On the Recovery of a Band-Limited Signal, After Instantaneous Companding and Subsequent Band Limiting

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If $f(t)$ is a band-limited function, with band limit $-\Omega$ to Ω, the result of instantaneously companding $f(t)$ is in general no longer band-limited. Nevertheless, it has been proved that knowledge of merely those frequencies of the compandor output which lie in the band from $-\Omega$ to Ω is sufficient to recover the original signal $f(t)$. An iteration formula has been proposed that, in theory, performs the desired recovery. In this paper we study in detail some of the practical questions raised by that formula. We show that the successive approximations converge to the solution $f(t)$ at a geometric rate, uniformly for all t, and that the iteration procedure is stable. We then describe a method of performing the recovery in real time and a successful simulation of it on a general-purpose analog computer. The circuit used in the simulation serves as a first approximation to a practical realization of the recovery scheme.

I. INTRODUCTION

When a signal, $f(t)$, is transmitted over a channel there is a tendency for the low-amplitude part of $f(t)$ to become masked by the presence of channel noise and for the high-amplitude part of $f(t)$ to become distorted by the nonlinearity of components in those ranges. It would be valuable, therefore, to find a way of assigning to $f(t)$ another signal from which $f(t)$ could be recovered, but which would have the property that its amplitude lay more nearly in the middle ranges than did that of $f(t)$. This second signal is then transmitted, instead of the original $f(t)$. One relatively simple way of obtaining such a signal is by instantaneous companding: The signal sent is $\varphi[f(t)]$, where $\varphi(x)$ is a monotonic function (to allow recovery of $f(t)$ from $\varphi[f(t)]$), which has a large slope around $x = 0$ so as to magnify signals of low amplitude, and which approaches a constant value for large x so as to cut down on signals of high amplitude.

The drawback of instantaneous companding is that it destroys the property of band-limitedness: if $f(t)$ is a band-limited signal with band limit $-\Omega$ to Ω, the signal $\varphi[f(t)]$ is not in general so band-limited. Thus, if the signals are being sent over an idealized band-limiting channel, the function $\varphi[f(t)]$ is distorted in the process of transmission, even though the original $f(t)$ would not have been. This would have put a serious theoretical obstacle in the path of instantaneous companding, were it not for a theorem by Beurling,[1] which shows that, to recover a band-limited function $f(t)$ with band limit $-\Omega$ to Ω from the companded function $\varphi[f(t)]$, it is not necessary to know the complete spectrum of $\varphi[f(t)]$ but only that part of its spectrum that lies in the frequency interval $-\Omega$ to Ω. More precisely, Beurling has shown that, if $f_1(t)$ and $f_2(t)$ are two band-limited signals with band limit $-\Omega$ to Ω, and if the spectra of $\varphi[f_1(t)]$ and $\varphi[f_2(t)]$ agree in the interval $-\Omega$ to Ω only, then $f_1(t)$ must coincide when $f_2(t)$. This may be interpreted as saying that "no information is lost" in transmitting $\varphi[f(t)]$ over an idealized band-limiting channel since the result, although bearing no simple relation to $\varphi[f(t)]$, is still sufficient to determine $f(t)$ uniquely. Beurling's proof, however, is nonconstructive, and gives no indication of how the band-limited function $f(t)$ might be obtained from a knowledge of only the part of the spectrum of $\varphi[f(t)]$ between $-\Omega$ and Ω.

In another paper,[2] an iteration formula has been given, by means of which, in theory, the recovery could be performed under the hypothesis (somewhat more restrictive than Beurling's) that $\varphi'(x)$ is bounded, and bounded away from 0. In this paper we will study in detail some of the practical questions raised by that formula. We will show that the successive approximations converge uniformly for all t to the solution $f(t)$ at a geometric rate, and that the iteration procedure is stable. We will then describe a method of instrumenting the recovery in real time and a successful simulation of it on a general-purpose analog computer. The circuit used in the simulation serves as a first approximation to a practical realization of the present recovery scheme.

