Theory of Current-Carrier Transport and Photoconductivity in Semiconductors with Trapping

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(Manuscript received September 30, 1959)

Fundamental differential equations are derived under the unrestricted approximation of electrical neutrality that admits trapping. Extension is made for applied magnetic field. The transport equations derived hold without explicit reference to detailed trapping and recombination statistics. Modified ambipolar diffusivity, drift velocity and lifetime function apply in the steady state. The same diffusion length is shown to hold for both carriers, and a general "diffusion-length lifetime" is defined. Mass-action statistics are considered for cases of (one or) two energy levels. Certain "effective" — rather than physically proper — electron and hole capture and release frequencies or times that apply to concentration increments are defined. Criteria are given for minority-carrier trapping, recombination and majority-carrier trapping, and for "shallow" and "deep" traps. Applications of the formulation include: the diffusion-length lifetime for the Shockley-Read electron and hole lifetimes; linear and nonlinear steady-state and transient photoconductivity; negative photoconductivity; the photoconductive decay observed by Hornbeck and Haynes in p-type silicon; the photomagnetoelectric effect; and drift of an injected pulse. Photomagnetoelectric current is found to be decreased by minority-carrier trapping, through an increase in diffusion length. A simple general criterion is given for the local direction of drift of a concentration disturbance. With trapping, there may be "reverse drift," whose direction is normally that for the opposite conductivity type, and also local regions of carrier depletion that may extend in practice over appreciable distances.

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photoconductivity also is analyzed critically and in some detail, a procedure facilitated by results from the present formulation of comparative formal simplicity. This analysis involves a formalism that recurs in theory of time-dependent transport. A detailed treatment of the drift with trapping of an injected pulse is given. These applications of the formulation constitute an illustrative selection.

In Section 1.2 is assembled descriptive material intended to be read for further preliminary orientation as to the contents of the paper, and also to be read piecemeal with corresponding portions of the main sections.

1.2 *Outline of Procedures and Results*

The formulation is accomplished in two stages. By treating concentrations of added electrons and holes formally as unrelated variables, differential equations for the transport are derived in Section 2.1 along the lines of previous treatments.[10,11]† Extension for applied magnetic field is included.[11] These equations involve no specific reference to the detailed trapping and recombination statistics. Specialized to the steady state, the ambipolar continuity equation is formally the no-trapping equation, but with the sum of fixed and mobile positive (or negative) charges as dependent variable, and with suitably modified ambipolar diffusivity, drift velocity, and lifetime function, which depend in general on two (concentration-dependent) phenomenological differential "trapping ratios." The same diffusion length is shown to apply for both electrons and holes, and a general "diffusion-length lifetime," τ_0, based on the unmodified ambipolar diffusivity, is defined. The formulation is completed in Section 2.2 with equations for the time rates of change of concentrations of carriers trapped in centers of each type.

These rates are written in accordance with mass action, which provides a simple‡ and general§ basis for trapping and recombination.‖ Two

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† In Ref. 11 small Hall angles are assumed, in part because appreciable magnetoresistance is otherwise involved. As indicated in this reference, arbitrary Hall angles (and injection levels) could suitably be taken into account by theory involving the phenomenological magnetoresistance without added carriers.

‡ See Hoffmann.[2] The mass-action approach, now widely used in semiconductor theory, is essentially that used in early theory of metal-semiconductor junctions: see Schottky and Spenke.[12]

§ Boltzmann statistics, assumed for the transport equations, imply mass-action relationships at equilibrium: see Spenke.[13] But, with definitions of equilibrium parameters suitably extended, mass-action equations apply also for degenerate semiconductors: see Rose.[14]

‖ A treatment based on Fermi statistics that allows for degeneracy and includes dependence of occupation probabilities on applied magnetic field has been given by Landsberg.[15]

energy levels (as well as a single one) are considered; equations are written in Section 2.2.1 for two types of trapping centers and, through simple formal modification, in Section 2.2.2 for two levels from centers of a single type.† Partly by way of notational convention, the levels are taken as acceptor and donor levels, which give negative and positive fixed charges. This case is the simplest for which both steady-state trapping ratios occur, these ratios being the respective changes in concentration of all fixed negative charges and all fixed positive charges divided by the change in concentration of all negative or positive charges. With suitable interpretation of the notation, the equations apply to one- or two-level cases in general; results written for centers of the acceptor type, for example, are not restricted to this type. Moreover, it will appear that, in the analysis of transient (or steady-state) photoconductivity for a given multilevel model, the trapping at a given time need usually be considered in detail in no more than two successive levels. Levels appreciably lower and higher than these may contribute to recombination, but will not contribute to trapping, in the sense that the lower levels may be assumed to remain completely full (or else saturated) and the higher levels completely empty.‡

To facilitate analysis and interpretations, in Section 2.2.1.2 "effective" capture and release frequencies and times that apply to concentration increments are defined *a priori* from the mass-action equations. The four effective frequencies or times for each energy level differ from the physically proper ones, which depend on the trapped concentrations and thus on the detailed solution of the particular problem. They satisfy a fundamental restriction, used extensively in the theory, which is derived from thermal-equilibrium relationships involving detailed balance. With this restriction, quantitative criteria are established in Section 2.2.1.3 for ranges of minority-carrier trapping, recombination and majority-carrier trapping. These ranges may be specified in terms of the location of the equality level§ relative to the Fermi level $\mathcal{E}_F$ and the "reflected Fermi level" $\mathcal{E}_F'$, the reflection of $\mathcal{E}_F$ about $\mathcal{E}$, its location for intrinsic material. If spins are taken into account, quantities of the mass-action theory serve to locate the trapping level relative to $\mathcal{E}$. It is shown

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† Theory for multilevel centers is given in Landsberg,[15,16] Champness,[17] Okada,[18] Shockley and Last,[19] Mercouroff,[20] Khartsiev,[21] Sah and Shockley,[22] Bernard,[23] Kalashnikov and Tissen,[24] and Kalashnikov.[25]

‡ The influence of trapping at a given level on recombination at another has been calculated for the near-equilibrium steady state by Kalashnikov.[26] See also Mashovets.[27]

§ This is the Fermi level for which the (equilibrium) rates of electron and hole capture and release are all equal.[22,28] The equality level is similar in purport to the demarcation level of Rose,[29,30] which is the trapping level for which the rates are equal.

that a proper criterion for "shallow" or "deep" minority-carrier trapping levels is that $\mathcal{E}_F'$ separates these levels. Thus, levels in extrinsic material considerably shallower than the midgap may still be "deep" levels.

Detailed theory is given in Section III through various applications of the general formulation, and consequences of the mass-action statistics are examined. In Section 3.1.1 diffusion length and diffusion-length lifetime, as well as the trapping ratios, are evaluated from equations written for the limiting linear small-signal steady state. A "capture concentration" is introduced, use of which is found to simplify formally much of the detailed theory, including that for time-dependent cases. This concentration is the concentration of (single-level) centers multiplied by the respective equilibrium fractions of centers occupied and unoccupied. Values of it that are small or large result, respectively, in negligible capture frequencies or in large capture frequencies with negligible release frequencies. For the case of a single energy level, the general (equilibrium) Shockley-Read electron and hole lifetimes[31] are obtained in forms involving the capture concentration. These lifetimes are shown to correspond to a diffusion-length lifetime τ_0 whose general expression is formally the same as that for the common (equilibrium) lifetime[31,32] in the limit of small concentration of centers. This common lifetime otherwise applies as such only under a condition restricting the capture concentration, which is frequently severe: In the minority-carrier trapping range, it is that this concentration be small compared with the equilibrium minority-carrier concentration. From conditions for the neglect of quadratic terms in the mass-action equations, the linear approximation is shown to imply a restriction of injection level that may be much more severe than the familiar small-signal condition[10] based on the conductivity change.

The general single-level trapping ratios and lifetime functions for the nonlinear steady state are obtained in Section 3.1.2. These and the mobile-carrier concentrations, as well as the volume recombination rate, can be expressed in terms of trapped-carrier concentration as single concentration variable. The lifetime functions reduce to the Shockley-Read lifetimes in the linear small-signal limit and to a single limiting large-signal value. The familiar common lifetime function[31] for small concentration of centers usually does not apply in the small-signal range unless it is substantially constant in this range. The differing general lifetime functions otherwise usually apply, and small-signal minority-carrier trap saturation obtains. The apparent diffusion-length lifetime then increases to a small-signal saturation-range value equal to the majority-carrier release time.† Further increase occurs in the approach to a large-signal lifetime, which, in this case, is also the (small-signal) lifetime in the limit of strongly extrinsic material of the opposite conductivity type. Such increases of lifetime can account for certain cases of superlinearity, or the more-rapid-than-linear increase of photoconductivity with injection level, on the basis of a single trapping level.‡

Transient decay of photoconductivity is analyzed in Section 3.2. In the linear small-signal case, the decay is given by a sum of exponential modes with (real and positive) decay constants whose number exceeds by one the number of types of centers present.[2,28,37-41]§ For nonrecombinative trapping by centers of two types, the decay constants and equilibrium concentrations after injection are evaluated in Section 3.2.1 for electron and hole traps present together and for electron (or hole) traps only. With the latter, carriers released from one type may be captured in the other. The general linear case for centers of one type, including recombination, is analyzed in detail. The two time constants are given in forms involving the capture concentration. If one is large compared with the other, then the larger may be identified as the lifetime, while the smaller represents a trapping transient during which approach to the steady-state trapping ratio takes place. This transient has small amplitude for small concentration of centers, for which capture rates in the ratio of capture frequencies and release rates in the ratio of release frequencies decay with the concentration in the lifetime mode. It does not occur if the steady-state trapping ratio obtains initially, or if "critical recombination" obtains, with which, because of equal capture frequencies, trapped concentration does not change from initial value zero.

Sufficiently small capture concentration gives, with the comparatively short trapping transient, a lifetime substantially equal to[38,39,40] the common steady-state electron, hole and diffusion-length lifetime. The required condition is frequently severe: In the minority-carrier trapping range, it is the same as the common-lifetime condition. Capture concentration large results in decay times equal to[38,40] the steady-state electron and hole lifetimes and given by the electron and hole capture times. If one of these is large compared with the other, then the smaller represents the transient for practically complete trapping of the carriers of one kind, and the larger represents the recombinative decay of the

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† Approximate steady-state solutions which exhibit small-signal nonlinearity have been given by Tolpygo and Rashba.[33]

‡ A multilevel model for superlinearity has been given by Rose,[29,34,35] (and Ref. 1, Ch. 1A). See also Bube.[35]

§ See also Ref. 1, Ch. 3A. This chapter also includes some nonlinear cases.

carriers in traps and of the carriers of the other kind as these are captured. In all these cases, the lifetime decreases monotonically as concentration of centers increases. Under a condition that is usually met in the minority-carrier trapping range, this decrease occurs primarily in two ranges of concentration of centers, with approximate constancy of lifetime in an intermediate range.†

The photoconductive decay is governed in the general case by nonlinear differential equations. These are considered for centers of a single type in Section 3.2.2. The general single-level problem is rather intractable analytically.‡ Solutions of the nonlinear equations are given for two special cases, namely, nonrecombinative trapping and sufficiently small concentration of centers or large concentrations of mobile excess carriers such that the steady-state lifetimes are substantially equal. The latter solution§ has the rather restricted general application of the common steady-state lifetime function,[31] since it is the integrated form corresponding to this function.‖ By solving suitably linearized equations, the decay times associated with a small-amplitude pulse of added carriers above a steady generation level are evaluated. If, as is often permissible, direct recombination may be neglected,[48,49,50] then the decay in the general large-signal limit is exponential with lifetime equal to the steady-state large-signal lifetime. During this decay, the concentrations of carriers in traps remain constant. This lifetime and the corresponding concentrations in traps are evaluated for centers of a single type and for the two-level cases. A differential equation that is invariant under interchange of quantities pertaining to electrons and to holes is derived for centers of a single type. It provides a first integral under a condition that holds for sufficiently large concentration of centers or concentrations of mobile carriers. With this first integral, the decay problem may be formulated as a first-order (rather than second-order) nonlinear differential equation. The large-signal condition, obtained in this connection, differs from the familiar one[10] in that, as a condition for equal electron and hole lifetimes, it entails not only relatively large change in conductivity but

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† The approach to constancy with increasing concentration of centers is discussed by Wertheim.[41]

‡ Certain analytical approximations have been considered by Isay.[42] A treatment which includes numerically computed solutions has been given by Nomura and Blakemore.[43]

§ It is equivalent to ones given by Rittner, in Ref. 1, Ch. 3A, and by Guro.[44]

‖ The decay lifetime has been evaluated as this function by Okada.[45] That the nonlinearity according to this function does not account for (small-signal) decay in silicon has been observed by Blakemore.[46] This author has fitted dependences of lifetime on injection level and temperature assuming two-level recombination from one type of center or from two types. The common lifetime function has been employed for centers in germanium by Iglitsyn, Kontsevoi and Sidorov.[47] It appears that these centers were in the recombination range.

also requires saturation of centers, which may be present in relatively large concentration.

Conditions are obtained for centers that give recombination with substantially constant lifetime for minority carriers and inappreciable trapping. Constant lifetime that applies in the small-signal range also applies in the large-signal range, provided the energy level of the centers is not too far from the Fermi level towards the majority-carrier band. It requires, however, sufficient strongly extrinsic material. In material of mixed conductivity type, the recombination rate cannot, in general, be specified in terms of a minority-carrier lifetime. But "linear recombination" may apply, characterized by a two-lifetime recombination rate that is the sum of contributions respectively proportional to the added minority- and majority-carrier concentrations. The assumption of general linear recombination is also a convenient notational device: In the analysis of models involving nonrecombinative traps in conjunction with the recombination centers, it permits deriving results in forms that apply for any conductivity of either type.

The phenomenon of negative photoconductivity, or the decrease in conductivity below the equilibrium value upon optical injection,† results essentially from excitation of minority carriers from traps with recombination in other centers. Theory for this effect is given in Section 3.2.3, a general expression for mobile-carrier concentrations being derived for the linear small-signal case. This result is of comparative formal simplicity and shows that the effect tends to be offset by recombination in the traps and to be enhanced with deep traps of small capture cross section.‡

A general procedure is outlined in Section 3.2.4 for analysis of trapping models with a number of discrete energy levels, which relates the various decay times to capture cross sections and these energy levels. This procedure is applied to observations of Hornbeck and Haynes[54] on electron trapping in p-type silicon.§ For the sample on which the most extensive measurements were made, the decay times ranged from 20 microseconds to 260 seconds. Their model, that of two kinds of nonrecombinative traps with recombination in other centers, is found to imply a hole-capture cross section of the deep traps and of the shallower

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† This has been analyzed by Stöckmann.[51] It has recently been observed in silicon by Collins.[52] Infrared quenching of photoconductivity or luminescence from short-wavelength excitation, discussed by Rose[29,34,35] and others, is a closely related effect.

‡ Excitations involving trapping levels may increase normal photoconductivity.[53]

§ See also Ref. 1, Ch. 3F.

(but still "deep") traps that is small compared with about 10^{-24} and 10^{-20} cm², respectively.

These cross sections are calculated from expressions for the decay times for nearly empty traps. Recombination in the deep traps cannot account for the observed decay: The hole-capture cross section that gives the decay time of 260 seconds for the nearly empty traps would give a considerably larger decay time, rather than the observed value of 1 second, for the traps nearly full. Recombination in the shallower traps, however, can account for the observed decay: The hole-capture cross section calculated from the decay time for these nearly empty traps is in close agreement with that which fits the entire decay in the deep traps. The lifetime with traps filled of 20 microseconds may then be ascribed to recombination in the higher level of two-level shallower traps. A recalculation of cross sections and energy levels on the basis of this model gives hole-capture cross sections large compared with 1.2×10^{-17} cm², equal to 2.4×10^{-20} cm² and small compared with 10^{-24} cm², respectively, for the recombination level and the shallower and deep trapping levels. The corresponding electron-capture cross sections† are 2.3×10^{-15}, 1.1×10^{-13} and 2.9×10^{-14} cm², the last two being half an order of magnitude smaller than the ones calculated by Hornbeck and Haynes. The shallower and deep trapping levels are found to lie 0.007 ev above and 0.23 ev below the Fermi level for intrinsic material; the latter trapping level is 0.78 ev below the conduction band.‡ Use is made of the observed straggle effect from the shallower traps, comparison being made with the theoretical expression derived in Section 3.4.3 for the limiting decay time at fixed location for the tail of the distribution from a pulse injected under applied field after the maximum has drifted past. It is shown that a model for which the trapping levels are levels of centers of a single type cannot account for the observations. While the levels found are close to two levels of gold,[56,57] it is thus unlikely that they result from a single metallic impurity. This conclusion bears on the indications that the deep traps are associated with the presence of oxygen as an impurity.

The steady-state photomagnetoelectric (PME) effect with trapping is analyzed in Section 3.3 on the basis of the general formulation with applied magnetic field.§ Equations formally similar to those for no trapping apply in terms of redefined quantities that involve the trapping

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† Theory to account for such large cross sections has been given by Lax.[55]

‡ An energy gap of 1.10 ev at 300°K is used rather than 1.00 ev as in Ref. 54.

§ A treatment of photoconductance and PME voltage with trapping under ac illumination is included in: Lashkarev, Rashba, Romanov and Demidenko.[58] Mironov[59] deals with the transient decays after removal of steady illumination.

ratios. The effect may exhibit small-signal nonlinearity with nonuniform lifetime if recombinative deep traps in the minority-carrier trapping range are involved. The influence of trapping as such is investigated. It is found that nonrecombinative minority-carrier traps (in conjunction with recombination centers) increase diffusion-length lifetime by an amount proportional to the capture concentration.† Thus, minority-carrier trapping decreases PME current. A comparatively slight decrease in τ_0 and increase in PME current results from majority-carrier trapping.

Detailed illustrative procedures and related theory are given for the determination of capture cross sections, concentrations and energy levels from suitable PME and photoconductivity measurements at given temperature. With trapping in recombinative traps of a single type, the PME current-conductance ratio involves light intensity implicitly through its dependence on the lifetime τ_c that is defined in terms of the change in conductivity for a given steady, uniform volume-generation rate. The ratio, however, determines a relationship between τ_0 and τ_c, a transcendental relationship with the preferred method[11] of the high-recombination-velocity dark surface. This relationship, in conjunction with suitable additional conductance measurements also independent of light intensity, suffices to determine both τ_0 and τ_c, then the capture (and release) frequencies and capture concentration, and finally the quantities sought. The linear small-signal theory‡ is given for recombinative traps of a single type and also for nonrecombinative traps with recombination centers, for which the results are essentially similar though somewhat simpler.

Preliminary to analysis of transport problems, the general ambipolar continuity equation is specialized to the linear small-signal case in Section 3.4.1. Then, for trapping (and recombination) in centers of a

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† Jonscher[60] gives an increase of diffusion length with trap concentration which is bounded and always essentially negligible, a result at variance with that given here. In Jonscher's nonambipolar treatment, the continuity equation does not include a term in the second space derivative of trapped-carrier concentration. Though this term is relatively small for sufficiently strongly extrinsic material, its neglect significantly affects the higher-order differential equation for concentration of mobile minority carriers, obtained by eliminating trapped-carrier concentration, in that it gives a coefficient of the term in the second space derivative that is too small by just the factor by which diffusion-length lifetime is increased.

‡ Zitter[61] discusses the phenomenological dependence for any model of electron and hole lifetimes on τ_c and lifetime derived from the PME effect (in the thick slab). The latter is the same as τ_0, and Zitter relates it to a diffusion length. Amith,[62,63] has presented the effect of nonrecombinative traps on the PME current-conductance ratio, and has pointed out that the predominant effect is usually on conductance. That on PME current is generally negligible in comparison if the traps are minority-carrier traps and are present in not too large concentration in sufficiently strongly extrinsic material.

single type and linear recombination in other centers, the respective concentrations are shown to satisfy certain third-order partial differential equations, which are of the second order in time. These reduce to the same equation if there is no volume generation. They otherwise each contain a term proportional to the volume generation function, the equations for the mobile-carrier concentrations containing the time derivative of this function as well.

The case analyzed in Section 3.4.2, that of injection into a filament in the steady state with applied field, yields qualitative information of interest. If a certain frequency ν_v, the "straggle constant," is positive, then the field-opposing and field-aiding solutions in the regions separated by the point of injection are sharply varying and gradually varying exponentials, as in the no-trapping case.[64] But with negative ν_v, gradually varying field-opposing and sharply varying field-aiding solutions obtain. In the limit of no diffusion, these give added carrier concentrations only in the direction opposite to the direction of drift normally determined by conductivity type. This "reverse drift" is explained by a simple and entirely general criterion, obtained from the fundamental equations, for the local direction of drift of a concentration disturbance: Normal or reverse drift occurs according to whether injection results in proportionately more or fewer minority carriers than there are at thermal equilibrium. For no trapping, for example, the concentrations are increased locally by the same increments, so that proportionately more minority carriers result if the material is extrinsic; and zero drift[10,64] obtains if the material is intrinsic. Conditions for the sign of ν_v are given. It is shown from these that reverse drift, which occurs for sufficiently large trap concentration in not too strongly extrinsic material, occurs with nonrecombinative trapping if minority carriers are trapped so that the fraction of the time they are free is smaller than the equilibrium minority-carrier to majority-carrier concentration ratio.

Drift of a pulse of carriers injected into a filament, with trapping by centers of a single type,† is analyzed in detail in Section 3.4.3. Bilateral

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† Fan[37,65] has given a solution of this drift problem which applies for negligible majority-carrier capture frequency. Clarke[40] has, in effect, pointed out this restriction, to which solutions for the decay of photoconductivity given by Fan[37] and Rittner[1] are also subject. Jonscher[66] has given solutions for drift of minority carriers with recombination and nonrecombinative trapping at variance with solutions given here. The otherwise plausible neglect by Jonscher, in a nonambipolar treatment for strongly extrinsic material, of a term in the continuity equation involving the gradient of trapped-carrier concentration is apparently not justified. In the differential equation for concentration of mobile minority carriers, it results in minority-carrier release frequency only as a factor in the concentration-gradient term instead of ν_v, which, for this case, is substantially the sum of the capture and release frequencies. This neglect of the capture frequency is tantamount to neglect of the capture concentration compared to the equilibrium minor-

or two-sided Laplace transforms derived in Appendix A are used to obtain solutions of the differential equation for negligible diffusion. These solutions are of two main types, according to whether a frequency unit, ν, connected with the introduction of dimensionless variables and parameters, is real or imaginary. From theory concerning ν and the parameters, it is found that real ν implies either the minority-carrier trapping range or the recombination range.† Illustrative solutions of real ν for which nonrecombinative trapping is assumed are presented graphically. These show that carriers that remain untrapped appear in a comparatively rapidly attenuated pulse that drifts at the ambipolar velocity. This remnant of the initial pulse leads a continuous distribution, which, as a result of multiple trapping, ultimately spreads as a time-dependent gaussian distribution and exhibits a maximum that drifts at a fraction of the ambipolar velocity. For nonrecombinative trapping, this fraction approaches comparatively slowly a limiting value that does not exceed the fraction of the time the carriers are free,‡ and (for imaginary ν as well) the fraction of carriers trapped, obtained by integrating over the drift range, approaches comparatively rapidly the fraction of the time carriers are trapped. Recombination in other centers reduces the distance for a maximum at given time and thus the apparent mobility of the distribution. The decay constant for the straggle effect is found to be the (positive) straggle constant, accordingly so named.

Imaginary ν obtains over the majority-carrier trapping range and, for nonrecombinative trapping, over the reverse-drift range. With recombinative trapping, it obtains also for zero drift and over a normal-drift range other than that of majority-carrier trapping. Illustrative solutions, for which nonrecombinative trapping is assumed, are presented graphically for reverse drift and for majority-carrier trapping. It appears that in the reverse-drift range an attenuated pulse of untrapped carriers, which drifts at the ambipolar velocity, leads a continuous distribution

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ity-carrier concentration. It presumably leads, for example, to the conclusion of this reference that a very short pulse is transmitted without distortion and is only attenuated. Also, the solution given for the steady state of continuous injection should properly include the fraction of the time minority carriers are free as a factor in the exponent. That is, trapping results in a more gradual decay with distance; for this case of no diffusion, a lifetime applies that is equal to the sum of the lifetime proper and the (generally much larger) lifetime for multiple trapping, which is discussed in Section 3.2.4.

† As shown in Section 2.2.1.2, these ranges in their entirety together constitute the "minority-carrier capture range," for which the equilibrium minority- to majority-carrier capture frequency ratio exceeds unity. It is shown in Section 3.4.3 that there is a minority-carrier capture range of imaginary ν which includes the reverse-drift range.

‡ Fan[37] has shown from his solution that this limiting value is, for relatively small trap concentration, equal to the free-time fraction.

of added mobile minority carriers, which crowds towards the injection point as its maximum excursions both above and below the axis increase with time. There is local carrier depletion, the distribution being negative over part of the drift range after a certain time. The distribution approaches a pulse at the injection point of strength equal, for nonrecombinative trapping, to the initial strength times the free-time fraction. It does not exhibit essentially unidirectional drift: The drift of added carriers, initially in the direction of the ambipolar velocity, is largely in the opposite direction after some trapping has taken place. A numerical estimate of the effect of diffusion indicates that negative added-carrier concentrations can occur over appreciable distances under conditions that can be realized in practice.† The illustrative solution for majority-carrier trapping shows that negative added-carrier concentrations occur in this case also. Majority-carrier trapping, however, results essentially in drift at the ambipolar velocity and, if it is nonrecombinative, the fraction of carriers trapped approaches the trapped-time fraction.

The solution is given for "critical trapping," the borderline case between cases of real and imaginary ν. For nonrecombinative trapping, it is the same as that for zero drift and gives exponential continuous distributions that are established progressively as the drift range increases and otherwise do not change with time. For trapping in intrinsic material without diffusion, drift does not start, and the initial pulse results simply in pulses for the concentration increments that remain at the injection point, where they change as trapping and recombination proceed. With diffusion, ambipolar drift occurs, since the condition for zero drift no longer holds in the intrinsic material as carriers are trapped away from the injection point. Further physical interpretations for the various cases are obtained by evaluating the current density of added carriers, which represents the equal departures for given total current density of the electron and hole flow densities from their values for no added carriers.

1.3 *List of Symbols*

The following list includes most of the symbols to be employed, and is largely consistent with previous notation.[10,11,64]

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† Kaiser[67] has suggested that negative added-carrier concentrations that were observed with localized optical injection in silicon under applied field may be accounted for through these results. A theoretical discussion of carrier depletion is included in Ref. 10.

$a \equiv$ parameter in distribution of (153).

$A_{nj}, A_{pj} \equiv$ capture cross sections for electrons and holes for the jth energy level or type of center.

$b \equiv \mu_n/\mu_p$, drift mobility ratio.

$C_i \equiv$ coefficient for direct electron-hole recombination given in (37).

$C_{nj}, C_{pj} \equiv$ electron and hole capture coefficients for the jth energy level or type of center.

$D \equiv kT\mu_n\mu_p(n + p)/\sigma = (n + p)/(n/D_p + p/D_n)$, ambipolar diffusivity for no trapping.

$D' \equiv$ modified ambipolar diffusivity, defined in (31).

$D_i \equiv$ diffusivity for intrinsic material.

$D_n, D_p \equiv$ diffusion constants for electrons and holes.

$\hat{D}_n, \hat{D}_p \equiv$ diffusivities defined in (133).

$D_0 \equiv$ value of D at thermal equilibrium.

$D_0' \equiv$ value of D' at thermal equilibrium.

$e \equiv$ electronic charge.

$\mathbf{E} \equiv$ electrostatic field.

$\mathcal{E} \equiv$ Fermi level for intrinsic material.

$\mathcal{E}_F \equiv$ Fermi level.

$\mathcal{E}_F' \equiv$ "reflected Fermi level," the reflection of $\mathcal{E}_F$ about $\mathcal{E}$.

$\mathcal{E}_j \equiv$ electron energy for the jth energy level or type of center.

$F_n, \hat{F}_n, F_p \equiv$ fractions of mobile electrons, trapped electrons, and mobile holes for drift of a pulse, given by (164).

$g \equiv$ rate of volume generation of electron-hole pairs.

$\Delta g \equiv g - g_0$.

$g_0 \equiv$ value of g at thermal equilibrium.

$\mathbf{G} \equiv$ quantity defined by (9).

$\Delta G \equiv$ conductance increase of slab per unit width, given by (116), (118) and (129).

$G_0 \equiv$ dark conductance of slab per unit width.

$\mathbf{I} \equiv \mathbf{I}_p + \mathbf{I}_n$, total current density.

$\Delta\mathbf{I} \equiv$ current density of added carriers, defined by (19).

$\mathbf{I}_D \equiv$ diffusion current density, defined by (7).

$\mathbf{I}_n, \mathbf{I}_p \equiv$ electron and hole current densities.

$I_0, I_1 \equiv$ modified Bessel functions, in the notation of Watson.

$J_0, J_1 \equiv$ Bessel functions, in the notation of Watson.

$k \equiv$ Boltzmann's constant.

$\mathbf{k} \equiv$ unit vector in the direction of magnetic field.

$\mathbf{K}$ ≡ quantity defined by (10).
K_G ≡ factor in (129) for conductance change of illuminated slab, evaluated in (130).
K_τ ≡ factor defined in (119) by which diffusion-length lifetime with nonrecombinative trapping exceeds recombination lifetime.
L ≡ $v_0\tau$, length unit defined in (143).
$\mathcal{L}$ ≡ surface rate of generation of electron-hole pairs from strongly absorbed radiation.
L_0 ≡ $(D_0\tau_0)^{\frac{1}{2}}$, diffusion length.
$\mathfrak{L}$ ≡ operator symbol for two-sided Laplace transform.
m ≡ $p + \hat{p} = n + \hat{n}$.
Δm ≡ $m - m_0$.
m_0 ≡ value of m at thermal equilibrium.
n ≡ electron concentration.
Δn ≡ $n - n_0$.
$\overline{\Delta n}$ ≡ two-sided Laplace transform of Δn.
ΔN ≡ $\Delta n/(\mathcal{P}/L)$.
$\overline{\Delta N}$ ≡ $\overline{\Delta n}/(\mathcal{P}/L)$.
$\hat{n}$ ≡ concentration of fixed negative charges.
$\Delta\hat{n}$ ≡ $\hat{n} - \hat{n}_0$.
$\overline{\Delta\hat{n}}$ ≡ two-sided Laplace transform of $\Delta\hat{n}$.
$\Delta\hat{N}$ ≡ $\Delta\hat{n}/(\mathcal{P}/L)$.
$\mathcal{N}$, $\mathcal{N}_j$ ≡ total concentrations of centers.
n_j ≡ electron concentration for the Fermi level at the jth trapping level.
$n_j{}^*$ ≡ equality density, defined in (54).
$\mathcal{N}_j{}^*$ ≡ "capture concentration," defined (for $j = 1$) in (63).
n_s ≡ $n_0 - p_0$.
n_0 ≡ value of n at thermal equilibrium.
$\hat{n}_0$ ≡ value of $\hat{n}$ at thermal equilibrium.
N_1, N_2 ≡ dimensionless decay constants defined by (147).
p ≡ hole concentration.
Δp ≡ $p - p_0$.
$\overline{\Delta p}$ ≡ two-sided Laplace transform of Δp.
ΔP ≡ $\Delta p/(\mathcal{P}/L)$.
$\overline{\Delta P}$ ≡ $\overline{\Delta p}/(\mathcal{P}/L)$.
$\hat{p}$ ≡ concentration of fixed positive charges.
$\Delta\hat{p}$ ≡ $\hat{p} - \hat{p}_0$.

$\mathcal{P} \equiv$ number per unit area of carrier pairs injected over cross section of filament.

$p_j \equiv$ hole concentration for the Fermi level at the jth trapping level.

$p_j^* \equiv$ equality density, defined in (54).

$p_0 \equiv$ value of p at thermal equilibrium.

$\hat{p}_0 \equiv$ value of $\hat{p}$ at thermal equilibrium.

$r_n \equiv d\hat{n}/dm$, steady-state trapping ratio.

$r_{nj}\, , r_{pj} \equiv$ trapping ratios for transient photoconductive decay modes, evaluated in Section 3.2.1; also similar quantities that are given in (149).

$r_p \equiv d\hat{p}/dm$, steady-state trapping ratio.

$\mathcal{R}_m\, , \mathcal{R}_n\, , \mathcal{R}_p \equiv$ functions specifying rates of decrease of m, n and p through trapping and recombination.

$s \equiv$ Laplace transform variable.

$s_m\, , s_n\, , s_p \equiv$ surface recombination velocities for m, n and p, related by (115).

$T \equiv$ temperature in degrees absolute.

$U \equiv t/\tau$, dimensionless time variable.

$\mathbf{v} \equiv$ ambipolar drift velocity, defined by (8).

$\mathbf{v}' \equiv$ modified ambipolar drift velocity, defined in (31).

$\hat{\mathbf{v}}_n\, , \hat{\mathbf{v}}_p \equiv$ velocities defined in (133).

$\mathbf{v}_0 \equiv$ value of $\mathbf{v}$ at thermal equilibrium.

$\mathbf{v}_0' \equiv$ value of $\mathbf{v}'$ at thermal equilibrium.

$V \equiv$ electrostatic potential.

$X \equiv x/L$, dimensionless distance.

$\alpha_1\, , \alpha_2 \equiv$ quantities defined in (78).

$\alpha_{10} \equiv n_0/n_1 = p_1/p_0\, .$

$\alpha_{20} \equiv p_0/p_2 = n_2/n_0\, .$

$\Delta_1 \equiv$ quantity defined by (62).

$\zeta \equiv$ dimensionless parameter defined in (145).

$\eta \equiv$ dimensionless parameter defined by (152).

$\theta \equiv \theta_p - \theta_n = \theta_p + | \theta_n |\, .$

$\theta_n\, , \theta_p \equiv$ Hall angles for electrons and holes.

$\Theta \equiv$ variable defined by (157).

$\kappa \equiv$ dimensionless parameter defined in (145).

$\mu_n\, , \mu_p \equiv$ drift mobilities for electrons and holes.

$\nu \equiv$ frequency unit (real or imaginary) defined in (143).

$\nu_D \equiv$ quantity defined in (136).

$\nu_{gnj}\, , \nu_{gpj} \equiv$ "effective" release frequencies for electrons and holes, defined for acceptor-type ($j = 1$) and donor-type ($j = 2$) centers in (50), and for two-level centers in Section 2.2.2.

$\nu_{ij} \equiv$ decay constants defined by (47).

ν_{n3} , $\nu_{p3} \equiv$ decay constants for "linear recombination," defined in Section 3.2.2.

$\nu_s \equiv$ sum of hole and electron capture and release frequencies defined by (83).

ν_{tnj} , $\nu_{tpj} \equiv$ "effective" capture frequencies for electrons and holes, defined for acceptor-type ($j = 1$) and donor-type ($j = 2$) centers in (50), and for two-level centers in Section 2.2.2.

$\nu_v \equiv$ "straggle constant," defined in (136).

ν_1 , $\nu_2 \equiv$ decay constants for photoconductivity.

$\xi \equiv$ dimensionless parameter defined by (151).

$\sigma \equiv \sigma_n + \sigma_p$, total conductivity.

σ_n , $\sigma_p \equiv$ partial conductivities for electrons and holes.

$\tau \equiv$ time unit defined in (143).

$\tau_c \equiv$ conductivity lifetime, defined by (117).

τ_{gnj} , $\tau_{gpj} \equiv$ "effective" release times for electrons and holes, the reciprocals of ν_{gnj} , ν_{gpj} .

$\tau_m \equiv$ steady-state lifetime or lifetime function for Δm.

$\bar{\tau}_m \equiv$ lifetime for Δm with recombination centers as well as traps, evaluated in Section 3.3.

τ_n , $\tau_p \equiv$ steady-state electron and hole lifetimes or lifetime functions.

τ_{n0} , $\tau_{p0} \equiv (C_{nj}\mathcal{N}_j)^{-1}$, $(C_{pj}\mathcal{N}_j)^{-1}$ for a particular j.

$\tau_r \equiv$ apparent lifetime from PME current-conductance ratio, evaluated in Section 3.3.

