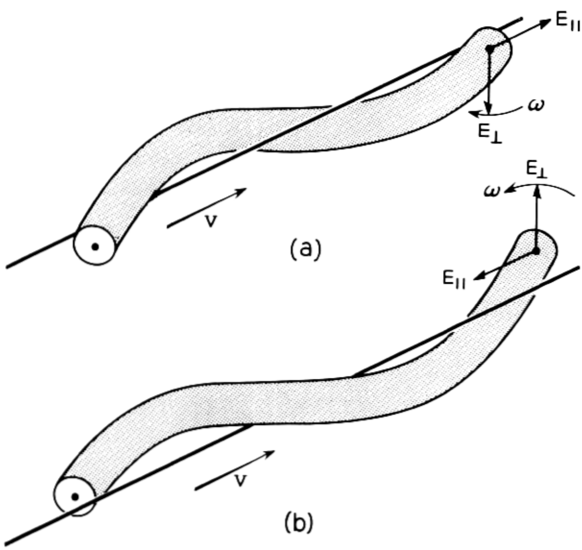

Fig. 3 — Beam motion for (b) fast and (a) slow synchronous waves excited at a frequency ω.

R_g^2. In this case the guiding center plays the role that the radius vector plays in the cyclotron wave. For either wave the important quantity is the transverse displacement associated with the disc as has been shown by Bobroff[7] [see Eq. (50)], and more recently by Siegman[8] and Klüver.[9]

III. AMPLIFICATION IN SPATIALLY VARYING MAGNETIC FIELDS

Suppose that the initial excitation of the beam at $z = z_0$ consists of a cyclotron wave of frequency ω_s with normalized amplitude $a_{c0}(\omega_s)$,

$$\begin{aligned} a_{c0}(\omega_s) &= | \omega_s \omega_{c0} |^{\frac{1}{2}} R_{b0} e^{i\omega_s t_0} \\ &= | \omega_s \omega_{c0} |^{\frac{1}{2}} \mathbf{R}_{b0} \end{aligned} \tag{8}$$

and a synchronous wave of frequency ω_m with amplitude $a_{s0}(\omega_m)$,

$$\begin{aligned} a_{s0}(\omega_m) &= | \omega_m \omega_{c0} |^{\frac{1}{2}} R_{g0} e^{-i\omega_s t_0} \\ &= | \omega_m \omega_{c0} |^{\frac{1}{2}} \mathbf{R}_{g0} . \end{aligned} \tag{9}$$

The phase of a_{c0} and a_{s0} at $z = z_0$ is included. The longitudinal magnetic field is assumed to be azimuthally symmetric but varying with both z and t. The cylotron frequency at $z = z_0$ has the value ω_{c0}. The purpose of the following analysis will be to find the amplitude of the cyclotron and synchronous waves at some plane $z > z_0$. It is not difficult to see that the cyclotron and synchronous waves are coupled by a changing magnetic field. In Ref. 6 it is shown that the radius vector and guiding center are coupled by field variations. In particular, if we make use of (29) of the Appendix of this paper, which describes the transverse trajectory of the disc for an azimuthally symmetric but otherwise arbitrarily varying magnetic field, and the definition of the instantaneous radius vector and guiding center, $\mathbf{R}_b = \dot{\mathbf{R}}/i\omega_c$ and $\mathbf{R}_g = \mathbf{R} - \mathbf{R}_b$. (see Fig. 1), it follows that

$$\begin{aligned} \dot{a}_c(\omega_s) - i\omega_c a_c(\omega_s) &= \frac{\dot{\omega}_c}{2\omega_c} \left| \frac{\omega_s}{\omega_m} \right|^{\frac{1}{2}} a_s(\omega_m), \\ \dot{a}_s(\omega_m) &= \frac{\dot{\omega}_c}{2\omega_c} \left| \frac{\omega_m}{\omega_s} \right|^{\frac{1}{2}} a_c(\omega_s), \end{aligned} \tag{10}$$

in which $a_c = | \omega_s \omega_c |^{\frac{1}{2}} \mathbf{R}_b$ and $a_s = | \omega_m \omega_c |^{\frac{1}{2}} \mathbf{R}_g$. The dot denotes the total time derivative. It is assumed that $\omega_c \neq 0$. These equations are in the coupled-mode form. The coupling disappears for a constant magnetic field, $\dot{\omega}_c = 0$.