II. MATHEMATICAL FUNDAMENTALS

Throughout the subsequent discussions we will be concerned with functions that are square-integrable; this restriction is imposed so that we may pass freely, by means of the Fourier transform, between the time and frequency domains. For a function $f(t)$ that is square-integrable, that is, one for which

$$\int_{-\infty}^{\infty} |f(t)|^2 \, dt < \infty ,$$

the Fourier transform

$$F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt \tag{1}$$

is defined and has an inverse given by

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)e^{i\omega t}\, d\omega. \tag{2}$$

Furthermore,

$$\int_{-\infty}^{\infty} |f(t)|^2\, dt = \int_{-\infty}^{\infty} |F(\omega)|^2\, d\omega. \tag{3}$$

We will say that a signal $f(t)$ is *band-limited with band* $-\Omega$ *to* Ω if its Fourier transform $F(\omega)$ vanishes for $|\omega| > \Omega$. Then, by (2), the band-limited signal has a representation as an integral with finite limits

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\Omega}^{\Omega} F(\omega)e^{i\omega t}\, d\omega. \tag{4}$$

Let us next consider an instantaneous compandor, which we will describe by $\varphi(x)$; that is, if $f(t)$ is the input signal to the compandor, the output signal is $\varphi[f(t)]$. We require that the function $\varphi(x)$ satisfy

$$\varphi(0) = 0 \tag{5}$$

and

$$0 < b < \varphi'(x) < B < \infty \qquad (\text{or } -B < \varphi'(x) < -b < 0) \tag{6}$$

in the range of operation for x, and we consider the effect of companding on a band-limited function. If $f(t)$ is band-limited with band $-\Omega$ to Ω, the companded signal $\varphi[f(t)]$ need not be band-limited. Nevertheless, it is proved in Ref. 1 that one can compute the original band-limited signal $f(t)$ from a knowledge of merely those frequencies of $\varphi[f(t)]$ which lie in the band from $-\Omega$ to Ω. In order to describe the method of computation, and to enable us to examine the problem in more detail, we introduce the following notation:

i. Let us denote by T the operation of taking the Fourier transform [that is, $Tf(t)$ is the function heretofore denoted by $F(\omega)$], and denote by T^{-1} the operation of taking the inverse Fourier transform [that is, $T^{-1}F(\omega)$ is the function $f(t)$], and let $\chi(\omega)$ be the function that equals 1 for $|\omega| \leqq \Omega$ and equals 0 for $|\omega| > \Omega$. With this notation, we may describe the operation of a low-pass filter on a function $g(t)$ as simply $T^{-1}\chi Tg$, for the action of the filter may be thought of as decomposing $g(t)$ into its frequency components (performing the operation T), preserving without change those frequencies in the band $-\Omega$ to Ω while eliminating all others [multiplying Tg by $\chi(\omega)$], and lastly recomposing the results back into a function of time (performing the operation T^{-1}).

ii. For brevity's sake, let us denote by B the space of all band-limited functions with band limit $-\Omega$ to Ω.

iii. Let $Sf = T^{-1}\chi T\varphi[f]$; that is, the operation S applied to a function $f(t)$ consists of companding it and subsequently band-limiting the compandor output. We should observe that, in forming the function Sf, we use only those frequencies of the compandor output $\varphi[f]$ which lie in the band $|\omega| \leqq \Omega$. We should also note that Sf is always a band-limited function.

iv. We need some way of measuring distance between two functions. Since we are dealing with square-integrable functions, we choose as our measure the quantity

$$\| f - g \| = \left[\int_{-\infty}^{\infty} | f(t) - g(t) |^2 \, dt \right]^{\frac{1}{2}}.$$

We refer to $\| f \|$ as "*the norm of the function* f". This norm has many of the properties of ordinary distance; in particular, the triangle inequality

$$\| f + g \| \leqq \| f \| + \| g \|$$

is valid in it. For general square-integrable functions, convergence in norm need not imply ordinary pointwise convergence; that is, we may have functions $f_n(t)$ for which $\| f_n \| \to 0$ but which themselves do not approach 0 at a point (for example, functions with high but thin spikes). It is very important, however, that, for functions in the space B, convergence in norm does imply uniform convergence on the whole t-axis; indeed we have, if $f(t)$ is in B,

$$| f(t) | \leqq \sqrt{\frac{\Omega}{\pi}} \, \| f \|, \qquad \text{for all } t. \tag{7}$$

The proof of (7) is straightforward and is given in Appendix B.