τ_{tnj} , $\tau_{tpj} \equiv$ "effective" capture times for electrons and holes, the reciprocals of ν_{tnj} , ν_{tpj} .

$\tau_0 \equiv$ "diffusion-length lifetime," evaluated in (35) and (65).

$\bar{\tau}_0 \equiv$ "diffusion-length lifetime" with nonrecombinative traps and recombination centers, given by (119).

τ_1 , $\tau_2 \equiv$ time constants for photoconductive decay, the reciprocals of ν_1 , ν_2 .

$\tau_3 \equiv$ lifetime for decay through recombination centers, the reciprocal of ν_{n3} or ν_{p3} .

$\tau_\infty \equiv$ photoconductive decay time for nearly empty traps.

$\Phi \equiv -e^{-1}\mathcal{E}$.

$\Psi \equiv$ potential defined by (15).

$\Psi_j \equiv -e^{-1}\mathcal{E}_j$.

II. GENERAL FORMULATION

2.1 *The Transport Equations*

The general neutrality condition may be written as

$$m \equiv p + \hat{p} = n + \hat{n}, \tag{1}$$

which states that the total concentration of positive charges, the sum of the concentrations of mobile holes and fixed positive charges, is equal to the corresponding total concentration of negative charges. It is, as will appear, of advantage to deal with the total concentration m of charges of either kind.

By way of extension of the familiar (nonambipolar) continuity equations for holes and for electrons that apply for no trapping, two forms of the continuity equation for m may be written:

$$\begin{aligned} \partial m/\partial t &= \partial p/\partial t + \partial \hat{p}/\partial t = -e^{-1} \operatorname{div} \mathbf{I}_p + g - \mathcal{R}_m \\ &= \partial n/\partial t + \partial \hat{n}/\partial t = e^{-1} \operatorname{div} \mathbf{I}_n + g - \mathcal{R}_m . \end{aligned} \tag{2}$$

Here, for simplicity, the same volume-generation-rate function g is assumed for both holes and electrons; generalization to include excitation to or from trapping levels (as well as interband excitation) is given in Section 2.2.3. The volume rate $\mathcal{R}_m$ is associated with trapping and recombination. It depends directly on the various concentrations and not explicitly on coordinates and time; $\partial \hat{p}/\partial t$ and $\partial \hat{n}/\partial t$ contribute only to $\mathcal{R}_m$, and, if these are respectively subtracted from (2), then continuity equations for holes and electrons, namely,

$$\begin{aligned} \partial p/\partial t &= -\epsilon^{-1} \operatorname{div} \mathbf{I}_p + g - \mathcal{R}_p , \qquad & \mathcal{R}_p &= \mathcal{R}_m + \partial \hat{p}/\partial t, \\ \partial n/\partial t &= e^{-1} \operatorname{div} \mathbf{I}_n + g - \mathcal{R}_n , \qquad & \mathcal{R}_n &= \mathcal{R}_m + \partial \hat{n}/\partial t \end{aligned} \tag{3}$$

result. The same volume rate $\mathcal{R}_m$ is properly used in each of the equations (2) since it depends directly only on concentrations; it must apply, in particular, in the case of zero current densities. As (2) shows, this use of $\mathcal{R}_m$ is consistent with the neutrality condition and with the condition[64]

$$\operatorname{div} \mathbf{I} = 0, \qquad \mathbf{I} = \mathbf{I}_p + \mathbf{I}_n , \tag{4}$$

which applies in regions containing no sources or sinks of current. Differing volume rates for $p + \hat{p}$ and $n + \hat{n}$ are properly introduced only if there is appreciable space charge.

The familiar current-density equations,

$$\mathbf{I}_p = \sigma_p \mathbf{E} - eD_p \operatorname{grad} p$$

and (5)

$$\mathbf{I}_n = \sigma_n \mathbf{E} + eD_n \operatorname{grad} n,$$

apply under the assumption of Boltzmann statistics, which imply also proportionality of the hole and electron mobilities μ_p and μ_n to the corresponding diffusion constants D_p and D_n in accordance with Einstein's relation. By use of these equations and the neutrality condition, (1), a continuity equation for m of ambipolar form may be derived: The hole and electron current densities $\mathbf{I}_p$ and $\mathbf{I}_n$ are eliminated from (2); the electrostatic field $\mathbf{E}$ is also eliminated by means of the expression for $\mathbf{E}$ involving the total current density $\mathbf{I}$ that is obtained by adding the equations in (5); and use is made of (4), the condition of solenoidal $\mathbf{I}$. This procedure is similar to that previously employed in the no-trapping case[10] except that, for the required generality, p and n are treated formally as unrelated variables. The single continuity equation for m that results from (2) may be written in various forms as follows:†

$$\begin{aligned} \partial m/\partial t - g + \mathcal{R}_m &= -e^{-1} \operatorname{div} \mathbf{I}_D - \mathbf{v} \cdot \mathbf{G} \\ &= -e^{-1}(\operatorname{div} \mathbf{I}_D + \mathbf{K} \cdot \mathbf{I}) \\ &= -e^{-1} \operatorname{div} \mathbf{I}^*, \end{aligned} \tag{6}$$

where

$$\begin{aligned} \mathbf{I}_D &\equiv -e\sigma^{-1}(\sigma_p D_n \operatorname{grad} n + \sigma_n D_p \operatorname{grad} p) \\ &= -e[D \operatorname{grad} m - \sigma^{-1}(\sigma_p D_n \operatorname{grad} \hat{n} + \sigma_n D_p \operatorname{grad} \hat{p})] \\ &= -eD(n + p)^{-1} \operatorname{grad} np = - ekT\mu_n\mu_p\sigma^{-1} \operatorname{grad} np, \end{aligned} \tag{7}$$

$$\mathbf{v} \equiv e\mu_n\mu_p(n - p)\sigma^{-2}\mathbf{I}, \tag{8}$$

$$\begin{aligned} \mathbf{G} &\equiv (n - p)^{-1}(n \operatorname{grad} p - p \operatorname{grad} n) \\ &= \operatorname{grad} m - (n - p)^{-1}(n \operatorname{grad} \hat{p} - p \operatorname{grad} \hat{n}), \end{aligned} \tag{9}$$

$$\begin{aligned} \mathbf{K} &\equiv (D_n - D_p)^{-1} \operatorname{grad} D = e^2\mu_n\mu_p\sigma^{-2}(n \operatorname{grad} p - p \operatorname{grad} n) \\ &= e(\mu_p^{-1} - \mu_n^{-1})^{-1} \operatorname{grad} [\sigma^{-1}(n + p)] \\ &= -e(\mu_p^{-1} + \mu_n^{-1})^{-1} \operatorname{grad} [\sigma^{-1}(n - p)], \end{aligned} \tag{10}$$

and

$$\mathbf{I}^* \equiv \alpha\mathbf{I}' + \beta\mathbf{I}'', \qquad \alpha + \beta = 1, \tag{11}$$

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† This equation specialized to the case of $\Delta\hat{p} = 0$ can be shown to be consistent with a continuity equation for Δp derived by Rittner[1] under the assumption of a common lifetime function for electrons and holes.

with

$$\begin{aligned}
\mathbf{I}' &\equiv \mathbf{I}_D - e({\mu_p}^{-1} + {\mu_n}^{-1})^{-1}\sigma^{-1}(n - p)\mathbf{I} \\
&= -e({\mu_p}^{-1} + {\mu_n}^{-1})^{-1}[(n - p)\mathbf{E} + (kT/e)\ \mathrm{grad}\ (n + p)] \\
&= (b + 1)^{-1}(b\mathbf{I}_p - \mathbf{I}_n) \qquad (12) \\
\mathbf{I}'' &\equiv \mathbf{I}_D + e({\mu_p}^{-1} - {\mu_n}^{-1})^{-1}\sigma^{-1}(n + p)\mathbf{I} \\
&= e({\mu_p}^{-1} - {\mu_n}^{-1})^{-1}[(n + p)\mathbf{E} + (kT/e)\ \mathrm{grad}\ (n - p)] \\
&= (b - 1)^{-1}(b\mathbf{I}_p + \mathbf{I}_n).
\end{aligned}$$

The diffusivity D, a definition of which is contained in the last of the equations in (7), is the general ambipolar concentration-dependent diffusivity, which occurs in the theory for no trapping. It is used here simply for notational convenience. The velocity $\mathbf{v}$ of (8) is properly interpreted in the continuity equation as two velocities, $n\mathbf{v}/(n - p)$ for drift of Δp and $p\mathbf{v}/(p - n)$ for drift of Δn. It is otherwise formally similar to the ambipolar velocity of the theory for no trapping, except that now $n - p$ is not a constant concentration. The first two right-hand forms of the continuity equation, (6), exhibit terms associated with diffusion and drift, respectively, as comparison with the continuity equation for the no-trapping case shows. The current density $\mathbf{I}^*$, as given by (11), is introduced for generality; with solenoidal $\mathbf{I}$, the divergences of $\mathbf{I}'$ and $\mathbf{I}''$ are equal.† From the expressions for these current densities in terms of $\mathbf{I}_p$, $\mathbf{I}_n$ and the drift mobility ratio b given in (12), it may be verified that $\mathbf{I}^*$ may be chosen as $\mathbf{I}_p$ or $-\mathbf{I}_n$, as in (2). Indeed, as is otherwise evident, $\mathbf{I}^*$ may be written simply as a linear combination of $\mathbf{I}_p$ and $-\mathbf{I}_n$, normalized as in (11), since a linear combination, so normalized, of any two $\mathbf{I}^*$ is also an $\mathbf{I}^*$. The current densities $\mathbf{I}'$ and $\mathbf{I}''$ are introduced because their use is frequently convenient.

The mobile-carrier concentrations n and p are, in accordance with (1), properly written as $m - \hat{n}$ and $m - \hat{p}$ where they occur explicitly and in the diffusivity D, in the electron and hole conductivities $\sigma_n = e\mu_n n$ and $\sigma_p = e\mu_p p$ and in the total conductivity $\sigma = \sigma_n + \sigma_p$. For acceptor and donor centers of single types, $\partial\hat{n}/\partial t$ and $\partial\hat{p}/\partial t$ in terms of the various concentrations provide, with the continuity equation, three simultaneous differential equations in the dependent variable m, $\hat{n}$ and $\hat{p}$. For more than single types of acceptor and donor centers, $\hat{n}$ and $\hat{p}$ are sums of fixed-charge concentrations. Equations are then written for the rates of increase of each of these concentrations, and the number of

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† Note that these divergences equal that of $\mathbf{I}_p$ and $(-\mathbf{I}_n)$; also, $\mathbf{I}'' - \mathbf{I}'$ equals $2b(b^2 - 1)^{-1}\mathbf{I}$.

simultaneous differential equations exceeds by one the total number of types of centers present. These are the differential equations for the general transport problem with trapping and recombination provided **I** is a known function of the space coordinates and time.

In some cases, **I** must be determined from boundary conditions. Use is then made of the fundamental differential equation, (4), which may be written to involve the electrostatic potential V as additional dependent variable. With

$$\sigma \mathbf{E} = -\sigma \operatorname{grad} V = \mathbf{I} - e \operatorname{grad} (D_n n - D_p p), \tag{13}$$

it follows that (4) may be written in the form

$$\operatorname{div} [\sigma \operatorname{grad} V - (kT/e) \operatorname{grad} (\sigma_n - \sigma_p)] = 0, \tag{14}$$

in which σ_n, σ_p and σ are to be expressed in terms of m, $\hat{n}$ and $\hat{p}$. In this formulation, V is introduced into the continuity equation through the elimination of **I** by means of (13). Another procedure, of advantage in some connections, involves use of the potential

$$\Psi \equiv V - (kT/e)(b - 1)(b + 1)^{-1} \ln (\sigma/\sigma_0) \tag{15}$$

instead of V as dependent variable. Then **I** is given by

$$\mathbf{I} = -\sigma \operatorname{grad} \Psi + eD_i \operatorname{grad}(\hat{p} - \hat{n}), \tag{16}$$

where $D_i \equiv 2(D_p^{-1} + D_n^{-1})^{-1}$ is the diffusivity in intrinsic material. Aside from the effect of trapping on **I**, as given by the second term of (16), Ψ is the potential that "drives" the total current density. This may be described as the electrostatic potential modified by the Dember potential. The latter gives the field associated with diffusion of carriers of differing diffusion constants.

Electrostatic field given by

$$\mathbf{E} = e^{-1}(n + p)^{-1}[I_n/\mu_n + \mathbf{I}_p/\mu_p - kT \operatorname{grad}(\hat{p} - \hat{n})], \tag{17}$$

an equation somewhat analogous to (16), is a result obtained by solving for **E** in the equations for **I**″ in (12). As (17) shows, **E** in the absence of trapping (and of appreciable space charge) may be written in a form that does not involve concentration gradients explicitly.

In the ambipolar form of the present treatment, the equations of (5) are

$$\mathbf{I}_p = (\sigma_p/\sigma)\mathbf{I} + \mathbf{I}_D = (\sigma_{p0}/\sigma_0)\mathbf{I} + \Delta\mathbf{I}$$

and (18)

$$\mathbf{I}_n = (\sigma_n/\sigma)\mathbf{I} - \mathbf{I}_D = (\sigma_{n0}/\sigma_0)\mathbf{I} - \Delta\mathbf{I},$$

in which zero subscripts denote values at thermal equilibrium; $\Delta\mathbf{I}$ is defined by

$$\begin{aligned}\Delta\mathbf{I} &\equiv e^2\mu_n\mu_p\sigma_0^{-1}\sigma^{-1}(n_0p - p_0n)\mathbf{I} + \mathbf{I}_D \\ &= \tfrac{1}{2}e\sigma_0^{-1}[(\mu_n + \mu_p)(n_0 + p_0)\mathbf{I}' + (\mu_n - \mu_p)(n_0 - p_0)\mathbf{I}''] \\ &= \sigma_0^{-1}(\sigma_{n0}\mathbf{I}_p - \sigma_{p0}\mathbf{I}_n). \qquad (19)\end{aligned}$$

Application of one or more of (13) or (17) and (18) is frequently required in connection with boundary conditions. The ambipolar diffusion current density $\mathbf{I}_D$ includes the effect of the Dember field and contributes the same particle flow density to both $\mathbf{I}_n$ and $\mathbf{I}_p$. Use of the expressions for $\mathbf{I}_n$ and $\mathbf{I}_p$ that involve $\Delta\mathbf{I}$ is of particular advantage for physical interpretations and in small-signal cases, since $\Delta\mathbf{I}$ is the current density of excess mobile carriers.[10] For given total current density $\mathbf{I}$, it represents the equal electron and hole flow densities, that are the departures from the thermal-equilibrium flow densities and that do not contribute to $\mathbf{I}$. Note that $\mathbf{I}^*$ may also be chosen as $\Delta\mathbf{I}$.

2.1.1 *Extension for Applied Magnetic Field*

The current densities for Hall angles θ_n and θ_p small are given in general by Equations (10) and (13) of a previous paper.[11] These result in

$$\begin{aligned}e(\partial m/\partial t - g + \mathcal{R}_m) &= -\operatorname{div}\mathbf{I}_p = \operatorname{div}\mathbf{I}_n \\ &= -\operatorname{div}(\sigma_p\mathbf{E}) + eD_p \operatorname{div}\operatorname{grad} p \\ &\qquad - \theta_p\llbracket\operatorname{grad}\sigma_p, \mathbf{E}, \mathbf{k}\rrbracket \\ &= \operatorname{div}(\sigma_n\mathbf{E}) + eD_n \operatorname{div}\operatorname{grad} n \\ &\qquad + \theta_n\llbracket\operatorname{grad}\sigma_n, \mathbf{E}, \mathbf{k}\rrbracket, \qquad (20)\end{aligned}$$

in which $\mathbf{k}$ is a unit vector in the direction of the magnetic field and the heavy brackets denote scalar triple products. With n and p treated formally as unrelated variables, multiplying respectively by σ_n and σ_p, adding and simplifying gives

$$\begin{aligned}\partial m/\partial t - g + \mathcal{R}_m = &-e^{-1}(\operatorname{div}\mathbf{I}_D + \mathbf{K}\cdot\mathbf{I}) \\ &- e^2\mu_n\mu_p\sigma^{-3}[\theta\llbracket(\mu_p p^2 \operatorname{grad} n + \mu_n n^2 \operatorname{grad} p), \mathbf{I}, \mathbf{k}\rrbracket \\ &+ (\theta_p\sigma_n + \theta_n\sigma_p)(D_n n - D_p p)\llbracket\operatorname{grad} n, \operatorname{grad} p, \mathbf{k}\rrbracket], \qquad (21)\end{aligned}$$

where $\mathbf{I}_D$ and $\mathbf{K}$ are defined in (7) and (10) and θ is the sum of the magnitudes of the Hall angles, $\theta_p - \theta_n$. In deriving this continuity equation, use is made of curl $\mathbf{E} = 0$, which holds for steady applied magnetic field; time dependence of $\mathbf{I}$ generally has a quite negligible effect.[11] Use is also made of the relationships

$$\sigma\mathbf{E} = \mathbf{I} - e\,\mathrm{grad}(D_n n - D_p p) - \sigma^{-1}(\theta_p\sigma_p + \theta_n\sigma_n)\mathbf{I} \times \mathbf{k} - \theta\mathbf{I}_D \times \mathbf{k} \tag{22}$$

and

$$\begin{aligned} \llbracket(\theta_p n\,\mathrm{grad}\,p - \theta_n p\,\mathrm{grad}\,n),\mathbf{E},\mathbf{k}\rrbracket = {} & \sigma^{-1}\llbracket(\theta_p n\,\mathrm{grad}\,p - \theta_n p\,\mathrm{grad}\,n),\mathbf{I},\mathbf{k}\rrbracket \\ & + e\sigma^{-1}(\theta_p D_n n - \theta_n D_p p)\llbracket\mathrm{grad}\,n,\,\mathrm{grad}\,p,\mathbf{k}\rrbracket, \end{aligned} \tag{23}$$

which hold with the neglect of terms quadratic in Hall angles. Equation (23) is obtained in a straightforward manner from (22), which is obtained by writing total current density as

$$\begin{aligned} \mathbf{I} = {} & \sigma\mathbf{E} + e\,\mathrm{grad}(D_n n - D_p p) + (\theta_p\sigma_p + \theta_n\sigma_n)\mathbf{E} \times \mathbf{k} \\ & - e\,\mathrm{grad}(\theta_p D_p p - \theta_n D_n n) \times \mathbf{k} \end{aligned} \tag{24}$$

and then solving for $\mathbf{E}$. The terms on the right-hand side of (24) represent, respectively, drift, Dember, Hall and PME contributions.

A differential equation that expresses the solenoidal property of $\mathbf{I}$ is, from (24),

$$\begin{aligned} & \mathrm{div}[\sigma\mathbf{E} + (kT/e)\,\mathrm{grad}(\sigma_n - \sigma_r)] \\ & \quad + \llbracket\mathrm{grad}(\theta_p\sigma_p + \theta_n\sigma_n),\mathbf{E},\mathbf{k}\rrbracket = 0. \end{aligned} \tag{25}$$

If direct use must be made of this fundamental equation, then it is well to eliminate $\mathbf{I}$ from (21) by means of (24), and to employ the electrostatic potential V as one of the dependent variables.

The current densities are given in ambipolar form by

$$\mathbf{I}_p = (\sigma_p/\sigma)\mathbf{I} + \mathbf{I}^{\equiv}$$

and (26)

$$\mathbf{I}_n = (\sigma_n/\sigma)\mathbf{I} - \mathbf{I}^{\equiv},$$

where, if terms quadratic in Hall angles are neglected,

$$\mathbf{I}^{\equiv} \equiv \mathbf{I}_D + \theta(\sigma_n\sigma_p/\sigma^2)\mathbf{I} \times \mathbf{k} - \sigma^{-1}(\theta_p\sigma_n + \theta_n\sigma_p)\mathbf{I}_D \times \mathbf{k}. \tag{27}$$

Components of total current density perpendicular to the applied magnetic field are

$$I_x = \sigma E_x + e \frac{\partial}{\partial x} (D_n n - D_p p) + \sigma^{-1}(\theta_p \sigma_p + \theta_n \sigma_n) I_y + \theta I_{Dy} \quad (28)$$

and I_y, which is given by a similar expression obtained by interchanging x's and y's and (to retain a right-handed coordinate system by effectively reversing the direction of the z axis) changing the signs of the Hall angles. One way of deriving (28) is to substitute the expression obtained by solving for E_y in the equation for I_y, for E_y in the equation for I_x obtained from (24), and to neglect terms quadratic in Hall angles.

2.1.2 *Formulation for the Steady State in Terms of Trapping Ratios*

A number of results for the steady state can be established from the general differential equations without specifying in detail the trapping and recombination statistics. Differential "trapping ratios"

$$r_n \equiv d\hat{n}/dm, \qquad r_p \equiv d\hat{p}/dm \quad (29)$$

are introduced. These apply since, in the steady state, $\hat{n}$ and $\hat{p}$ each depend directly only on total concentration m of negative or positive charges. In the immediate context, r_n and r_p will be considered simply as factors that depend in general on m, which, multiplying grad m, give grad $\hat{n}$ and grad $\hat{p}$, respectively. They apply, of course, for any number of types of centers present. Their evaluation for particular models is given in Section 3.1 in connection with the more detailed analysis of the steady state.

With (29), it follows from (6) through (9) that the continuity equation for the steady state may be written as

$$\operatorname{div}\,(D' \operatorname{grad} \Delta m) - \mathbf{v}' \cdot \operatorname{grad} \Delta m + \Delta g - \Delta m/\tau_m = 0, \quad (30)$$

in which D' and $\mathbf{v}'$ are modified ambipolar diffusivity and drift velocity that are given by

$$\begin{aligned} D' &\equiv kT\mu_n\mu_p\sigma^{-1}[(1 - r_p)n + (1 - r_n)p] \\ &= [1 - (r_p n + r_n p)/(n + p)]D, \\ \mathbf{v}' &\equiv e\mu_n\mu_p\sigma^{-2}[(1 - r_p)n - (1 - r_n)p]\mathbf{I} \\ &= [1 - (r_p n - r_n p)/(n - p)]\mathbf{v}, \end{aligned} \quad (31)$$

and in which the net generation rate $g - \mathcal{R}_m$ has been written as the increment in this rate over thermal equilibrium, $\Delta g - \Delta m/\tau_m$, with Δg and Δm being the corresponding increments in g and m and τ_m a lifetime function for Δm. The modified diffusivity and velocity do not apply to time-dependent cases; $\mathbf{v}'$ would, for example, give the effect of applied field on apparent diffusion length, but is not, as will appear in Section 3.4.3, drift velocity for an injected pulse.

Expressing the concentration gradients for the steady state in terms of r_n or r_p and grad Δm also formally simplifies (14), the differential equation that must also be used if the current-flow geometry is not known. In connection with (18), the current-density equations, the procedure results in ambipolar diffusion current density given by

$$\mathbf{I}_D = -eD' \operatorname{grad} \Delta m \tag{32}$$

for the steady state.

The trapping ratios defined by (29) can assume negative as well as positive values: If centers of a given type trap mostly carriers of the opposite charge, then a negative trapping ratio obtains. Consider, for example, trapping in centers of the acceptor type, which are neutral or negatively charged. For these, positive r_n cannot exceed unity; it nearly equals unity if electron trapping is the predominant process, so that the excess trapped electron and mobile hole concentrations are substantially equal. If, however, hole trapping is the predominant process, then r_n is a large negative number, the increment in concentration of fixed negative charges being negative and balanced by the excess mobile electron concentration, so that m retains substantially its thermal-equilibrium value. Similar considerations apply to r_p for centers of the donor type. Thus, the trapping ratio is close to unity or a large negative number according to whether the centers predominantly trap carriers of the same charge or of the opposite charge.†

For a large negative trapping ratio, the comparatively small increments in m are associated with large magnitudes of D' and $\mathbf{v}'$, as (31) shows. A concentration variable other than Δm may then be more suitable. The equation in the linear combination $A\Delta n + B\Delta p$ (with constant A and B) of the excess mobile-carrier concentrations that results from (30) has diffusivity and velocity equal to D' and $\mathbf{v}'$ each divided by $A(1 - r_n) + B(1 - r_p)$, since, from (29), grad Δm is $[A(1 - r_n) + B(1 - r_p)]^{-1}$ grad $(A\Delta n + B\Delta p)$. In general, they are bounded in magnitude for all values of the trapping ratios that can occur.‡ In this equa-

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† For acceptor centers, say, of total concentration $\mathfrak{N}_1$, the trapping ratio $r_p' \equiv d(\mathfrak{N}_1 - \hat{n})/d(p + \mathfrak{N}_1 - \hat{n})$ for holes may be defined. The two ratios are symmetrically related: They may be interchanged in $r_p' = r_n/(r_n - 1)$ and, as one increases to unity, the other becomes negatively infinite.

‡ Note that $A(1 - r_n) + B(1 - r_p)$ equals $r_p - r_n$ for $A = 1$ and $B = -1$ and, if r_n and r_p are constants (as obtains under suitable small-signal restriction), also for $A = r_p$ and $B = -r_n$. It follows that $n - p$ and $r_pn - r_np$ are (under this restriction) both subject to diffusivity $D_0'/(r_p - r_n)$ and velocity $\mathbf{v}_0'/(r_p - r_n)$.

tion, the recombination term is properly written as $-(A\Delta n + B\Delta p)/(A\tau_n + B\tau_p)$, where τ_n and τ_p are lifetime functions for Δn and Δp. These functions are respectively equal to $\Delta n/\mathcal{R}_m$ and $\Delta p/\mathcal{R}_m$ since, from (3), $\mathcal{R}_n = \mathcal{R}_p = \mathcal{R}_m$ holds for the steady state.

The equilibrium lifetimes for electrons and holes differ in general, but are nevertheless always associated with the same diffusion length. This result follows readily from (30), whose linear small-signal form is

$$D_0' \operatorname{div} \operatorname{grad} \Delta m - \mathbf{v}_0' \cdot \operatorname{grad} \Delta m + \Delta g - \Delta m/\tau_m = 0, \tag{33}$$

the zero subscripts denoting thermal-equilibrium values.† The lifetime function τ_m is here constant; and, since Δn and Δp equal $(1 - r_n)\Delta m$ and $(1 - r_p)\Delta m$, with r_n and r_p the thermal-equilibrium trapping ratios, (33) implies

$$\begin{aligned}(1 - r_n)^{-1}D_0' \operatorname{div} \operatorname{grad} \Delta n &- (1 - r_n)^{-1}\mathbf{v}_0' \cdot \operatorname{grad} \Delta n \\ &+ \Delta g - \Delta n/(1 - r_n)\tau_m = 0\end{aligned} \tag{34}$$

for electrons and a similar equation for holes. Thus, for Δn the lifetime is τ_m multiplied by $(1 - r_n)$, while — as may be established in greater generality from (29) and (30) — the diffusivity and velocity are those for Δm multiplied by the reciprocal of this factor, and similarly for Δp. It follows, in particular, that the product of equilibrium diffusivity and lifetime, which is the square of L_0, the diffusion length, is the same for Δn, Δp and Δm, independently of the particular trapping and recombination statistics.[9] A "diffusion-length lifetime" τ_0, based on the unmodified ambipolar diffusivity D_0, may accordingly be defined by[61]

$$\begin{aligned}\tau_0 \equiv L_0^2/D_0 = (D_0'/D_0)\tau_m &= [1 - (r_p n_0 + r_n p_0)/(n_0 + p_0)]\tau_m \\ &= (n_0\tau_p + p_0\tau_n)/(n_0 + p_0),\end{aligned} \tag{35}$$

in which τ_p and τ_n are the equilibrium lifetimes for Δp and Δn. The more detailed analysis of Section 3.1.1 includes evaluation of the single diffusion length and lifetime τ_0 that correspond to the (equilibrium) Shockley-Read electron and hole lifetimes. Diffusion-length lifetime for recombination in the presence of nonrecombinative traps is evaluated in Section 3.3.

For the steady-state formulation that includes applied magnetic field, it is readily shown that (21), the continuity equation, assumes the form of (30) if $\mathbf{v}'$ is redefined in accordance with

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† As shown in Section 3.1.1, the required small-signal restriction may be more severe than that given in Ref. 10 for the no-trapping case.

$$\mathbf{v}' \equiv e\mu_n\mu_p\sigma^{-2}\{[(1 - r_p)n - (1 - r_n)p]\mathbf{I} + \theta(e/\sigma)[(1 - r_p)\mu_n n^2 + (1 - r_n)\mu_p p^2]\mathbf{I} \times \mathbf{k}\}. \tag{36}$$

Note that the second scalar product in (21) vanishes, since steady-state concentration gradients are collinear vectors. For this case, use of the trapping ratios formally simplifies (25), as well as (27) and (28), which involve the form for $\mathbf{I}_D$ of (32).

2.2 *Mass-Action Theory*

2.2.1 *Single-Level Centers of Two Types*

In this section, centers of both the acceptor and donor types are assumed to be present, namely centers that can have respectively single negative or positive charges or be neutral. By use of a suitable convention, the equations apply, in effect, to the more general model of two types of centers each of which has two states of charge (which differ by one electronic charge). On the basis of equations of this section, theory for centers of a single type but with two energy levels or three states of charge is given in Section 2.2.2.

Under the assumption of mass-action interactions, the equations

$$\begin{aligned} g - \mathcal{R}_m &= g - C_i np - C_{p1}[p\hat{n} - p_1(\mathcal{N}_1 - \hat{n})] \\ &\qquad - C_{n2}[n\hat{p} - n_2(\mathcal{N}_2 - \hat{p})], \\ \partial\hat{n}/\partial t &= \mathcal{R}_n - \mathcal{R}_m = C_{n1}[n(\mathcal{N}_1 - \hat{n}) - n_1\hat{n}] \\ &\qquad - C_{p1}[p\hat{n} - p_1(\mathcal{N}_1 - \hat{n})], \\ \partial\hat{p}/\partial t &= \mathcal{R}_p - \mathcal{R}_m = -C_{n2}[n\hat{p} - n_2(\mathcal{N}_2 - \hat{p})] \\ &\qquad + C_{p2}[p(\mathcal{N}_2 - \hat{p}) - p_2\hat{p}] \end{aligned} \tag{37}$$

hold. The first equation gives $\mathcal{R}_m$, and it (as well as the other two) is obtained by considering the photoconductive case of uniform concentration and no transport, $g - \mathcal{R}_m$ being the contribution to $\partial m/\partial t$ that does not involve transport. Four processes are taken into account for each type of center. In the second equation, for example, the term $C_{p1}p\hat{n}$ is the volume rate of neutralization of fixed negative charges by holes; C_{p1} is a phenomenological capture coefficient,† which depends in general on temperature and not on concentration. The second term in the same brackets gives the rate for the inverse process, $C_{p1}p_1$ being the emission

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† In the terminology of Sah and Shockley[22] this quantity is called a capture probability.

coefficient for hole emission from a neutral acceptor center. Here $\mathfrak{N}_1$ is the total concentration of the acceptor centers, and the concentration p_1, constant at given temperature, is defined by the condition that the quantity in brackets vanishes at thermal equilibrium, in accordance with detailed balance. The preceding brackets relate to the interactions of the same centers with electrons, the term $C_{n1}n(\mathfrak{N}_1 - \hat{n})$ being the volume rate of capture of electrons by the neutral acceptor centers and $C_{n1}n_1$ being the coefficient for electron emission from the charged ones. The concentrations n_1 and p_1 are those of the Hall-Shockley-Read theory, and are here introduced without explicit reference to Boltzmann statistics.[2,13,14] The third equation expresses the dependence of $\partial\hat{p}/\partial t$ on the analogous processes for the donor centers. In the first equation, which includes the rate C_inp of direct electron-hole recombination, only interactions that change the total concentration m are involved.

The sign or magnitude of the charge that a center can assume is not of material significance in the analysis of this section; although written symmetrically for fixed charges of both signs, (37) may formally be transformed so as to apply to two types of donor or acceptor centers. This possibility is related to the circumstance that the fixed charges are not properly considered as trapped carriers, since the trapping processes are manifest through changes in fixed-charge concentrations rather than in these concentrations themselves. For example, centers of the acceptor type function as electron or hole traps according to whether the concentration of the charged centers increases or decreases with carrier injection. Consistent with the discussion in Section 2.1.2 of the steady-state trapping ratios, either type of center may be considered alternatively as an electron trap or a hole trap, under the convention that a change in fixed-charge concentration resulting from trapping may be negative as well as positive. To establish this result from (37), write the two equations that apply for, say, acceptor centers only. Then transform these so that the concentration $\mathfrak{N}_1 - \hat{n}$ of neutral centers becomes concentration of fixed positive charges $\hat{p}$, and the concentration $\hat{n}$ of charged centers becomes concentration $\mathfrak{N}_2 - \hat{p}$ of neutral centers; note that a given increase in the original $\hat{n}$ is equivalent to the corresponding decrease in the new $\hat{p}$. New equations then result that (with the replacement of C_{n1} and C_{p1} respectively by C_{n2} and C_{p2}) are the ones that follow directly from (37) for donor centers only.

2.2.1.1 *Thermal-Equilibrium Relationships.* The definitions

$$\begin{aligned} n_1 &\equiv n_0(\mathfrak{N}_1 - \hat{n}_0)/\hat{n}_0\,, \qquad & p_1 &\equiv p_0\hat{n}_0/(\mathfrak{N}_1 - \hat{n}_0), \\ n_2 &\equiv n_0\hat{p}_0/(\mathfrak{N}_2 - \hat{p}_0), \qquad & p_2 &\equiv p_0(\mathfrak{N}_2 - \hat{p}_0)/\hat{p}_0 \end{aligned} \tag{38}$$

are required by detailed balance. It is evident from these equations that

$$n_1p_1 = n_2p_2 = n_0p_0 = n_i^2 \tag{39}$$

holds, where n_i is the thermal-equilibrium electron or hole concentration in intrinsic material. Note that (39) states, in effect, that the product — $(C_{n1}n_1)(C_{p1}p_1)$ or $(C_{n2}n_2)(C_{p2}p_2)$ — of the electron and hole emission coefficients equals n_i^2 times the product of the corresponding capture coefficients.[2,22,28]

If the concentrations in the right-hand members of (38) are known, then n_1, p_1, n_2 and p_2 are, of course, determined. Certain relationships hold between the concentrations. Since $n_0 - p_0 \equiv n_s$ equals $\hat{p}_0 - \hat{n}_0$ from (1), the neutrality condition, this condition and the last equation of (39) give

$$n_0 = \tfrac{1}{2}\{[(\hat{p}_0 - \hat{n}_0)^2 + 4n_i^2]^{\frac{1}{2}} + (\hat{p}_0 - \hat{n}_0)\}$$

and (40)

$$p_0 = \tfrac{1}{2}\{[(\hat{p}_0 - \hat{n}_0)^2 + 4n_i^2]^{\frac{1}{2}} - (\hat{p}_0 - \hat{n}_0)\}.$$

It is readily found from (38) that fractions of charged acceptor and donor centers are given respectively by

$$\hat{n}_0/\mathfrak{N}_1 = (1 + \alpha_{10}^{-1})^{-1}, \qquad \hat{p}_0/\mathfrak{N}_2 = (1 + \alpha_{20}^{-1})^{-1}, \tag{41}$$

with

$$\begin{aligned} \alpha_{10} &\equiv n_0/n_1 = p_1/p_0 = \tfrac{1}{2}n_1^{-1}[(n_s^2 + 4n_i^2)^{\frac{1}{2}} + n_s], \\ \alpha_{20} &\equiv p_0/p_2 = n_2/n_0 = \tfrac{1}{2}p_2^{-1}[(n_s^2 + 4n_i^2)^{\frac{1}{2}} - n_s]; \end{aligned} \tag{42}$$

the final expressions on the right follow by use of (40). For given semiconductor material at given temperature, n_i is known and n_s is determined by conductivity type and conductivity σ_0, and n_1 and n_2 (and hence p_1 and p_2 also) are accordingly determined by the fractions of charged centers. Expressing the thermal-equilibrium concentrations of mobile carriers and fixed charges in terms of each other (with other concentrations as parameters) thus involves roots of quadratic equations. The relationships given apply regardless of the number of kinds of centers present, since (40) contains no quantities pertaining to particular centers, and each equation of (41) and (42) contains quantities pertaining only to a single kind of center.

