Analysis of Quality Factor of Annular Core Inductors

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(Manuscript received August 25, 1959)

This paper summarizes and formally presents methods in use during the past 25 years at Bell Telephone Laboratories for the design of high-quality inductors. The quality factor, Q, of annular core inductors can be maximized by adapting core dimensions, winding details, etc., to the specific properties of the magnetic core. Commercial core materials include 2–81 Mo-permalloy powder and carbonyl iron powder. Analysis of the relationships of dimensions and magnetic properties is given first in terms of the bare fundamentals of dc copper resistance and magnetic core losses. The paper then derives expressions for, and considers the effects of, additional losses due to the windings, such as eddy currents in the copper, dielectric losses and the contribution to effective resistance due to parallel distributed capacitance. Finally, a graph is supplied showing the maximum Q and optimum frequency for many commercial sizes of annular cores.

I. INTRODUCTION

The analysis of annular magnetic core inductors in terms of the desired characteristics and the properties of their components entails adjustments of core and winding dimensions, and the selection of best arrangements for each set of requirements. The present paper summarizes the methods in use during the past 25 years at Bell Telephone Laboratories. Such analysis involves consideration of numerous properties of core and winding, some of which may occasionally be negligible. It will be illuminating first to make an analysis based upon the fewest basic properties. Subsequently the contributions and complications of secondary properties will be treated, as their need becomes apparent.

II. SIMPLE ANALYSIS

2.1 *Magnetic Core*

The *inductance* of a winding of N turns on an annular core is

$$L = \frac{4N^2 \mu w h \times 10^{-9}}{D} \quad \text{henry,} \tag{1}$$

where μ is the permeability of the ring core material, which is of mean diameter D, radial thickness w, and axial height h, all in centimeters.[1]

For simplicity, the "air inductance", due to the windings on the space outside of the core, will be neglected. It represents a very small contribution to the total inductance of coils with high permeability cores.

Energy losses in the core material show themselves as an *effective resistance* increase in the winding:

$$R_m = (aB_m + c + ef)\mu Lf \quad \text{ohm}, \tag{2}$$

where a is the hysteresis loop area constant, B_m is the peak induction in the magnetic material, c is the "residual" loss constant, e is the eddy current coefficient and f is the ac frequency at which the measurement is made.[1] If the hysteresis loss is not negligible, the induction B_m must be calculated in terms of the rms current I (in amperes) in the winding

$$B_m = 0.4\sqrt{2}\,\frac{\mu N I}{D},$$

or[2]

$$B_m = \frac{I}{5}\sqrt{\frac{2\mu L \times 10^9}{whD}} \quad \text{gauss}. \tag{3}$$

2.2 *Copper Winding*

The copper winding is assumed to be applied by means of a shuttle that passes through the center of the annulus, leaving a circular opening unfilled with wire, as illustrated in Fig. 1, The depth of the winding is d, on the inside diameter of the core. This represents the basic limitation on copper winding area. The available winding area is therefore

$$\pi d(D - w - d) = A_c .$$

On a per-turn basis, the copper area is $s\pi d(D - w - d)/N$, where s is the packing factor, or the fraction of the available area occupied by copper conductor.

The average length of turn in the winding can be calculated closely on the assumption that the winding depth is uniformly equal to d both inside and outside of the core ring.* It amounts to $2(w + h + 2d)$.

The *total copper resistance* is thus

$$R_c = \frac{2\rho(w + h + 2d)N^2}{s\pi d(D - w - d)},$$

* This assumption is conservative, since it yields a slightly larger value than is obtained by more rigorous analysis.

or

$$R_c = \frac{\rho LD(w + h + 2d) \times 10^9}{2\mu s\pi dwh(D - w - d)} \quad \text{ohm}, \tag{4}$$

where ρ is the resistivity of the copper, in ohm-centimeters. This is the dc resistance, which may be considered as the entire copper resistance at frequencies low enough to make copper eddy current losses negligible. It is assumed to depend solely upon core dimensions and to increase smoothly, without discontinuities due to wire size changes. This assumption is based on the proposal to keep the present analysis simple, reserving more complicated analyses to subsequent sections.