The theorem proved in Ref. 2 asserts that, if we choose a constant c so that

$$| 1 - c\varphi'(x) | \leqq r < 1 \tag{8}$$

for x in the range of operation of the compandor, then, for any two functions $f_1(t)$ and $f_2(t)$ both in B, we have

$$\| cSf_1 - cSf_2 - (f_1 - f_2) \| \leqq r \, \| f_1 - f_2 \|. \tag{9}$$

Inequality (9) has many consequences. One of these, established in Ref. 2, is that, given any function $a(t)$ in B, the sequence of functions $g_k(t)$, defined iteratively by

$$g_{k+1}(t) = ca(t) + g_k(t) - cSg_k, \qquad \text{with} \quad g_0(t) = 0, \tag{10}$$

converges uniformly on the whole t-axis to a limit $g(t)$, which is also in the space B. By taking the limit on both sides of (10) we then obtain

$$g(t) = ca(t) + g(t) - cSg,$$

or

$$Sg = a(t). \tag{11}$$

It follows from (9) as well that $g(t)$ is the only function B for which (11) can hold.

Let us interpret this result in physical terms. We may think of a compandor into which is sent a signal $f(t)$ of B, and whose output $\varphi[f(t)]$ (which is not in general band-limited) is transmitted over equipment that acts as a pure band-limiter. We may thus describe the received signal as $a(t) = T^{-1}\chi T\varphi[f] = Sf$, and our objective is to recover from $a(t)$ the original compandor input $f(t)$. The iteration formula (10) applied to $a(t)$ does precisely this, for the functions $g_k(t)$ generated by it converge uniformly to a function $g(t)$ in B for which $Sg = a(t)$, and, since there can be only one such function, $g(t)$ must be precisely the desired $f(t)$. The iteration process itself is interpretable in physical terms: the operation S, which has to be applied to g_k in order to compute g_{k+1}, consists of companding g_k and sending the resulting signal through a filter whose action duplicates that of the transmission network. In essence, g_{k+1} consists of g_k corrected by an appropriate constant multiple of the difference between the received signal $a(t)$ and the signal that would be received, if $g_k(t)$ were companded and transmitted.

In thinking of applying an iteration scheme, the questions of rapidity of convergence and of stability at once present themselves. Let us next consider these.

III. RAPIDITY OF CONVERGENCE

We will begin by showing that the approximating functions $g_k(t)$ converge to their limit $g(t)$ at a geometric rate. Since the function $g(t)$ in B to be recovered is given by

$$g(t) = \lim_{n\to\infty} g_n(t),$$

we may write

$$\| g_k(t) - g(t) \| = \| g_k(t) - g_{k+1}(t) + g_{k+1}(t) - g_{k+2}(t) + \cdots \|$$
$$\leqq \| g_k(t) - g_{k+1}(t) \| + \| g_{k+1}(t) - g_{k+2}(t) \| + \cdots . \quad (12)$$

Now, from the definition (10), we have

$$g_{i+1} = ca(t) + g_i(t) - cSg_i ,$$
$$g_i = ca(t) + g_{i-1}(t) - cSg_{i-1} ,$$

whence, by subtraction,

$$\| g_{i+1}(t) - g_i(t) \| = \| cSg_i - cSg_{i-1} - (g_i - g_{i-1}) \|, \quad (13)$$

and, since all the functions g_i are in B, the inequality (9) may be applied to the right side of (13) to yield

$$\| g_{i+1} - g_i \| \leqq r \| g_i - g_{i-1} \|, \quad (14)$$

for all i. By applying the above relation in turn to $\| g_i - g_{i-1} \|$ and so on down to $\| g_1 - g_0 \| = \| a(t) \|$, we may replace (14) by

$$\| g_{i+1} - g_i \| \leqq r^i \| a \|, \quad (15)$$

which, together with (12), yields

$$\| g_k - g \| \leqq \frac{r^k}{1 - r} \| a \|.$$

Since the function $g_k - g$ is in B we may take advantage of the relation between the norm and absolute value that holds (see point iv in Section II) to conclude that actually

$$| g_k(t) - g(t) | \leqq \frac{\sqrt{\Omega/\pi} \, \| a \|}{1 - r} r^k, \qquad r < 1, \qquad \text{for all } t.$$

This establishes that the convergence of g_k to g is geometric in rapidity over the whole t-axis. The constant r, which determines the actual convergence rate, comes from (8) and depends only on the companding curve $\varphi(x)$. In order to obtain the fastest convergence, c should be chosen so as to make r as small as possible.