On the other hand, the fixed-charge and mobile-carrier concentrations for centers of two kinds are obtainable in general in terms of $\mathfrak{N}_1$, $\mathfrak{N}_2$, n_1, n_2 and n_i^2. It will suffice to indicate that the concentrations are roots of biquadratics that follow readily from the equations

$$\begin{aligned} m_0 &= \hat{n}_0 + n_1\hat{n}_0/(\mathfrak{N}_1 - \hat{n}_0) = \hat{p}_0 + p_1(\mathfrak{N}_1 - \hat{n}_0)/\hat{n}_0 , \\ m_0 &= \hat{n}_0 + n_2(\mathfrak{N}_2 - \hat{p}_0)/\hat{p}_0 = \hat{p}_0 + p_2\hat{p}_0/(\mathfrak{N}_2 - \hat{p}_0) \end{aligned} \tag{43}$$

for $\hat{n}_0$ and $\hat{p}_0$ and

$$\begin{aligned} m_0 &= \alpha_{10}[n_1 + \mathfrak{N}_1/(1 + \alpha_{10})] = \alpha_{20}[p_2 + \mathfrak{N}_2/(1 + \alpha_{20})], \\ \alpha_{10}\alpha_{20} &= n_2/n_1 = p_1/p_2 \end{aligned} \tag{44}$$

for α_{10} and α_{20}, and hence n_0 and p_0. Equations (43) and (44) are obtained from (38) by eliminating, respectively, the concentrations of mobile carriers and fixed charges by use of the neutrality condition.† They are equivalent to combining (40) with (41) and (42), which are accordingly subject to a requirement of mutual consistency. For example, temperature determines n_i^2 for a given semiconductor; specifying conductivity also then determines n_0 and p_0; specifying further n_1 and n_2 determines $\hat{n}_0/\mathfrak{N}_1$ and $\hat{p}_0/\mathfrak{N}_2$ from (41) and (42), but only one of $\mathfrak{N}_1$ and $\mathfrak{N}_2$ can now be independently specified, since $\hat{p}_0 - \hat{n}_0$ must equal $n_0 - p_0$.

Through familiar considerations involving equilibrium Boltzmann statistics, the concentration n_1 or p_1 (and n_2 or p_2) has been shown to equal electron concentration in the conduction band or hole concentration in the valence band for the Fermi level coincident with the energy level of the centers.[31] The relationship

$$n_1 = n_i^2/p_1 = n_i e^{(\mathcal{E}_1 - \mathcal{E})/kT} = n_i e^{[e(\Phi - \Psi_1)/kT]} \tag{45}$$

for acceptor centers is here employed, and a similar one is used for donor centers. Here $\Psi_1 \equiv -e^{-1}\mathcal{E}_1$ and $\Phi \equiv -e^{-1}\mathcal{E}$ are the equivalent electrostatic potentials of the energy level $\mathcal{E}_1$ of the centers and the Fermi energy $\mathcal{E}$ for intrinsic material. This relationship is more phenomenological than those involving the energies of the conduction-band and valence-band edges and which give n_1 and p_1 in units of the effective densities of states in the bands. Note that the temperature dependence of the energy gap is involved through n_i, while the difference between the effective densities of states or the effective masses with nonspherical energy surfaces in momentum space is reflected simply in a difference between Φ and the midgap potential. If statistical weights associated with spin degeneracy are taken into account, then the definitions of (38) are of course retained, but (45) is modified. The right-hand members (for n_1) are multiplied

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† It is easily seen that cubics result for centers of one kind only, or if complete ionization obtains for one of two kinds of centers.

by two; the exponentials for p_1 are multiplied by one-half. In the similar result for donor centers, the exponentials for n_2 and p_2 are multiplied by one-half and two, respectively. For given n_1 and n_2, these modifications[17,68,69] produce comparatively minor changes in $\mathcal{E}_1$ and $\mathcal{E}_2$ or Ψ_1 and Ψ_2.

2.2.1.2 *Equations in Concentration Increments; Trapping and Release Frequencies and Times.* For detailed analysis, it is advantageous to replace (37) by equations in the increments Δm, $\Delta \hat{n}$ and $\Delta \hat{p}$ in m, $\hat{n}$ and $\hat{p}$ over their thermal-equilibrium values. By subtracting from (37) the corresponding thermal-equilibrium equations, in which the time derivatives and the quantities in the various square brackets are zero, the result

$$\begin{aligned}
\Delta g - \Delta \mathcal{R}_m &= \Delta g + \nu_{11}\Delta m + \nu_{12}\Delta \hat{n} + \nu_{13}\Delta \hat{p} - C_i \Delta n \Delta p \\
&\qquad - C_{p1}\Delta p \Delta \hat{n} - C_{n2}\Delta n \Delta \hat{p}, \\
\partial \Delta \hat{n}/\partial t &= \Delta \mathcal{R}_n - \Delta \mathcal{R}_m = \nu_{21}\Delta m + \nu_{22}\Delta \hat{n} + \nu_{23}\Delta \hat{p} \\
&\qquad - (C_{n1}\Delta n + C_{p1}\Delta p)\Delta \hat{n}, \\
\partial \Delta \hat{p}/\partial t &= \Delta \mathcal{R}_p - \Delta \mathcal{R}_m = \nu_{31}\Delta m + \nu_{32}\Delta \hat{n} + \nu_{33}\Delta \hat{p} \\
&\qquad - (C_{n2}\Delta n + C_{p2}\Delta p)\Delta \hat{p}
\end{aligned} \tag{46}$$

follows, in which the decay constants of what will be referred to as the "ν_{ij} notation" are given by

$$\begin{aligned}
\nu_{11} &\equiv -C_i(n_0 + p_0) - C_{p1}\hat{n}_0 - C_{n2}\hat{p}_0, \\
\nu_{12} &\equiv C_i p_0 - C_{p1}(p_0 + p_1) + C_{n2}\hat{p}_0, \\
\nu_{13} &\equiv C_i n_0 - C_{n2}(n_0 + n_2) + C_{p1}\hat{n}_0, \\
\nu_{21} &\equiv C_{n1}(\mathcal{N}_1 - \hat{n}_0) - C_{p1}\hat{n}_0, \\
\nu_{22} &\equiv -C_{n1}(\mathcal{N}_1 - \hat{n}_0 + n_0 + n_1) - C_{p1}(p_0 + p_1), \\
\nu_{23} &\equiv C_{p1}\hat{n}_0, \\
\nu_{31} &\equiv C_{p2}(\mathcal{N}_2 - \hat{p}_0) - C_{n2}\hat{p}_0, \\
\nu_{32} &\equiv C_{n2}\hat{p}_0, \\
\nu_{33} &\equiv -C_{n2}(n_0 + n_2) - C_{p2}(\mathcal{N}_2 - \hat{p}_0 + p_0 + p_2).
\end{aligned} \tag{47}$$

Zero subscripts denote thermal equilibrium values. Note that $\Delta \mathcal{R}_m$, $\Delta \mathcal{R}_n$ and $\Delta \mathcal{R}_p$ are respectively $\mathcal{R}_m$, $\mathcal{R}_n$ and $\mathcal{R}_p$ minus $g_0 = C_i n_i^2$. In (46), in which Δm, $\Delta \hat{n}$ and $\Delta \hat{p}$ are to be considered as dependent variables, the quadratic terms have been written compactly with Δn and Δp, which may be replaced by $\Delta m - \Delta \hat{n}$ and $\Delta m - \Delta \hat{p}$; and n_0 and p_0 in (47) may be replaced by $m_0 - \hat{n}_0$ and $m_0 - \hat{p}_0$.

It is desirable to supplement the ν_{ij} notation with another notation, which, although it often results in less compact expressions, facilitates physical interpretations. If volume generation and direct recombination are neglected for the present, the respective contributions to $\partial \Delta n/\partial t$ and $\partial \Delta p/\partial t$ other than the terms involving transport processes as such may be written as

$$\begin{aligned} -\Delta \mathcal{R}_n = -\Delta \mathcal{R}_m - \partial \Delta \hat{n}/\partial t &= -\nu_{tn1}\Delta n + \nu_{gn1}\Delta \hat{n} \\ &\quad - \nu_{tn2}\Delta n + \nu_{gn2}\Delta(\mathcal{N}_2 - \hat{p}), \\ -\Delta \mathcal{R}_p = -\Delta \mathcal{R}_m - \partial \Delta \hat{p}/\partial t &= -\nu_{tp1}\Delta p + \nu_{gp1}\Delta(\mathcal{N}_1 - \hat{n}) \\ &\quad - \nu_{tp2}\Delta p + \nu_{gp2}\Delta \hat{p}. \end{aligned} \tag{48}$$

The top and bottom rows of the forms on the right give the respective contributions of the acceptor and donor centers. The decay constants may be identified as certain capture and release frequencies by comparison with the equations

$$\begin{aligned} -\Delta \mathcal{R}_n &= -C_{n1}[(\mathcal{N}_1 - \hat{n}_0)\Delta n - n_0\Delta \hat{n} - \Delta n \Delta \hat{n}] + C_{n1}n_1\Delta \hat{n} \\ &\quad - C_{n2}[\hat{p}_0\Delta n + n_0\Delta \hat{p} + \Delta n \Delta \hat{p}] + C_{n2}n_2\Delta(\mathcal{N}_2 - \hat{p}), \\ -\Delta \mathcal{R}_p &= -C_{p1}[\hat{n}_0\Delta p + p_0\Delta \hat{n} + \Delta p \Delta \hat{n}] + C_{p1}p_1\Delta(\mathcal{N}_1 - \hat{n}) \\ &\quad - C_{p2}[(\mathcal{N}_2 - \hat{p}_0)\Delta p - p_0\Delta \hat{p} - \Delta p \Delta \hat{p}] + C_{p2}p_2\Delta \hat{p}, \end{aligned} \tag{49}$$

which follow from (37). In (49), the magnitudes of the contributions involving brackets are capture rates, while the other terms on the right are release rates.

Expression of the capture rates in terms of capture frequencies would require writing them with Δn or Δp as a factor, and would thus necessitate solution of the particular problem. These physical capture frequencies would depend in general on coordinates and time. The contributions to the capture rates that contain $\Delta \hat{n}$ and $\Delta \hat{p}$ as factors are associated, however, with trap saturation: These contributions, for carriers of given charge, represent the decreases and increases in capture rate with the filling of centers that assume, respectively, the same and the opposite charges. They may, in a phenomenological sense, be deleted from the capture rates and assigned to the release rates. The "effective" capture and release rates that result from this procedure are clearly rates in terms of which the capture and release frequencies of (48) and related times may be defined as follows:

Electron capture by neutral acceptors:

$$\nu_{tn1} \equiv \tau_{tn1}^{-1} \equiv C_{n1}(\mathfrak{N}_1 - \hat{n}_0) = C_{n1}\mathfrak{N}_1/(1 + \alpha_{10}),$$

Electron release from charged acceptors:

$$\nu_{gn1} \equiv \tau_{gn1}^{-1} \equiv C_{n1}(n + n_1),$$

Hole capture by charged acceptors:

$$\nu_{tp1} \equiv \tau_{tp1}^{-1} \equiv C_{p1}\hat{n}_0 = C_{p1}\mathfrak{N}_1\alpha_{10}/(1 + \alpha_{10}),$$

Hole release from neutral acceptors:

$$\nu_{gp1} \equiv \tau_{gp1}^{-1} \equiv C_{p1}(p + p_1), \tag{50}$$

Electron capture by charged donors:

$$\nu_{tn2} \equiv \tau_{tn2}^{-1} \equiv C_{n2}\hat{p}_0 = C_{n2}\mathfrak{N}_2\alpha_{20}/(1 + \alpha_{20}),$$

Electron release from neutral donors:

$$\nu_{gn2} \equiv \tau_{gn2}^{-1} \equiv C_{n2}(n + n_2),$$

Hole capture by neutral donors:

$$\nu_{tp2} \equiv \tau_{tp2}^{-1} \equiv C_{p2}(\mathfrak{N}_2 - \hat{p}_0) = C_{p2}\mathfrak{N}_2/(1 + \alpha_{20}),$$

Hole release from charged donors:

$$\nu_{gp2} \equiv \tau_{gp2}^{-1} \equiv C_{p2}(p + p_2).$$

The second forms given for capture frequencies follow by use of (41). Note, for example, that ν_{tn1} is an average frequency per electron of electron capture by a neutral acceptor center and hence the reciprocal of the corresponding electron capture or trapping time, τ_{tn1}; and that ν_{gn1} is an average frequency per charged center of electron release from a charged acceptor center and hence the reciprocal of the corresponding electron release time, τ_{gn1}. The saturation terms that originate from the true capture rates appear as the contributions from n and p in the "effective" release frequencies, while the "effective" capture frequencies do not depend on the injection level.

It is readily seen that, if direct recombination is neglected, then an alternative procedure† for including the quadratic terms in (46) is to

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† Another alternative procedure is to replace $\hat{n}_0$ by $\hat{n}$ and $\hat{p}_0$ by $\hat{p}$ or, more generally, by increasing $\hat{n}_0$ and $\hat{p}_0$ by a fraction γ of $\Delta\hat{n}$ and of $\Delta\hat{p}$ and n_0 and p_0 by the fraction $1 - \gamma$ of Δn and of Δp. The definitions of (50), which correspond to $\gamma = 0$, are then modified, and depend on γ, the fraction of the quadratic terms assigned to capture.

generalize the ν_{ij} by replacing n_0 and p_0 in (47) by n and p. The formal equivalence of (46) and (48) that results then permits expressing these generalized ν_{ij} in terms of the effective capture and release frequencies of (50), as follows:

$$\begin{aligned}
\nu_{11} &= -\nu_{tp1} - \nu_{tn2} \,, \\
\nu_{12} &= -\nu_{gp1} + \nu_{tn2} \,, \\
\nu_{13} &= \nu_{tp1} - \nu_{gn2} \,, \\
\nu_{21} &= \nu_{tn1} - \nu_{tp1} \,, \\
\nu_{22} &= -\nu_{tn1} - \nu_{gn1} - \nu_{gp1} \,, \\
\nu_{23} &= \nu_{tp1} \,, \\
\nu_{31} &= -\nu_{tn2} + \nu_{tp2} \\
\nu_{32} &= \nu_{tn2} \,, \\
\nu_{33} &= -\nu_{gn2} - \nu_{tp2} - \nu_{gp2} \,.
\end{aligned} \tag{51}$$

Note that

$$\nu_{11} + \nu_{23} + \nu_{32} = 0 \tag{52}$$

holds for this case of no direct recombination.

The four effective trapping and release times or frequencies for each type of center satisfy a fundamental restriction, namely:

$$\frac{\tau_{gnj}\tau_{tpj}}{\tau_{tnj}\tau_{gpj}} = \frac{\nu_{tnj}\nu_{gpj}}{\nu_{gnj}\nu_{tpj}} = \frac{p_0}{n_0}\,\frac{1 + \Delta p/(p_0 + p_j)}{1 + \Delta n/(n_0 + n_j)}\,, \qquad j = 1, 2. \tag{53}$$

Thus, only three are independent. As will appear, this restriction is widely useful for calculations and physical interpretations. It is essentially a consequence of detailed balance: For thermal equilibrium, it follows readily from relationships tantamount to this principle, such as (41) and (42) or the definitions of (38). The factor on the right that depends on Δn and Δp results simply from the concentration dependence of the effective release frequencies.

A property easily established from (42) and (50) is the following: For centers of given capture coefficients and energy level, if the electron and hole capture frequencies are equal for a given conductivity, then the equilibrium release frequencies are equal for material of the opposite conductivity type and the same value of $| n_0 - p_0 |$, that is, for material such that the values of n_0 and p_0 are, in effect, interchanged.

2.2.1.3 *Trapping and Recombination Ranges; Shallow and Deep Traps.* Three linear small-signal ranges, characterized respectively primarily by minority-carrier trapping, recombination and majority-carrier trapping, may be defined for each type of center by use of (53). The "minority-carrier trapping range" is defined by the condition that the equilibrium minority-carrier to majority-carrier release frequency ratio exceeds unity. In p-type material, this ratio, ν_{gnj}/ν_{gpj}, is $C_{nj}n_j/C_{pj}p_0 = C_{nj}n_0/C_{pj}p_j$, from (39), (41), (42) and (50); and, from (53), ν_{tnj}/ν_{tpj} is larger by the factor p_0/n_0. The "majority-carrier trapping range" is defined by the condition that the majority- to minority-carrier capture frequency ratio exceeds unity, for which the equilibrium majority- to minority-carrier release frequency ratio is larger by the factor p_0/n_0 for p-type material, or by n_0/p_0 for n-type. The "recombination range" is defined as that not included in either trapping range. Thus, the recombination range is given by $n_0/n_j = p_j/p_0 \leqq C_{nj}/C_{pj} \leqq p_j/n_0 = p_0/n_j$ for p-type material, the electron-trapping range by $C_{nj}/C_{pj} > p_j/n_0 = p_0/n_j$, and the hole-trapping range by $C_{nj}/C_{pj} < p_j/p_0 = n_0/n_j$. A "minority-carrier capture range", which includes the trapping and recombination ranges, may be defined by $\nu_{tnj}/\nu_{tpj} > 1$. Similar results, obtainable by interchanging n and p, hold for n-type material. Ranges of minority-carrier-dominated and majority-carrier-dominated transitions[22,28] are those parts of the trapping ranges here considered for which strong inequalities hold. Equal capture frequencies, which occur at the boundary between the recombination and majority-carrier trapping ranges, result in what will be termed "critical recombination", with which, as will be seen, $\Delta\hat{n}$ or $\Delta\hat{p}$ is zero.

The three ranges may be specified in terms of the equality densities. These are the equilibrium carrier concentrations for the Fermi level coincident with the equality level. They are defined in the present context by

$$\begin{gathered} n_j^* \equiv C_{pj}p_j/C_{nj} = p_0\nu_{tpj}/\nu_{tnj} = n_0\nu_{gpj}/\nu_{gnj} \\ \text{and} \\ p_j^* \equiv C_{nj}n_j/C_{pj} = n_0\nu_{tnj}/\nu_{tpj} = p_0\nu_{gnj}/\nu_{gpj}, \end{gathered} \tag{54}$$

in which the release frequencies are equilibrium values. Thus, the recombination range is given by $n_0 \leqq p_j^* \leqq p_0$ or $p_0 \geqq n_j^* \geqq n_0$ for p-type material, the electron-trapping range by $n_j^* < n_0$ or $p_j^* > p_0$ and the hole-trapping range by $n_j^* > p_0$ or $p_j^* < n_0$, and similarly for n-type material. The ranges may evidently also be specified in terms of the equality level, the Fermi level $\mathcal{E}$ for intrinsic material, the actual Fermi level $\mathcal{E}_F$ and the "reflected Fermi level," $\mathcal{E}_F' \equiv 2\mathcal{E} - \mathcal{E}_F$, the reflection of $\mathcal{E}_F$ about $\mathcal{E}$: For the recombination range, the equality level is between $\mathcal{E}_F$ and $\mathcal{E}_F'$; for the minority-carrier trapping range, it is between $\mathcal{E}_F$ and the edge of the majority-carrier band; and for the majority-carrier trapping range, it is between $\mathcal{E}_F'$ and the edge of the minority-carrier band. Note that, if the capture coefficients are equal, then $n_j^* = p_j$ (or $p_j^* = n_j$) holds and the respective trapping ranges are given by conditions on the trapping level $\mathcal{E}_j$ obtained by interchanging those on the equality level.

The volume rates of electron and of hole transitions at equilibrium are respectively $C_{n1}n_0(\mathfrak{N}_1 - \hat{n}_0) = C_{n1}n_1\hat{n}_0 = n_0\nu_{tn1}$ and $C_{p1}p_0\hat{n}_0 = C_{p1}p_1(\mathfrak{N}_1 - \hat{n}_0) = p_0\nu_{tp1}$ for acceptor-type centers. From (53), these rates are proportional to[†] ν_{gn1} and ν_{gp1}. Hence each definition given for a trapping range insures that the transition rate at equilibrium for the particular carriers is the larger, and also that the transition rate ν_{tn1} or ν_{tp1} per mobile carrier is the larger too. The asymmetrical relationship between the definitions for minority- and majority-carrier trapping reflects the circumstance that a transition rate will be the larger if either the cross section or the concentration of the particular carriers is sufficiently large. The recombination range is that for which a larger transition rate per mobile minority carrier is associated with a total transition rate for majority carriers which is the larger.

For shallow minority-carrier traps, since relatively few are occupied by minority carriers at equilibrium so that they can capture majority carriers, the condition for the minority-carrier trapping range may be met even though the capture coefficients are comparable in magnitude. For deep traps, since relatively few can capture minority carriers, the minority-carrier trapping generally requires a minority-carrier capture coefficient considerably the larger. Suitable condition for "shallow" and "deep" traps are, in view of the condition on C_{nj}/C_{pj} for the electron-trapping range, respectively $p_j \ll n_0$ (or $n_j \gg p_0$) and $n_j \ll p_0$ (or $p_j \gg n_0$) in p-type material. That is, "shallow" and "deep" traps for minority carriers are appreciably removed from the reflected Fermi level $\mathcal{E}_F'$, towards the edges of the minority- or majority-carrier band. Similarly, for majority-carrier trapping, "shallow" and "deep" traps are appreciably removed from the Fermi level $\mathcal{E}_F$, towards the edges of the majority- or minority-carrier band, respectively.

[†] They equal ν_{gn1} and ν_{gp1} times the capture concentration, as shown by (63) in Section 3.1.1.

2.2.2 *Centers with Two Energy Levels*

The formalism for centers of two types is readily modified to yield equations for one type of center with two energy levels. With the assumption that the centers can each assume single negative or positive charge or be neutral, $\hat{n}$ and $\hat{p}$ denote concentrations of centers in the respective charged states. It is thus clear that the fundamental mass-action equations for this case are formally the same as (37), with the modification that both $\mathfrak{N}_1 - \hat{n}$ and $\mathfrak{N}_2 - \hat{p}$ are replaced by $\mathfrak{N} - \hat{n} - \hat{p}$, where $\mathfrak{N}$ is the total concentration of the centers.

For thermal equilibrium, definitions of n_1, p_1, n_2 and p_2 apply that are equations of (38) with both $\mathfrak{N}_1 - \hat{n}_0$ and $\mathfrak{N}_2 - \hat{p}_0$ replaced by $\mathfrak{N} - \hat{n}_0 - \hat{p}_0$. It follows that the restriction

$$n_0^2/n_1n_2 = p_1p_2/p_0^2 = \hat{n}_0/\hat{p}_0 \tag{55}$$

holds for this two-level case. As is easily verified, (39) and (40) still apply, while the fractions of charged centers are

$$\begin{aligned}
\hat{n}_0/\mathfrak{N} &= (1 + n_1/n_0 + n_1n_2/n_0^2)^{-1} \\
&= (1 + p_0/p_1 + p_0^2/p_1p_2)^{-1} \\
&= \alpha_{10}/(1 + \alpha_{10} + \alpha_{20}), \\
\hat{p}_0/\mathfrak{N} &= (1 + p_2/p_0 + p_1p_2/p_0^2)^{-1} \\
&= (1 + n_0/n_2 + n_0^2/n_1n_2)^{-1} \\
&= \alpha_{20}/(1 + \alpha_{10} + \alpha_{20}),
\end{aligned} \tag{56}$$

with α_{10} and α_{20} given by (42). The modifications of (43) and (44) for the biquadratics are the replacement of $\mathfrak{N}_1 - \hat{n}_0$ and $\mathfrak{N}_2 - \hat{p}_0$ by $\mathfrak{N} - \hat{n}_0 - \hat{p}_0$ and of $\mathfrak{N}_1/(1 + \alpha_{10})$ and $\mathfrak{N}_2(1 + \alpha_{20})$ by $\mathfrak{N}/(1 + \alpha_{10} + \alpha_{20})$. Note that (55) is not an independent equation, in that it is implicit in the modified (43).† Relationships formally identical with (45) give n_1 and n_2 in terms of the two energy levels.

The fundamental two-level mass-action equations for no direct recombination yield the equations in concentration increments

$$\begin{aligned}
\Delta g - \Delta\mathcal{R}_m &= \Delta g - (\nu_{tp1} + \nu_{tn2})\Delta m + (\nu_{tn2} - \nu_{gp1} - C_{n2}n_2)\Delta\hat{n} \\
&\quad + (\nu_{tp1} - \nu_{gn2} - C_{p1}p_1)\Delta\hat{p}, \\
\partial\Delta\hat{n}/\partial t &= (\nu_{tn1} - \nu_{tp1})\Delta m - (\nu_{tn1} + \nu_{gn1} + \nu_{gp1})\Delta\hat{n} \\
&\quad + (\nu_{tp1} - C_{n1}n - C_{p1}p_1)\Delta\hat{p}, \\
\partial\Delta\hat{p}/\partial t &= (\nu_{tp2} - \nu_{tn2})\Delta m + (\nu_{tn2} - C_{p2}p - C_{n2}n_2)\Delta\hat{n} \\
&\quad - (\nu_{tp2} + \nu_{gp2} + \nu_{gn2})\Delta\hat{p},
\end{aligned} \tag{57}$$

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† The corresponding restriction for two types of centers has $(\hat{n}_0/\hat{p}_0)\ (\mathfrak{N}_2 - \hat{p}_0)/(\mathfrak{N}_1 - \hat{n}_0)$ as right-hand member.

from which appropriate ν_{ij} can immediately be identified. Effective capture and release frequencies are here employed whose definitions are provided by (50) if $\mathfrak{N}_1 - \hat{n}_0$ and $\mathfrak{N}_2 - \hat{p}_0$ are replaced by $\mathfrak{N} - \hat{n}_0 - \hat{p}_0$, and $\mathfrak{N}_1/(1 + \alpha_{10})$ and $\mathfrak{N}_2/(1 + \alpha_{20})$ replaced by $\mathfrak{N}_2/(1 + \alpha_{10} + \alpha_{20})$. Aside from these modified definitions, the equations of (57) are formally identical with equations for two kinds of centers except for the additional terms in which the capture coefficients appear. These are "constraint" terms. The ones in $\partial\Delta\hat{n}/\partial t$ represent the decrease in this rate that results from the decrease in the concentration of neutral centers associated with an increase in $\hat{p}$; neutral centers capturing electrons and emitting holes are the two processes that increase $\hat{n}$. The rate decrease $C_{n1}n\Delta\hat{p}$ is that associated with the electron capture, while $C_{p1}p_1\Delta\hat{p}$ is that associated with the hole emission. The condition that the rate decreases for these two processes be the same is clearly $n_1^* = n$. Similarly, the constraint terms in $\partial\Delta\hat{p}/\partial t$ represent the respective decreases $C_{p2}p\Delta\hat{n}$ and $C_{n2}n_2\Delta\hat{n}$ in the neutral-center hole capture and electron emission rates associated with an increase in $\hat{n}$; these decreases are equal if $p_2^* = p$ holds. The third forms of (54) show that a pair of equal constraint terms implies an equilibrium hole-to-electron release-frequency or transition-rate ratio for the acceptor or donor levels equal, respectively, to n/n_0 or p_0/p, which are substantially unity near thermal equilibrium.

For this two-level case, the four effective trapping and release times or frequencies associated with each energy level satisfy the fundamental restriction that is formally identical with (53). It is also easily verified that the various conditions given for the recombination and trapping ranges and for shallow and deep traps apply without formal modification.

By suitable notational generalization of the fundamental mass-action equations, the results of this section can be shown to apply to two-level centers in general, whose states (differing successively by one electronic charge) may include ones that are multiply charged, either positively or negatively. Through use of the phenomenological capture coefficients, statistical weights associated with multiply charged states do not enter explicitly. For example, the concentration $\mathfrak{N} - \hat{n} - \hat{p}$ of neutral centers may be replaced by concentration $\tilde{p}$ of centers with single positive charge, and $\hat{p}$ used to denote concentration of centers with double positive charge. Then $\hat{n}$ is replaced by the new concentration $\mathfrak{N} - \tilde{p} - \hat{p}$ of neutral centers.† Thus, with obvious modifications in the physical descrip-

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† Note that these transformations applied to (1) give $\Delta m = \Delta n - \Delta\tilde{p} - \Delta\hat{p} = \Delta p + \Delta\hat{p}$. While the correct neutrality condition holds, Δm is no longer the increment in concentration of total negative or positive charges.

tion of capture and release frequencies and other quantities, the theory is essentially unchanged.

2.2.3 *Volume Generation with Excitations Involving Trapping Levels*

The excitations associated with the absorption of radiation of wavelengths beyond the limit of intrinsic absorption may be taken into account phenomenologically through suitable generation terms in the differential equations. To the volume rate g of interband excitations in the differential equation for n is added $g_{c1} + g_{c2}$, where g_{c1} is the volume rate of hole excitations from the conduction band to centers of type 1 — that is, electron excitations from these centers to the conduction band — and g_{c2} is the similar quantity for centers of type 2. Similarly, in the differential equation for p, to g is added $g_{v1} + g_{v2}$, each term of which is the volume rate of electron excitations from the valence band to the centers or hole excitations from the centers to the valence band. To g in the differential equation for m is added $g_{v1} + g_{c2}$, and not g_{c1} or g_{v2}, since g_{c1} increases n as it decreases $\hat{n}$, while g_{v2} increases p as it decreases $\hat{p}$. The generation terms Δg_m, Δg_n and Δg_p in the differential equations for Δm, Δn and Δp are thus†

$$\begin{aligned} \Delta g_m &\equiv \Delta g + \Delta g_{v1} + \Delta g_{c2}, \\ \Delta g_n &\equiv \Delta g + \Delta g_{c1} + \Delta g_{c2}, \\ \Delta g_p &\equiv \Delta g + \Delta g_{v1} + \Delta g_{v2}, \end{aligned} \tag{58}$$

and the generation terms that the equations for $\Delta\hat{n}$ and $\Delta\hat{p}$ now contain are respectively $\Delta g_m - \Delta g_n = \Delta g_{v1} - \Delta g_{c1}$ and $\Delta g_m - \Delta g_p = \Delta g_{c2} - \Delta g_{v2}$.

The additional generation terms clearly represent the same processes as do the emission terms of (37). The distinction implicit in the notation is valid, however, consistent with zero values of these additional generation terms at thermal equilibrium. Each generation rate of (37) is determined at equilibrium by the phonons and radiation associated with the equal corresponding capture rate. Since detailed balance applies also to the radiative part separately, there is no net radiation at equilibrium from any given process of capture and the corresponding generation.

III. DETAILED THEORY AND APPLICATIONS

3.1 *Diffusion Length and Steady-State Lifetime Functions*

3.1.1 *Linear Theory*

The equations of (46) for two types of centers, when written for the steady state and linearized by neglect of the quadratic terms, give con-

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† The excitations involving trapping levels only, which may occur for large concentrations of centers (presumably with concomitant impurity-band conduction), are here neglected.

centration increments that are proportional, and solving for $\Delta\hat{n}/\Delta m$ and $\Delta\hat{p}/\Delta m$ provides the thermal-equilibrium trapping ratios. These and the corresponding lifetime τ_m are thus given by

$$\begin{aligned}
r_n &= \frac{\nu_{23}\nu_{31} - \nu_{21}\nu_{33}}{\nu_{22}\nu_{33} - \nu_{23}\nu_{32}}, \\
r_p &= \frac{\nu_{21}\nu_{32} - \nu_{22}\nu_{31}}{\nu_{22}\nu_{33} - \nu_{23}\nu_{32}}, \\
\tau_m &= -(\nu_{11} + \nu_{12}r_n + \nu_{13}r_p)^{-1} \qquad (59) \\
&= \frac{-(\nu_{22}\nu_{33} - \nu_{23}\nu_{32})}{\nu_{11}(\nu_{22}\nu_{33} - \nu_{23}\nu_{32}) + \nu_{12}(\nu_{23}\nu_{31} - \nu_{21}\nu_{33}) + \nu_{13}(\nu_{21}\nu_{32} - \nu_{22}\nu_{31})},
\end{aligned}$$

in terms of which, with the thermal-equilibrium diffusivity D_0' from (31), the diffusion length can be expressed and the diffusion-length lifetime evaluated. These results apply also for two-level centers if equilibriums ν_{ij} are defined in accordance with (57).

The case of single-level centers of one type lends itself to more detailed analysis. Results from the linearized equations for, say, acceptor centers only, for which r_p is zero, are:

$$\begin{aligned}
r_n &= -\nu_{21}/\nu_{22} = \frac{\nu_{tn1} - \nu_{tp1}}{\nu_{tn1} + \nu_{gn1} + \nu_{gp1}} \\
&= \frac{\tau_{p0}(\mathfrak{N}_1 - \hat{n}_0) - \tau_{n0}\hat{n}_0}{\tau_{p0}(\mathfrak{N}_1 - \hat{n}_0 + n_0 + n_1) + \tau_{n0}(p_0 + p_1)} \\
&= \frac{\mathfrak{N}_1^*(\tau_{tp1} - \tau_{tn1})}{(\mathfrak{N}_1^* + n_0)\tau_{tp1} + p_0\tau_{tn1}} \\
&= 1 - \tau_n/\tau_p\,, \\
\tau_n &= (1 - r_n)\tau_m = -(\nu_{21} + \nu_{22})/\Delta_1 \\
&= (\nu_{tp1} + \nu_{gp1} + \nu_{gn1})/\Delta_1 \\
&= \frac{\tau_{n0}(\hat{n}_0 + p_0 + p_1) + \tau_{p0}(n_0 + n_1)}{\mathfrak{N}_1^* + n_0 + p_0} \qquad (60) \\
&= \frac{(\mathfrak{N}_1^* + p_0)\tau_{tn1} + n_0\tau_{tp1}}{\mathfrak{N}_1^* + n_0 + p_0}, \\
\tau_p &= \tau_m = -\nu_{22}/\Delta_1 \\
&= (\nu_{tn1} + \nu_{gn1} + \nu_{gp1})/\Delta_1 \\
&= \frac{\tau_{p0}(\mathfrak{N}_1 - \hat{n}_0 + n_0 + n_1) + \tau_{n0}(p_0 + p_1)}{\mathfrak{N}_1^* + n_0 + p_0} \\
&= \frac{(\mathfrak{N}_1^* + n_0)\tau_{tp1} + p_0\tau_{tn1}}{\mathfrak{N}_1^* + n_0 + p_0}.
\end{aligned}$$

The ν_{ij} and release frequencies are equilibrium values; τ_{n0} and τ_{p0}, given by

$$\tau_{n0} \equiv (C_{n1}\mathfrak{N}_1)^{-1} = (1 - \hat{n}_0/\mathfrak{N}_1)\tau_{tn1} = (1 + p_1/p_0)^{-1}\tau_{tn1}$$

and (61)

$$\tau_{p0} \equiv (C_{p1}\mathfrak{N}_1)^{-1} = (\hat{n}_0/\mathfrak{N}_1)\tau_{tp1} = (1 + n_1/n_0)^{-1}\tau_{tp1},$$

are the respective limiting lifetimes† in strongly extrinsic p- and n-type materials (in which they are also τ_{tn1} and τ_{tp1}); Δ_1, given by

$$\begin{aligned}\Delta_1 \equiv \nu_{11}\nu_{22} - \nu_{12}\nu_{21} &= \nu_{tn1}\nu_{gp1} + \nu_{tn1}\nu_{tp1} + \nu_{tp1}\nu_{gn1} \\ &= C_{n1}C_{p1}\mathfrak{N}_1(\mathfrak{N}_1^* + n_0 + p_0),\end{aligned} \tag{62}$$

is always positive if neither C_{n1} nor C_{p1} is zero; and $\mathfrak{N}_1^*$, which will be referred to as the "capture concentration," is given variously by

$$\begin{aligned}\mathfrak{N}_1^* &\equiv \nu_{tn1}\nu_{tp1}/C_{n1}C_{p1}\mathfrak{N}_1 = n_i^2\mathfrak{N}_1/(n_0 + n_1)(p_0 + p_1) \\ &= n_0\nu_{tn1}/\nu_{gn1} = p_0\nu_{tp1}/\nu_{gp1} = n_i(\tau_{gn1}\tau_{gp1}/\tau_{tn1}\tau_{tp1})^{\frac{1}{2}} \\ &= \alpha_{10}(\mathfrak{N}_1 - \hat{n}_0)/(1 + \alpha_{10}) = \hat{n}_0/(1 + \alpha_{10}) \\ &= \mathfrak{N}_1(\hat{n}_0/\mathfrak{N}_1)(1 - \hat{n}_0/\mathfrak{N}_1) \\ &= \mathfrak{N}_1\alpha_{10}/(1 + \alpha_{10})^2.\end{aligned} \tag{63}$$

The different forms for these results are obtained by use of (51) for the ν_{ij}, definitions of (50) for the capture and release frequencies, and equilibrium relationships of (38), (41), (42) and (53). The middle term of the second form for Δ_1 is the one that gives rise to $\mathfrak{N}_1^*$, and it follows from (53) that the first term is large or small compared with the third according to whether p_0 is large or small compared with n_0. Capture concentration $\mathfrak{N}_1^*$ large compared with $n_0 + p_0$ is, as will be shown in Section 3.2.1, the condition that capture frequencies predominate over release frequencies. The volume rates of electron and hole transitions at equilibrium (see Section 2.2.1.2), $n_0\nu_{tn1}$ and $p_0\nu_{tp1}$, are equal to $\mathfrak{N}_1^*$ times the corresponding release frequencies. The equations of (63) show that $\hat{n}_0$ and $\mathfrak{N}_1 - \hat{n}_0$ may be written in terms of $\mathfrak{N}_1^*$ and α_{10}. The concentration $\mathfrak{N}_1^*$ is small if the centers are nearly all ionized or un-ionized; the last form shows that its largest value is $\frac{1}{4}\mathfrak{N}_1$, which it assumes for $\alpha_{10} = 1$ or $\hat{n}_0/\mathfrak{N}_1 = \frac{1}{2}$, that is, for the Fermi level coincident with the energy level of the centers. Entirely similar results, for which obvious notational changes are required in some forms, hold for donor centers only.