The solutions of (10) become somewhat more transparent if use is made of the solutions of (29) for the quantities $\mathbf{R}_b$ and $\mathbf{R}_g$, which are given in Ref. 6 and are included in the Appendix:

$$\begin{aligned}\mathbf{R}_b &= (\omega_{c0}/\omega_c)^{\frac{1}{2}}\,[\mathbf{R}_{b0}F^*e^{i\varphi} - \mathbf{R}_{g0}XF],\\ \mathbf{R}_g &= (\omega_{c0}/\omega_c)^{\frac{1}{2}}[-\mathbf{R}_{b0}(XF)^*e^{i\varphi} + \mathbf{R}_{g0}F].\end{aligned} \tag{11}$$

For a static field the quantities X, F and φ are functions of z which are dependent only on the spatial variation of ω_c between z_0 and z and the drift velocity of the discs. Their values are determined from (31), (32) and (33). For a spatially varying field, (31) takes the form

$$\frac{dX}{dz} - \frac{i\omega_c(z)X}{v(z)} + \tfrac{1}{2}(1 - X^2)\,\frac{d\ \ln\ [\omega_c(z)/\omega_{c0}]}{dz} = 0 \tag{12}$$

with $X(z_0) = 0$. The quantities F and φ are obtained from the integrals

$$F = \exp\left(-\frac{1}{2}\int_{z_0}^{z}\left\{X(z')\,\frac{d\ln\,[\omega_c(z')/\omega_{c0}]}{dz'}\right\}dz'\right) \tag{13}$$

and

$$\varphi = \int_{z_0}^{z}\left[\frac{\omega_c(z')}{v(z')}\right]dz'. \tag{14}$$

In (12), (13) and (14), v can be considered to be independent of z for the small-signal case in the absence of static longitudinal electric fields. If we choose $\omega_m = -\omega_s$, then we may rewrite (11) as

$$\begin{aligned}|\,\omega_s\omega_c\,|^{\frac{1}{2}}\,\mathbf{R}_b &= |\,\omega_s\omega_{c0}\,|^{\frac{1}{2}}\,[F^*e^{i\varphi}R_{b0}e^{i\omega_s t_0} - XFR_{g0}e^{i\omega_s t_0}],\\ |\,\omega_s\omega_c\,|^{\frac{1}{2}}\,\mathbf{R}_g &= |\,\omega_s\omega_{c0}\,|^{\frac{1}{2}}\,[-(XF)^*e^{i\varphi}R_{b0}e^{i\omega_s t_0} + FR_{g0}e^{i\omega_s t_0}],\end{aligned} \tag{15}$$

which can finally be written in the form

$$\begin{vmatrix} a_c(\omega_s) \\ a_s(-\omega_s) \end{vmatrix} = \begin{vmatrix} F^*e^{i\varphi} & -XF \\ -(XF)^*e^{i\varphi} & F \end{vmatrix}\begin{vmatrix} a_{c0}(\omega_s) \\ a_{s0}(-\omega_s) \end{vmatrix}. \tag{16}$$

We see that, if $\omega_s > 0$, the fast cyclotron wave is coupled to the slow synchronous wave. The determinant of the transformation has the value $|\,F\,|^2(1 - |\,X\,|^2)e^{i\varphi}$, which, in view of (34), has the value $e^{i\varphi}$. Equation (35) shows that the total kinetic power, $|\,a_c\,|^2 - |\,a_s\,|^2 = |\,a_{c0}\,|^2 - |\,a_{s0}\,|^2$, of the two waves is conserved. The quantity $|\,F\,|^2$ represents the kinetic power gain of each wave and $|\,XF\,|^2$ represents the kinetic power gain of one of the waves as a result of an initial excitation of the other. Equation (34) expresses the conservation of kinetic power and the fact that $|\,F\,|^2 \geqq 1$ and $|\,X\,|^2 < 1$. Thus, one may conclude that variations of any kind will always amplify existing cyclotron or synchronous waves. If both waves exist coherently, one fast and the other slow, then field variations may deamplify the waves. Inspection of (12) shows that, when ω_c is a slow function of z, then X will remain very small and F must remain very close to unity. Hence, for slow changes in the magnetic field strength the kinetic power of each wave is conserved.

It follows that the noise temperature of the cyclotron and synchronous waves is invariant under slow changes in the magnetic field strength. Solutions for X and F for monotonically changing fields are given in Ref. 6. Under the appropriate conditions, X returns periodically to zero. Hence, the noise temperature of the waves may be preserved even for rapid changes in the field if the contour is designed properly.