IV. STABILITY

The stability of an iteration scheme refers to its sensitivity to error. In the case at hand, the solution $g(t)$ is the limit of the functions $g_k(t)$ defined by (10), where we are interpreting $a(t)$ as the signal received when the band-limited signal $g(t)$ is companded and subsequently transmitted over a noiseless band-limiting channel, and where the operation S consists of companding followed by band-limiting. With this model in mind, it is easy to imagine that, in a real application, the received signal would not be, because of noise, precisely $a(t)$; or that the compandor, when applied to $g(t)$, had not acted on the precise curve $\varphi(x)$; or that the channel was not precisely an ideal band-limiting channel. The iteration procedure is *stable* if the function $g^*(t)$ that it produces under each of these three conditions of error differs from the true $g(t)$ by an amount commensurate with the error.

Let us first take up the case that the received function $a^*(t)$ is not equal to $a(t)$. The iteration scheme (10) applied to $a^*(t)$ yields a function $g^*(t)$ in B for which

$$g^* = ca^* + g^* - cSg^*,$$

while the true $g(t)$ in B satisfies

$$g = ca + g - cSg.$$

Subtracting the two equations above and taking the norm of both sides, we obtain

$$\| g^* - g \| \leqq c \| a^* - a \| + \| cSg - cSg^* - (g - g^*) \| .$$

We may now apply (9), obtaining

$$\| g^* - g \| \leqq c \| a^* - a \| + r \| g^* - g \| ,$$

from which

$$\| g^* - g \| \leqq \frac{c}{1 - r} \| a^* - a \|,$$

or, passing to absolute values (as in point iv of Section II),

$$| g^*(t) - g(t) | \leqq \frac{c\sqrt{\Omega/\pi}}{1 - r} \| a^* - a \| , \qquad \text{for all } t. \tag{16}$$

This is precisely a statement of stability, for it asserts that the maximum deviation of g^* from g is bounded by a fixed constant multiple of the norm (in our case the square root of the energy) of the error $a^* - a$.

Let us consider next the effect of a compandor error on the iteration; that is, the possibility that the compandor output is not $\varphi[g(t)]$ but rather $\varphi^*[g(t)]$, where $\varphi^*(x)$ is a curve not identical with $\varphi(x)$. Let us assume that the companding itself is stable; i.e., that the quantity

$$\| \varphi^*[g(t)] - \varphi[g(t)] \| ,$$

which represents the square root of the energy of the difference of the two outputs, is commensurate with the compandor error, which we may measure by the quantity

$$\sup_x \mid \varphi^*(x) - \varphi(x) \mid .$$

What we mean by this precisely is that there exists a fixed constant K such that

$$\| \varphi^*[g] - \varphi[g] \| \leqq K \sup_x \mid \varphi^*(x) - \varphi(x) \mid$$

for all functions $g(t)$ under consideration in the problem. As we have seen in point iii of Section II, the results of transmitting the two outputs are, respectively,

$$a^*(t) = T^{-1}\chi T\varphi^*[g] \qquad \text{and} \qquad a(t) = T^{-1}\chi T\varphi[g]$$

so that, utilizing the stability shown above of the recovery formula with respect to received signals, we have, from (16),

$$\begin{aligned} \mid g^*(t) - g(t) \mid &\leqq \frac{c\sqrt{\Omega/\pi}}{1 - r} \| a^* - a \| \\ &= \frac{c\sqrt{\Omega/\pi}}{1 - r} \| T^{-1}\chi T\varphi^*[g] - T^{-1}\chi T\varphi[g] \| , \end{aligned} \tag{17}$$

where $g^*(t)$ is the function yielded by the iteration scheme on the basis of the erroneous signal $a^*(t)$. Since, by (3), the Fourier transform of a function has the same norm as the function, we have

$$\| T^{-1}\chi T\varphi^*[g] - T^{-1}\chi T\varphi[g] \| = \| \chi T\varphi^*[g] - \chi T\varphi[g] \|$$

and

$$\| T\varphi^*[g] - T\varphi[g] \| = \| \varphi^*[g] - \varphi[g] \| ,$$

while

$$\| \chi T\varphi^*[g] - \chi T\varphi[g] \| \leqq \| T\varphi^*[g] - T\varphi[g] \| ,$$