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† Conditions for these lifetimes are $p_0 \gg \mathfrak{N}_1^* + p_1 + p_1^*$, $n_0 \gg \mathfrak{N}_1^* + n_1 + n_1^*$.

The diffusion length L_0 and lifetime τ_0 corresponding to the electron and hole lifetimes of (60), which are the Shockley-Read lifetimes, may be evaluated from (31) or (35) and (60). These equations give

$$\begin{aligned} L_0^2 = D_0'\tau_m = D_0\tau_0 &= D_0[1 - r_n p_0/(n_0 + p_0)]\tau_p \\ &= kT\mu_n\mu_p\sigma_0^{-1}(n_0\tau_{tp1} + p_0\tau_{tn1}) \qquad (64) \\ &= \sigma_0^{-1}(\sigma_{p0}D_n\tau_{tn1} + \sigma_{n0}D_p\tau_{tp1}), \end{aligned}$$

where σ_{n0} and σ_{p0} are $e\mu_n n_0$ and $e\mu_p p_0$. Other forms may be written by expressing τ_{tn1} and τ_{tp1} in terms of τ_{n0} and τ_{p0} by use of (61). The diffusion-length lifetime for this case,

$$\begin{aligned} \tau_0 &= (\nu_{gn1} + \nu_{gp1})/(\nu_{tn1}\nu_{gp1} + \nu_{tp1}\nu_{gn1}) \\ &= [\tau_{p0}(n_0 + n_1) + \tau_{n0}(p_0 + p_1)]/(n_0 + p_0) \qquad (65) \\ &= (n_0\tau_{tp1} + p_0\tau_{tn1})/(n_0 + p_0), \end{aligned}$$

is formally similar to the familiar common lifetime[31,32] for both electrons and holes for the limiting case of $\mathcal{N}_1$ small, as inspection of (60) serves to verify.† Thus, L_0 and τ_0 are, for given τ_{n0} and τ_{p0} or τ_{tn1} and τ_{tp1}, independent of $\mathcal{N}_1$. For given energy level and capture coefficients, τ_0 is proportional to $\mathcal{N}_1^{-1}$. The true L_0 and τ_0 apply, of course, in the linear part of the small-signal range, in which no appreciable trap saturation occurs. With small-signal trap saturation, diffusion length and lifetime that are usually considerably larger apply in the saturation range. These are evaluated in Section 3.1.2.

It can be shown that the electron and hole lifetimes of (60) are substantially equal to τ_0 if

$$| \nu_{tn1} - \nu_{tp1} | = | \nu_{gn1}/n_0 - \nu_{gp1}/p_0 | \mathcal{N}_1^* \ll (1 + \epsilon)(\nu_{gn1} + \nu_{gp1}) \qquad (66)$$

holds, in which ϵ is the smaller of $(\mathcal{N}_1^* + n_0)/p_0$ and $(\mathcal{N}_1^* + p_0)/n_0$, as given by the respective conditions $| \tau_p - \tau_0 |/\tau_0 \ll 1$ and $| \tau_n - \tau_0 |/\tau_0 \ll 1$. For extrinsic material, ϵ may usually be neglected; including it provides an appreciably weaker condition only if $\mathcal{N}_1^*$ is larger than the equilibrium majority-carrier concentration. The condition of (66) may be severe: It is essentially $\mathcal{N}_1^*$ small compared with the equilibrium minority-carrier concentration for the minority- to majority-carrier release frequency ratio of order unity or larger in extrinsic material, that is, for the minority-carrier trapping range defined in Section 2.2.1.2.

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† This formal similarity holds for any number M of kinds of centers, τ_0 being given by $[\sum_{j=1}^{M} (n_0\tau_{tpj} + p_0\tau_{tnj})^{-1}]^{-1}/(n_0 + p_0)$, as may be shown from the first form for $\mathcal{R}_m$ of (71) and the observation that, if two or more different kinds of centers are present, then $\mathcal{R}_m$ is the sum of similar terms for each kind.

General conditions for the validity of this linear analysis may be formulated as conditions for the neglect of the quadratic terms. For this purpose, assume uniform concentrations and volume-generation rate. Then (46) and (51) yield

$$d\Delta n/dt = \Delta g - \nu_{tn1}\Delta n + \nu_{gn1}\Delta\hat{n} + C_{n1}\Delta n\Delta\hat{n} = 0$$

and (67)

$$d\Delta p/dt = \Delta g - \nu_{tp1}\Delta p - \nu_{gp1}\Delta\hat{n} - C_{p1}\Delta p\Delta\hat{n} = 0$$

for acceptor centers only in the steady state and no direct recombination. The conditions may be derived in a self-consistent manner by first obtaining, with the neutrality condition, the concentrations from the linearized forms of (67). These concentrations, namely

$$\begin{aligned} \Delta n &= (\nu_{tp1} + \nu_{gp1} + \nu_{gn1})\Delta g/\Delta_1 , \\ \Delta\hat{n} &= (\nu_{tn1} - \nu_{tp1})\Delta g/\Delta_1 , \\ \Delta p &= (\nu_{tn1} + \nu_{gn1} + \nu_{gp1})\Delta g/\Delta_1 , \end{aligned} \tag{68}$$

are then substituted in (67), so that conditions for negligible quadratic terms may be obtained as restrictions on (positive) Δg and, by use of (68), as corresponding restrictions on the concentrations. It will suffice to give the former restrictions, which for the neglect of $C_{n1}\Delta n\Delta\hat{n}$ and $C_{p1}\Delta p\Delta\hat{n}$ are, respectively,

$$\Delta g/\Delta_1 \ll \nu_{tn1}/C_{n1}(\nu_{tn1} - \nu_{tp1}) = (\mathfrak{N}_1 - \hat{n}_0)/(\nu_{tn1} - \nu_{tp1})$$

and (69)

$$\Delta g/\Delta_1 \ll \Delta_1/C_{p1}(\nu_{tn1} - \nu_{tp1})(\nu_{tn1} + \nu_{gn1} + \nu_{gp1})$$

for $\nu_{tn1} > \nu_{tp1}$. If $\nu_{tp1} > \nu_{tn1}$, the restriction for neglect of one of the quadratic terms turns out to be that of (69) for the other one, but with subscripts n and p interchanged — an interchange that does not affect Δ_1. This distinction arises because the signs of the approximate linear terms in $\Delta\hat{n}$ depend on the sign of $\nu_{tn1} - \nu_{tp1}$. It is easily shown that† for this quantity zero, or the case of "critical recombination," $\Delta\hat{n}$ is identically zero and (67) are linear for all Δg. For trapping without recombination, (68) does not apply because Δ_1 is zero, but the conditions may properly be written as the restrictions on the concentrations obtained by use of these equations. For example, for electron trapping with C_{p1} zero, the condition $\Delta n \ll n_0 + n_1$ results, which may be a severe condition for p-type material.

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† A solution is excluded that does not admit thermal equilibrium, for which Δg or $\Delta\hat{n}$ is zero for certain negative values of Δn and Δp or Δp and Δg, respectively.

3.1.2 *Nonlinear Theory*

For added carrier concentrations resulting from arbitrary injection levels, steady-state lifetime functions τ_n and τ_p may be evaluated from $\mathcal{R}_m$. For acceptor centers only and no direct recombination, $\mathcal{R}_m$ is given by

$$\begin{aligned}\mathcal{R}_m &= C_{n1}[n(\mathcal{N}_1 - \hat{n}) - n_1\hat{n}] \\ &= C_{p1}[p\hat{n} - p_1(\mathcal{N}_1 - \hat{n})],\end{aligned} \tag{70}$$

which results in

$$\begin{aligned}\mathcal{R}_m &= \frac{np - n_i^2}{\tau_{p0}(n + n_1) + \tau_{n0}(p + p_1)} \\ &= \frac{\hat{n}(\mathcal{N}_1 - \hat{n}) + (n_0 + n_1)\hat{n} + (p_0 + p_1)(\mathcal{N}_1 - \hat{n})}{\tau_{p0}(\mathcal{N}_1 - \hat{n}) - \tau_{n0}\hat{n}}\,\frac{\Delta\hat{n}}{\mathcal{N}_1} \\ &= \frac{\hat{n}(1 - \hat{n}/\mathcal{N}_1) + (n_0 + n_1 - p_0 - p_1)\Delta\hat{n}/\mathcal{N}_1 + n_0 + p_0}{\tau_{p0}(\mathcal{N}_1 - \hat{n}) - \tau_{n0}\hat{n}}\,\Delta\hat{n} \\ &= \Delta n/\tau_n = \Delta p/\tau_p .\end{aligned} \tag{71}$$

Eliminating $\hat{n}$ by means of the second equation of (70) and the use of (61) results in the first form† for $\mathcal{R}_m$ of (71). This familiar form[31] furnishes τ_n or τ_p in terms of Δn or Δp alone if one of these concentrations is eliminated by solving the second equation of (70) written with $\hat{n}$ replaced by $p - n$; and Δn or Δp may at the same time be related to, say, the generation rate $\Delta g = \mathcal{R}_m$ for steady-state photoconductivity. The algebra involves radicals. A better procedure for such analysis employs the second or the third form for $\mathcal{R}_m$; these result from (70) by elimination of n and p with the neutrality condition. Then Δn and Δp are, with $\mathcal{R}_m$, written in term of $\hat{n}$ or $\Delta\hat{n}$ as independent parameter in accordance with‡

$$\Delta n = \frac{\tau_{p0}(n_0 + n_1) + \tau_{n0}(\hat{n} + p_0 + p_1)}{\tau_{p0}(\mathcal{N}_1 - \hat{n}) - \tau_{n0}\hat{n}}\,\Delta\hat{n}$$

and (72)

$$\Delta p = \Delta m = \frac{\tau_{n0}(p_0 + p_1) + \tau_{p0}(\mathcal{N}_1 - \hat{n} + n_0 + n_1)}{\tau_{p0}(\mathcal{N}_1 - \hat{n}) - \tau_{n0}\hat{n}}\,\Delta\hat{n},$$

so that the lifetime functions are given by

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† If two or more different kinds of centers are present, then $\mathcal{R}_m$ is the sum of similar terms for each kind. For the corresponding lifetimes in terms of Δn and Δp, see Okada.[45]

‡ Note that $\Delta\hat{n}$ has the sign of the denominator, which is proportional to $\nu_{tn1} - \nu_{tp1}$ for $\Delta\hat{n}$ small.

$$\tau_n = \frac{\tau_{p0}(n_0 + n_1) + \tau_{n0}(\hat{n} + p_0 + p_1)}{\hat{n}(1 - \hat{n}/\mathfrak{N}_1) + (n_0 + n_1 - p_0 - p_1)\Delta\hat{n}/\mathfrak{N}_1 + n_0 + p_0}$$

and (73)

$$\tau_p = \frac{\tau_{n0}(p_0 + p_1) + \tau_{p0}(\mathfrak{N}_1 - \hat{n} + n_0 + n_1)}{n(1 - \hat{n}/\mathfrak{N}_1) + (n_0 + n_1 - p_0 - p_1)\Delta\hat{n}/\mathfrak{N}_1 + n_0 + p_0}.$$

The functions reduce to the Shockley-Read lifetimes of (60) for the equilibrium value $\hat{n}_0$ of $\hat{n}$, as may be verified by use of (41) and (42). The range of $\hat{n}$ is from $\hat{n}_0$ to the limiting large-signal value given by $\hat{n}/\mathfrak{N}_1 = \tau_{p0}/(\tau_{n0} + \tau_{p0})$, for which the denominator in (72) vanishes, and for which τ_n and τ_p both equal[31] $\tau_{n0} + \tau_{p0}$.

The trapping ratio r_n corresponding to the lifetime functions of (73) is given by

$$\begin{aligned} r_n^{-1} &= \frac{\tau_{p0}(\mathfrak{N}_1 - \hat{n} + n + n_1) + \tau_{n0}(p + p_1)}{\tau_{p0}(\mathfrak{N}_1 - \hat{n}) - \tau_{n0}\hat{n}} \\ &= \left[1 + \frac{\tau_{p0}\Delta n + \tau_{n0}\Delta p}{\tau_{p0}(\mathfrak{N}_1 - \hat{n} + n_0 + n_1) + \tau_{n0}(p_0 + p_1)}\right]\frac{\Delta p}{\Delta\hat{n}} \qquad (74) \\ &= \frac{\Delta p}{\Delta\hat{n}} + \frac{\begin{array}{l}\tau_{p0}^{\;2}(n_0 + n_1) + \tau_{n0}^{\;2}(p_0 + p_1) \\ \qquad + \tau_{p0}\tau_{n0}(\mathfrak{N}_1 + n_0 + n_1 + p_0 + p_1)\end{array}}{[\tau_{p0}(\mathfrak{N}_1 - \hat{n}) - \tau_{n0}\hat{n}]^2}\,\Delta\hat{n}, \end{aligned}$$

which is obtained from (72) by differentiating with respect to Δm. The equilibrium value of r_n, which is that of $\Delta\hat{n}/\Delta p$, is the r_n given in (60), while the limiting large-signal value is zero, as may be expected. By means of (72) and (74), the steady-state continuity equation, (30), may be written (for acceptor centers only), with $\Delta\hat{n}$ as independent variable and the components of grad $\Delta\hat{n}$ as dependent variables. The second or third form for r_n^{-1} of (74), with $\Delta p/\Delta\hat{n}$ given by (72), lends itself to this purpose; note that $d\Delta n/d\Delta\hat{n}$ and $d\Delta p/d\Delta\hat{n}$ equal $r_n^{-1} - 1$ and r_n^{-1}, respectively.

The lifetime function[31] for $|\Delta\hat{n}| \ll \Delta n \sim \Delta p = \Delta m$,

$$\tau_n \sim \tau_p \sim \frac{\tau_{p0}(n_0 + n_1 + \Delta p) + \tau_{n0}(p_0 + p_1 + \Delta p)}{n_0 + p_0 + \Delta p}, \qquad (75)$$

may be derived most directly from the first form for $\mathcal{R}_m$ of (71). By solving the second equation of (70) for $\Delta\hat{n}$, the condition $|\Delta\hat{n}| \ll \Delta n \sim \Delta p$ may be written as

$$\Delta p + \frac{\nu_{gn1} + \nu_{gp1}}{C_{n1} + C_{p1}} \gg \frac{|\nu_{tn1} - \nu_{tp1}|}{C_{n1} + C_{p1}} = \frac{|\nu_{gn1}/n_0 - \nu_{gp1}/p_0|}{C_{n1} + C_{p1}}\mathfrak{N}_1^*. \qquad (76)$$

Equilibrium release probabilities are here employed. Equation (76) with Δp set equal to zero is the condition that $\Delta \hat{n}$ be relatively small for all Δp. Since this condition is (66) with ϵ set equal to zero, it subsumes the condition for equilibrium lifetimes substantially τ_0, to which it is usually equivalent.

It is readily shown from (75) that, with† $\Delta p \ll n_0 + p_0$ (and equilibrium lifetimes τ_0), the lifetimes are substantially τ_0 if Δp is small compared with $(\nu_{gn1} + \nu_{gp1})/(C_{n1} + C_{p1})$. This condition and the one of (66) may be severe conditions under essentially the same circumstances. That is, in the minority-carrier trapping range defined in Section 2.2.1.2, lifetimes are τ_0 for $\mathfrak{N}_1^*$ small compared with minority-carrier concentration n_0 or p_0; and then, consistent also with the condition of Section 3.1.1 for the neglect of $C_{n1}\Delta n\Delta\hat{n}$ or $C_{p1}\Delta p\Delta\hat{n}$ suitably specialized, for Δp small compared with $n_0 + n_1$ or $p_0 + p_1$. If the condition on Δp is not met, then, with the condition on $\mathfrak{N}_1^*$, (75) gives a lifetime that increases rapidly with injection level at low injection levels.‡ But such observed behavior with extrinsic material, as these considerations indicate, cannot usually be properly analyzed by use of (75). The steady-state lifetimes in the small-signal range generally either result primarily from recombination or majority-carrier trapping and are both τ_0 and substantially constant, or else have distinct equilibrium values given by (60) with dependences on (small-signal) injection level obtainable by the general procedure described. It will be shown that, in the latter case, substantially constant apparent diffusion-length lifetimes given by τ_{gn1} for n-type material or τ_{gp1} for p-type generally apply in the small-signal range above a certain injection level. Thus, unless trap concentration is quite small, (75) has significant application in the former case only to the transition from τ_0 to the lifetime $\tau_{n0} + \tau_{p0}$ for the large-signal range.

Nonconstant small-signal lifetime functions are associated with deep traps in the minority-carrier trapping range. Such traps will be saturated (in the steady state) even in the presence of a concentration of mobile minority carriers that is relatively quite small. From (67), by equating $d\Delta n/dt$ and $d\Delta p/dt$ (which are zero also in the immediate context), $\Delta\hat{n}$ may be written as

$$\Delta\hat{n} = (\nu_{tn1}\Delta n - \nu_{tp1}\Delta p)/(\nu_{gn1} + \nu_{gp1} + C_{n1}\Delta n + C_{p1}\Delta p), \quad (77)$$

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† The more general condition without this restriction includes $\tau_0 \sim \tau_{n0} + \tau_{p0}$ for large values of Δp.

‡ As a result of saturation of centers available for minority-carrier capture, this lifetime increases essentially linearly in the small-signal range from the equilibrium value $\tau_{p0}(n_0 + n_1)/p_0$ or $\tau_{n0}(p_0 + p_1)/n_0$ and asymptotically to the large-signal value τ_{p0} or τ_{n0}.

in which the concentration-dependent contributions to the release frequencies are exhibited separately. Suppose that the transitions in p-type material are electron-dominated, so that $\nu_{gn1} \gg \nu_{gp1}$. For deep traps, as defined in Section 2.2.1.2, $n_1 \ll p_0$ (or $p_1 \gg n_0$) holds, which gives $n_0 + n_1 \ll p_0 + p_1$ and therefore implies $C_{n1} \ggg C_{p1}$ or $\tau_{p0} \ggg \tau_{n0}$ for the present case. Then, for $\Delta n \gg n_0 + n_1$, the denominator in (77) is $C_{n1}\Delta n$. Furthermore, $\nu_{tn1} \equiv C_{n1}(\mathcal{N}_1 - \hat{n}_0) \ggg \nu_{tp1} \equiv C_{p1}\hat{n}_0$ holds from (53). Hence $\Delta\hat{n} \sim \mathcal{N}_1 - \hat{n}_0$ follows† for Δn large compared with $n_0 + n_1$ and not too small compared with Δp. If trap concentration is not too large, small-signal saturation evidently occurs under the conditions assumed. If it is large, then a large conductivity increase is associated with the majority-carrier concentration corresponding to the saturated traps. Nonconstant small-signal lifetime functions apply in either case, whether saturation occurs in the small-signal range or not.

The lifetime functions in the saturation range approach the limiting large-signal lifetime, $\tau_{n0} + \tau_{p0}$, substantially equal to τ_{p0}. Though τ_{p0} is otherwise the minority-hole-capture-limited hole lifetime in strongly extrinsic n-type material, in this case it is a lifetime limited by majority-carrier capture. For small-signal saturation, $\Delta\hat{n}$ changes relatively slightly from the small-signal saturation range to its limiting value, $(\nu_{tn1} - \nu_{tp1})/(C_{n1} + C_{p1}) = \mathcal{N}_1 - \hat{n}_0 - \tau_{n0}\mathcal{N}_1/(\tau_{n0} + \tau_{p0})$, for the large-signal range. This circumstance might suggest that τ_{p0} applies over both ranges. In general, it does not: The denominators in (73) are comparatively small (reducing, for example, to $n_0 + n_1$ for $\Delta\hat{n} = \mathcal{N}_1 - \hat{n}_0$) and are sensitive to very small changes in $\Delta\hat{n}$; a very small change in concentration of unsaturated traps can affect lifetimes appreciably. As will be shown in Section 3.2.2, large-signal lifetime implies relatively large increase in conductivity. The equations of (67) for the steady state, simplified for relatively small departure of $\Delta\hat{n}$ from $\mathcal{N}_1 - \hat{n}_0$ still (necessarily) nonlinear, may be solved, in terms of Δn and for the saturation range, for the lifetime functions $\tau_n = \Delta n/\Delta g$ and $\tau_p = \Delta p/\Delta g$. With $\nu_{gn1} \gg \nu_{gp1} + \tau_{p0}^{-1}$, which will still apply in the present case even if $\mathcal{N}_1$ is of order $p_0 + p_1$, it is found that $\tau_n \sim \tau_{p0}/[1 + (p_0 + \mathcal{N}_1 - \hat{n}_0)/\Delta n]$ holds for the saturation range, specified by $\Delta n \gg n_0 + n_1$. Thus,

$$\tau_n \sim \tau_{p0}\Delta n/(p_0 + \mathcal{N}_1 - \hat{n}_0),$$

proportional to Δn, holds for the saturation range of relatively small Δn. An apparent diffusion-length lifetime, τ_0', may be found by evaluating

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† The equivalent condition $\Delta p \gg \mathcal{N}_1 - \hat{n} + n_0 + n_1$ from (72) takes into account $\Delta p > \Delta n$. The small-signal saturation value of $\mathcal{N}_1 - \hat{n}$ may be appreciably larger than $n_0 + n_1$, but its limiting large-signal value is small compared with $n_0 + n_1$ if $\mathcal{N}_1 \ll (\nu_{gn1}/\nu_{gp1})(p_0 + p_1)$ holds.

$(D'/D)\tau_p = (1 - r_n)\tau_p$ for small-signal saturation.† With $r_n^{-1} = 1 + d\Delta n/d\Delta\hat{n}$, the expression $1 + (n_0 + n_1)(\mathcal{N}_1 - \hat{n}_0)/\Delta n^2$ is found for $1 - r_n$; and dividing $\Delta p \sim \Delta n + \mathcal{N}_1 - \hat{n}_0$ by Δg gives

$$\begin{aligned}\tau_p &\sim [1 + (\mathcal{N}_1 - \hat{n}_0)/\Delta n]\tau_n \\ &\sim \tau_{p0}[1 + (\mathcal{N}_1 - \hat{n}_0)/\Delta n]/[1 + (p_0 + \mathcal{N}_1 - \hat{n}_0)/\Delta n].\end{aligned}$$

It is easily seen that, for small $\mathcal{N}_1 - \hat{n}_0$ of order $n_0 + n_1$ or less, the lifetime function for τ_0' so obtained is $\tau_{p0}\Delta n/p_0$, as are τ_n and τ_p . As may be expected, this result is consistent with (75). For $\mathcal{N}_1 - \hat{n}_0 \ggg n_0 + n_1$, however, the lifetime function gives $\tau_0' \sim \tau_{p0}/(1 + \tau_{p0}/\tau_{gp1})$ for $(\mathcal{N}_1 - \hat{n}_0)^2 \gg \Delta n^2 \gg (n_0 + n_1)(\mathcal{N}_1 - \hat{n}_0)$. The condition $\mathcal{N}_1 \ll p_0 + p_1$ then gives $\tau_0' \sim \tau_{gp1}$ for small-signal saturation. If this inequality is reversed, then $\tau_0' \sim \tau_{p0}$ results, and saturation occurs with relatively large increase in hole concentration.

From (60), the equilibrium electron, hole and diffusion-length lifetimes are, for these cases, generally small compared with τ_{p0} . They are given by $\tau_0 \sim \tau_n \sim [(n_0 + n_1)/p_0]\tau_{p0} = [(n_0 + n_1)/(\mathcal{N}_1 - \hat{n}_0)]\tau_{gp1}$ and $\tau_p \sim [(\mathcal{N}_1 - \hat{n}_0)/p_0]\tau_{p0} = \tau_{gp1}$ if small-signal saturation occurs, for which τ_0 is also small compared with $\tau_0' = \tau_{gp1}$. The minority-carrier and apparent diffusion-length lifetime functions increase with injection level, most rapidly as Δn becomes comparable with $n_0 + n_1$ and the traps fill. These results clearly provide a simple model, based on a single trapping level, for superlinearity,[29,34,35,36] the more-rapid-than-linear increase of photoconductivity with injection level.‡ With small-signal saturation, two superlinear ranges may occur, the first as diffusion-length lifetime increases from τ_0 to τ_{gp1} , and the second as it increases from τ_{gp1} to τ_{p0} in the large-signal range. With large-signal saturation resulting from large concentration of traps, only one superlinear range occurs, since a nearly linear intermediate range is absent. Only one range occurs also under the condition of (66) for small trap concentration. It is evident, however, that with superlinearity this condition is generally quite severe.

For the majority-carrier-dominated case of $\nu_{tp1} \gg \nu_{tn1}$ (or $\nu_{gp1} \ggg \nu_{gn1}$) in p-type material, there can be no small-signal saturation. With small trap concentration, lifetime $\tau_0 \sim (1 + p_1/p_0)\tau_{n0} = \tau_{tn1}$, which is limited

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† The second form follows since $\Delta n/(n_0 + n_1) \gg (\mathcal{N}_1 - \hat{n}_0)/(p_0 + \mathcal{N}_1 - \hat{n}_0)$ holds *a fortiori*.

‡ For small concentration of centers, Δp may exhibit a less-rapid-than-linear, a linear, or a superlinear dependence on Δg, as Rittner[1] has shown using a lifetime function tantamount to that of (75). From this equation, superlinearity results, as may be expected, if $\tau_{n0} + \tau_{p0}$ exceeds τ_0 , so that the numerator increases more rapidly with Δp than the denominator. See also Ridout,[70] Newman, Woodbury and Tyler[71] and Sandiford.[7]

by minority-carrier capture and obtains over the entire small-signal range, then changes in accordance with (75) to $\tau_{n0} + \tau_{p0}$ in the large-signal range. This change is a decrease to $\tau_{n0} \gg \tau_{p0}$ if the trapping level is near the Fermi level or higher.

The steady-state fractions of ionized centers can be represented by simple formal generalizations of the equilibrium relationships of (41) and (56). In these equations, $\hat{n}_0$ and $\hat{p}_0$ are replaced by $\hat{n}$ and $\hat{p}$ and α_{10} and α_{20} by

$$\begin{aligned} \alpha_1 &\equiv \frac{C_{n1}\, n + C_{p1}p_1}{C_{p1}p + C_{n1}n_1} = \alpha_{10}\, \frac{1 + \Delta n/(n_0 + n_1{}^*)}{1 + \Delta p/(p_0 + p_1{}^*)}, \\ \alpha_2 &\equiv \frac{C_{p2}p + C_{n2}n_2}{C_{n2}n + C_{p2}p_2} = \alpha_{20}\, \frac{1 + \Delta p/(p_0 + p_2{}^*)}{1 + \Delta n/(n_0 + n_2{}^*)}, \end{aligned} \tag{78}$$

as can readily be shown† by solving for the ionized fractions from (37) and also from the corresponding two-level equations of Section 2.2.2.

3.2 *Photoconductivity*

A number of results for steady-state photoconductivity being implicit in Section 3.1, the present section will deal principally with the transient decay.

3.2.1 *Linear Theory*

For two types of centers in the linear small-signal case the time derivatives of Δm, $\Delta\hat{n}$ and $\Delta\hat{p}$ for photoconductive decay are given respectively by (46) without Δg and the quadratic terms. The general solution is accordingly

$$\begin{aligned} \Delta m &= \sum_{j=1}^{3} A_j e^{-\nu_j t}, \\ \Delta\hat{n} &= \sum_{j=1}^{3} r_{nj} A_j e^{-\nu_j t}, \\ \Delta\hat{p} &= \sum_{j=1}^{3} r_{pj} A_j e^{-\nu_j t}, \end{aligned} \tag{79}$$

in which the A_j are constants determined by the initial conditions, and the r_{nj} and r_{pj} are trapping ratios for the respective decay modes determined by

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† The equation given in the abstract of the paper of Sah and Shockley[22] rewritten in the present notation yields $\hat{n}/(\mathcal{N} - \hat{n} - \hat{p}) = \alpha_1$ and $(\mathcal{N} - \hat{n} - \hat{p})/\hat{p} = \alpha_2{}^{-1}$, from which the ionized fractions for the two-level case here given follow as solutions of simultaneous linear equations.

$$\begin{aligned}(\nu_{11} + \nu_j) + \nu_{12}r_{nj} + \nu_{13}r_{pj} &= 0,\\ \nu_{21} + (\nu_{22} + \nu_j)r_{nj} + \nu_{23}r_{pj} &= 0,\\ \nu_{31} + \nu_{32}r_{nj} + (\nu_{33} + \nu_j)r_{pj} &= 0,\end{aligned} \tag{80}$$

with the decay constants ν_j being the roots of the equation obtained by equating to zero the determinant of (80). The A_j are found in terms of the trapping ratios and the initial concentrations Δm_1, $\Delta\hat{n}_1$ and $\Delta\hat{p}_1$ by setting t equal to zero in (79) and solving. The solution so obtained applies as well to the two-level case, for which the ν_{ij} are defined in accordance with (57). The decay constants ν_j, which are roots of a cubic, are always real and (since the coefficients alternate in sign) positive, except that one of them may be zero. Establishing these properties involves expressing the coefficients in terms of the capture and release frequencies by means of (51) or (57) and making use of (53).

The constant term of the cubic and one decay constant are zero if there is trapping only and no recombination. This case can occur essentially in two ways: The two types of center may trap, respectively, the two kinds of carriers, or they may both trap only one kind. If, say, the acceptor centers trap only electrons and the donor centers trap only holes, then the ν_j are readily found to be zero, $\nu_{tn1} + \nu_{gn1}$ and $\nu_{tp2} + \nu_{gp2}$. The last two decay constants characterize the respective exponential increases with time of $\Delta\hat{n}$ and $\Delta\hat{p}$ to new equilibrium concentrations after injection that correspond to the zero decay constant. These equilibrium concentrations are fractions of Δm (which remains constant) equal to the fractions of the time the electrons and the holes are trapped. Indeed, since the two types of centers trap independently in this case, the solution consists of solutions written independently for each. But if, for example, electrons only are trapped by both types of centers, then this independence does not obtain; electrons released from centers of one type may be trapped by centers of the other type. With the convention here employed, concentration of electrons trapped by donor centers may then be written as negative $\Delta\hat{p}$. The decay constants are found to be zero and $\frac{1}{2}\{\nu_{tn1} + \nu_{gn1} + \nu_{tn2} + \nu_{gn2} \pm [(\nu_{tn1} + \nu_{gn1} - \nu_{tn2} - \nu_{gn2})^2 + 4\nu_{tn1}\nu_{tn2}]^{\frac{1}{2}}\}$, with an equilibrium value after injection of total trapped electron concentration equal to $\Delta p/[1 + \nu_{gn1}\nu_{gn2}/(\nu_{tn1}\nu_{gn2} + \nu_{tn2}\nu_{gn1})]$.

The general linear small-signal case for one type of center is readily evaluated in detail. For acceptor centers only, the solution is given by the first two equations of (79), all terms with $j = 3$ being omitted. The trapping ratios are given by

$$\begin{aligned}r_{nj} &= -\nu_{21}/(\nu_{22} + \nu_j) = (\nu_{tn1} - \nu_{tp1})/(\nu_{tn1} + \nu_{gn1} + \nu_{gp1} - \nu_j)\\ &= -(\nu_{11} + \nu_j)/\nu_{12} = (\nu_j - \nu_{tp1})/\nu_{gp1},\end{aligned} \tag{81}$$

and the ν_j are the roots of

$$\nu_j^2 - \nu_s\nu_j + \Delta_1 = 0, \qquad j = 1, 2, \tag{82}$$

where ν_s is defined by

$$\nu_s \equiv -(\nu_{11} + \nu_{22}) = \nu_{tn1} + \nu_{gn1} + \nu_{tp1} + \nu_{gp1} \tag{83}$$

and Δ_1 by (62). The decay constants are thus

$$\nu_j = \tfrac{1}{2}[(-1)^{j-1}\nu_r + \nu_s], \qquad j = 1, 2, \tag{84}$$

with

$$\nu_r \equiv (\nu_s^2 - 4\Delta_1)^{\frac{1}{2}} = [(\nu_{tn1} + \nu_{gn1} - \nu_{tp1} - \nu_{gp1})^2 + 4\nu_{gn1}\nu_{gp1}]^{\frac{1}{2}}. \tag{85}$$

The corresponding time constants $\tau_1 \equiv \nu_1^{-1}$ and $\tau_2 \equiv \nu_2^{-1}$ are also equal respectively to ν_2/Δ_1 and ν_1/Δ_1. Nonoscillatory decay is easily verified for this case: The second form for ν_r shows that the ν_j are real; and, since $\nu_r < \nu_s$, the ν_j are positive.

A subcase that provides some physical interpretations is that of $\mathfrak{N}_1$ sufficiently small so that capture frequencies are small compared with release frequencies. As (62) and (83) show, the condition $\nu_s^2 \gg 4\Delta_1$ then holds, and expansion of the radical in (85) gives

$$\begin{aligned} \tau_1 \sim \nu_s^{-1} &= \tau_{gn1}\tau_{gp1}/(\tau_{gn1} + \tau_{gp1}) \\ &\ll \tau_2 \sim \nu_s/\Delta_1 \sim \tau_0 . \end{aligned} \tag{86}$$

Thus, for this subcase, τ_2 is the steady-state lifetime τ_0 of (65). It is large compared with τ_1, the time constant for the adjustment of $\Delta\hat{n}$ to a fixed fraction of Δp substantially equal to the equilibrium trapping ratio, r_n. This interpretation of τ_1 follows from solutions for the concentrations: The last form for the r_{nj} of (81) and r_n from (60) give

$$\begin{aligned} r_{n1} &\sim 1 + \nu_{gn1}/\nu_{gp1} , \\ r_{n2} &\sim (\nu_{tn1} - \nu_{tp1})/(\nu_{gn1} + \nu_{gp1}) \sim r_n ; \end{aligned} \tag{87}$$

and the result

$$\begin{aligned} \Delta n/\Delta p_1 &= [r_n\nu_{gn1}/(\nu_{gn1} + \nu_{gp1})]e^{-t/\tau_1} \\ &\qquad + [1 - r_n\nu_{gn1}/(\nu_{gn1} + \nu_{gp1})]e^{-t/\tau_2}, \\ \Delta\hat{n}/\Delta p_1 &= r_n(-e^{-t/\tau_1} + e^{-t/\tau_2}), \\ \Delta p/\Delta p_1 &= -[r_n\nu_{gp1}/(\nu_{gn1} + \nu_{gp1})]e^{-t/\tau_1} \\ &\qquad + [1 + r_n\nu_{gp1}/(\nu_{gn1} + \nu_{gp1})]e^{-t/\tau_2} \end{aligned} \tag{88}$$

holds for initial concentration $\Delta\hat{n}_1$ zero. Since r_n is small, the mobile-carrier concentrations are mostly in the second or lifetime decay mode and differ only slightly. For this subcase, if the initial trapping ratio is r_n rather than zero, then the first decay modes are not present and $\Delta\hat{n}$ is $r_n\Delta p = r_n\Delta p_1 e^{-t/\tau_2}$. The first decay modes do not occur either for "critical recombination," with which $\Delta\hat{n}$ remains identically zero as a result of equal capture frequencies ν_{tn1} and ν_{tp1} or, for this subcase, equal capture rates for Δn and Δp. For small $\mathfrak{N}_1$, the capture rates are in all cases substantially in the ratio ν_{tn1}/ν_{tp1}. In linear cases, they also decay in the lifetime mode after this mode predominates. The release rates behave similarly, their ratio being equal to ν_{gn1}/ν_{gp1}, or to $(n_0/p_0)(\nu_{tn1}/\nu_{tp1})$ in accordance with (53).