Sinusoidal field variations of the form $\omega = \omega_{c0}[1 + \Delta \sin 2\pi z/L]$ lead to exponentially growing values of F if $2\pi v/L = \omega_{c0}$. Thus, large amplification is also possible.

IV. AMPLIFICATION IN TIME-VARYING FIELDS

In time-varying fields the quantities X and F will depend upon the initial or entry phase of the disc. An example will be given to demonstrate the phase dependence. It is assumed that the varying component of the magnetic field is a traveling wave, so that one may write

$$\omega_c = \omega_{c0}\{1 + \Delta \sin[\omega_p t - \beta_p(z - z_0)]\}, \tag{17}$$

in which ω_p is the pump frequency and β_p the pump propagation constant. The field near the axis of a cylindrical TE_{01} mode structure is appropriate for this case. It is assumed that $\Delta \ll 1$, so that one may rewrite (31) as

$$\dot{X} - i\omega_{c0}X + \tfrac{1}{2}\Delta(\omega_p - \beta_p v)\cos[\omega_p t - \beta_p(z - z_0)] = 0, \tag{18}$$

in which terms in Δ^2 have been neglected. This corresponds to very weak coupling. Equation (16) has the solution, using the boundary condition $X(t_0) = 0$ and $z - z_0 = v(t - t_0)$,

$$X(\tau) = -\tfrac{1}{2}\Delta(\omega_p - \beta_p v)e^{i\omega_{c0}\tau}\int_0^\tau \cos[\omega_p(\tau' + t_0) - \beta_p v\tau']e^{-i\omega_{c0}\tau'}\,d\tau', \tag{19}$$

in which $\tau = t - t_0$ is the elapsed time in the pump field. Performing the integration, one obtains

$$X(\tau) = -\tfrac{1}{4}\Delta(\omega_p - \beta_p v)\left[\frac{e^{i(\omega_p - \beta_p v - \omega_{c0})\tau} - 1}{i(\omega_p - \beta_p v - \omega_{c0})}\,e^{i\omega_p t_0} - \frac{e^{-i(\omega_p - \beta_p v + \omega_{c0})\tau} - 1}{i(\omega_p - \beta_p v + \omega_{c0})}\,e^{-i\omega_p t_0}\right]e^{i\omega_{c0}\tau}. \tag{20}$$

In the limiting case $\omega_p - \beta_p v = \omega_{c0}$, $\omega_{c0}\tau \gg 1$, one can approximate (20) by

$$X \cong -\tfrac{1}{4}\Delta\omega_{c0}\tau e^{i\omega_{c0}\tau} e^{i\omega_p t_0} . \tag{21}$$

It can be seen that X is a function of the time the electron enters the pump field and increases linearly with drift time. In this limit it can be shown that

$$F \cong e^{\frac{1}{2}(\Delta\omega_{c0}\tau/4)^2}, \tag{22}$$

which satisfies (34), and that $\varphi \cong \omega_{c0}\tau$,

In this case, choosing $\omega_m = \omega_p - \omega_s$, (11) takes the form

$$\begin{aligned} | \omega_s\omega_c |^{\frac{1}{2}}\mathbf{R}_b &= F^* e^{i\varphi} | \omega_s\omega_{c0} |^{\frac{1}{2}} R_{b0} e^{i\omega_s t_0} \\ &\quad - | \omega_s/\omega_m |^{\frac{1}{2}} XF, R_{g0} e^{-i\omega_m t_0} | \omega_m\omega_{c0} |^{\frac{1}{2}}, \\ | \omega_m\omega_c |^{\frac{1}{2}} \mathbf{R}_g &= - | \omega_m/\omega_s |^{\frac{1}{2}} (XF)^* e^{i\varphi} | \omega_s\omega_{c0} |^{\frac{1}{2}} R_{b0} e^{i\omega_s t_0} \\ &\quad + F | \omega_m\omega_{c0} |^{\frac{1}{2}} R_{g0} e^{-i\omega_m t_0} . \end{aligned} \tag{23}$$