since the two sides of the inequality represent integrals of the same positive function, over a finite and an infinite interval respectively. Combining these with (17), we obtain

$$\mid g^*(t) - g(t) \mid \leqq \frac{c\sqrt{\Omega/\pi}}{1 - r} \| \varphi^*[g] - \varphi[g] \| ,$$

and, by the assumption of compandor stability, the right-hand side above is commensurate with

$$\sup_x \mid \varphi^*(x) - \varphi(x) \mid .$$

We conclude that the iteration procedure is stable with respect to a compandor error whenever the companding process itself is thus stable.

Lastly we take up the question of stability under a variation of the channel characteristic. That is, we suppose that the compandor output $\varphi[g(t)]$ is transmitted over a channel whose effect on it is $T^{-1}\chi^*T\varphi[g]$ rather than $T^{-1}\chi T\varphi[g]$, where the function $\chi^*(\omega)$ differs from the ideal $\chi(\omega)$ of point i in Section II. We have, from (16),

$$\begin{aligned} | g^*(t) - g(t) | &\leqq \frac{c\sqrt{\Omega/\pi}}{1 - r} \| a^* - a \| \\ &= \frac{c\sqrt{\Omega/\pi}}{1 - r} \| T^{-1}\chi^*T\varphi[g] - T^{-1}\chi T\varphi[g] \|, \end{aligned} \tag{18}$$

and, by (3),

$$\| T^{-1}\chi^*T\varphi[g] - T^{-1}\chi T\varphi[g] \| = \| \chi^*T\varphi[g] - \chi T\varphi[g] \| = \| (\chi^* - \chi)T\varphi[g] \| .$$

Now from the definition of the norm (point iv in Section II), we may estimate the quantity $\| (\chi^* - \chi)T\varphi[g] \|$ in various ways. We may choose to say that

$$\| (\chi^* - \chi)T\varphi[g] \| \leqq \sup_{\omega} | \chi^*(\omega) - \chi(\omega) | \, \| T\varphi[g] \| , \tag{19}$$

from which we can show that, for signals of bounded energy, the recovery computation is stable with respect to a departure of the transmission characteristic from the ideal $\chi(\omega)$, when the deviation is measured by the quantity

$$\sup_{\omega} | \chi^*(\omega) - \chi(\omega) | .$$

That is, we will be able to conclude that the error $| g^*(t) - g(t) |$ will be small if

$$\sup_{\omega} | \chi^*(\omega) - \chi(\omega) |$$

is sufficiently small. The weakness of this sort of stability statement lies in its requirement that $| \chi^*(\omega) - \chi(\omega) |$ be everywhere small; it yields no information when $| \chi^*(\omega) - \chi(\omega) |$ is small everywhere, except on a small segment of the ω-axis. In those cases, the quantity

$$\sup_{\omega} | \chi^*(\omega) - \chi(\omega) |$$

ceases to be an adequate measure of closeness, and we would prefer to have a stability statement involving $\| \chi^* - \chi \|$, for this may be small even when $| \chi^*(\omega) - \chi(\omega) |$ is occasionally large. We may find such a statement, for functions whose frequency spectrum, after companding, is bounded, by using

$$\| (\chi^* - \chi)T\varphi[g] \| \leqq \| \chi^* - \chi \| \sup_{|\omega| < \Omega} | T\varphi[g] | . \tag{20}$$

The weakness in turn of

$$\| \chi^* - \chi \| = \left[\int_{-\Omega}^{\Omega} | \chi^* - \chi |^2 \, d\omega \right]^{\frac{1}{2}}$$

as a measure of closeness is that it may become large, even when

$$| \chi^* - \chi |$$

is mostly small, simply because the interval of integration $| \omega | \leqq \Omega$ is large. To find an expression for the size of error that has the virtues of both of those above without the disadvantages of either — that is, one which is not sensitive either to occasionally large values of $| \chi^* - \chi |$, or to the length of the band $| \omega | \leqq \Omega$ — we may apply a combination of estimates (19) and (20) to the quantity $\| (\chi^* - \chi) T\varphi [g] \|$. Let us divide the interval $-\Omega$ to Ω into two complementary sets, A and $A' = [(-\Omega, \Omega) - A]$, and let us define

$$\epsilon_A = \sup_{\omega \varepsilon A} | \chi^* - \chi |,$$

$$\epsilon_A' = \left[\int_{\omega \varepsilon A'} | \chi^* - \chi |^2 \, d\omega \right]^{\frac{1}{2}}.$$