The condition for neglect of the capture frequencies may be severe. The approximate form of ν_s applies if $\nu_{tn1} + \nu_{tp1} \ll \nu_{gn1} + \nu_{gp1}$ holds, which implies that

$$\begin{aligned}\mathfrak{N}_1{}^* &\ll (\nu_{gn1} + \nu_{gp1})/(\nu_{gn1}/n_0 + \nu_{gp1}/p_0) \\ &= (n_0\nu_{tn1} + p_0\nu_{tp1})/(\nu_{tn1} + \nu_{tp1}),\end{aligned} \tag{89}$$

a condition which subsumes

$$\mathfrak{N}_1{}^* \ll n_0 + p_0 \tag{90}$$

for neglect of $\nu_{tn1}\nu_{tp1}$ in Δ_1. The conditions of (89) and that for steady-state lifetimes equal to τ_0 of (66) are the same for the minority-carrier trapping range defined in Section 2.2.1.2, for which they are $\mathfrak{N}_1{}^*$ small compared with the equilibrium minority-carrier concentration. The condition $\nu_s{}^2 \gg 4\Delta_1$ is

$$\begin{aligned}\mathfrak{N}_1{}^* &\ll \tfrac{1}{4}n_i{}^2(n_0 + p_0)^{-1}(\nu_{gn1}/\nu_{gp1} + \nu_{gp1}/\nu_{gn1} + 2) \\ &= \tfrac{1}{4}n_i{}^2(n_0 + p_0)^{-1}(\tau_{gn1} + \tau_{gp1})^2/\tau_{gn1}\tau_{gp1}\end{aligned} \tag{91}$$

if (89) holds, and it can be shown to be weaker than (89) in general if the minority- to majority-carrier release frequency ratio exceeds a number that is about three for extrinsic material and about six for intrinsic material.[†] Equations (89) and (91) are both subsumed *a fortiori* by $\mathfrak{N}_1{}^* \ll n_i{}^2/(n_0 + p_0)$, which is $\mathfrak{N}_1 \ll (n_0 + n_1)(p_0 + p_1)/(n_0 + p_0)$.

The release frequencies may be neglected under the condition of (90) but with the inequality signs reversed. The solution is then simply[‡] $\Delta n/\Delta n_1 = e^{-t/\tau_{tn1}}$ and $\Delta p/\Delta p_1 = e^{-t/\tau_{tp1}}$. For $\mathfrak{N}_1{}^*$ large, (60) shows that

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[†] Equation (91) gives a stronger or weaker condition according to whether $n_0\nu_{gp1}/\nu_{gn1} + p_0\nu_{gn1}/\nu_{gp1}$ is smaller or larger than $3(n_0 + p_0)$.

[‡] This result easily follows directly from the differential equations. Or, note that the radicand in (85) is $(\nu_{tn1} - \nu_{tp1})^2$.

τ_{tn1} and τ_{tp1} are respectively the steady-state lifetimes τ_n and τ_p. The condition $\nu_s^2 \gg 4\Delta_1$ is accordingly $\frac{1}{4}(\tau_n/\tau_p + \tau_p/\tau_n) + \frac{1}{2} \gg 1$ — namely, that one of τ_n or τ_p be small compared with the other. If τ_n or τ_p is the smaller, then substantially all of Δn or Δp, respectively, is transformed comparatively rapidly into positive or negative $\Delta \hat{n}$, after which a slower recombinative decay of $\Delta \hat{n}$ and the concentration of the other mobile carriers takes place as these carriers are captured.

The condition $\nu_s^2 \gg 4\Delta_1$ implies $\tau_1 \ll \tau_2$, with τ_1 essentially a characteristic time for trapping and τ_2 essentially a lifetime. This interpretation does not apply if ν_s^2 and $4\Delta_1$ are comparable so that τ_1 and τ_2 do not differ by much. For small $\mathfrak{N}_1$ and the majority-carrier trapping range, for example, $\tau_1 \sim \tau_2$ may hold; (89) may apply, but not (91) (see footnote on previous page). The case of $\nu_s^2 \sim 4\Delta_1$ for $\mathfrak{N}_1$ large, for which τ_1, τ_2, τ_{tn1}, τ_{tp1}, τ_n and τ_p are all substantially equal, is a case of recombination with but slight trapping.

The general trapping time and lifetime obtained from (84) and related equations are

$$\begin{aligned} \tau_1 = \nu_s^{-1} &= \tau_{gn1}\tau_{gp1}/[(\mathfrak{N}_1^*/p_0 + 1)\tau_{gn1} + (\mathfrak{N}_1^*/n_0 + 1)\tau_{gp1}] \\ &= \tau_{tn1}\tau_{tp1}/[(1 + p_0/\mathfrak{N}_1^*)\tau_{tn1} + (1 + n_0/\mathfrak{N}_1^*)\tau_{tp1}] \\ \ll \tau_2 = \nu_s/\Delta_1 &= (\mathfrak{N}_1^* + n_0 + p_0)^{-1} \\ &\quad \cdot [n_0\tau_{gn1} + p_0\tau_{gp1} + n_i^2(\tau_{gn1} + \tau_{gp1})/\mathfrak{N}_1^*] \\ &= (\mathfrak{N}_1^* + n_0 + p_0)^{-1} \\ &\quad \cdot [(\mathfrak{N}_1^* + p_0)\tau_{tn1} + (\mathfrak{N}_1^* + n_0)\tau_{tp1}]. \end{aligned} \tag{92}$$

Comparison with (60) and (65) shows that this lifetime τ_2 is larger than the steady-state lifetimes τ_n, τ_p and τ_0; all are equal in the limit of $\mathfrak{N}_1$ small. For $\mathfrak{N}_1$ large in intrinsic material, τ_2 equals $2\tau_0$. Furthermore, these lifetimes all decrease monotonically to zero as $\mathfrak{N}_1$ increases indefinitely.

The decrease of τ_2 with increasing $\mathfrak{N}_1$ may, however, proceed essentially in two ranges, with approximate constancy of τ_2 in an intermediate range.[41] From the first form for τ_2 of (92), this intermediate range occurs provided there are capture concentrations $\mathfrak{N}_1^*$ that are small compared with $n_0 + p_0$ and also large compared with

$$(\nu_{gn1} + \nu_{gp1})/(\nu_{gn1}/n_0 + \nu_{gp1}/p_0);$$

that is, if the strong inequality

$$\tau_{gn1} + \tau_{gp1} \lll (n_0/p_0)\tau_{gn1} + (p_0/n_0)\tau_{gp1} \tag{93}$$

holds. It can hold for sufficiently strongly extrinsic material if the majority-carrier release time is not too small. For small $\mathfrak{N}_1$, τ_2 varies inversely with $\mathfrak{N}_1$, as (65) for τ_0 shows. For large $\mathfrak{N}_1$ such that $\mathfrak{N}_1^* \gg n_0 + p_0$, τ_2 varies similarly, equalling the value $(n_0\tau_{gn1} + p_0\tau_{gp1})/(n_0 + p_0)$ of approximate constancy divided by $\mathfrak{N}_1^*/(n_0 + p_0)$. With the third or fourth form for $\mathfrak{N}_1^*$ of (63), this τ_2 reduces to $\tau_{tn1} + \tau_{tp1}$. Since τ_1 for large $\mathfrak{N}_1^*$ is the harmonic mean of τ_{tn1} and τ_{tp1}, τ_1 is the smaller of these capture times and τ_2 the larger, as previously discussed for this case. It can be shown that, for the minority-carrier trapping range, the inequalities that $\mathfrak{N}_1^*$ must satisfy for approximate constancy of τ_2 generally imply the condition $\nu_s^2 \gg 4\Delta_1$ on which the calculation is based.[†] A similar situation has been shown to obtain with the inequality for the case of negligible capture frequencies. But since this case involves a condition for neglect of the capture frequencies that is usually severe in the minority-carrier trapping range, it is the present case that would usually apply in practice in this range.

3.2.2 *Nonlinear Theory*

Although the general problem of photoconductive decay is intractable analytically, some special cases can be solved and certain techniques of approximation are effective. From (46) and (51), general equations that apply for centers of the acceptor type may be written as

$$\begin{aligned}
d\Delta n/dt &= \Delta g + \nu_{gn1}\Delta\hat{n} - [\nu_{tn1} - (C_{n1} - C_i)\Delta\hat{n}]\Delta n - C_i\Delta n^2, \\
d\Delta\hat{n}/dt &= [\nu_{tn1} - \nu_{tp1} - (C_{n1} + C_{p1})\Delta\hat{n}]\Delta n \\
&\qquad - (\nu_{tp1} + \nu_{gn1} + \nu_{gp1})\Delta\hat{n} - C_{p1}\Delta\hat{n}^2 \\
&= [\nu_{tn1} - \nu_{tp1} - (C_{n1} + C_{p1})\Delta\hat{n}]\Delta p \\
&\qquad - (\nu_{tn1} + \nu_{gn1} + \nu_{gp1})\Delta\hat{n} + C_{n1}\Delta\hat{n}^2, \\
d\Delta p/dt &= \Delta g - \nu_{gp1}\Delta\hat{n} - [\nu_{tp1} + (C_{p1} - C_i)\Delta\hat{n}]\Delta p - C_i\Delta p^2,
\end{aligned} \tag{94}$$

in which equilibrium values of release frequencies are employed. Since $|\Delta\hat{n}|$ is bounded (by a concentration that cannot exceed $\mathfrak{N}_1$), it is clear that, if the initial concentration Δp_1 is sufficiently large, then the decay proceeds with $\Delta\hat{n}$ after a short transient substantially equal to

$$(\nu_{tn1} - \nu_{tp1})/(C_{n1} + C_{p1}),$$

[†] One inequality is the reverse of that of (89); hence it is the condition for the neglect of the release frequencies in ν_s. The other inequality is (90). It follows that $\nu_s^2 \gg 4\Delta_1$ is $\mathfrak{N}_1^* \gg 4(n_0^{-1} + p_0^{-1})\nu_{gn1}\nu_{gp1}/(\nu_{gn1}/n_0 + \nu_{gp1}/p_0)^2$. This condition is weaker than the reverse of (89) under the same circumstances that make (91) a weaker condition than (89).

for which the coefficient of Δn or Δp in $d\Delta\hat{n}/dt$ is zero. This value of $\Delta\hat{n}$ corresponds to maximum (but not necessarily complete) saturation or neutralization in the traps. Although direct recombination, characterized by the quasi-hyperbolic decay law

$$\Delta p/\Delta p_1 = (C_i \Delta p_1 t + 1)^{-1} \tag{95}$$

predominates in principle for very large Δp_1, it can frequently be neglected.[48,49,50] The large-signal decay is then exponential with lifetime $(C_{n1} + C_{p1})/(C_{n1}\nu_{tp1} + C_{p1}\nu_{tn1}) = \tau_{n0} + \tau_{p0}$, the limiting large-signal steady-state lifetime. With the limiting value of $\Delta\hat{n}$, this result follows from, say, the last equation of (94), in which neglect of the release rate $\nu_{gp1}\Delta(\mathcal{N}_1 - \hat{n})$ is consistent with Δp large.

Examination in further detail of the large- and small-signal decay is facilitated by the equation for negligible direct recombination,

$$\begin{aligned} &\tau_{n0}\Delta n^{-1} d\Delta n/dt + \tau_{p0}\Delta p^{-1} d\Delta p/dt \\ &\quad + 1 + [(p_0 + p_1)(1 - \Delta n/\Delta p) + (n_0 + n_1)(1 - \Delta p/\Delta n)]/\mathcal{N}_1 \\ &\qquad\qquad = (\tau_{n0}\Delta n^{-1} + \tau_{p0}\Delta p^{-1})\Delta g, \end{aligned} \tag{96}$$

which is readily obtained from (67) or (94) as a linear combination of $d\Delta n/dt = d(\Delta p - \Delta\hat{n})/dt$ and $d\Delta p/dt$ that eliminates the quadratic terms. For example, consistent with results for the linear case of small $\mathcal{N}_1$, assuming in this equation the steady-state trapping ratio r_n of (87) and (so that Δn and Δp are proportional) a single decay time, this time is given as the lifetime $\tau_2 \sim \tau_0$ of (86). If either $\Delta n \sim \Delta p$ or $\mathcal{N}_1$ is sufficiently large, then the term with brackets, which arises from the terms involving release frequencies, may evidently be neglected. It is otherwise plausible that release does not appreciably affect large mobile carrier concentrations, while capture predominates with large trap concentration. For no volume generation, (96) may then be integrated, with the result

$$(\Delta n \Delta p)^{\frac{1}{2}} (\Delta n/\Delta p)^{-\frac{1}{2}(C_{n1} - C_{p1})/(C_{n1} + C_{p1})} = A\, e^{[-t/(\tau_{n0} + \tau_{p0})]}, \tag{97}$$

in which A is a constant determined by the initial concentrations. It is easily verified that, besides furnishing the large-signal lifetime, (97) is consistent with the linear solution for large $\mathcal{N}_1$, for which the release frequencies may be neglected and the decay times are τ_{tn1} and τ_{tp1}. This equation is a first integral of (94) for a case of large $\mathcal{N}_1$, one which can accordingly be formulated as a first-order (rather than second-order) nonlinear differential equation.

The condition under which (97) holds is, from (96),

$$\mathfrak{N}_1 \gg | \Delta\hat{n}[(n_0 + n_1)/\Delta n - (p_0 + p_1)/\Delta p] | . \tag{98}$$

From (96) and (98), the large-signal lifetime $\tau_{n0} + \tau_{p0}$ obtains if $\Delta n \sim \Delta p \gg \Delta\hat{n}$ holds and $\Delta p \gg \mathfrak{N}_1^{-1} | \Delta\hat{n}(n_0 + n_1 - p_0 - p_1) |$ also. With the $\Delta\hat{n}$ for $\Delta n \sim \Delta p$ from (77), these conditions are respectively that of (76) and

$$\begin{aligned} &\Delta p + (\nu_{gn1} + \nu_{gp1})/(C_{n1} + C_{p1}) \\ &\qquad\qquad = \Delta p + \tau_0(n_0 + p_0)/(\tau_{n0} + \tau_{p0}) \\ &\gg | n_0 + p_0 - (\nu_{gn1} + \nu_{gp1})/(C_{n1} + C_{p1}) | \\ &\qquad\qquad = (n_0 + p_0) | 1 - \tau_0/(\tau_{n0} + \tau_{p0}) | . \end{aligned} \tag{99}$$

Since the left-hand sides of (76) and (99) are the same, comparison of the right-hand sides will indicate which condition is the more restrictive in any particular case.† The condition that corresponds to (99) obtained from the lifetime for $\Delta n \sim \Delta p$ of (75) is similar except that the constant term on the left is replaced by $n_0 + p_0$. Setting Δp equal to zero in either gives the condition that $\tau_{n0} + \tau_{p0}$ apply for all Δp, which is that it equal τ_0.

The decay times associated with a small-amplitude pulse of added carriers above a steady generation level Δg are readily evaluated. The equations for $d\delta n/dt$ and $d\delta p/dt$, linear in the concentration increments δn and δp that result from the pulse, may be obtained from (67). Written with capture and release frequencies that are concentration-dependent, they are formally the same as the linear small-signal ones for $d\Delta n/dt$ and $d\Delta p/dt$. For the release frequencies, the definitions of (50) apply; for the capture frequencies, $\hat{n}_0$ in these definitions is replaced by $\hat{n}$. The condition $\nu_s^2 \gg 4\Delta_1$ of Section 3.2.1 generalized in this way is the condition for a lifetime $\tilde{\tau}_2$ for δn and δp equal to the generalized ratio ν_s/Δ_1 and large compared with the corresponding time constant for trapping. The lifetime $\tilde{\tau}_2$ depends on the steady-state values of Δn, $\Delta\hat{n}$ and Δp; (72) gives Δn and Δp in terms of $\Delta\hat{n}$, and (71) relates $\Delta\hat{n}$ to $\Delta g = \mathfrak{R}_m$. It reduces to τ_2 of (92) for the linear small-signal case and to $\tau_{n0} + \tau_{p0}$ for Δg large.

The approximation $\Delta n \sim \Delta p$ applied to (94) gives the differential equation

$$d\Delta p/dt + \frac{[1 + (n_0 + p_0)^{-1}\Delta p]\Delta p}{[1 + (\tau_{n0} + \tau_{p0})\tau_0^{-1}(n_0 + p_0)^{-1}\Delta p]\tau_0} = 0, \tag{100}$$

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† In the minority-carrier trapping range, for example, (76) and (99) may require approximately that Δp be large compared with trap concentration and with equilibrium majority-carrier concentration, respectively.

whose solution[1,44] may be written as

$$\frac{\Delta p}{\Delta p_1}\left[\frac{1 + (n_0 + p_0)^{-1}\Delta p}{1 + (n_0 + p_0)^{-1}\Delta p_1}\right]^{[(\tau_{n0}+\tau_{p0})/\tau_0]-1} = e^{-(t/\tau_0)}. \tag{101}$$

As may be expected, this solution is the integrated form that corresponds to the lifetime of (75); the approximation results in the steady-state $\Delta\hat{n}$ of (77), and the condition required is that of (76). Casual inspection of (101) might suggest that τ_0 and $\tau_{n0} + \tau_{p0}$ apply respectively for Δp small and large compared with $n_0 + p_0$. This conclusion is, of course, illusory: For the minority-carrier trapping range, the exponent in (101) is large and τ_0 applies only if Δp is restricted as explained in Section 3.1.2.

The limiting large-signal $\Delta\hat{n}$, $\Delta\hat{p}$ and lifetime for two kinds of centers are easily evaluated from (46) and (51), and the extension to any number of kinds of centers is obvious. As may be expected, the values $(\nu_{tn1} - \nu_{tp1})/(C_{n1} + C_{p1})$ or $(\nu_{tp2} - \nu_{tn2})/(C_{n2} + C_{p2})$ of $\Delta\hat{n}$ or $\Delta\hat{p}$ are as if the acceptor or donor centers alone were present; and the lifetime is the harmonic mean of lifetimes $\tau_{n0} + \tau_{p0}$ for each kind of center, the decay constant being the sum of the separate decay constants. This result does not apply to the two-level case: From (57), $\Delta\hat{n}$ and $\Delta\hat{p}$ are found to equal $C_{n1}C_{n2}\mathfrak{N}/(C_{n1}C_{n2} + C_{n2}C_{p1} + C_{p1}C_{p2}) - \hat{n}_0$ and $C_{p1}C_{p2}\mathfrak{N}/(C_{n1}C_{n2} + C_{n2}C_{p1} + C_{p1}C_{p2}) - \hat{p}_0$, with $(1 + C_{n1}/C_{p1} + C_{p2}/C_{n2})/(C_{n1} + C_{p2})\mathfrak{N}$ as the large-signal lifetime.

General solutions for trapping only and no recombination can be obtained without difficulty. For, say, electron trapping by acceptor centers, Δp maintains its initial value Δp_1, and the nonlinear equation for Δn that results from replacing $\Delta\hat{n}$ by $\Delta p_1 - \Delta n$ in the first equation of (94) has the solution

$$\Delta n = \Delta n_2 \frac{1 - \dfrac{\nu_t + \nu_{tn1} - \nu_{gn1} - C_{n1}\Delta p_1}{-\nu_t + \nu_{tn1} - \nu_{gn1} - C_{n1}\Delta p_1}\, e^{-\nu_t t}}{1 - \dfrac{-\nu_t + \nu_{tn1} + \nu_{gn1} + C_{n1}\Delta p_1}{\nu_t + \nu_{tn1} + \nu_{gn1} + C_{n1}\Delta p_1}\, e^{-\nu_t t}} \tag{102}$$

for initial value Δp_1, if direct recombination is neglected,[†] with

$$\begin{aligned} \nu_t &\equiv [(\nu_{tn1} - \nu_{gn1} - C_{n1}\Delta p_1)^2 + 4\nu_{tn1}\nu_{gn1}]^{\frac{1}{2}}, \\ \Delta n_2 &\equiv \tfrac{1}{2}C_{n1}^{-1}(\nu_t - \nu_{tn1} - \nu_{gn1} + C_{n1}\Delta p_1). \end{aligned} \tag{103}$$

The concentration Δn_2 is the new equilibrium concentration which Δn

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[†] The general form of this solution is not changed if direct recombination is included.

approaches asymptotically after injection. For Δp_1 large, these equations give

$$\Delta n = \left[\Delta p_1 - \left(1 - \frac{n_0 + n_1}{\Delta p_1}\right)(\mathfrak{N}_1 - \hat{n}_0)\right] \cdot \left(1 + \frac{\mathfrak{N}_1 - \hat{n}_0}{\Delta p_1} e^{-C_{n1}\Delta p_1 t}\right). \tag{104}$$

Thus, $\Delta\hat{n} = \Delta p_1 - \Delta n$ rapidly increases to the limiting value

$$[1 - (n_0 + n_1)/\Delta p_1](\mathfrak{N}_1 - \hat{n}_0) \sim \mathfrak{N}_1 - \hat{n}_0 = \nu_{tn1}/C_{n1},$$

which corresponds to substantially all traps charged. For Δp_1 small, the equations give

$$\Delta n/\Delta p_1 = [\nu_{gn1}/(\nu_{tn1} + \nu_{gn1})]\,[1 + (\nu_{tn1}/\nu_{gn1})\, e^{-(\nu_{tn1} + \nu_{gn1})t}], \tag{105}$$

which may be obtained also by suitable specialization of results for the general linear small-signal case. According to (105), $\Delta n/\Delta p_1$ decreases from unity to $\tau_{tn1}/(\tau_{tn1} + \tau_{gn1})$, the fraction of the time electrons are free, while $\Delta\hat{n}/\Delta p_1$ increases to $\tau_{gn1}/(\tau_{tn1} + \tau_{gn1})$, the fraction of the time electrons are trapped. An effect of slight recombination on Δn would in all cases be a comparatively slow decay from a value approximately equal to the equilibrium value Δn_2 for trapping only.

It is sometimes relevant to deal with a model involving centers that provide nonrecombinative trapping in conjunction with other centers, of a suitably idealized type, that provide only recombination that can be specified simply in terms of a constant lifetime. Such centers would in general be present in comparatively small concentration, so that the amplitude of their trapping transient is negligible. Furthermore, this transient would be comparatively brief, so that steady-state lifetime applies after negligible time.

With certain restrictions, the idealized centers may be centers that function in the recombination range or in the majority-carrier trapping range.† The $\Delta\hat{n}$ or $\Delta\hat{p}$ for these centers obtained by setting $d\Delta\hat{n}/dt$ or $d\Delta\hat{p}/dt$ equal to zero results in a contribution to both $d\Delta n/dt$ and $d\Delta p/dt$ that is the negative of a steady-state recombination rate similar to that of the first form for $\mathfrak{R}_m$ of (71). With subscripts "3" employed to denote the recombination centers, this recombination rate may be written as $\nu_{tn3}\Delta n/(1 + \nu_{gn3}/\nu_{gp3}) + \nu_{tp3}\Delta p/(1 + \nu_{gp3}/\nu_{gn3})$, in which the release frequencies are concentration-dependent. If now $\nu_{gp3} \gg \nu_{gn3}$ holds in p-

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† For the minority-carrier trapping range, the lifetime function $(\tau_{n0} + \tau_{p0})\Delta p/(n_0 + p_0)$ would apply for $\Delta n \sim \Delta p \ll n_0 + p_0$, as can be shown by use of (100).

type material, it may be a consequence of the condition $\nu_{tp3} > \nu_{tn3}$ for hole trapping; or, with $\nu_{tp3} < \nu_{tn3}$, it may imply the recombination range. In either case, from (53) the small signal recombination rate is substantially $\nu_{tn3}\Delta n$ provided Δn is not too small because of strong electron trapping in the trapping centers. Lifetime τ_n is then $\tau_{tn3} \sim \tau_0$. While the large-signal lifetime differs in principle, the assumed inequality implies $\tau_0 \sim \tau_{n0} \gg \tau_{p0}$ if the energy level of the recombination centers is not too far from the Fermi level towards the valence band. In general, $\nu_{gp3} \gg \nu_{gn3}$ gives a small-signal recombination rate equal to

$$C_{n3}\mathfrak{N}_3(p_0\Delta n + n_0\Delta p)/(p_0 + p_3)$$

and thus a lifetime that cannot properly be associated with either Δn or Δp alone. In intrinsic material, for example, it is $\Delta n + \Delta p$ with which a lifetime may be associated.

These considerations suggest the formal representation of "linear recombination" by including in $\partial\Delta m/\partial t$ or in both $\partial\Delta n/\partial t$ and $\partial\Delta p/\partial t$ the negative of a recombination rate $\nu_{n3}\Delta n + \nu_{p3}\Delta p$. This procedure is useful in deriving results in forms that apply symmetrically without reference to conductivity type. For the p-type case here discussed, ν_{n3} and ν_{p3} equal ν_{tn3} and $(n_0/p_0)\nu_{tn3}$. The one that corresponds to the majority carrier can usually be set equal to zero for sufficiently strong extrinsic material.

3.2.3 *Negative Photoconductivity*

Under certain conditions, optical generation with excitations involving trapping levels will cause a decrease in conductivity below the thermal-equilibrium value.[51,52] This negative photoconductivity will be considered for a simple model — that of two types of centers, of which one gives trapping and the other only recombination. For traps of the acceptor type, (94) gives $d\Delta n/dt$ and $d\Delta p/dt$, except that suitable generation and recombination terms must be included. From Section 2.2.3, generation terms are respectively Δg_n and Δg_p; and, from Section 3.2.2, the linear recombination term $-(\nu_{n3}\Delta n + \nu_{p3}\Delta p)$ may be included in both equations. For simplicity, direct recombination and the quadratic terms will be neglected, and the concentrations evaluated for the steady state. The result is

$$\Delta p = \frac{(\nu_{gp1} - \nu_{n3})\Delta g_n + (\nu_{tn1} + \nu_{gn1} + \nu_{n3})\Delta g_p}{\Delta_1 + (\nu_{n3} + \nu_{p3})(\nu_{gn1} + \nu_{gp1}) + \nu_{n3}\nu_{tp1} + \nu_{p3}\nu_{tn1}}, \qquad (106)$$

with Δ_1 defined by (62); a similar expression for Δn is obtainable by interchanging subscripts n and p, a transformation that does not change the denominator.

This result verifies the conclusion that Δp may be negative as a result of excitations from the traps to the conduction band in conjunction with recombination, and similarly for Δn, with excitations from the valence band to the traps. As a simple case, consider p-type material with ν_{p3} zero. If there is also trapping only of electrons and excitation only of electrons from traps to the conduction band, then C_{p1} and Δg_p are zero, whence Δn is zero and Δp is $-\Delta g_n/\nu_{gn1}$, with Δg_n equal to Δg_{c1} from (58). Recombination, in this case, produces negative Δp, which compensates the reduction by the excitation of the concentration of (negatively) charged traps, and the effect tends to be enhanced with deep traps of small capture cross section.

3.2.4 *Further Theory with an Application to Experiment*

Illustrative application will be made to observations of Hornbeck and Haynes on electron trapping in p-type silicon.[54] In this work, techniques were devised to measure the various time constants in the decay of photoconductivity, which, for certain samples, covered a range of about 10^7 in relative value. Evidence for two trapping levels was found, and electron capture cross sections and energy levels were estimated from the data, the model employed being that of two types of traps that capture only electrons, a lifetime being associated with recombination in centers of another type. The sample† for which there is most detailed information exhibited a 20-microsecond photoconductive decay, attributed to recombination, for sufficiently high injection levels; a decay of time constant about 10 milliseconds, attributed to decay in comparatively shallow traps that were initially filled in concentration of 2×10^{12} cm^{-3}; and decay in deep traps that were initially filled in concentration of 10^{13} cm^{-3} whose time constant varied from 1 second for the traps nearly full to 260 seconds for the traps nearly empty. Both types of traps are "deep" traps, as defined in Section 2.2.1.2. The present theory will be used to calculate the upper limits for the hole-capture cross sections implied by this model, and it will be shown how the conclusions are modified if an alternative model is assumed.

In outline, the general procedure here employed involves first assigning trial values to the energy levels of the traps, and then calculating expressions for decay constants from the equations, suitably linearized for particular ranges. These decay constants are roots of algebraic equa-

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† Data and results for sample 223B are given in the text and various figures of the Hornbeck and Haynes paper.[54]

tions and, assuming them well separated, may be obtained as the magnitudes of ratios of successive coefficients. The coefficients are homogeneous expressions in capture and release frequencies (and also "constraint frequencies," if multiple-level traps are involved) — that of the highest power being unity, followed by linear, quadratic and higher-order forms for the successively lower powers. With assumed trapping levels and known equilibrium concentrations of carriers and unoccupied traps, the coefficients provide corresponding homogeneous forms in the capture coefficients. These coefficients remain to be found from observed decay constants. To each product of capture coefficients that occurs, a number of products of frequencies generally contribute, but, for the particular semiconductor material and trapping model, these usually differ by orders of magnitude and a single one predominates. With this considerable simplification, decay constants can be expressed in terms of the frequencies so that physical mechanisms involved can often be readily identified. If a sufficient number of distinct decay constants are known from experiment, the energies of the trapping levels may also be determined. The consistency of the assumed trapping model may then be checked; the energies found should clearly not differ from those assumed by so much that the particular simplified expressions employed for the decay constants do not apply.

Consistent with the notation here employed, the deep traps may be assumed to be of the donor type and the shallower traps of the acceptor type. Trial values of the energy levels will be taken as 0.23 ev below the Fermi level $\mathcal{E}$ in intrinsic material and at $\mathcal{E}$. These levels are approximately 0.78 ev below the conduction band for an energy gap of 1.10 ev and at midgap, substantially in accord with the locations determined by Hornbeck and Haynes† The values[72,73] at 300°K of 1500 and 570 cm^2 volt^{-1} sec^{-1} for the electron and hole mobilities and 1.73×10^{20} cm^{-6} for n_i^2 give $n_0 = 4.3 \times 10^5$ cm^{-3} and $p_0 = 4.1 \times 10^{14}$ cm^{-3} for the 27-ohm-cm p-type sample, with $n_1 = p_1 = 1.32 \times 10^{10}$ cm^{-3}, $n_2 = 2.8 \times 10^6$ cm^{-3} and $p_2 = 6.1 \times 10^{13}$ cm^{-3}.

For the two kinds of traps with recombination at the rate $\nu_{n3}\Delta n \equiv \Delta n/\tau_3$ only in other centers of the idealized type discussed in Section 3.2.2, the outlined procedure applied to the equations written for the linear small-signal range gives the longest decay time τ_∞ as

$$\tau_\infty = \tau_3 + \tau_{gn1} + \tau_{gn2} + (\tau_{gn1}/\tau_{tn1})\tau_3 + (\tau_{gn2}/\tau_{tn2})\tau_3 \,. \quad (107)$$

The fourth and fifth terms represent recombination with multiple trap-

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† An energy gap of 1.10 ev at 300°K is employed rather than 1.00 ev as in Ref. 54. The trial values employed originated in a two-level analysis (which later appeared inapplicable), the n_2 being that for $\mathfrak{N} = 1.15 \times 10^{13}$, or $\hat{p}_0/\mathfrak{N} = 0.87$.

ping in the shallower and deep traps, respectively, and the latter predominates for the case under examination.[†] In (107), τ_{gn1} and τ_{gn2} are, of course, the "effective" equilibrium release times, and are not correctly interpreted as the physically proper ones,[‡] namely $(C_{n1}n_1)^{-1}$ and $(C_{n2}n_2)^{-1}$.

An upper limit for the coefficient C_{p2} for the capture of holes by occupied deep traps may be obtained by assuming recombination in deep traps only and then calculating τ_∞ from the linear small-signal equations. The result is

$$\tau_\infty = \frac{\nu_{tn2} + \nu_{gp2}}{\nu_{tn2}\nu_{gp2}} = \frac{\hat{p}_0 + \nu_{gp2}/C_{n2}}{\hat{p}_0\nu_{gp2}} \sim \tau_{gp2} \qquad (108)$$

in the simplified form obtained by the outlined procedure. The final approximation on the right[§] applies with $\hat{p}_0 = 10^{13}$ cm^{-3} and $\tau_\infty = 260$ seconds, provided merely that $C_{n2} \gg 4 \times 10^{-16}$ cm^3 seconds^{-1} holds. Then $C_{p2} = 8 \times 10^{-18}$ cm^3 seconds^{-1} follows, since $p_0 + p_2$ is 4.7×10^{14} cm^{-3}; and the cross section for hole capture A_{p2}, obtained by dividing by mean thermal velocity, is 8×10^{-25} cm^2. Recombination in the deep traps that gives τ_∞ cannot account for the observed decay. It can be shown[‖] that the decay time for the traps nearly full would then be large compared with $\tau_{gp2} = 260$ seconds rather than 1 second. The actual A_{p2} may thus be considered small compared with about 10^{-24} cm^2.

For an upper limit to C_{p1}, recombination only in the shallower traps is assumed. For this case, the rather lengthy general expression for τ_∞ simplifies to give

$$\tau_\infty = \tau_{gn2} + (\tau_{gn2}/\tau_{tn2})[\tau_{tn1}(1 + \tau_{gp1}/\tau_{gn1})]. \qquad (109)$$

The contribution τ_{gn2} is the time constant for the initial decay in the deep traps, obtainable as the longest decay time from the equations linearized for nearly full deep traps and nearly empty shallower ones. This release time represents recombination of electrons in the shallower

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[†] See footnote 20 of Ref. 54.

[‡] The τ_g of Ref. 54 should be identified as τ_{gn1} and τ_{gn2}. For the deep traps, $S\tau_g$ is accordingly $(n_0 + n_2)^{-1}$, which increases with the p-type conductivity and is not a property of the traps alone. The formula employed for locating trapping levels relative to a band edge holds if τ_g in it is the physically proper release time. With τ_g the "effective" release time, it holds only if n_0 is negligible compared with n_1 or n_2. In Equations (1) of Ref. 54, dn/dt lacks the term $C_{n1}n_0\Delta_n$.

[§] The equilibrium τ_p of (60) and the lifetime τ_2 of (92) written for the deep traps, for which $\mathfrak{N}_2^* \sim 0.1\mathfrak{N}_2 \sim 0.1\hat{p}_0$, also reduce to τ_{gp2} if C_{n2} is not too small.

[‖] The lifetime $\bar{\tau}_2$ of Section 3.2.2 evaluated for the deep traps nearly saturated is, by use of approximations for near-saturation of Section 3.1.2, found to be given by $\bar{\tau}_2 \sim (C_{p2}\mathfrak{N}_2)^{-1}\Delta n^2/(n_0 + n_2)p_0 \gg \tau_{gp2}$ for $(n_0 + n_2)p_0 \gg \Delta n^2 \gg (n_0 + n_2)\,\hat{p}_0$. The inequality on the right is equivalent to $\Delta n \gg \hat{p}$; the one on the left is largely consistent numerically with $\Delta n \ll n_0 + n_1$, for which the shallower traps are substantially empty.

traps without their recapture in the deep ones. The contribution $(\tau_{gn2}/\tau_{tn2})\tau_{tn1}$ represents recombination in the shallower traps with multiple trapping in the deep traps. The contribution with τ_{gp1}/τ_{gn1} as a factor represents recombination in the shallower traps with multiple trapping involving both levels.† This contribution predominates since, as shown in Section 3.1.2, the small-signal saturation of the shallower traps implies $\tau_{gp1} \gg \tau_{gn1}$. The only capture coefficient it contains is C_{p1}, which is found to equal 2.5×10^{-13} cm^3 seconds^{-1} for $\tau_\infty = 260$ seconds. The corresponding hole-capture cross section A_{p1} is 2.4×10^{-20} cm^2. The actual cross section is small compared with this value if recombination occurs primarily in centers of a third kind.