Notice that the combination $Xe^{-i\omega_m t_0}$ has a dependence on t_0 of the form $e^{i\omega_s t_0}$ and $X^* e^{i\omega_s t_0}$ has a dependence of the form $e^{-i\omega_m t_0}$. Consequently, one may write

$$\begin{vmatrix} a_c(\omega_s) \\ a_s(\omega_m) \end{vmatrix} = \begin{vmatrix} F^* e^{i\varphi} & - | \omega_s/\omega_m |^{\frac{1}{2}} XF \\ - | \omega_m/\omega_s |^{\frac{1}{2}} (XF)^* e^{i\varphi} & F \end{vmatrix} \cdot \begin{vmatrix} a_{c0}(\omega_s) \\ a_{s0}(\omega_m) \end{vmatrix}, \tag{24}$$

and one may note that the cylotron wave of frequency ω_s is coupled to the synchronous wave of frequency $\omega_m = \omega_p - \omega_s$. If $\omega_p > \omega_s > 0$ then $\omega_m > 0$ and both waves may carry positive kinetic power. Both may also carry negative power when $\omega_p < \omega_s < 0$. If $| \omega_p | < | \omega_s |$ one wave must carry negative kinetic power. Notice first that the transformation represented by (24) reduces to the transformation for the space-varying case given in (16) in the limit $\omega_p = 0$, $\omega_m = -\omega_s$. The transformation still has a determinant of magnitude unity because of the condition given in (34). The significance of this fact can be understood from the following considerations. One may write for the kinetic power output at the signal frequency for unit power input, $P_{ks} = | F |^2$ and the power output at the idler frequency as $P_{km} = | \omega_m/\omega_s | | XF |^2$. Note that the Manley-Rowe relationship[10] for a three-frequency reactive enegry converter

$$(P_{ks} - 1) - | \omega_s/\omega_m | P_{km} = 0 \tag{25}$$

is satisfied as a result of the relationship given in (34).

Equation (34) has been seen to play a major role in both stationary and time-varying field pumping. As a result of this condition the kinetic power is conserved in the coupling of a fast (slow) cyclotron wave and a slow (fast) synchronous wave by a spatially varying static magnetic field. In the case of high-frequency pumping, (34) is equivalent to the Manley-Rowe relationship. In fact, conservation of kinetic power and the Manley-Rowe relationship are the same in the special case $\omega_p = 0$. In Ref. 6 it is shown that the condition expressed by (34) is merely a statement of the fact that the momentum canonically conjugate with the angular coordinate is time independent. This implies that the conservation of angular momentum of the total system, beam plus field, and the Manley-Rowe condition are analogous. In line with this fact one may note that the constraints which were placed on the solution, $\omega_p - \beta_p v = \omega_{c0}$ and $\omega_m = \omega_p - \omega_s$, in combination with the unperturbed propagation constants of the cyclotron and synchronous waves, $\beta_s = (\omega_s - \omega_{c0})/v$ and $\beta_m = \omega_m/v$, yield the more commonly known constraints

$$\begin{aligned} \omega_s + \omega_m &= \omega_p , \\ \beta_s + \beta_m &= \beta_p . \end{aligned} \tag{26}$$

It is interesting to note that there are only two coupled modes in the space-varying ($\omega_p = 0$) pumping independent of the degree of coupling, whereas in the high-frequency case there are two modes only in the limit of weak coupling. As the coupling is made stronger, inspection shows that sidebands of mixing between multiples of the pump frequency and the signal frequency are introduced. The idler waves will be both synchronous and cyclotron waves, but the most important waves will be those discussed there. Clearly the difference arises because there can be no multiples of the pump frequency in $\omega_p = 0$ pumping.

V. INFLUENCE OF SPACE CHARGE

In the presence of space charge the motion of a beam electron in the center-of-mass system of a uniform beam can be written approximately as

$$\ddot{\rho} - i(\omega_c \dot{\rho} + \tfrac{1}{2}\dot{\omega}_c \rho) - \tfrac{1}{2}\omega_q^{\,2} \rho = 0, \tag{27}$$

in which ω_q is the plasma frequency of the beam. Compare this to (34) for the beam disc. In the limit $\dot{\omega}_c = 0$ this leads to two natural motions of the electron, one at a frequency $\omega_c' = \frac{1}{2}\omega_c[1 - (1 - 2\omega_q^{\,2}/\omega_c^{\,2})^{\frac{1}{2}}]$ and the other at $\omega_c'' = \frac{1}{2}\omega_c[1 + (1 - 2\omega_q^{\,2}/\omega_c^{\,2})^{\frac{1}{2}}]$. In the limit of zero space charge, $\omega_q = 0$, these motions correspond to the guiding center and orbital motion.