If $| T\varphi[g] |$ for $| \omega | \leqq \Omega$ and $\| \varphi[g] \|$ are bounded by M, then, using (19) on the set A and (20) on the set A', we obtain

$$\| (\chi^* - \chi) T\varphi[g] \| \leqq M \min_A \max(\epsilon_A , \epsilon_A').$$

Hence, from (18),

$$| g^*(t) - g(t) | \leqq \frac{c\sqrt{\Omega/\pi}}{1 - r} M \inf_A \max (\epsilon_A , \epsilon_A').$$

This establishes that, for signals whose energy and frequency spectrum after companding, are bounded by a fixed constant, the iteration procedure is stable with respect to a departure of the transmission characteristic from the ideal $\chi(\omega)$; our present measure of deviation is the best combination of $| \chi^*(\omega) - \chi(\omega) |$ and $\| \chi^* - \chi \|$, in the sense of minimizing over all sets A the quantity $\max(\epsilon_A , \epsilon_A')$. The two boundedness restrictions we have imposed do not seem unduly severe.

V. INSTRUMENTATION OF THE ITERATION FORMULA

With the stability of the recovery computation thus established, we will now describe a way of instrumenting the iteration formula in real time, and a simulation of the resulting recovery process on a general-purpose analog computer. The iteration formula is

$$g_{n+1} = ca(t) + g_n - cSg_n , \qquad \text{with} \quad g_0 = 0,$$

and we have already interpreted $a(t)$ as the signal received after the function $g(t)$ in B has been companded and transmitted over a band-limiting channel, and the operation S as companding, followed by band-limiting. The iteration is thus performable by analog methods, with the aid of a compandor and of a band-limiting filter, used to carry out the operation S. A filter of this type has a delay, however. Thus, in order to add its output, Sg_n, to the function $ca(t) + g_n(t)$, as required by the iteration, the latter would have to be delayed by an amount equal to the filter's delay. We may obviate the necessity for an additional delay network by observing that, since the function $ca(t) + g_n(t)$ is in B, passing it through the filter does not distort it, so that the addition may be performed before filtering. Thus, a possible circuit for performing the iteration is that of Fig. 1. By connecting s of these circuits in series, and by supplying as input the function $g_1(t) = a(t)$, the output will be the (delayed) approximation $g_{s+1}(t)$ to $g(t)$, for which

$$| g(t) - g_{s+1}(t) | \leqq \frac{c\sqrt{\Omega/\pi}}{1 - r} \| a \| r^{s+1}.$$

The circuit of Fig. 1 served as the basis for a simulation on a general-purpose analog computer. The companding curve $\varphi(x)$ was chosen to be of the type described by Mallinckrodt[3] and consisted of a straight line of slope 10 for $-0.2 \leqq x \leqq 0.2$ that had joined to it at $x = \pm 0.2$ a logarithmic curve which matched it in slope. The range of interest for x was $| x | < 2.5$. The constant c was chosen as 1/12, yielding for r the relatively large value of $r = 14/15$. The band-limiting filter was simulated from an expression kindly supplied by J. Bangert. Since it required 13 integrators, it was not possible to simulate more than one stage of the circuit, so that the iteration was performed step by step; the output $g_{n+1}(t)$ was recorded at every step and served, along with $a(t)$, as the input for the next iteration. Since the simulation was per-

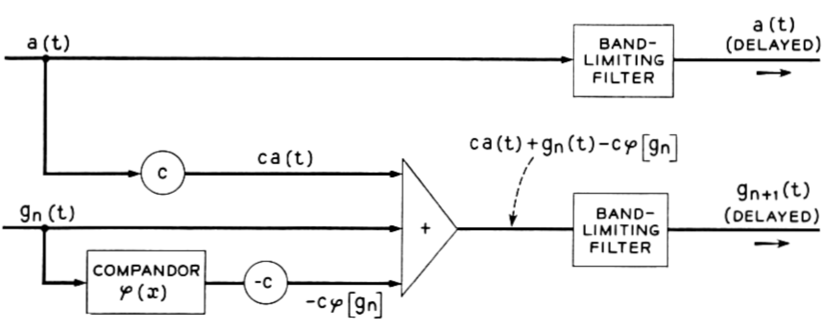

Fig. 1 — Circuit for simulation on a general-purpose analog computer.