Recombination in the shallower traps can account for the observed deep-trap decay. Indeed, as may be expected, if the equations are linearized for small departures from a concentration $\hat{p}_1$ of unoccupied deep traps, then a (longest) decay time

$$\begin{aligned} \tau_d &= \tau_{gn2} + (\tau_{gn2}/\tau_{tn2})[\tau_{tn1}(1 + \tau_{gp1}/\tau_{gn1})](\hat{p}_1/\hat{p}_0) \\ &= \tau_{gn2} + (\tau_\infty - \tau_{gn2})(\hat{p}_1/\hat{p}_0) \end{aligned} \tag{110}$$

results, which increases from τ_{gn2} to τ_∞ as $\hat{p}_1$ increases from zero to $\hat{p}_0$ and is of the same form as that employed by Hornbeck and Haynes[54] to fit their data.‡ The observed decay in the shallower traps can also be accounted for through C_{p1}. The equations for nearly full deep traps and nearly empty shallower ones give $\tau_{gp1}(1 + \tau_{tn1}/\tau_{gn1}) \sim \tau_{gp1}$ for decay in the shallower traps as intermediate time constant, τ_{tn1} for electron capture being the shortest and τ_{gn2} for the initial deep-trap decay being the longest.§ The C_{p1} obtained by setting τ_{gp1} equal to 10^{-2} second is 3 per cent smaller than the value obtained from τ_∞ and is thus in rather fortuitously close agreement.

If a model with this C_{p1} is to account for experiment, then the assumption that the shallower traps are two-level traps, which gave the observed lifetime‖ of 20 microseconds through recombination in the higher

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† The quantity in brackets in (109) can be shown, from (53), to be the τ_0 of (65) for the shallower traps in the p-type material. Thus, τ_0 itself may be said to entail multiple trapping through $(\tau_{gp1}/\tau_{gn1})\tau_{tn1}$, the major contribution to τ_0 in the minority-carrier trapping range.

‡ The interpretation differs, since τ_{gn2} is an "effective" release time. In the notation of Hornbeck and Haynes, $\hat{p}_1/\hat{p}_0$ is $1 - y$.

§ Note that τ_{gp1} results also from assuming filled deep traps and negligible Δn, or $\Delta p \sim \hat{p}_0 + \Delta \hat{n}$.

‖ For 33 microseconds, as given in Table I of Hornbeck and Haynes,[54] calculated capture cross section for recombination would be smaller in proportion.

level, seems necessary. Otherwise, a near-saturation lifetime much larger even than τ_{gp1} would obtain.† From (60), (65) and (66) written for the higher level, the recombination lifetime will be $\tau_0 \sim \tau_{n0}$ throughout the small-signal range and with no small-signal saturation if $p_0 \gg C_{n3}n_3/C_{p3} = p_3^*$ and $p_0 \gg C_{n3}\mathfrak{N}_1/C_{p3} = (C_{p3}\tau_{n0})^{-1}$ hold. The latter condition‡ is $A_{p3} \gg 1.2 \times 10^{-17}$ cm^2. The coefficient C_{n3} and cross section A_{n3} for electron capture in the higher level are found to equal 2.5×10^{-8} cm^3 second^{-1} and 2.3×10^{-15} cm^2.

For the further analysis on the hypothesis that this model applies, the energy levels are properly treated as unknowns. The contribution $\tau_{gp1}(\tau_{tn1}/\tau_{gn1})$ to the time constant for decay in the shallower traps equals $\tau_\infty(\tau_{tn2}/\tau_{gn2}) = \tau_\infty(n_0 + n_2)/\hat{p}_0$ from the expression for τ_∞, and will accordingly be small compared with 10^{-2} second for deep traps sufficiently deep so that $n_0 + n_2 \ll 4 \times 10^8$ cm^{-3} holds.§ Then τ_{gp1} is 10^{-2} second and, with $p_1 \ll p_0$, C_{p1} has the value already found. An additional datum is available from experiment, namely the decay constant for the straggle effect: With the shallower traps nearly filled, multiple trapping results in an extended tail in the distribution of carriers from an injected pulse that are caused to drift past a fixed detector, at which the decay with time is measured. As shown in Section 3.4.3, the decay constant is the "straggle constant" ν_v, which is substantially $\nu_{gn1} + \nu_{tp1} + \nu_{gp1}$ for $p_0 \gg n_0$. Since ν_{gp1} is 10^2 second^{-1}, the observed value, 2×10^4 second^{-1}, is to be equated to $\nu_{gn1} \sim C_{n1}n_1$. With this result, the value for τ_{gp1} and $\mathfrak{N}_1 - \hat{n}_0 \sim \mathfrak{N}_1 = 2 \times 10^{12}$ cm^{-3}, the equation for τ_∞ contains only C_{n1} or n_1 and n_2 as unknowns. It fixes, say, $n_1/(n_0 + n_2)$ and thus approximately the separation between the energy levels, but there are not sufficient data with the model assumed to determine each level separately. It appears, however, from measurements relating to deep traps in samples of various conductivities,‖ that the location considered for these traps is substantially correct. With the trial value 2.8×10^6 cm^{-3} of n_2, the value obtained for C_{n1} is 1.2×10^{-6} cm^3 second^{-1}, and the value for n_1 is 1.7×10^{10} cm^{-3}, corresponding to an energy level for the shallower traps 0.007 ev above the trial location at the Fermi level in intrinsic material. With $\nu_{gn2} = 1$ second^{-1}, the value obtained for C_{n2} is 3.1×10^{-7} cm^3 second^{-1}. The cross sections $A_{n1} = 1.1 \times 10^{-13}$ cm^2 and $A_{n2} =$

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† With appropriate notational changes, the result in the footnote on page 577 for (recombinative) deep traps applies to the shallower traps.

‡ This controls if the higher energy level is further than about 0.42 ev from the conduction band.

§ This condition holds by a factor of about 10^2 for the trial value of n_2.

‖ See Fig. 13 of Ref. 54.

2.9×10^{-14} cm^2 that result are half an order of magnitude smaller than the ones calculated by Hornbeck and Haynes.†

It seems likely that the photoconductivity under illumination intense enough to give the shorter decay times was quite appreciably offset as a result of heating of the sample.‡ Significant error from this source seems unlikely, however: The time constant for cooling was very probably comparable with the longer decay times, and measurements concerning these were made with considerably less intense initial illumination.§ There were presumably no pronounced effects of nonuniform generation in the thickness of the sample, the generation rate at the dark surface being at least 40 per cent of that at the illuminated surface, as calculated from the diffusion length for the shortest decay time.‖

Work has been done towards the identification of the impurities in silicon that occasion these trapping effects.[76,77] It might be noted that the energy levels suggest gold.[56,57] But there is evidence that gold gives a single center with two (or possibly more) levels, and such a center cannot account for the saturation of the shallower traps at a concentration less than that of the deep traps. Consider the assumption that a two-level model does apply, with shallow traps only partly filled in the experiments. Then τ_{gp1} applies for the decay at the shallower level and $\tau_{tn1}' \equiv 1/C_{n1}(\mathfrak{N} - \hat{n}_0)$ for the time constant of 20 microseconds observed with the spark source.¶ It follows that C_{n1} is 4×10^{-9} cm^3 second^{-1} for the $\mathfrak{N}$ of 1.15 $\hat{p}_0$ and the negligible $\hat{n}_0$ that the trial levels give. But, with this C_{n1}, the initial Δn immediately after the steady illumination that is shut

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† Compared with the value from ten samples that they calculated in connection with Table I of their paper, A_{n2} is one order of magnitude smaller.

‡ Perhaps this heating accounts for apparent concentrations of normally empty traps determined from Fig. 4 of Ref. 54, which are about 0.7 of the values quoted.

§ Buck[74] has found a positive temperature coefficient of resistance in 38 ohm-cm p-type and 350 ohm-cm n-type silicon of 0.8 per cent per °C at room temperature, and has observed time constants for the cooling of the samples, similarly supported by wire leads and of comparable size and geometry, of the order of 100 seconds. The thermal time constant equals the heat capacity divided by the thermal dissipation constant, or power input per unit temperature elevation. For the sample here considered, 0.2 cm square and 2 cm long, power input is 8.7×10^{-4} watt for 10^{16} photons per cm^2 second absorbed, since 1 microwatt corresponds to $(5.1 \times 10^{16})\,\lambda$ photons per second of wavelength λ, and effective λ for the tungsten illumination is about 9×10^{-5} cm. The dissipation constant for a temperature elevation of 1°C with this power input in conjunction with the heat capacity of the sample of 0.14 joule per °C gives a thermal time constant of about 160 seconds. Haynes[75] has estimated a temperature elevation of no more than a few degrees for the more intense illuminations employed; heating of 3°C would decrease conductivity by an amount comparable with the total photoconductivity of Fig. 4 of Ref. 54.

‖ This diffusion length is 0.17 cm. In measurements on n-type silicon, a silicon filter and a constant-temperature enclosure were used.

¶ Note that τ_{tn1}' and $\tau_{tn1} \equiv 1/C_{n1}(\mathfrak{N} - \hat{n}_0 - \hat{p}_0)$ are the times for electron capture at the shallower level respectively for filled and empty deep traps.

off by the shutter is two orders of magnitude smaller than the apparent saturation value 2×10^{12} cm^{-3} of $\Delta\hat{n}$, and thus appreciably smaller than the initial value 6×10^{11} cm^{-3} of Δn estimated from the initial conductivity change.[†] Also, the apparent saturation value of $\Delta\hat{n}$ is proportional to Δg, which is not the fairly well-defined saturation observed.[‡] Moreover, the value of C_{n1} gives a decay constant for the straggle effect, evaluated as $\nu_{gn1} + \nu_{gp1}$, equal to 157 second^{-1} and the same two orders of magnitude smaller than that observed. Inconsistencies largely similar result from the assumption of two-level trapping with recombination entirely in other centers.

No evidence has, indeed, been found that the observed trapping effects result through metallic impurities or through lattice defects produced either by mechanical deformation or by bombardment with high-energy electrons.[4,76,77,78] Present indications are that the deep traps in silicon are associated with the presence of oxygen as an impurity;[4,67,78] but these traps, as well as the specific reactions instrumental in their formation, have not yet been physically identified. Concentrations of the traps and of certain donor centers due to oxygen[79,80] have been found to be correlated.[§] Both traps[76] and donors[85,86,87] are much more numerous in crystals grown (from quartz crucibles) with rotation of the seed than in those grown without, may be considerably increased in concentration by comparatively prolonged heating at 450°C, and may be largely removed quite rapidly by heating at temperatures above 500°C.[‖] Concentrations of the shallower traps do not exhibit this dependence. The correlation is qualitative in that donor concentration is the more dependent on heating at 450°C; appreciable trap concentration may occur in an untreated crystal grown with rotation, and may assume a value considerably smaller than the donor concentration after heating at the lower temperature.[79,80,88][¶] It should further be noted that, while these observations have been mostly confined to n-type silicon (because the donors tend to convert p-type to n-type), observations concerning the deep and shallow traps which occur in p-type silicon indicate that a common mechanism is operative.[76]

[†] See Figs. 4 and 5 of Ref. 54.

[‡] The steady-state equations give initial concentrations $\Delta n = \tau_{tn1}'(1 + \tau_{gp1}/\tau_{gn1})\Delta g$ and $\Delta\hat{n} = \tau_{gp1}\Delta g$. The apparent saturation value of $\Delta\hat{n}$ is the sum of these concentrations, since Δn is trapped rapidly, with time constant τ_{tn1}'.

[§] Determinations of oxygen content from infrared absorption at 9 microns in combination with resistivity measurements on crystals heat treated at 450°C and 1000°C have shown that formation of these donors is associated with oxygen.[80-84]

[‖] There seems to be an indication that the trap concentration is increased by water vapor but not by oxygen in the gaseous ambient.

[¶] Deep traps originally present have been largely removed by heating only 5 seconds at 700°C and subsequently have been introduced in a concentration larger

3.3 *The Photomagnetoelectric Effect*

The steady-state effect with trapping will be analyzed on the basis of the general theory of Sections 2.1.1 and 2.1.2. For the PME field along an infinite slab to the faces of which the applied magnetic field is parallel and the y axis perpendicular, (28) and (32) give

$$E_x = \sigma^{-1}(I_x - \theta I_{Dy}) = \sigma^{-1}(I_x + \theta e D' d\Delta m/dy). \tag{111}$$

The total short-circuit current per unit width of slab along the magnetic field is accordingly given by

$$I_{sc} = \int_{-y_0}^{y_0} I_x \, dy = -\theta e \int_{-y_0}^{y_0} D' \frac{d\Delta m}{dy} \, dy, \tag{112}$$

and the field along the slab under the open-circuit condition is related to I_{sc} as previously derived.† To evaluate the integral in (112), Δm is first found from the continuity equation

$$d(D' d\Delta m/dy)/dy - \Delta m/\tau_m = 0, \tag{113}$$

which follows from (30) and (36); the drift term is either zero or of order θ^2 for the short-circuit or open-circuit condition. Since, for the slab, $I_{py} = -I_{ny} = I_{Dy}$, boundary conditions are

$$\begin{aligned} \mathcal{L} - D' d\Delta m/dy &= s_{n1}\Delta n = s_{p1}\Delta p = s_{m1}\Delta m, \qquad y = y_0, \\ D' d\Delta m/dy &= s_{n2}\Delta n = s_{p2}\Delta p = s_{m2}\Delta m, \qquad y = -y_0, \end{aligned} \tag{114}$$

in which $\mathcal{L}$ is the surface rate of generation of electron-hole pairs by strongly absorbed radiation and the right-hand members give surface recombination rates. For the linear small-signal case, the velocity functions s_n and s_p (with second subscripts "1" and "2" for the respective surfaces) are constants, with

$$s_m = (1 - r_n)s_n = (1 - r_p)s_p \tag{115}$$

the surface recombination velocity for Δm.

The increase in conductance of the slab is given by

$$\Delta G = e \int_{-y_0}^{y_0} (\mu_n \Delta n + \mu_p \Delta p) \, dy = e(\mu_n + \mu_p) \int_{-y_0}^{y_0} (\tau_c/\tau_m)\Delta m \, dy. \tag{116}$$

The second form follows from $\Delta n/\tau_n = \Delta p/\tau_p = \Delta m/\tau_m = \mathcal{R}_m$, with

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than the original one by heating 16 hours at 470°C.[76] It is not yet known whether prolonged heating at 1000°C, which prevents appreciable subsequent introduction of donors at the lower temperature[82,85,86,87] would similarly prevent the introduction of deep traps.

† See Ref. 11, Equation (39).

$$\tau_c \equiv \Delta\sigma/e(\mu_n + \mu_p)\Delta g = (\mu_n\tau_n + \mu_p\tau_p)/(\mu_n + \mu_p) \tag{117}$$

a lifetime function that determines the conductivity increase $\Delta\sigma$ for the uniform volume generation rate $\Delta g = \mathcal{R}_m$. For the linear small-signal case, the lifetime functions are constants and

$$\begin{aligned} \Delta G &= e(\mu_n + \mu_p)\tau_c D_0'(d\Delta m/dy)\,\Big|_{y=-y_0}^{y=y_0} \\ &= e(\mu_n + \mu_p)\tau_c(\mathcal{L} - s_{m1}\Delta m\,|_{y=y_0} - s_{m2}\Delta m\,|_{y=-y_0}) \end{aligned} \tag{118}$$

follows by use of (113) and (114).

These results show how the theory previously given for the PME effect without trapping[11] is readily generalized to include trapping by writing equations in terms of Δm and the diffusivity D', lifetime function τ_m and surface recombination function s_m, and employing suitably generalized ΔG. Experiment may determine D_0', τ_m and s_m. In accordance with (115) and the results of Section 2.1.2, each of these gives rise, as determined by the trapping ratios, to corresponding quantities for electrons and for holes.

In its dependence on trapping and recombination in centers of a single type, the PME effect is generally nonlinear if deep traps in the minority-carrier trapping range are involved. Then trap saturation occurs in the small-signal range, as described in Section 3.1.2, and the lifetime may be nonuniform: From the illuminated surface into the slab, it may decrease from a saturation value to a much smaller linear small-signal value, a transition value at a given depth being sharply dependent on light intensity.

The influence on the PME effect of trapping as such may be investigated by assuming traps that may be nonrecombinative in conjunction with recombination on the dark surface, or with recombination in the volume of the idealized type discussed in Section 3.2.2. With the latter procedure, the linear recombination term $(-\nu_{n3}\Delta n - \nu_{p3}\Delta p)$ is included in the continuity equation. For the linear small-signal case, τ_m for the traps is thus replaced by $\bar{\tau}_m \equiv [\tau_m^{-1} + (1 - r_n)\nu_{n3} + (1 - r_p)\nu_{p3}]^{-1}$. For p-type material, ν_{p3} is set equal to zero, and $\tau_3 \equiv \nu_{n3}^{-1}$ is introduced. Then, for nonrecombinative electron traps of the acceptor type, $r_n = \nu_{tn1}/(\nu_{tn1} + \nu_{gn1})$, $r_p = 0$, $D_0' = [1 - p_0\mathfrak{N}_1^*/(n_0 + p_0)(\mathfrak{N}_1^* + n_0)]D_0$ and $\bar{\tau}_m = (1 + \nu_{tn1}/\nu_{gn1})\tau_3$ are obtained by use of the first of (31) and (60). If the traps are of the donor type, then $r_n = 0$, $r_p = -\nu_{tn2}/\nu_{gn2}$, $D_0' = [1 + \mathfrak{N}_2^*/(n_0 + p_0)]D_0$ and $\bar{\tau}_m = \tau_3$ are obtained. Essentially the same diffusion-length lifetime associated with D_0, namely

$$\bar{\tau}_0 = [1 + \mathfrak{N}_j^*/(n_0 + p_0)]\tau_3 \equiv K_\tau\tau_3, \qquad j = 1, 2, \tag{119}$$

results for both types of traps. Thus, minority-carrier trapping increases the diffusion-length lifetime and hence decreases PME current. The effect is appreciable in cases for which the capture concentration $\mathfrak{N}_j^*$ is at least comparable with the equilibrium concentration of majority carriers.† Similar analysis for nonrecombinative majority-carrier traps gives a K_τ which is that of (119) modified by division by $1 + \mathfrak{N}_j^*/n_0$ for n-type material or by $1 + \mathfrak{N}_j^*/p_0$ for p-type. Thus, majority-carrier trapping decreases the diffusion-length lifetime and increases PME current, but this increase is only that for a K_τ no smaller than $(1 + p_0/n_0)^{-1}$ or $(1 + n_0/p_0)^{-1}$, respectively.

Capture cross sections, concentrations and energy levels of traps may be found from suitable PME and photoconductivity measurements at a single temperature. Theory for trapping and recombination in traps of a single type, which holds whatever the method be for determining diffusion length L_0 and lifetime τ_0 will be considered first; while the PME method has certain advantages, any one of a number of other methods may also be employed.‡ In view of the fundamental restriction of (53) to which the four capture and release frequencies are subject, it will be convenient to deal with the capture frequencies ν_{tn1} and ν_{tp1} and the capture concentration $\mathfrak{N}_1^*$ of (63) as independent parameters. To determine these parameters, three quantities must be measured. Suppose, for example, that from suitable linear small-signal measurements, τ_0, τ_c and the lifetime τ_2 of (92) for decay of photoconductivity are known. Solving (60) and (92) for τ_{tn1}, τ_{tp1} and $\mathfrak{N}_1^*$ gives§

$$\begin{aligned} \tau_{tn1} &= [(\tau_2 - \tau_p)/(\tau_2 - \tau_a)]\tau_0 , \\ \tau_{tp1} &= [(\tau_2 - \tau_n)/(\tau_2 - \tau_a)]\tau_0 , \\ \mathfrak{N}_1^* &= (n_0 + p_0)(\tau_2 - \tau_a)/(\tau_n + \tau_p - \tau_2), \end{aligned} \tag{120}$$

in which τ_a is defined by

$$\tau_a \equiv (n_0\tau_n + p_0\tau_p)/(n_0 + p_0). \tag{121}$$

Then, with

$$\begin{aligned} \tau_n &= [(\mu_n + \mu_p)n_0\tau_c - \mu_p(n_0 + p_0)\tau_0]/(\mu_n n_0 - \mu_p p_0), \\ \tau_p &= [\mu_n(n_0 + p_0)\tau_0 - (\mu_n + \mu_p)p_0\tau_c]/(\mu_n n_0 - \mu_p p_0), \end{aligned} \tag{122}$$

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† It can be shown that, if different types of traps are present, the $\mathfrak{N}_j^*$ in (119) is replaced by the sum of the respective capture concentrations.

‡ See, for example, van Roosbroeck and Buck.[89]

§ Note, from (60) and (92), that τ_2 is larger than τ_n, τ_p and τ_a, and smaller than $\tau_n + \tau_p$.

subject to† $\mu_n n_0 \neq \mu_p p_0$, which are obtained by solving the equations defining τ_0 and τ_c, (35) and (117), τ_{tn1}, τ_{tp1} and $\mathfrak{N}_1^*$ can be found from experiment. It should be noted that (117) does not represent an advantageous method for determining τ_c; an indirect method will be given that obviates the necessity of knowing the intensity of absorbed radiation.

An additional independent datum is required to determine C_{n1}, C_{p1}, $\mathfrak{N}_1$ and n_1 or p_1. As (41), (42), (50) and (63) show, τ_{n0}, τ_{p0}, or equilibrium concentration of empty traps would serve.‡ Thus, a measurement involving the saturation range is required in addition to those in the linear small-signal range. It is, in fact, desirable that two such measurements be made, for reasons that will be discussed. Suppose, for example, that there is small-signal trap saturation in p-type material and that the decay frequency $C_{p1}\mathfrak{N}_1 = \tau_{p0}^{-1}$ in the saturation range and $\mathfrak{N}_1 - \hat{n}_0$ are known in addition to τ_0 and τ_2. It follows then, from the first equation of (63), that $\mathfrak{N}_1^*$ is ν_{tp1} times a known constant:

$$\begin{aligned} \mathfrak{N}_1^* &= (C'_{p1}\tau_{tp1})^{-1}, \\ C'_{p1} &\equiv \tau_{p0}^{-1}(\mathfrak{N}_1 - \hat{n}_0)^{-1} = (1 + p_1/p_0)C_{p1}. \end{aligned} \tag{123}$$

Eliminating $n_0\tau_{tp1} + p_0\tau_{tn1}$ from τ_2 of (92) by use of the third form for τ_0 of (65) results in an equation linear in τ_{tn1} and τ_{tp1}, after $\mathfrak{N}_1^*$ has been eliminated by use of (123). This linear equation and the one for τ_0 may be solved for τ_{tn1} and τ_{tp1}, and, with (50), this solution gives

$$\begin{aligned} \hat{n}_0/\mathfrak{N}_1 &= (1 + n_1/n_0)^{-1} \\ &= \tau_{p0}\,\frac{(p_0 - n_0)/(p_0 + n_0) - C'_{p1}p_0(\tau_2 - \tau_0)}{p_0\tau_2/(p_0 + n_0) - \tau_0}, \\ C_{n1} &= (\mathfrak{N}_1 - \hat{n}_0)^{-1} \\ &\quad \cdot \frac{(p_0 - n_0)/(p_0 + n_0) - C'_{p1}p_0(\tau_2 - \tau_0)}{\tau_0 - n_0\tau_2/(p_0 + n_0) - C'_{p1}(p_0 + n_0)(\tau_2 - \tau_0)\tau_0}, \\ C_{p1} &= C'_{p1} - \tau_{p0}^{-1}(\mathfrak{N}_1 - \hat{n}_0)^{-1}\hat{n}_0/\mathfrak{N}_1. \end{aligned} \tag{124}$$

With trapping, the PME current-conductance ratio does not determine τ_0 but depends also on τ_c (which differs from τ_0 because of trapping), and direct determination of τ_c requires knowledge of light intensity.

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† With the denominator $\mu_n n_0 - \mu_p p_0$ equal to zero, $\tau_0 = \tau_c$ follows, and the numerators are also zero.

‡ The neutrality condition would serve in cases of trapping by acceptors which determine the (p-type) conductivity, for which $\hat{n}_0 = p_0 - n_0$ holds; but $\hat{n}_0$ and $p_0 - n_0$ that differ phenomenologically must generally be considered to obtain.

This circumstance need not, however, really vitiate what in the no-trapping case is a primary motivation for dealing with this ratio: What is determined independently of light intensity is a relationship between τ_0 and τ_c, and a further relationship between these lifetimes will serve to determine both. For C'_{p1} known, the relationship

$$C'_{p1} = (\tau_{tp1}\mathfrak{N}_1^*)^{-1} = (\tau_n + \tau_p - \tau_2)/(p_0 + n_0)(\tau_2 - \tau_n)\tau_0 \quad (125)$$

obtained from (120) is, with (122), linear in τ_c and readily solved for this lifetime.

The PME method of the high-recombination-velocity dark surface is best employed, since it generally provides better accuracy for the conductance change than does the thick-slab method which it otherwise subsumes as a limiting case.† Optimum slab thickness is about one or two diffusion lengths. For large dark-surface recombination velocity, the small-signal results for no trapping‡ give, for the present case,

$$\begin{aligned} I_{sc} &= -\theta e \mathfrak{L} L_0 (S_1 + \coth 2Y_0)^{-1} \\ &= -\theta(\mu_n + \mu_p)^{-1}(L_0/\tau_c)(\coth Y_0)\Delta G, \end{aligned} \quad (126)$$

in which diffusion length L_0 is $(D_0\tau_0)^{\frac{1}{2}}$, Y_0 is y_0/L_0 and S_1 is $s_{m1}L_0/D_0'$; note that ΔG now involves τ_c as a factor. Thus, τ_c is given by

$$\tau_c = \frac{2Y_0 \coth Y_0}{[\mathcal{I}/(\Delta G/G_0)]}\tau_0, \quad (127)$$

where

$$\mathcal{I}/(\Delta G/G_0) \equiv -2y_0(\mu_n + \mu_p)I_{sc}/\theta D_0 \Delta G \quad (128)$$

is the dimensionless PME current-conductance ratio. In (127), τ_0 enters also through Y_0, and, with (125), both τ_0 and τ_c may be found. Note that apparent lifetime τ_r on the assumption of no trapping, obtained by equating $\mathcal{I}/(\Delta G/G_0)$ to $[2y_0/(D_0\tau_r)^{\frac{1}{2}}] \coth [y_0/(D_0\tau_r)^{\frac{1}{2}}]$, is related to τ_0 and τ_c by $\tau_r \tanh^2 [y_0/(D_0\tau_r)^{\frac{1}{2}}] = (\tau_c^2/\tau_0) \tanh^2 Y_0$, and equals τ_c^2/τ_0 only for the thick slab, for which the hyperbolic tangents are unity.

If the model that applies is that of nonrecombinative traps with recombination in other centers, then (119) gives the lifetime $\bar{\tau}_0$ upon which the linear small-signal I_{sc} for minority-carrier trapping depends. For ΔG, (116) holds for the linear small-signal case, for which τ_c/τ_m is $[(1 - r_n)\mu_n + (1 - r_p)\mu_p]/(\mu_n + \mu_p)$. The solution for Δm is readily obtained by comparison with that for the corresponding no-trapping

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† See Ref. 11, Section 3.42.

‡ See Ref. 11, Equation (50).

case.† With the factor L_0^2/D_0' replaced by $\bar{\tau}_m$, it is found that

$$\Delta G = \frac{K_G e(\mu_n + \mu_p)\tau_3 \mathcal{L}(\cosh 2Y_0 - 1)}{S_1 \sinh 2Y_0 + \cosh 2Y_0} \tag{129}$$

is the linear small-signal ΔG for large dark-surface recombination velocity, in which Y_0 and S_1 are defined as above, but in terms of $L_0 \equiv (D_0\bar{\tau}_0)^{\frac{1}{2}}$. The factor $K_G \equiv (\tau_c/\tau_m)(\bar{\tau}_m/\tau_3)$ is τ_c/τ_n in general for p-type material, for which $\bar{\tau}_m$ is $\tau_3/(1 - r_n)$; for n-type material, K_G is τ_c/τ_p. The expression that K_G multiplies also depends on trapping, since L_0 does. Equations (60) and (117) give, for p-type material,‡

$$K_G = 1 + (b + 1)^{-1}\mathfrak{N}_j^*/n_0 \qquad \text{(electron trapping)}$$

$$K_G = \frac{1 + (1 + b^{-1})^{-1}\mathfrak{N}_j^*/p_0}{1 + \mathfrak{N}_j^*/p_0} \qquad \text{(hole trapping).} \tag{130}$$

For hole and electron trapping, respectively, in n-type material, n_0 and p_0 in these equations are interchanged and $b \equiv \mu_n/\mu_p$ replaced by its reciprocal.

With I_{sc} for this model given by the first form of (126) with the redefined L_0,

$$\mathcal{I}/(\Delta G/G_0) = (K_\tau/K_G)\, 2Y_0 \coth Y_0 \tag{131}$$

follows by use of (119), (128) and (129). Apparent lifetime τ_r is accordingly given by $\tau_r \tanh^2 y_0/(D_0\tau_r)^{\frac{1}{2}} = (K_G^2/K_\tau)\tau_3 \tanh^2 Y_0$, and equals $(K_G^2/K_\tau)\tau_3$ for the thick slab. As trap concentration increases, diffusion length increases and a slab of any given thickness becomes a "thin" slab, for which $Y_0 \coth Y_0 \sim 1$; and $\mathcal{I}/(\Delta G/G_0)$ approaches a constant value that is independent of the thickness. For example, if the half-thickness y_0 is of order $(D_0\tau_3)^{\frac{1}{2}}$, then $K_\tau \gg 1$ or $\mathfrak{N}_j^* \gg n_0 + p_0$ also gives small Y_0. From the expressions for K_τ and (130) for K_G it is found that $\mathcal{I}(\Delta G/G_0)$ approaches $2(b + 1)n_0/(n_0 + p_0)$ for electron trapping and $2(b + 1)p_0/b(n_0 + p_0)$ for hole trapping, regardless of conductivity type. On the other hand, if the slab is so thick that $y_0^2 \gg D_0\tau_3\mathfrak{N}_j^*/(n_0 + p_0)$ holds, then the condition $\mathfrak{N}_j^* \gg n_0 + p_0$ for large

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† In Equation (44) of Ref. 11, Δp is replaced by Δm; the D_0 that appears explicitly originates from the boundary conditions and is replaced by D_0'; and S_1 and S_2 are the velocities for Δm multiplied by L_0/D_0'.

‡ Note that K_τ and K_G are equal (for electron or hole trapping) in p-type material for which $\mu_n n_0 = \mu_p p_0$ holds.

trap concentration gives† τ_3/τ_r equal to K_τ/K_G^2 or $(b + 1)^2 n_0^2/(n_0 + p_0)\mathcal{N}_j^*$ for electron trapping and $(b + 1)^2 p_0/b^2(n_0 + p_0)$ for hole trapping in p-type material, with similar results for n-type obtained as in connection with (130).

With the aid of suitable saturation-range measurements, the cross section, concentration and energy level of the nonrecombinative traps are readily found. For traps of the acceptor type, C_{n1}, $\mathcal{N}_1$ and n_1 or p_1 are to be determined, and these can easily be calculated from values from experiment of $\mathcal{N}_1 - \hat{n}_0$, saturation-range lifetime τ_3, lifetime $\tau_{gn1} \equiv [C_{n1}(n_0 + n_1)]^{-1}$ for the traps nearly full, and any one of τ_0, τ_∞ or $\mathcal{N}_1^*$. These last three quantities are not independent; from (107), lifetime τ_∞ for the traps nearly empty is $[1 + (\mathcal{N}_1 - \hat{n}_0)/(n_0 + n_1)]\tau_3 = [1 + \mathcal{N}_1^*/n_0]\tau_3$. Measurement of $\mathcal{G}/(\Delta G/G_0)$ serves to determine $\bar{\tau}_0$: By means of (119) and (130), K_τ/K_G may be written as $(b + 1)n_0[n_0 + p_0 - (p_0 - bn_0)\tau_3/\bar{\tau}_0]^{-1}$ for electron trapping in p-type material, or as an analogous expression for hole trapping in n-type, so that (131) involves only $\bar{\tau}_0$ as unknown.

3.4 *Transport of Injected Carriers*

3.4.1 *The Linear Differential Equations*

The general differential equations of Sections 2.1 and 2.2 are here specialized to the linear small-signal case of trapping (and recombination) in centers of a single type, for which certain specific transport problems will be considered. From (6) through (9) and (46), the linear continuity equation for centers of the acceptor type is

$$\begin{aligned} \partial\Delta m/\partial t &= \hat{D}_p \operatorname{div} \operatorname{grad} \Delta p + \hat{D}_n \operatorname{div} \operatorname{grad} \Delta n \\ &\qquad - \hat{\mathbf{v}}_p \cdot \operatorname{grad} \Delta p - \hat{\mathbf{v}}_n \cdot \operatorname{grad} \Delta n + \Delta g - \mathcal{R}_m \\ = \partial\Delta p/\partial t &= D_0 \operatorname{div} \operatorname{grad} \Delta p - \hat{D}_n \operatorname{div} \operatorname{grad} \Delta\hat{n} \\ &\qquad - \mathbf{v}_0 \cdot \operatorname{grad} \Delta p + \hat{\mathbf{v}}_n \cdot \operatorname{grad} \Delta\hat{n} + \Delta g + \nu_{11}\Delta p + \nu_{12}\Delta\hat{n}, \end{aligned} \tag{132}$$

the first form being that which applies for the linear case in general. The diffusivity and velocity with minority-carrier subscripts of those defined by

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† It has been shown by Amith[62,63,90] that, for minority-carrier trapping in the thick slab, τ_3/τ_r is proportional to $\mathcal{N}_j^{-2}$ for large $\mathcal{N}_j$, if K_τ is taken as unity. This dependence obtains in the intermediate range in which $\mathcal{N}_j^*$ is large compared with minority-carrier concentration n_0 or p_0 but small compared with p_0 or n_0 so that the change in diffusion length may be neglected. For majority-carrier trapping in general, τ_3 and τ_r are substantially equal in this range.

$$\begin{aligned} \hat{D}_p &\equiv n_0 D_0/(n_0 + p_0), & \hat{D}_n &\equiv p_0 D_0/(n_0 + p_0), \\ \hat{\mathbf{v}}_p &\equiv n_0 \mathbf{v}_0/(n_0 - p_0), & \hat{\mathbf{v}}_n &\equiv -p_0 \mathbf{v}_0/(n_0 - p_0) \end{aligned} \tag{133}$$

are, in sufficiently strongly extrinsic material, substantially the minority-carrier diffusivity and velocity, as are D_0 and $\mathbf{v}_0$; while those with majority-carrier subscripts are comparatively small. The linear equation

$$\partial \Delta \hat{n}/\partial t = \nu_{21} \Delta p + \nu_{22} \Delta \hat{n} \tag{134}$$

holds for interactions with the traps. Eliminating $\Delta \hat{n}$ from (132) and (134), substituting from (133) and making use of (51), (62) and (63) results in

$$\begin{aligned} \partial^2 \Delta p/\partial t^2 - D_0 \operatorname{div} \operatorname{grad}(\partial \Delta p/\partial t) + \mathbf{v}_0 \cdot \operatorname{grad}(\partial \Delta p/\partial t) + \nu_s \partial \Delta p/\partial t \\ - \nu_D D_0 \operatorname{div} \operatorname{grad} \Delta p + \nu_v \mathbf{v}_0 \cdot \operatorname{grad} \Delta p + \Delta_1 \Delta p \\ = \partial \Delta g/\partial t + (\nu_s - \nu_{tp1}) \Delta g, \end{aligned} \tag{135}$$

with ν_s defined by (83), and ν_D and ν_v by

$$\begin{aligned} \nu_D &\equiv [1 + \mathfrak{N}_1^*/(n_0 + p_0)](\nu_{gn1} + \nu_{gp1}) \\ \nu_v &\equiv \nu_{gn1} + \nu_{gp1} + (\nu_{gn1} - \nu_{gp1})\mathfrak{N}_1^*/(n_0 - p_0). \end{aligned} \tag{136}$$

The frequency ν_v will be referred to as the "straggle constant". It is readily shown that the linear differential equations that Δn and $\Delta \hat{n}$ satisfy are entirely similar to (135) except for suitable modifications of the right-hand member; all the concentrations satisfy the same equation if there is no volume generation. For Δn it suffices to replace ν_{tp1} where it occurs explicitly by ν_{tn1} , while for $\Delta \hat{n}$ only the generation term $(\nu_{tn1} - \nu_{tp1})\Delta g$ occurs, $\partial \Delta g/\partial t$ being absent. It is also readily shown that linear recombination in other centers can be taken into account by adding $\nu_{n3} + \nu_{p3}$ to the coefficient ν_s of $\partial \Delta n/\partial t$ and $\partial \Delta p/\partial t$ and $(\nu_s - \nu_{tn1})\nu_{n3} + (\nu_s - \nu_{tp1})\nu_{p3}$ to the coefficient Δ_1 of Δn and Δp.