In order to amplify the internal motion of the electrons it is necessary to pump at either ω_c' or ω_c'' rather than ω_c . Hence, if $2\omega_q^2/\omega_c^2$ is sufficiently close to one, then both ω_c' and ω_c'' are appreciably different from ω_c and pumping at a frequency equal to ω_c will amplify the beam motion and consequently the beam waves without amplifying the electron orbits as has been shown by Adler et al. for quadrupole pump fields.[11]

VI. CONCLUSION

The cyclotron wave associated with the rotational motion of the beam electrons and the synchronous wave associated with the spatial configuration of the beam have been shown to be coupled by varying magnetic fields. Spatial variations of any kind in the strength of the magnetic field always amplify existing cyclotron or synchronous waves. One of the coupled modes is fast and the other slow, and the total kinetic power is conserved. Sinusoidal variations in the field strength such that $2\pi v/L = \omega_{c0}$, in which v is the drift velocity and L is the periodicity, are particularly effective in coupling the modes. The kinetic power in each mode is conserved when the field varies slowly.

In magnetic pumping with a time-varying field, a fast cyclotron wave and a fast synchronous wave are coupled if the signal frequency is lower than the pump frequency. If the signal frequency is larger than the pump frequency one of the modes must be slow. The Manley-Rowe condition is satisfied by this interaction. The space-varying magnetic field is seen to be a special case of parametric pumping at zero pump frequency. Conservation of kinetic power in this case is a special case of the Manley-Rowe relationship.

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APPENDIX

The azimuthally symmetric fields are derived from a vector potential, **A**, with rectangular components $(-\frac{1}{2}yH(z,t),\ \frac{1}{2}xH(z,t),\ 0)$ which give rise to an electric field, $\mathbf{E} = -c^{-1}\partial\mathbf{A}/\partial t$, with components $(\frac{1}{2}c^{-1}y\partial H/\partial t,\ -\frac{1}{2}c^{-1}x\partial H/\partial t, 0)$ and a magnetic field, $\mathbf{H} = \text{curl}\ \mathbf{A}$, with components $(-\frac{1}{2}x\partial H/\partial z,\ -\frac{1}{2}y\partial H/\partial z,\ H)$. The equation of motion for the beam disc is

$$\begin{aligned} m\ddot{x} &= -(e/c)[\tfrac{1}{2}y(\partial H/\partial t + \dot{z}\partial H/\partial z) + \dot{y}H], \\ m\ddot{y} &= (e/c)[\tfrac{1}{2}x(\partial H/\partial t + \dot{z}\partial H/\partial z) + \dot{x}H]. \end{aligned} \tag{28}$$

Multiplying the second equation by i and writing $\mathbf{R} = x + iy$ yields

$$\ddot{\mathbf{R}} - i(\omega_c\dot{\mathbf{R}} + \tfrac{1}{2}\dot{\omega}_c\mathbf{R}) = 0, \tag{29}$$

in which $\omega_c(z,t) = eH(z,t)/mc$. Assuming that the time dependence of the z-coordinate of the disc can be specified the solution for (29) may be written

$$\begin{aligned} \mathbf{R}_b(t) &= \dot{\mathbf{R}}(t)/i\omega_c = (\omega_{c0}/\omega_c)^{\frac{1}{2}}[\mathbf{R}_{b0}F^*e^{i\varphi} - \mathbf{R}_{g0}XF], \\ \mathbf{R}_g(t) &= \mathbf{R}(t) - \mathbf{R}_b(t) = (\omega_{c0}/\omega_c)^{\frac{1}{2}}[-\mathbf{R}_{b0}(XF)^*e^{i\varphi} + \mathbf{R}_{g0}F], \end{aligned} \tag{30}$$

in which X satisfies the first-order differential equation

$$\dot{X} - i\omega_c X + \tfrac{1}{2}\dot{\omega}_c(1 - X^2)/\omega_c = 0 \tag{31}$$

with initial conditions $X(t_0) = 0$, $\mathbf{R}_{b0} = \mathbf{R}_b(t_0)$, $\mathbf{R}_{g0} = \mathbf{R}_g(t_0)$,

$$\varphi(t) = \int_{t_0}^{t} \omega_c(t')\, dt' \tag{32}$$

and

$$F(t) = \exp\left\{-\frac{1}{2}\int_{t_0}^{t} [\dot{\omega}_c X/\omega_c]\, dt'\right\}. \tag{33}$$

Multiplying (31) by X^* and the complex conjugate equation by X, adding and integrating yields

$$|F|^2(1 - |X|^2) = 1. \tag{34}$$

Combining (30) and (34) yields

$$\omega_c({R_g}^2 - {R_b}^2) = \omega_{c0}({R_{g0}}^2 - {R_{b0}}^2). \tag{35}$$

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