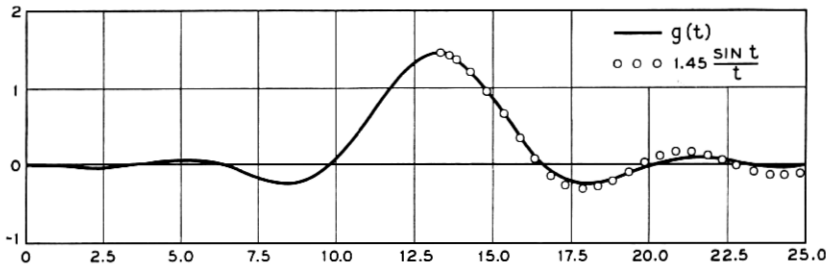

Fig. 2. — The figure $g(t)$ to be recovered.

formed to test the efficacy of the recovery process, the function $g(t)$, to which the approximations $g_n(t)$ were expected to converge geometrically, was picked in advance to be 1.45 sin t/t, which is in B, as required.

The function actually used for $g(t)$ was an approximation to this, generated on the computer as the step response of the filter, and appears in Fig. 2. Since the filter is not ideal, the $g(t)$ used does not coincide precisely with 1.45 sin t/t but it is a good approximation; the closeness of the two curves provides, further, a measure of the filter's performance. The curve $a(t)$, obtained as a result of companding and filtering the $g(t)$ of Fig. 2, appears as the bottom-most of the curves of Fig. 3. The remaining curves of Fig. 3 represent the odd approximants: g_3, g_5, g_7, g_9, g_{11} and g_{13}, yielded by the iteration. They are seen to converge well, although their limit is not quite the function $g(t)$. To test whether the error was due simply to the various inherent machine insensitivities, the last approximation, $g_{13}(t)$, was companded and band-limited, and the result was compared with the original $a(t)$. The difference of these two functions appears in Fig. 4 and is seen to be very small. The simulation consequently appears to be successful, in that it verifies, in a special case and within the limits of machine accuracy, the theoretical predictions of convergence and stability for the recovery process.

As we have mentioned before, an obvious way of mechanizing this

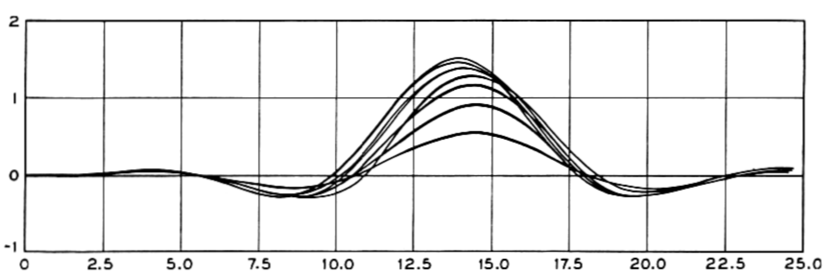

Fig. 3. — The sequence of approximations produced by the iteration.

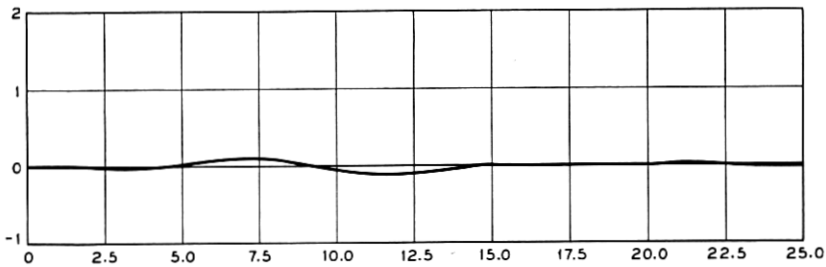

Fig. 4 — Difference between the original $a(t)$ and companded band-limited approximation $g_{13}(t)$.

recovery procedure is to connect s of the circuits of Fig. 1 in series, and to supply as input the function $g_1(t) \equiv a(t)$; the output will then be the approximation $g_{s+1}(t)$ to $g(t)$, but delayed by s times the delay of the band-limiting filter. This delay is an undesirable feature in practice, and may perhaps be decreased, at the expense of some error in the recovery, by using a filter with a smoother cutoff.