3.4.2 *Steady-State Transport; Reverse Drift*

A simple case that yields qualitative information of interest is that of injection into a filament in the steady state with applied field. For this case,

$$\nu_D D_0 d^2 \Delta p/dx^2 - \nu_v v_0 d\Delta p/dx - \Delta_1 \Delta p = 0 \tag{137}$$

is to be solved for, say, Δp zero for distance x along the filament negatively or positively infinite and continuous at the origin at which there is carrier injection with zero injected total-current density. Equation (137) is easily shown to be equivalent to (30) and (31) specialized for no volume generation and acceptor centers only; ν_D and ν_v are $(-\nu_{22}) = \nu_s - \nu_{tp1}$ times D_0'/D_0 and v_0'/v_0, respectively, and τ_p, from (60), is $(\nu_s - \nu_{tp1})/\Delta_1$.

The solutions in the semi-infinite regions separated by the origin are $e^{r_1 x}$ and $e^{r_2 x}$, where r_1 and r_2 are given by

$$\begin{aligned}\begin{pmatrix} r_1 \\ r_2 \end{pmatrix} &= \tfrac{1}{2}\left[\nu_v v_0/\nu_D D_0 \pm \left[(\nu_v v_0/\nu_D D_0)^2 + 4\Delta_1/\nu_D D_0\right]^{\frac{1}{2}}\right] \\ &\sim \begin{pmatrix} \nu_v v_0/\nu_D D_0 \\ -\Delta_1/\nu_v v_0 \end{pmatrix},\end{aligned} \tag{138}$$

as obtained from (137). The case of recombination without appreciable trapping[64] presents no unfamiliar features; the approximation given, which is that for Δ_1 small, as may result from one of the capture coefficients small, will accordingly be considered. The magnitude of r_1 is thus large compared with that of r_2. With the condition $v_0 > 0$, which may be assumed without loss of generality, $e^{r_1 x}$ gives the familiar sharply varying field-opposing solution to the left of the origin and $e^{r_2 x}$ gives the corresponding gradually varying field-aiding solution to the right, provided ν_v is positive; then, r_1 and r_2 are respectively positive and negative. But negative ν_v can occur, for which an anomalous behavior obtains, the field-opposing and field-aiding solutions then being respectively the gradually and sharply varying exponentials $e^{r_2 x}$ and $e^{r_1 x}$. For this case, in the limit of no diffusion, added carrier concentration appears only in the direction opposite to that of the ambipolar drift velocity, that is, opposite to the direction of drift normally determined by conductivity type.

This "reverse drift" associated with trapping may be understood in terms of properties of the current density $\Delta\mathbf{I}$ of added carriers. From (19), added carriers drift in the direction of the total current density, or the contribution to $\Delta\mathbf{I}$ from drift has the sign of $\mathbf{I}$ if $n_0\Delta p - p_0\Delta n$ or $\Delta p/\Delta n - p_0/n_0$ is positive; that is, if injection results in proportionately more holes than electrons than is the case at thermal equilibrium. This behavior is, of course, that which normally occurs in n-type material; with no trapping, $\Delta p/\Delta n$ equals unity and added carriers drift with or opposite to $\mathbf{I}$ according to whether the semiconductor is n-type or p-type, with no drift in intrinsic material.[10,64] Thus, the normal behavior requires the conditions that $\Delta p/\Delta n - p_0/n_0$ be positive in n-type and negative in p-type. It is easily shown, by writing these conditions by means of (60) for the steady-state value $(1 - r_n)^{-1}$ of $\Delta p/\Delta n$, that both are tantamount in the steady state to the single condition, $\nu_v > 0$. This condition clearly always holds for the majority-carrier trapping range, while reverse drift results for sufficient minority-carrier trapping in not too strongly extrinsic material. From (136), $\nu_v > 0$ gives

$$p_0 - n_0 > (\nu_{gn1} - \nu_{gp1})\mathfrak{N}_1^*/(\nu_{gn1} + \nu_{gp1}) \qquad (139)$$

for p-type material, and a similar inequality for n-type is obtainable by changing the sign of each side. Equating the two sides gives the condition for no drift, which, for no trapping, holds for intrinsic material. From (54), the right-hand side of (139) may be written as

$$(n_0 - n_1^*)\mathfrak{N}_1^*/(n_0 + n_1^*) = (p_1^* - p_0)\mathfrak{N}_1^*/(p_1^* + p_0).$$

It reduces to $\mathfrak{N}_1^*$ for electron trapping without recombination, since then ν_{gp1} and n_1^* are zero. For this case, since $\mathfrak{N}_1^*$ equals $n_0\nu_{tn1}/\nu_{gn1}$ from (63), reverse drift obtains if n_0/p_0 in p-type material exceeds $\tau_{tn1}/(\tau_{tn1} + \tau_{gn1})$, the fraction of the time electrons are free. A similar result holds for hole trapping in n-type material.

3.4.3 *Drift of an Injected Pulse*

The differential equation for drift with negligible diffusion and no volume generation in one cartesian dimension with trapping by centers of a single type is

$$\partial^2\Delta p/\partial t^2 + v_0\partial^2\Delta p/\partial x\partial t + \nu_s\partial\Delta p/\partial t + \nu_v v_0\partial\Delta p/\partial x + \Delta_1\Delta p = 0, \qquad (140)$$

from (135). For a pulse of carriers injected into a doubly infinite filament, a suitable technique of solution is that of the bilateral or two-sided Laplace transform[91,92] with respect to the distance variable, for which the notations

$$F(s, U) \equiv \int_{-\infty}^{\infty} e^{-s\lambda} f(\lambda, U)\, d\lambda \equiv \mathfrak{L}\{f(X, U)\} \equiv \overline{f(X, U)} \qquad (141)$$

are here employed. Dimensionless independent variables

$$X \equiv x/L, \qquad U \equiv t/\tau \qquad (142)$$

are introduced, and with distance and time units given by†

$$\begin{aligned} L &\equiv v_0\tau, \\ \tau &\equiv (\,|\,\nu^2\,|\,)^{-\frac{1}{2}}, \\ \nu^2 &\equiv 4[\nu_v(\nu_s - \nu_v) - \Delta_1] \qquad (143) \\ &\equiv 4n_i^2(\nu_{tn1} - \nu_{tp1})^2(p_0 - n_0)^{-2} \\ &\quad \cdot[(\nu_{tn1} + \nu_{tp1})(p_0 - n_0)/(\nu_{tn1} - \nu_{tp1})\mathfrak{N}_1^* - 1], \end{aligned}$$

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† The second form for ν^2 follows by use of (52), (62) and (63).

subject to the restrictions $\nu^2 \neq 0$ and $n_0 \neq p_0$, the reduced equation

$$\begin{aligned} &\partial^2 \Delta p/\partial U^2 + \partial^2 \Delta p/\partial X \partial U + \zeta \partial \Delta p/\partial U \\ &\qquad + \tfrac{1}{2}(\zeta + \kappa)\partial \Delta p/\partial X + \tfrac{1}{4}(\zeta^2 - \kappa^2 \mp 1)\Delta p = 0 \end{aligned} \tag{144}$$

results, where κ and ζ are the parameters

$$\begin{aligned} \kappa &\equiv (2\nu_v - \nu_s)\tau \\ &= [\nu_{gn1} + \nu_{gp1} + (n_0 + p_0)(\nu_{tn1} - \nu_{tp1})/(n_0 - p_0)]\tau, \\ \zeta &\equiv \nu_s \tau = (\nu_{tn1} + \nu_{gn1} + \nu_{tp1} + \nu_{gp1})\tau > 0. \end{aligned} \tag{145}$$

Coefficient unity for the second term of (144) results from the definition of L. The double sign in the last term of the equation results from the necessity of defining a real (and positive) τ, the upper and lower signs applying respectively for positive and negative ν^2.

Laplace transformation of (144) gives

$$\begin{aligned} &d^2\overline{\Delta p}/dU^2 + (s + \zeta)d\overline{\Delta p}/dU \\ &\qquad + [\tfrac{1}{2}(\zeta + \kappa)s + \tfrac{1}{4}(\zeta^2 - \kappa^2 \mp 1)]\overline{\Delta p} = 0. \end{aligned} \tag{146}$$

As has governed the choices of L and τ, the roots $(-N_1)$ and $(-N_2)$ of the associated quadratic reduce to

$$\begin{pmatrix} -N_1 \\ -N_2 \end{pmatrix} = \tfrac{1}{2}\{\pm[(s - \kappa)^2 \pm 1]^{\frac{1}{2}} - (s + \zeta)\}, \tag{147}$$

in which the double sign inside the radical here and in what follows relates only to the sign of ν^2. Equation (146) holds for each of the transformed concentrations, as does (140) for each of the original ones. General solutions are thus

$$\begin{aligned} \overline{\Delta p} &= \sum_{j=1}^{2} A_j e^{-N_j U}, \\ \overline{\Delta \hat{n}} &= \sum_{j=1}^{2} r_{nj} A_j e^{-N_j U}. \end{aligned} \tag{148}$$

From these solutions in conjunction with the Laplace transform of $\partial \Delta \hat{n}/\partial U$ obtained from (134), it is found that the r_{nj} are given by

$$\begin{aligned} r_{nj} &= -\nu_{21}/(\nu_{22} + N_j/\tau) \\ &= (\nu_{tn1} - \nu_{tp1})/(\nu_{tn1} + \nu_{gn1} + \nu_{gp1} - N_j/\tau), \qquad j = 1,2. \end{aligned} \tag{149}$$

With ν_j replaced by N_j/τ, these are formally the same as the trapping ratios for the decay of photoconductivity given in the first line of (81).†

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† Other forms for the r_{nj} obtainable through Laplace transformation of $\partial \Delta p/\partial t$ from (132) written for one dimension and no diffusion are not similarly related to the forms of the second line of (81), though, for $s = 0$, the N_j/τ reduce to the ν_j and all r_{nj} to those for photoconductivity.

The A_j are easily found in terms of the r_{nj} and transforms $\overline{\Delta p_1}$ and $\overline{\Delta \hat{n}_1}$ of the initial distributions. For no carriers trapped initially, the case that will be considered, (147) to (149) give

$$\begin{pmatrix} A_1/\overline{\Delta p_1} \\ A_2/\overline{\Delta p_1} \end{pmatrix} = \tfrac{1}{2}\,\{1 \mp (s - \xi)/[(s - \kappa)^2 \pm 1]^{\frac{1}{2}}\}, \qquad (150)$$

in which the parameter ξ is defined by

$$\xi \equiv (\nu_{11} - \nu_{22})\tau = (\nu_s - 2\nu_{tp1})\tau. \qquad (151)$$

Corresponding coefficients for the solution for $\Delta\bar{n}$ are $(1 - r_{n1})A_1$ and $(1 - r_{n2})A_2$, and these are similar to the ones for $\Delta\bar{p}$ if ξ in (150) is replaced by

$$\eta \equiv (\nu_{11} - \nu_{22} - 2\nu_{21})\tau = (\nu_s - 2\nu_{tn1})\tau. \qquad (152)$$

Solutions for an injected gaussian delta pulse are advantageously derived as limiting forms as a approaches zero of solutions for the gaussian initial distribution whose transform is given by

$$\overline{\Delta p_1}/(\mathcal{P}/L) = \mathfrak{L}\{\tfrac{1}{2}\pi^{-\frac{1}{2}}a^{-1}e^{-X^2/4a^2}\} = e^{a^2s^2} \qquad (153)$$

for $\mathcal{P}$ carrier pairs injected per unit area of cross section. From (147) through (153), $\overline{\Delta p}$ for this initial gaussian distribution is given by

$$\overline{\Delta P} \equiv \overline{\Delta p}/(\mathcal{P}/L) = e^{a^2s^2-\frac{1}{2}U(s+\zeta)}\,(\cosh\{\tfrac{1}{2}U[(s - \kappa)^2 \pm 1]^{\frac{1}{2}}\} - (s - \xi)$$
$$\cdot[(s - \kappa)^2 \pm 1]^{-\frac{1}{2}}\sinh\{\tfrac{1}{2}U[(s - \kappa)^2 \pm 1]^{\frac{1}{2}}\}), \qquad (154)$$

from which $\overline{\Delta N} \equiv \overline{\Delta n}/(\mathcal{P}/L)$ is obtained by replacing ξ by η.

Certain inverse Laplace transforms that are needed are derived in Appendix A, and Appendix B includes some details of their use in calculation of the solutions from (154). Solutions for the initial delta pulse at the origin are found to be

$$\Delta P \equiv \Delta p/(\mathcal{P}/L) = [e^{\kappa X-\frac{1}{2}(\zeta+\kappa)U}]$$
$$\cdot\left(\delta(U - X) + \frac{1}{2}\left\{(\xi - \kappa)\genfrac{}{}{0pt}{}{I_0}{J_0}[\sqrt{X(U - X)}]\right.\right.$$
$$\left.\left.\genfrac{}{}{0pt}{}{+\,I_1}{-\,J_1}[\sqrt{X(U - X)}]\,\frac{X}{\sqrt{X(U - X)}}\right\}\mathbf{1}\,[X(U - X]\right), \qquad (155)$$

$$\Delta\hat{N} \equiv \Delta\hat{n}/(\mathcal{P}/L) = \tfrac{1}{2}(\xi - \eta)[e^{\kappa X-\frac{1}{2}(\zeta+\kappa)U}]$$
$$\cdot\genfrac{}{}{0pt}{}{I_0}{J_0}[\sqrt{X(U - X)}]\,\mathbf{1}\,[X(U - X)];$$

for $\Delta N \equiv \Delta n/(\mathcal{P}/L)$, ξ in ΔP is replaced by η. The modified Bessel functions I_0 and I_1 apply for the upper sign in (144) and (146), that is, for ν real, while the Bessel functions J_0 and J_1 apply for ν imaginary. The term in ΔP and ΔN with the delta function $\delta(U - X) = v_0\tau\delta(v_0t - x)$ represents a contribution that drifts at velocity v_0. The continuous distributions are confined to the interval $0 \leqq x \leqq v_0t$, $\mathbf{1}[X(U - X)] = \mathbf{1}[x(v_0t - x)]$ being the step function that is respectively zero and unity for negative and positive values of its argument. Note that κ and

$$\begin{aligned} \tfrac{1}{2}(\zeta + \kappa) &= \nu_v\tau, \\ \tfrac{1}{2}(\xi - \kappa) &= (1 - n_0/p_0)^{-1}(\nu_{tn1} - \nu_{tp1})\tau, \\ \tfrac{1}{2}(\eta - \kappa) &= (p_0/n_0 - 1)^{-1}(\nu_{tn1} - \nu_{tp1})\tau, \\ \tfrac{1}{2}(\xi - \eta) &= (\nu_{tn1} - \nu_{tp1})\tau \end{aligned} \tag{156}$$

are not restricted as to sign. For numerical computation of solutions and further analytical study, it is well to transform (155) by eliminating X in accordance with

$$X \equiv U \sin^2 \tfrac{1}{2}\Theta = \tfrac{1}{2}U(1 - \cos\Theta), \qquad U > 0, \tag{157}$$

which gives

$$\begin{aligned} \Delta P &= e^{-\frac{1}{2}U(\zeta + \kappa\cos\Theta)}\Big\{[\tfrac{1}{2}U(\pi - \Theta)]^{-1}\delta(\pi - \Theta) \\ &\qquad + \tfrac{1}{2}[(\xi - \kappa)\, {\textstyle\substack{I_0 \\ J_0}}\, (\tfrac{1}{2}U\sin\Theta)\ {\textstyle\substack{+I_1 \\ -J_1}}\, (\tfrac{1}{2}U\sin\Theta)\tan\tfrac{1}{2}\Theta] \\ &\qquad \cdot\mathbf{1}[\Theta(\pi - \Theta)]\Big\} \\ \Delta\hat{N} &= \tfrac{1}{2}(\xi - \eta)[\,e^{-\frac{1}{2}U(\zeta + \kappa\cos\Theta)}]\, {\textstyle\substack{I_0 \\ J_0}}\, (\tfrac{1}{2}U\sin\Theta) \\ &\qquad \cdot\mathbf{1}[\Theta(\pi - \Theta)]. \end{aligned} \tag{158}$$

For ΔN, ξ in ΔP is replaced by η. The use of Θ as a variable implies the step function of (155), while the step function of (158) simply restricts Θ as defined by (157) to the interval $0 \leqq \Theta \leqq \pi$.

Interpretation in descriptive terms of cases of imaginary ν requires further analysis. Illustrative cases of minority-carrier trapping in strongly extrinsic material, for which ν is real and whose interpretation is comparatively straightforward, will be presented first. For strongly extrinsic material, since the parameter ξ or η for minority carriers is substantially equal to κ, the minority-carrier concentration does not include the term with the Bessel function I_0. If, also, the trapping is nonrecombinative, then $\zeta = (\nu_g + \nu_t)\tau$ and $\kappa = (\nu_g - \nu_t)\tau$ hold with $\nu^2 = 4\nu_t\nu_g$, where ν_t and ν_g are ν_{tn1} or ν_{tp1} and ν_{gn1} or ν_{gp1}, respectively, and refer to the minority carrier. Figs. 1 and 2 show distributions of mobile minority

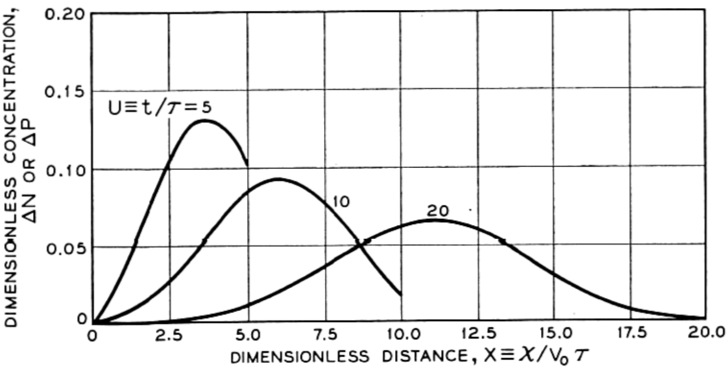

Fig. 1 — Continuous concentration distributions at different times of mobile minority carriers from an injected delta pulse for drift with trapping. A strongly extrinsic semiconductor and $\zeta = 1$, $\kappa = 0$ are assumed. For nonrecombinative trapping, these assumptions imply trapping time τ_t and release time τ_g both equal to twice the time unit τ. The pulse at the limit of the drift range is attenuated by the factor $e^{-\frac{1}{2}(\zeta-\kappa)U} = e^{-\frac{1}{2}U}$.

carriers for this case. For Fig. 1, ζ is unity and κ zero, as for trapping and release times equal,† and the continuous distributions are shown for different times after injection at the origin of the neutral delta pulse. These distributions are led by a delta pulse, which drifts at the ambipolar velocity v_0. This remnant of the initial pulse is composed of un-

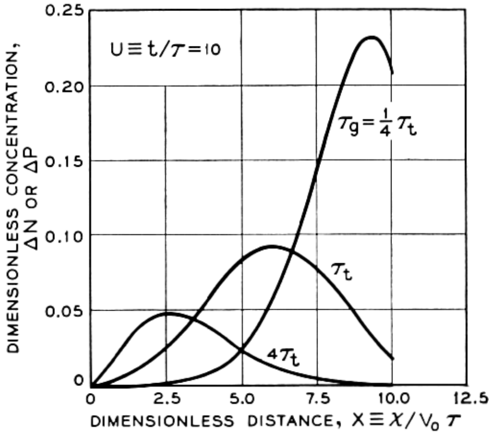

Fig. 2 — Continuous concentration distributions of mobile minority carriers from an injected delta pulse, for drift with trapping, all at time 10τ and for release time respectively 4, 1 and $\frac{1}{4}$ times trapping time. A strongly extrinsic semiconductor and nonrecombinative trapping are assumed; ζ equals $\frac{5}{4}$, 1, $\frac{5}{4}$; κ equals $-\frac{3}{4}$, 0, $\frac{3}{4}$; τ equals τ_t, $\frac{1}{2}\tau_t = \frac{1}{2}\tau_g$, τ_g, respectively.

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† Fig. 1 applies more generally: The values of the parameters do not rule out recombination, but imply merely $\nu = \nu_s = 2\nu_{tn1} = 2[(\nu_{gn1}/\nu_{tn1})(\nu_{tn1}^2 - \nu_{tp1}^2)]^{\frac{1}{2}}$ for minority electrons and similar relations for minority holes.

trapped carriers and is rapidly attenuated by the exponential factor, which is $e^{-\frac{1}{2}U}$ for the particular values of the parameters. The abrupt fronts of the distributions for the shorter times result from most carriers having been trapped only slightly — at least once, but not much more. A relative maximum† appears in the case of the figure for times greater than $2^{\frac{3}{2}}\tau$. For the longer times, the abrupt front disappears as a result of multiple trapping. Furthermore, there is a reduction of apparent mobility: The maximum ultimately drifts at velocity $\frac{1}{2}v_0$, the fraction of v_0 equal to the fraction of the time the carriers are free; this equality will be shown to apply for nonrecombinative minority-carrier trapping in strongly extrinsic material.[37] This limiting behavior sets in rather slowly, as the distribution for $t = 20\tau$ shows; its maximum occurs somewhat beyond the middle of the drift range.

Fig. 2 shows distributions all at time $t = 10\tau$ for release time respectively 4, 1 and $\frac{1}{4}$ times trapping time. The increasing areas under these distributions are associated with decreasing fractions of carriers trapped; it will be shown that, for nonrecombinative trapping, this trapped fraction rapidly approaches the fraction of the time carriers are trapped. The distributions have maxima appreciably beyond the respective values for large U of one-fifth, one-half and four-fifths of the drift range, and the distribution for the comparatively small release time still exhibits a high abrupt front at the time 10τ.

The parameters on which the solutions depend have certain general properties. From the first forms for κ and ζ of (145) and the definitions of τ and ν^2 of (143),

$$\zeta = (\kappa^2 \pm 1 + 4\Delta_1\tau^2)^{\frac{1}{2}} \geqq (\kappa^2 \pm 1)^{\frac{1}{2}} \tag{159}$$

follows. The inequality sign is associated with recombination, Δ_1 being zero for nonrecombinative trapping. The parameter ζ is real and never negative. For ν imaginary, so that the lower sign applies, a similar calculation gives

$$\kappa^2 = 1 + (\nu_s^2 - 4\Delta_1)\tau^2 \geqq 1, \qquad \nu^2 < 0; \tag{160}$$

the condition $\nu_s^2 - 4\Delta_1 \geqq 0$ implies real decay constants and holds from (85). For ν real, κ is not restricted. For example, for nonrecombinative trapping in strongly extrinsic material κ is $\frac{1}{2}[(\nu_g/\nu_t)^{\frac{1}{2}} - (\nu_t/\nu_g)^{\frac{1}{2}}]$ and can be zero (as for Fig. 1) or have any positive or negative value. Thus, $\zeta \geqq 1$ holds for ν real and $\zeta \geqq 0$ holds for ν imaginary. With (151) and

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† Expressions for $d\Delta N/dX = (\frac{1}{2}U \sin \Theta)^{-1} d\Delta N/d\Theta$ from the Maclaurin's expansions for the modified Bessel functions reduce, for $\zeta = 1$ and $\kappa = \eta = 0$, to $\frac{1}{4}e^{-\frac{1}{2}U}$ and $\frac{1}{4}(1 - \frac{1}{8}U^2)\, e^{-\frac{1}{2}U}$ at the origin and at the end of the drift range.

(152), which define ξ and η, it is readily seen that $\xi \sim \eta \sim \kappa$ holds under the condition for the validity in general of the lifetime function $\tau_n \sim \tau_p$ of (75); and that ξ or η is approximately κ for $n_0 \gg p_0$ or $p_0 \gg n_0$, respectively. These properties are consistent with the easily verified relationship,

$$n_0\xi - p_0\eta = (n_0 - p_0)\kappa. \tag{161}$$

Also, for nonrecombinative trapping, the parameters do not depend on the capture coefficient and $\zeta = (\kappa^2 \pm 1)^{\frac{1}{2}}$ equals ξ or η according to whether electrons or holes are trapped.

Qualitative and physical distinctions are related principally to the signs of ν^2 and κ. The general condition for real ν is

$$(\nu_{tn1} + \nu_{tp1})(p_0 - n_0)/(\nu_{tn1} - \nu_{tp1}) > \mathfrak{N}_1^*, \tag{162}$$

from (143), and ν is imaginary if the inequality is reversed. From (54), the left-hand side may be written as $(p_0 + n_1^*)(p_0 - n_0)/(p_0 - n_1^*) = (p_1^* + n_0)(p_0 - n_0)/(p_1^* - n_0)$. The condition $\kappa > 0$ is, for p-type material,

$$p_0 - n_0 > (\nu_{tn1} - \nu_{tp1})(p_0 + n_0)/(\nu_{gn1} + \nu_{gp1}), \tag{163}$$

from the first equation of (145); changing signs of both sides or reversing the inequality gives $\kappa > 0$ for n-type or $\kappa < 0$ for p-type. From (54) and (63), the right-hand side may be written as

$$\frac{(1 + n_0/p_0)(p_0 - n_1^*)\mathfrak{N}_1^*}{n_0 + n_1^*} = \frac{(1 + p_0/n_0)(p_1^* - n_0)\mathfrak{N}_1^*}{p_1^* + p_0}.$$

If recombinative trapping is excluded, then ν is evidently real in the limit of strongly extrinsic material. Note, for example, that as p_0 increases indefinitely, ν_{tp1} approaches zero while ν_{tn1} approaches $C_{n1}\mathfrak{N}_1 = \tau_{n0}^{-1}$, so that ν approaches $2C_{n1}(n_1\mathfrak{N}_1)^{\frac{1}{2}}$. Also, since ν_{gp1} increases indefinitely and ν_{gn1} approaches $C_{n1}n_1$, κ becomes positively infinite. If ν_{tn1} and ν_{tp1} are equal for given p_0, then $\nu_{tn1} > \nu_{tp1}$ holds for all larger p_0. Suppose first that these capture frequencies are equal for some p_0 of the n-type material. Then, as p_0 decreases from a large value, κ does likewise. With $\nu_{tn1} > \nu_{tp1}$, (162) and (163) show that κ decreases to zero and becomes negative for ν still real. Further decrease of p_0 results in κ becoming negatively infinite, since ν approaches zero as $p_0 - n_0$ approaches $(\nu_{tn1} - \nu_{tp1})\mathfrak{N}_1^*/(\nu_{tn1} + \nu_{tp1})$, following which κ increases to -1 with ν imaginary. While ν is not defined for intrinsic material, the equations show that, in the approach to the intrinsic limit, ν is imaginary and κ approaches ± 1, the sign being that of $(\nu_{tn1} - \nu_{tp1})/(n_0 - p_0)$.

It is evident from (162) and (163) that, for n-type material and $\nu_{tn1} > \nu_{tp1}$ — that is, in the majority-carrier trapping range — ν is imaginary and κ positive, hence greater than unity. With increasing n_0, ν_{tn1} and ν_{tp1} ultimately approach equality, ν approaches zero and κ becomes positively infinite. For p-type material and $\nu_{tn1} > \nu_{tp1}$, or the minority-carrier capture range, it is likewise evident that positive κ implies real ν and that imaginary ν implies $\kappa < -1$. This latter case includes reverse drift. From a result of Section 2.2.1.2, since ν_{tn1} equals ν_{tp1} in n-type material, ν_{gn1} equals ν_{gp1} in p-type. Decrease of p_0 from the value for $\nu_{gn1} = \nu_{gp1}$ gives $\nu_{gn1} > \nu_{gp1}$, and, from (139), as p_0 approaches n_0, negative ν_v occurs. In the limit of negatively infinite κ, for which ν is zero (with $\nu_{tn1} \neq \nu_{tp1}$), ν_v is in general positive. Thus, the reverse-drift range is in general the portion of the minority-carrier capture range of imaginary ν that results if a certain infinite range of large negative values of κ is excluded. For nonrecombinative trapping, (139) and (162) both yield $| p_0 - n_0 | < \mathfrak{N}_1^*$, so that the two ranges coincide.

If the capture frequencies are equal for some p_0 of the p-type material, the initial decrease of κ as p_0 decreases from a large value still obtains; but the p_0 for equal capture frequencies is approached for ν still real and κ positive, and κ again becomes positively infinite as ν approaches zero. Imaginary ν results with further decrease of p_0 so that $\nu_{tp1} > \nu_{tn1}$ results, κ decreasing from large positive values to unity and then from -1 to large negative values, the majority-carrier trapping and minority-carrier capture ranges, respectively, being realized (for hole-capture frequency the larger) in p- and n-type material. It is easily shown, as before, that the reverse-drift range applies, with recombination, for a finite range of negative values of κ less than -1 in this minority-carrier capture range of imaginary ν. Increase of n_0 beyond the value for negatively infinite κ given (as before) by $n_0 - p_0 = (\nu_{tp1} - \nu_{tn1})/(\nu_{tp1} + \nu_{tn1})$ results in real ν, with which κ ranges from large negative values and becomes positively infinite as the material becomes strongly n-type.

For nonrecombinative trapping, κ for real ν and strongly extrinsic material is, as has been noted, a positive or negative constant. For electron trapping, for example, ν is $2C_{n1}(n_1\mathfrak{N}_1)^{\frac{1}{2}}$ and κ is $\frac{1}{2}[(n_1/\mathfrak{N}_1)^{\frac{1}{2}} - (\mathfrak{N}_1/n_1)^{\frac{1}{2}}]$ in the limit of large p_0; κ is positively infinite in the limit of large n_0. As p_0 decreases in the minority-carrier capture range of real ν, κ becomes negatively infinite as $p_0 - n_0$ approaches $\mathfrak{N}_1^*$. With further decrease of p_0, κ increases to -1 in the minority-carrier capture range of imaginary ν, which (for nonrecombinative trapping) is the reverse-drift range. In the majority-carrier trapping range in general, ν is imaginary and κ greater than unity.

Maxima for large U of each of the continuous distributions for cases of real ν occur substantially together. From (158), the Θ for a maximum is found to be given by† $\tan \Theta = -\kappa^{-1}$, which gives‡ $X/U = x/v_0t = \frac{1}{2}[1 + \kappa/(\kappa^2 + 1)^{\frac{1}{2}}]$. For nonrecombinative traps, real ν implies minority-carrier trapping with positive ν_v and $(\kappa^2 + 1)^{\frac{1}{2}}$ equal to ζ. From (63), (136) and (145), X/U accordingly reduces to $\nu_v/(\nu_t + \nu_g)$, which is $[\tau_{tn1} - n_0\tau_{gn1}/(p_0 - n_0)]/(\tau_{tn1} + \tau_{gn1})$ for electron trapping or $[\tau_{tp1} - p_0\tau_{gp1}/(n_0 - p_0)]/(\tau_{tp1} + \tau_{gp1})$ for hole trapping. Hence X/U, the factor by which the apparent mobility is smaller than the magnitude of the ambipolar pseudomobility,[10] is in general less than $\tau_t/(\tau_t + \tau_g)$, the fraction of the time minority carriers are free; but X/U is substantially equal to this fraction[37] under the condition $| n_0 - p_0 | \gg \mathfrak{N}_1^*$, obtained by use of (63). As $| n_0 - p_0 |$ approaches $\mathfrak{N}_1^*$ in the nonrecombinative case, X/U approaches zero. Recombination reduces the distance for a maximum at given time, and thus reduces the apparent mobility, since, for nonrecombinative traps with recombination of lifetime τ_3 in other centers, the distribution of the mobile carriers subject to trapping is simply that for no recombination multiplied by the decay factor $e^{-x/v_0\tau_3}$. This factor applies because the carriers which arrive at x at whatever time have drifted in the conduction band for time x/v_0.

The decay constant for the straggle effect[54] is that of the limiting decay of the tail of the distribution at fixed x after the maximum has passed. It is given by the exponent in (158), and is accordingly ν_v. This result follows from (156), since, from (157), $x \ll v_0t$ implies $\cos \Theta \sim 1$. Real ν and hence positive ν_v obtain in this connection. By use of (63), it is easily shown that ν_v for strongly extrinsic material is substantially $\nu_{gn1} + \nu_{gp1}$ plus either ν_{tn1}, for $n_0 \gg p_0$, or ν_{tp1}, for $p_0 \gg n_0$.

Integrals of the solutions of (158) over the drift range are evaluated in Appendix C. These integrals give

$$\begin{aligned} F_p &\equiv \mathcal{P}^{-1} \int_0^{v_0t} \Delta p \, dx = \int_0^U \Delta P \, dX \\ &= e^{-\frac{1}{2}\zeta U}[\xi(\kappa^2 \pm 1)^{-\frac{1}{2}} \sinh \tfrac{1}{2}(\kappa^2 \pm 1)^{\frac{1}{2}}U + \cosh \tfrac{1}{2}(\kappa^2 \pm 1)^{\frac{1}{2}}U], \\ \hat{F}_n &\equiv \mathcal{P}^{-1} \int_0^{v_0t} \Delta \hat{n} \, dx = \int_0^U \Delta \hat{N} \, dX \\ &= e^{-\frac{1}{2}\zeta U}(\xi - \eta)(\kappa^2 \pm 1)^{-\frac{1}{2}} \sinh \tfrac{1}{2}(\kappa^2 \pm 1)^{\frac{1}{2}}U \end{aligned} \tag{164}$$

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† Use is made of the approximations $I_0(z) \sim I_1(z) \sim (2\pi z)^{-\frac{1}{2}}e^z$ for $|z|$ large. The distributions for large U are substantially proportional and gaussian in shape. For nonrecombinative minority-carrier trapping, they are as if the excess majority carriers were subject only to drift and diffusion with diffusivity $v_0L/4\zeta^3$.

‡ Note that, for the maximum, $\Theta - \pi/2$ has the sign of κ, so that $\cos \Theta$ has the opposite sign.

as mobile and trapped fractions at given time of carriers initially injected. For

$$F_n \equiv \mathcal{P}^{-1} \int_0^{v_0 t} \Delta n \, dx = F_p - \hat{F}_n ,$$

ξ in F_p is replaced by η. Equation (159) serves to verify that, with recombination, F_p, $\hat{F}_n$ and F_n all approach zero as U increases indefinitely. For nonrecombinative electron trapping, $\xi = \zeta = (\kappa^2 \pm 1)^{\frac{1}{2}}$ gives $F_p = 1$, as may be expected, and $\hat{F}_n = \frac{1}{2}(1 - \eta/\zeta)(1 - e^{-\zeta U})$. Thus, from (145) and (152), the trapped fraction approaches $\tau_{gn1}/(\tau_{tn1} + \tau_{gn1})$, the fraction of the time electrons are trapped, with time constant $\tau_{gn1}\tau_{tn1}/(\tau_{tn1} + \tau_{gn1})$; the mobile fraction approaches the fraction of the time electrons are free. For hole trapping, entirely similar results apply. All of these results evidently apply for ν imaginary as well as real.