VI. OPEN QUESTIONS

This study has concerned itself until now with the idealized version of the problem — one in which the effect of transmission on the companded signal has been to pass all frequencies $|\omega| \leqq \Omega$ without change and to eliminate all others. We have gone beyond this formulation only to show the stability of the recovery process with respect to a variation of the transmission characteristic from the ideal $\chi(\omega)$; that is, we have shown that, if this variation is not large, the error produced by applying the present recovery procedure will not be large. The problem of how we should proceed when given a signal $a^*(t) = T^{-1}\chi^*T\varphi\,[g]$, with $\chi^*(\omega) = 0$ for $|\omega| > \Omega$ but widely different from $\chi(\omega)$ in the band $|\omega| \leqq \Omega$, remains an open one. We may, of course, precede the recovery by passing $a^*(t)$ through a compensating network with characteristic $1/\chi^*(\omega)$; this would convert $a^*(t)$ to the ideal $a(t)$, to which our present iteration scheme could be applied without change. The question to be answered is whether there exists an alternative, which would not require compensation of the received signal; this is a matter worthy of further study.

APPENDIX A

We reproduce here A. Beurling's proof of uniqueness; we will use the notation of i through iv in Section II.

Let the companding function $\varphi(x)$ be monotonic and have the property that $\varphi[f(t)]$ is square-integrable whenever $f(t)$ is. Let us also suppose that $f_1(t)$ and $f_2(t)$ are both in B, and that $T\varphi[f_1] = T\varphi[f_2]$ for $|\omega| \leqq \Omega$ only. We will show that $f_1(t)$ and $f_2(t)$ must coincide identically.

By the Plancherel theorem for Fourier transforms,

$$\int_{-\infty}^{\infty} \{T\varphi[f_1] - T\varphi[f_2]\}\{\overline{Tf_1 - Tf_2}\}\, d\omega = \int_{-\infty}^{\infty} \{\varphi[f_1] - \varphi[f_2]\}\{\overline{f_1 - f_2}\}\, dt,$$

where the bar denotes complex conjugation. Now in the left-hand integral, by hypothesis, $T\varphi[f_1] = T\varphi[f_2]$ for $|\omega| \leqq \Omega$, and $Tf_1 = Tf_2 \equiv 0$ for $|\omega| > \Omega$, since f_1 and f_2 are in B. Thus,

$$\int_{-\infty}^{\infty} \{\varphi[f_1] - \varphi[f_2]\}\{f_1 - f_2\}\, dt = 0. \tag{21}$$

But, since φ is monotonic, the integrand of (21) is nonnegative, for if $f_1(t) \geqq f_2(t)$, then $\varphi[f_1(t)] \geqq \varphi[f_2(t)]$, and, similarly, if $f_1(t) \leqq f_2(t)$, then $\varphi[f_1(t)] \leqq \varphi[f_2(t)]$. Thus (21) implies that $f_1(t) \equiv f_2(t)$.

APPENDIX B

We will show here that, for functions in the space B, convergence in norm implies uniform convergence on the whole t-axis.

Let $f_n(t)$ be a sequence of functions in B, with $\| f_n(t) \| \to 0$. By applying Schwarz's inequality to the representation (4) we obtain

$$|f_n(t)| \leqq \frac{\sqrt{2\Omega}}{\sqrt{2\pi}} \left[\int_{-\Omega}^{\Omega} |F_n(\omega)|^2\, d\omega \right]^{\frac{1}{2}},$$

or

$$|f_n(t)| \leqq \sqrt{\Omega/\pi}\, \| F_n(\omega) \|, \tag{22}$$

where $F_n(\omega)$ is the Fourier transform of $f_n(t)$. But, by (3),

$$\| F_n(\omega) \| = \| f_n(t) \|,$$

so that (22) becomes

$$|f_n(t)| \leqq \sqrt{\Omega/\pi}\, \| f_n \|$$

whence we have

$$|f_n(t)| \to 0, \qquad \text{uniformly for all } t.$$

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