In Fig. 3 are shown continuous minority-carrier distributions for imaginary ν, in particular, for nonrecombinative trapping of minority carriers in the reverse-drift range and also of majority carriers. In the former case, an attenuated delta pulse of untrapped mobile carriers, which drifts at velocity v_0, leads a continuous distribution of minority carriers that crowds towards the origin as its maximum excursions above and below the axis both increase with time. The distribution of added minority carriers is negative over part of the drift range after a certain time.[†] In accordance with (164), it approaches a net positive delta pulse at the origin of strength $\mathcal{P}$, the initial strength, times the free-time fraction. In the case of majority-carrier trapping, the pulse of untrapped carriers increases in strength as it drifts at velocity v_0. This augmentation is appreciable with appreciable equilibrium minority-carrier concentration; it is negligible in strongly extrinsic material. The pulse leads a largely or entirely negative continuous distribution of minority carriers, which crowds towards the pulse as its excursion below the axis increases with time.[‡] This distribution approaches a negative delta pulse, which, with the minority carriers of the augmented pulse, gives a net pulse of strength $\mathcal{P}$. The corresponding distributions of mobile and trapped majority carriers approach net pulses of strengths equal to $\mathcal{P}$ times the free- and trapped-time fractions, respectively.

These results exhibit, with due allowance for the neglect of diffusion,

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[†] Negative concentrations of the trapped and majority carriers also occur. For this case, $\eta - \kappa$ is positive for p-type material, and, if it is not too large, negative Δn first appears (as for the case of the figure) at the end of the drift range for times greater than $2(\eta - \kappa)\tau$, since Δn equals $\frac{1}{2}[e^{-\frac{1}{2}(\zeta - \kappa)}]^U (\eta - \kappa - \frac{1}{2}U)$ for $\Theta = \pi$. With sufficiently large $\eta - \kappa$, negative Δn first appears within the range.

[‡] The majority carriers are similarly distributed: For the case of the figure, (negative) dimensionless majority-carrier concentration ΔP or ΔN is $\frac{3}{2}\Delta N$ or $\frac{3}{2}\Delta P$ for $X = U = 5$ and $\frac{14}{11}\Delta N$ or $\frac{14}{11}\Delta P$ for $X = U = 10$.

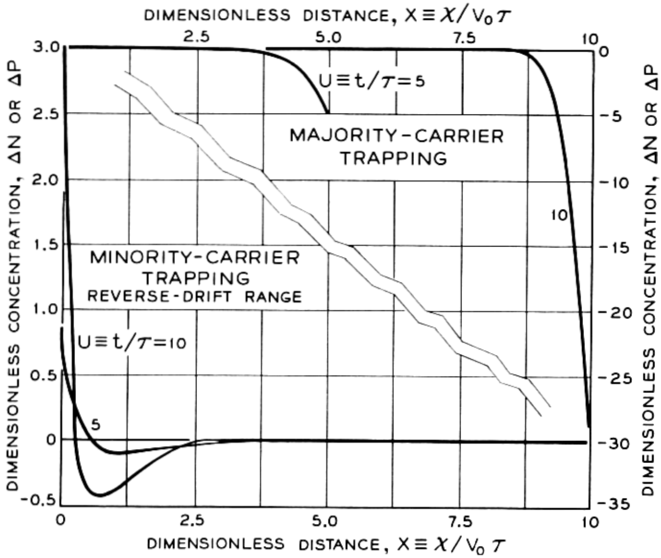

Fig. 3 — Continuous concentration distributions at different times of mobile minority carriers from an injected delta pulse, for drift with minority-carrier trapping in the reverse-drift range and for drift with majority-carrier trapping. Equilibrium majority-carrier concentration is taken as $2n_i$ for both cases, and nonrecombinative trapping is assumed, with τ_g/τ_t large compared with the equilibrium ratio of majority-carrier concentration excess to concentration of the (mobile) carriers that are subject to trapping. For minority-carrier trapping, a strong inequality for reverse drift accordingly holds, with $\tau_g \gg 3\tau_t = 4\tau$, $\zeta = \frac{3}{4}$, $\kappa = -\frac{5}{4}$, and attenuation of the pulse by the factor e^{-U}. For majority-carrier trapping, $\tau_g \gg \frac{3}{4}\tau_t = \tau$, $\zeta = \frac{3}{4}$, $\kappa = \frac{5}{4}$ hold, and the pulse is augmented by the factor $e^{\frac{1}{4}U}$. For both cases, $\xi = -\eta = \frac{3}{4}$ or $\eta = -\xi = \frac{3}{4}$ holds for electrons or holes trapped, so that the coefficients for the minority carriers of the terms in J_0 for the two cases are respectively $\frac{1}{4}$ and $-\frac{1}{4}$.

the essential behavior that would be realized in practice. In each case, the initial neutral pulse of mobile carriers will appear essentially as a diffuse pulse of trapped and mobile carriers in substantially the proportions that obtain for the steady state. Minority-carrier trapping in the reverse-drift range does not give largely unidirectional drift, while majority-carrier trapping results essentially in drift at the ambipolar velocity v_0. Increase of applied field, however, makes for the idealized behavior shown in the figure—in particular, for continuous distributions that are negative over increased actual distance at given time. Numerical estimate of the effect of diffusion in the reverse-drift case shows that negative added-carrier concentrations can occur over appreciable distances under conditions that can be realized in practice for reasonable values of trapping time.†

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† For Fig. 3, $\frac{1}{4}\tau_{gn1} \gg \tau = \frac{3}{4}\tau_{tn1}$ holds for p-type material, and if τ is 10^{-5} second, then $L = v_0\tau$ is about 0.4 cm for $p_0 = 2n_i$ and electron and hole mobilities of 1500

A simple descriptive interpretation of the crowding of the distributions towards the origin in the case of reverse drift is that the drift of added carriers, initially in the direction of v_0, is largely in the opposite direction, the direction of drift of majority carriers, after some trapping has taken place. For more detailed general interpretations, consider the current density $\Delta \mathbf{I}$ of added carriers of (19), which is given under the present assumptions by

$$\begin{aligned} \Delta I &= e^2 \mu_n \mu_p \sigma_0^{-2} I(\mathcal{P}/L)(n_0 \Delta P - p_0 \Delta N) \\ &= e v_0 (\mathcal{P}/L) [\mathrm{e}^{-\frac{1}{2}U(\zeta + \kappa \cos \Theta)} \Big\{ [\tfrac{1}{2} U(\pi - \Theta)]^{-1} \delta(\pi - \Theta) \\ &\quad \cdot \begin{matrix} +\frac{1}{2} I_1 \\ -\frac{1}{2} J_1 \end{matrix} (\tfrac{1}{2} U \sin \Theta) \tan \tfrac{1}{2} \Theta \, \mathbf{1}[\Theta(\pi - \Theta)] \Big\}, \end{aligned} \tag{165}$$

from (8), (158) and (161). With (158), this result shows that the Bessel functions of order zero are associated with carriers that neutralize the charge of trapped carriers or with the trapped carriers themselves, while those of order one are associated with the drift of, in effect, carrier pairs. The direction of drift of a mobile-carrier distribution considered in its entirety depends on the sign of the net ΔI, or ΔI integrated over the drift range. By use of (161) and (164),

$$\begin{aligned} \int_0^U \Delta I dX &= e^2 \mu_n \mu_p \sigma_0^{-2} I(\mathcal{P}/L)(n_0 F_p - p_0 F_n) \\ &= e v_0 (\mathcal{P}/L)(e^{-\frac{1}{2}\zeta U})[\kappa(\kappa^2 \pm 1)^{-\frac{1}{2}} \sinh \tfrac{1}{2}(\kappa^2 \pm 1)^{\frac{1}{2}} U \\ &\qquad + \cosh \tfrac{1}{2}(\kappa^2 \pm 1)^{\frac{1}{2}} U] \end{aligned} \tag{166}$$

results. For nonrecombinative electron trapping, this integral reduces to $e v_0 (\mathcal{P}/L)$, the initial ΔI, times† $\frac{1}{2}(1 + \kappa/\zeta) = \nu_v/(\nu_{tn1} + \nu_{gn1})$ in the limit for U infinite, from (136), (145) and (159). As may be expected, the limiting integral has the sign of v_0 or the opposite sign according to whether ν_v is positive or negative. That the distributions ultimately crowd towards the origin in the case of reverse drift is established on a more quantitative basis by this result.

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and 570 cm^2 volt^{-1} second^{-1} (as for silicon[72,73] at 300°K) and for an applied field of 100 volt cm^{-1}, which is reasonable for a filament of resistivity at least a few hundred ohm-cm and of area of cross section about 10^{-2} cm or less. The corresponding diffusion distance $(D_0 t)^{\frac{1}{2}}$ for $t = 10\tau$ is 0.054 cm, an order of magnitude smaller than the approximate distance $2L$ of 0.8 cm over which negative added-carrier concentrations occur. This difference is greater for larger τ_{tn1}.

† For real ν, which implies positive ν_v and p-type material, this factor is the limiting value of X/U for the maximum of the mobile-electron distribution.

It follows from (165) and is otherwise evident that ΔI is zero at the origin.† Also, for imaginary ν, the Bessel function with the minus sign applies and ΔI within the drift range is negative for positive v_0 if U is not too large. Hence, in the case of reverse drift, the mobile-carrier distributions become steeper even for small times.‡ This conclusion is consistent with the increase with U of the exponential factor for given small Θ, for which $\frac{1}{2}(\zeta + \kappa \cos \Theta) \sim \nu_v \tau$ is negative, and also with the decrease with increasing Θ of this factor for given U, which results because of $\kappa < -1$. With electron trapping, Δn becomes small for sufficiently large U at some Θ. But, with $\xi > \eta$, since $\Delta p \sim \Delta \hat{n}$ is still positive at this Θ, ΔI is still negative, from (165), I being negative for positive v_0. Thus, Δn becomes negative. As Δn becomes increasingly negative, electrons tend to be released from traps and $\Delta \hat{n}$ in turn becomes small [as a zero of $J_0(\frac{1}{2}U \sin \Theta)$ is approached]; but ΔI is still negative. Thus, $\Delta \hat{n}$ becomes negative too. To the right of the location at which $\Delta \hat{n}$ equals $(p_0/n_0 - 1)\Delta n < 0$, which is the condition $\Delta I = 0$ (for negative Δn, Δp and $\Delta \hat{n}$), ΔI is positive and the concentrations tend to increase algebraically. A progressive increase with time of the maximum excursion of Δn below the axis is associated with the presence of a (time-dependent) location at which ΔI changes sign.

In the case of majority-carrier trapping, the exponential factor for given U increases with Θ because of $\kappa > 1$; and $\zeta - \kappa$ being negative,§ this factor is smaller or greater than unity respectively for Θ small or $\Theta \sim \pi$. Thus, as U increases, the continuous minority-carrier distribution, which is negative for all Θ and U, crowds towards the pulse, which drifts at velocity v_0, and both increase in amplitude. For electron trapping in general, $\Delta \hat{n} > 0$ holds at (and near) the pulse. If the material is sufficiently strongly n-type, this $\Delta \hat{n}$ is largely compensated entirely by negative Δn. The pulse, in effect, removes electrons from the semiconductor and transfers them to traps, and its strength remains substantially constant; the continuous contribution to ΔI is negligible because of large τ. In less strongly n-type material, however, neutrality is maintained at the pulse in part by negative Δp and correspondingly more negative Δn, which makes for less trapping for given pulse strength. But holes removed from the semiconductor drift at velocity v_0 and thus (with an equal number of electrons) cause a progressive increase in the strength of the pulse; a $\Delta \hat{n}$ obtains that does not depend explicitly on n_0. The ΔI of the pulse likewise increases indefinitely. The net or inte-

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† Thus, at the origin, $\Delta n/\Delta p$ maintains the value n_0/p_0.

‡ Note also that (158) gives, for small $U > 0$, ΔP or ΔN at the origin equal to $\xi - \kappa$ or $\eta - \kappa$ times $(1 - \nu_v t)$, which increases with t in the case of reverse drift.

§ Note that (159) may be written as $(\zeta + \kappa)(\zeta - \kappa) = -1$ for nonrecombinative trapping and imaginary ν.

grated ΔI, however, increases to a limiting value, as given by the factor $\nu_v/(\nu_{tn1} + \nu_{gn1})$, which is greater than unity in the present case. The largest excursions of the continuous mobile-carrier distributions below the axis occur, of course, at the pulse, at which there is a discontinuity in ΔI with change of sign.

In cases of real ν, the exponential factor is less than unity and decreases as U increases.† With the monotonically increasing modified Bessel functions and ΔI everywhere in the direction of v_0, the continuous distributions, and, from (165), ΔI as well, ultimately possess relative maxima that drift in this direction with a common velocity. While the maxima occur substantially together, the distribution of the trapped minority carriers lags, consistent with the Bessel function of order zero dominating that of order one; the mobile minority-carrier distribution ultimately results entirely from carriers released from traps.

The case of "critical trapping," the borderline case between cases of real and imaginary ν, is one whose analysis furnishes further qualitative insight. For it, ν as defined in (143) is zero with $\nu_{tn1} \neq \nu_{tp1}$, and the condition is the equality corresponding to (162). The previous notation can be used for this case by choosing the time unit τ arbitrarily and noting that (144) through (154) then all apply if the terms with the double signs are omitted wherever they occur. A calculation essentially similar to that given in Appendix B provides the solution, which, for the initial delta pulse of $\mathcal{P}$ carrier pairs per unit area at the origin, is

$$\begin{aligned}
\Delta p &= (\mathcal{P}/L)\{\exp[-\tfrac{1}{2}(\zeta - \kappa)U]\cdot\delta(U - X) \\
&\qquad + \tfrac{1}{2}(\xi - \kappa)\exp[\kappa X - \tfrac{1}{2}(\zeta + \kappa)U]\cdot\mathbf{1}[X(U - X)]\} \\
&= \mathcal{P}\{\exp[-\tfrac{1}{2}(\nu_{tn1} + \nu_{tp1})(1 + (n_0 + p_0)/\mathfrak{N}_1^*)t]\cdot\delta(v_0 t - x) \qquad (167) \\
&\qquad + [p_0(\nu_{tn1} + \nu_{tp1})/v_0\mathfrak{N}_1^*]\exp[-(v_0\mathfrak{N}_1^*)^{-1} \\
&\qquad \cdot(p_0\nu_{tn1} + n_0\nu_{tp1})x - t/\tfrac{1}{2}(\tau_{tn1} + \tau_{tp1})]\cdot\mathbf{1}[x(v_0 t - x)]\}, \\
\Delta\hat{n} &= \tfrac{1}{2}(\mathcal{P}/L)(\xi - \eta)\exp[\kappa X - \tfrac{1}{2}(\kappa + \zeta)U]\cdot\mathbf{1}[X(U - X)] \\
&= \mathcal{P}[(p_0 - n_0)(\nu_{tn1} + \nu_{tp1})/v_0\mathfrak{N}_1^*]\exp[-(v_0\mathfrak{N}_1^*)^{-1} \\
&\qquad \cdot(p_0\nu_{tn1} + n_0\nu_{tp1})x - t/\tfrac{1}{2}(\tau_{tn1} + \tau_{tp1})]\cdot\mathbf{1}[x(v_0 t - x)].
\end{aligned}$$

For Δn, ξ is replaced by η in the first expression for Δp, or n_0 and p_0 as

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† In these cases, $\zeta + \kappa$ has the sign of ν_v and is positive, from (156), and (159) may be written for nonrecombinative trapping as $(\zeta + \kappa)(\zeta - \kappa) = 1$.

well as subscripts n and p are interchanged in the second. The concentration increments in terms of the dimensionless quantities are, of course, actually independent of τ. In the second expressions, quantities involved have been written in forms that apply under the condition for critical trapping. It follows readily from these equations that the mobile and trapped fractions are similar to those of (164) except that $(\kappa^2 \pm 1)^{\frac{1}{2}}$ is replaced by κ; they are independent of τ. For nonrecombinative trapping, the mobile and trapped fractions of carriers respectively approach, as before, the free- and trapped-time fractions, since κ equals $(-\zeta)$.†

As the second expression for $\Delta\hat{n}$ shows, critical trapping is a case of minority-carrier trapping. The continuous distributions are proportional and are equal to products of exponential decays with distance and with time. The amplitudes of the distributions at the origin are larger and the decay with distance is sharper the smaller is v_0. The time decay results from recombination, with which neither τ_{tn1} nor τ_{tp1} is infinite. For nonrecombinative trapping, the condition of critical trapping is the same as that of zero drift, and exponential distributions are established progressively over the variable range v_0t that otherwise do not change with time. It is well to note that the general case of zero ν_v is a case of imaginary ν, with ν^2 equal to $(-4\Delta_1)$. Furthermore, for trapping in intrinsic material, for which ν_v is not defined, it can be shown that the distributions are all identically zero except at the origin. As may be expected for this case of no diffusion, the initial pulse results simply in pulses for the concentration increments that change as trapping and recombination proceed.

IV. ACKNOWLEDGMENTS

The author is indebted to J. A. Burton and to W. H. Brattain for discussions to which this work owes its inception, and is pleased to acknowledge their helpful interest and that of many others, among whom are included G. Bemski, W. L. Brown, T. M. Buck, R. J. Collins, C. G. B. Garrett, G. W. Gobeli, J. R. Haynes, J. J. Hopfield, H. J. Hrostowski, W. Kaiser, M. Lax, C. A. Lee, F. J. Morin, W. T. Read, R. G. Shulman, W. G. Spitzer, G. H. Wannier, G. K. Wertheim and P. A. Wolff. Some were of assistance in particular connections, as was H. Y. Fan, with a communication concerning his treatment of drift with trapping, and H. O. Pollak, who provided a method for evaluating the integral over the drift range of mobile-carrier concentration.

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† Note that $\zeta = (\kappa^2 + 4\Delta_1)^{\frac{1}{2}}$ and $\kappa < 0$ hold.

APPENDIX A

Derivation of Two-Sided Laplace Transforms

The transforms

$$\mathfrak{L}\left\{\int_X^{\infty} \begin{matrix} I_0 \\ J_0 \end{matrix} [(\beta^2 - X^2)^{\frac{1}{2}}] f(\beta)\, d\beta\right\} = (s^2 \pm 1)^{-\frac{1}{2}} F[-\,(s^2 \pm 1)^{\frac{1}{2}}],$$

and

$$\mathfrak{L}\left\{f(X) \pm \int_X^{\infty} \beta(\beta^2 - X^2)^{-\frac{1}{2}} \begin{matrix} I_1 \\ J_1 \end{matrix} [(\beta^2 - X^2)^{\frac{1}{2}}] f(\beta) d\beta\right\} = F[-(s^2 \pm 1)^{\frac{1}{2}}] \tag{168}$$

entail restrictions on $f(X)$ and on the transform variable s for convergence of the integrals. In accordance with the definition of (141), the first transform is

$$\begin{aligned}
&\int_{-\infty}^{\infty} e^{-s\lambda} \int_{\lambda}^{\infty} \begin{matrix} I_0 \\ J_0 \end{matrix} [(\beta^2 - \lambda^2)^{\frac{1}{2}}] f(\beta)\, d\beta\, d\lambda \\
&\qquad = \int_{-\infty}^{\infty} f(\beta) \int_{-\infty}^{\beta} \begin{matrix} I_0 \\ J_0 \end{matrix} [(\beta^2 - \lambda^2)^{\frac{1}{2}}]\, e^{-s\lambda}\, d\lambda\, d\beta \\
&\qquad = \int_0^{\infty} [f(\beta) + f(-\beta)] \int_{\beta}^{\infty} \begin{matrix} J_0 \\ I_0 \end{matrix} [(\lambda^2 - \beta^2)^{\frac{1}{2}}]\, e^{s\lambda}\, d\lambda\, d\beta \\
&\qquad\qquad + \int_0^{\infty} f(\beta) \int_{-\beta}^{\beta} \begin{matrix} J_0 \\ I_0 \end{matrix} [(\lambda^2 - \beta^2)^{\frac{1}{2}}]\, e^{-s\lambda}\, d\lambda\, d\beta,
\end{aligned} \tag{169}$$

as obtained by changing the order of integration and changing from variable β to $-\beta$ in the contribution from the range of negative β. The first of the integrals with respect to λ is†

$$\begin{aligned}
&\int_{\beta}^{\infty} \begin{matrix} J_0 \\ I_0 \end{matrix} [(\lambda^2 - \beta^2)^{\frac{1}{2}}]\, e^{s\lambda}\, d\lambda \\
&\qquad = \int_0^{\infty} \mu(\mu^2 + \beta^2)^{-\frac{1}{2}} \begin{matrix} J_0 \\ I_0 \end{matrix} (\mu) \exp\,[s(\mu^2 + \beta^2)^{\frac{1}{2}}]\, d\mu \qquad (170) \\
&\qquad = (s^2 \pm 1)^{-\frac{1}{2}} \exp\,[-(s^2 \pm 1)^{\frac{1}{2}}\beta].
\end{aligned}$$

For convergence with the upper function and sign, s is taken as (real

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† Watson,[93] p. 416, Equation (2).

and) negative; for convergence with the lower function and sign, the real part of s is to be less than -1. The second is a Gegenbauer integral: With

$$\begin{aligned} \lambda &\equiv -\beta \cos \phi, \\ z \sin \psi &\equiv \begin{pmatrix} i \\ 1 \end{pmatrix} \beta, \\ z \cos \psi &\equiv -is\beta, \\ z &= i(s^2 \pm 1)^{\frac{1}{2}}\beta, \\ \cot \psi &= -\begin{pmatrix} 1 \\ i \end{pmatrix} s, \end{aligned} \tag{171}$$

the result

$$\begin{aligned} &\int_{-\beta}^{\beta} \begin{matrix} J_0 \\ I_0 \end{matrix} [(\lambda^2 - \beta^2)^{\frac{1}{2}}]\, e^{-s\lambda}\, d\lambda \\ &\qquad = \beta \int_0^{\pi} J_0(z \sin \phi \sin \psi)\, e^{iz \cos \phi \cos \psi} \sin \phi\, d\phi \\ &\qquad = 2(s^2 \pm 1)^{-\frac{1}{2}} \sinh\, [(s^2 \pm 1)^{\frac{1}{2}}\beta] \end{aligned} \tag{172}$$

follows.† By substituting from (170) and (172) in (169), the first transform of (168) is established. The second transform is established through

$$\begin{aligned} &\mathfrak{L}\left\{ -\int_X^{\infty} \begin{matrix} I_0 \\ J_0 \end{matrix} [(\beta^2 - X^2)^{\frac{1}{2}}] f'(\beta)\, d\beta \right\} \\ &\qquad = \int_{-\infty}^{\infty} f(\lambda) \exp\, [(s^2 \pm 1)^{\frac{1}{2}}\lambda]\, d\lambda \\ &\qquad = F[-(s^2 \pm 1)^{\frac{1}{2}}]. \end{aligned} \tag{173}$$

The original expression reduces to the form on the left upon integrating by parts for $f(\beta)$ such that $I_0(\beta)f(\beta)$ or $J_0(\beta)f(\beta)$ is zero for $\beta = \infty$. The result then follows by application of the first transform and another integration by parts, which entails $f(\lambda) \exp\, [(s^2 \pm 1)^{\frac{1}{2}}\lambda]$ equal to zero for positively and negatively infinite λ, a condition that subsumes the preceding one.

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† Watson[93] p. 379, Equation (1).

APPENDIX B

Solutions for Drift of an Injected Pulse

The dimensionless concentration $\Delta P \equiv \Delta p/(\mathcal{P}/L)$ is the inverse Laplace transform of $\overline{\Delta P}$ given by (154). It may be evaluated by use of

$$\mathfrak{L}^{-1} \exp\,(a^2s^2 - \tfrac{1}{2}Us) = \tfrac{1}{2}\pi^{-\frac{1}{2}}a^{-1} \exp\,[-\,(X - \tfrac{1}{2}U)^2/4a^2] \quad (174)$$

and the similar formula obtained by replacing U by its negative in conjunction with (168) and certain elementary rules. Equations (168) and (174) give

$$\mathfrak{L}^{-1}(s^2 \pm 1)^{-\frac{1}{2}} \exp\,[a^2(s^2 \pm 1) + \tfrac{1}{2}U(s^2 \pm 1)^{\frac{1}{2}}]$$
$$= \tfrac{1}{2}\pi^{-\frac{1}{2}}a^{-1} \int_X^\infty \begin{matrix} I_0 \\ J_0 \end{matrix}\,[(\beta^2 - X^2)^{\frac{1}{2}}]\, \exp\,[-\,(\beta - \tfrac{1}{2}U)^2/4a^2]\, d\beta,$$

$$\mathfrak{L}^{-1} \exp\,[a^2(s^2 \pm 1) + \tfrac{1}{2}U(s^2 \pm 1)^{\frac{1}{2}}] \quad (175)$$
$$= \tfrac{1}{2}\pi^{-\frac{1}{2}}a^{-1} \Big\{ \exp\,[-\,(X - \tfrac{1}{2}U)^2/4a^2] \pm \int_X^\infty \beta(\beta^2 - X^2)^{-\frac{1}{2}}$$
$$\cdot \begin{matrix} I_1 \\ J_1 \end{matrix}\,[(\beta^2 - X^2)^{\frac{1}{2}}]\, \exp\,[-\,(\beta - \tfrac{1}{2}U)^2/4a^2]\, d\beta \Big\}.$$

The exponents that occur in (154) may be transformed in accordance with

$$a^2s^2 - \tfrac{1}{2}U(s + \zeta) + \tfrac{1}{2}U[(s - \kappa)^2 \pm 1]^{\frac{1}{2}}$$
$$\equiv a^2(\kappa^2 \mp 1) - \tfrac{1}{2}(\kappa + \zeta)U + (2a^2\kappa - \tfrac{1}{2}U)(s - \kappa) \quad (176)$$
$$+\, a^2[(s - \kappa)^2 \pm 1] + \tfrac{1}{2}U[(s - \kappa)^2 \pm 1]^{\frac{1}{2}}$$

and the similar identity that holds with minus signs before the radicals. Consideration of the transformations that change the exponent in the transforms of (175) to that given by (176) shows that the inverse transforms or solutions sought contain $\exp\,[a^2(\kappa^2 \mp 1) - \frac{1}{2}(\kappa + \zeta)U + \kappa X]$ as factor, in which $\exp\,(\kappa X)$ results from replacing s by $s - \kappa$. They contain also $X + 2a^2\kappa - \frac{1}{2}U$ in place of X in (175), because of the term $(2a^2\kappa - \frac{1}{2}U)(s - \kappa)$ in (176). The solutions are obtained by straightforward application of these results to (154); corresponding to the s in the factor $s - \xi$ of the second terms, there are contributions obtained by differentiating with respect to X the expressions that originate from the first equation of (175). The solutions are:

$$
\begin{aligned}
\Delta P = {} & \tfrac{1}{2}\pi^{-\frac{1}{2}}a^{-1}\,\{\exp\,[a^2(\kappa^2 \mp 1) - \tfrac{1}{2}(\zeta + \kappa)U + \kappa X]\} \\
& \cdot\Big(\exp\,[-(X + 2a^2\kappa - U)^2/4a^2] - \tfrac{1}{2}(\xi - \kappa) \\
& \cdot\int_{X+2a^2\kappa-\frac{1}{2}U}^{\infty} \genfrac{}{}{0pt}{}{I_0}{J_0}\,\{[\beta^2 - (X + 2a^2\kappa - \tfrac{1}{2}U)^2]^{\frac{1}{2}}\} \\
& \cdot\{\exp\,[-(\beta + \tfrac{1}{2}U)^2/4a^2] - \exp\,[-(\beta - \tfrac{1}{2}U)^2/4a^2]\}\,d\beta \\
& \mp \frac{1}{2}\int_{X+2a^2\kappa-\frac{1}{2}U}^{\infty} [\beta^2 - (X + 2a^2\kappa - \tfrac{1}{2}U)^2]^{-\frac{1}{2}} \\
& \cdot\genfrac{}{}{0pt}{}{I_1}{J_1}\,\{[\beta^2 - (X + 2a^2\kappa - \tfrac{1}{2}U)^2]^{\frac{1}{2}}\} \\
& \cdot\{(X + 2a^2\kappa - \tfrac{1}{2}U - \beta)\exp\,[-(\beta + \tfrac{1}{2}U)^2/4a^2] \\
& - (X + 2a^2\kappa - \tfrac{1}{2}U + \beta)\exp\,[-(\beta - \tfrac{1}{2}U)^2/4a^2]\}\,d\beta\Big).
\end{aligned}
\tag{177}
$$

Replacing ξ by η in (177) gives the solutions for ΔN, and those for $\Delta\hat{N}$ are accordingly

$$
\begin{aligned}
\Delta\hat{N} = {} & \tfrac{1}{4}\pi^{-\frac{1}{2}}a^{-1}(\eta - \xi)\{\exp\,[a^2(\kappa^2 \mp 1) - \tfrac{1}{2}(\zeta + \kappa)U + \kappa X]\} \\
& \cdot\int_{X+2a^2\kappa-\frac{1}{2}U}^{\infty} \genfrac{}{}{0pt}{}{I_0}{J_0}\,\{[\beta^2 - (X + 2a^2\kappa - \tfrac{1}{2}U)^2]^{\frac{1}{2}}\} \\
& \cdot\{\exp\,[-(\beta + \tfrac{1}{2}U)^2/4a^2] - \exp\,[-(\beta - \tfrac{1}{2}U)^2/4a^2]\}\,d\beta.
\end{aligned}
\tag{178}
$$

The corresponding limiting solutions of (155) involve the step-function factor as result of the requirement that, for contributions in the limit of zero a, the gaussian factors in the integrands of (177) and (178) be centered at values within the range of integration.

APPENDIX C

Integrals Over the Drift Range

Consider first evaluation of $\hat{F}_n$, which may be written as

$$
\begin{aligned}
\hat{F}_n = {} & \tfrac{1}{4}(\xi - \eta)U\,[\exp\,(-\tfrac{1}{2}\,\zeta U)] \\
& \cdot\int_0^{\pi} [\exp\,(-\tfrac{1}{2}\kappa U\cos\Theta)]\,\genfrac{}{}{0pt}{}{I_0}{J_0}\,(\tfrac{1}{2}U\sin\Theta)\sin\Theta\,d\Theta
\end{aligned}
\tag{179}
$$

from (157) and (158). The transformations

$$
\begin{aligned}
z \sin \psi &\equiv \begin{pmatrix} i\frac{1}{2}U \\ \frac{1}{2}U \end{pmatrix}, \\
z \cos \psi &\equiv \begin{pmatrix} i\frac{1}{2}\kappa U \\ i\frac{1}{2}\kappa U \end{pmatrix}, \\
z &= i\tfrac{1}{2}(\kappa^2 \pm 1)^{\frac{1}{2}}U, \\
\cot \psi &= \begin{pmatrix} \kappa \\ i\kappa \end{pmatrix}
\end{aligned}
\tag{180}
$$

result in the single form,

$$
\begin{aligned}
\hat{F}_n = \tfrac{1}{4}(\xi - \eta)U[\exp(-\tfrac{1}{2}\zeta U)] \int_0^\pi [\exp(iz \cos \Theta \cos \psi)] \\
\cdot J_0(z \sin \Theta \sin \psi) \sin \Theta \, d\Theta;
\end{aligned}
\tag{181}
$$

the Gegenbauer's integral† reduces to $(2\pi/z)^{\frac{1}{2}} J_{\frac{1}{2}}(z) = 2z^{-1} \sin z$, and, with the definitions of z, the respective expressions for $\hat{F}_n$ of (164) follow.

The contribution to F_p from the Bessel function of order zero is clearly $[(\xi - \kappa)/(\xi - \eta)]\hat{F}_n$, while the contribution from the delta pulse at the limit of the drift range is $\exp[\frac{1}{2}(\kappa - \zeta)U]$. The contribution

$$
\begin{aligned}
&\pm\tfrac{1}{2}\{\exp[-\tfrac{1}{2}(\zeta + \kappa)U]\} \\
&\qquad \cdot \int_0^U \lambda[\lambda(U - \lambda)]^{-\frac{1}{2}} [\exp(\kappa\lambda)] \frac{I_1}{J_1} [(\lambda(U - \lambda))^{\frac{1}{2}}] \, d\lambda \\
&\quad = \pm\tfrac{1}{4}U[\exp(-\tfrac{1}{2}\zeta U)] \int_0^\pi [\exp(-\tfrac{1}{2}\kappa U \cos \Theta)] \\
&\qquad\qquad \cdot \frac{I_1}{J_1}(\tfrac{1}{2}U \sin \Theta)(1 - \cos \Theta) \, d\Theta \\
&\quad = \begin{pmatrix} -i \\ -1 \end{pmatrix} \tfrac{1}{4}U[\exp(-\tfrac{1}{2}\zeta U)] \\
&\qquad \cdot \int_0^\pi [\exp(iz \cos \Theta \cos \psi)] J_1(z \sin \Theta \sin \psi)(1 - \cos \Theta) \, d\Theta,
\end{aligned}
\tag{182}
$$

the last form for which follows from (180), remains to be evaluated. A method due to Pollak[94] involves first writing the integral in the last form, to be denoted by $\mathcal{J}$, in terms of

$$
\mathcal{U} \equiv z \cos \psi, \qquad \mathcal{V} \equiv z \sin \psi. \tag{183}
$$

---

† Watson[93] p. 379, Equation (1).

Then, the operator $(1 - i\partial/\partial\mathfrak{U})$ applied to $\mathcal{J}$ replaces, in effect, $1 - \cos\Theta$ in the integrand by $\sin^2\Theta$; it gives a Gegenbauer integral† which reduces to $(2\pi/z)^{\frac{1}{2}} \sin\psi\, J_{\frac{3}{2}}(z) = 2\mathfrak{V}z^{-2}(z^{-1}\sin z - \cos z)$. Since this is $-i[\exp(-i\mathfrak{U})]\partial(\mathcal{J}\exp i\mathfrak{U})/\partial\mathfrak{U}$,

$$\mathcal{J} = \mathcal{J}\,|_{\mathfrak{U}=0} \exp(-i\mathfrak{U}) + i2\mathfrak{V}\,[\exp(-i\mathfrak{U})] \cdot \int_0^{\mathfrak{U}} z^{-2}(z^{-1}\sin z - \cos z)\exp(i\mathfrak{U})\,d\mathfrak{U} \tag{184}$$

follows, in which z is $(\mathfrak{U}^2 + \mathfrak{V}^2)^{\frac{1}{2}}$. The integral in this equation may be evaluated by writing $\sin z$ and $\cos z$ in terms of exponentials and introducing $\mathfrak{U} + (\mathfrak{U}^2 + \mathfrak{V}^2)^{\frac{1}{2}}$ and $\mathfrak{U} - (\mathfrak{U}^2 + \mathfrak{V}^2)^{\frac{1}{2}}$ as variables of integration. It is found to equal

$$\tfrac{1}{2}i[\exp(i\mathfrak{U})]\{[\mathfrak{U}^2 + \mathfrak{V}^2 + \mathfrak{U}(\mathfrak{U}^2 + \mathfrak{V}^2)^{\frac{1}{2}}]^{-1}\exp[i(\mathfrak{U}^2 + \mathfrak{V}^2)^{\frac{1}{2}}] + [\mathfrak{U}^2 + \mathfrak{V}^2 - \mathfrak{U}(\mathfrak{U}^2 + \mathfrak{V}^2)^{\frac{1}{2}}]^{-1}\exp[-i(\mathfrak{U}^2 + \mathfrak{V}^2)^{\frac{1}{2}}]\} - i\mathfrak{V}^{-2}\cos\mathfrak{V}.$$

With $\mathcal{J}\,|\,\mathfrak{U}_{=0}$ equal to $2\mathfrak{V}^{-1}(1 - \cos\mathfrak{V})$ from an integral of Sonine,‡ $\mathcal{J}$ is thus found in terms of $\mathfrak{U}$ and $\mathfrak{V}$ and, by use of (180), in terms of κ and U. The respective expressions for F_p of (164) then follow.

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† Watson[93], p. 379, Equation (1).

‡ Watson[93], p. 374, Equation (3); p. 333, Equation 3.

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