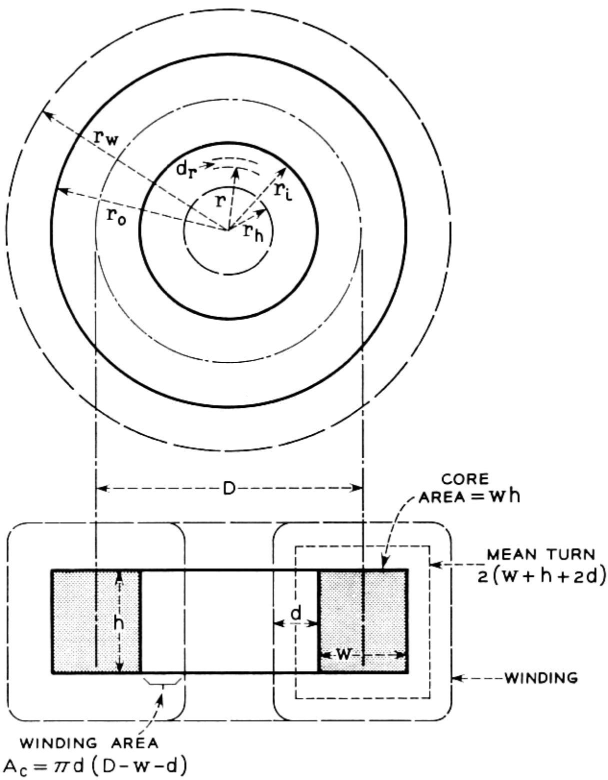

Fig. 1 — Idealized annular core and winding. Winding depth d is assumed constant for inside and outside of core ring. Core mean diameter is D, radial thickness w and axial height h.

2.3 *Coil Quality Factor, Q*

The quality factor of a coil, which enters into many important circuit computations, is $\omega L/R$, or

$$Q = \frac{2\pi fL}{R_c + R_m} = \frac{2\pi fL}{\frac{\zeta L}{\mu s} + \mu Lf(aB_m + c + ef)}, \tag{5}$$

where the dc resistance coefficient is

$$\zeta = \frac{\rho D(w + h + 2d) \times 10^9}{2\pi dwh(D - w - d)}. \tag{5a}$$

Simplifying (5) yields an expression that is independent of L:

$$Q = \frac{2\pi}{\frac{\zeta}{s\mu f} + \mu(aB_m + c + ef)}. \tag{6}$$

Inspection of this expression for Q shows that it is a maximizable function of the frequency f and the core permeability μ. Since frequency is the parameter that is always accessible for experimental control, it is chosen for initial analysis, with permeability optimization following, as explained below.* Inspection of the denominator of (6) shows that it will have a minimum when

$$\frac{\zeta}{\mu} = s\mu ef_0^2,$$

or

$$f_0 = \frac{1}{\mu}\sqrt{\frac{\zeta}{es}} \quad \text{cycles per second}, \tag{7}$$

where f_0 is the frequency at which the denominator of (6) will be a minimum and the coil Q will be a maximum. It is interesting to note in this expression that the optimum frequency is a function of the dc resistance coefficient and of the permeability and eddy current coefficient of the core, but that it is not explicitly a function of the hysteresis and residual losses. If a given core material is to be used at a specified optimum frequency, it is necessary to solve (7) for ζ, the dc resistance coefficient, which will fulfill the requirements.

Inserting this value of ζ into (6) gives the maximum Q of the coil, which is reached at the optimum frequency. Thus,

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* The choice of frequency for initial consideration is consistent with the procedure of Arguimbau.[3]

$$Q_0 = \frac{\pi}{\mu e f_0 + \mu \frac{(aB_m + c)}{2}}. \tag{8}$$

It will appear in further analysis that coil design is facilitated by focusing attention on Q_0, the maximum quality factor. This shifts around to various frequencies, as indicated by (7), for cores having different values of μ, ζ and e.

The quality factor at frequencies other than f_0 will be smaller than Q_0 by a factor related to the difference in frequency from f_0. If we take the ratio Q/Q_0 from (6) and (8) and recall, from (7), that $\zeta/\mu = s\mu e f_0^2$ and, from (8), that $(aB_m + c) = 2\pi/\mu Q_0 - 2ef_0$, we may insert these values, rearrange terms, and obtain

$$\frac{Q}{Q_0} = \frac{1}{1 + \frac{e\mu Q_0}{2\pi f}(f_0 - f)^2}. \tag{9}$$

This equation shows that the shape of the Q versus frequency curve is roughly parabolic with vertex at Q_0, located at frequency f_0. As pointed out previously, it will generally be profitable to design coils based on Q_0, and resort to (9) when behavior at nonoptimum frequency is sought. Fig. 2 gives a log-log graph of a typical Q versus f curve.

Equation (8) gives an explicit relationship between core loss constants and coil Q at the optimum frequency f_0, and reveals that the maximum Q is inversely proportional to μ. Equation (8) also shows that Q_0 and f_0 are inversely related, for a given core permeability and eddy current coefficient. It is thus evident that Q_0 and f_0 are not open to arbitrary selection and that, if such selection is required, adjustment of core permeability and/or eddy current loss coefficient will be necessary. Such arbitrary adjustments of characteristics of compressed powder cores are not available to coil designers under usual circumstances. Core manufacturers have met their needs by standardizing on a line of core characteristics[2] stepping from low permeability to permeability as high as practicable, together with eddy current coefficient as small as possible.

It was noted in discussion of (7) that the optimum frequency f_0 depends on ζ, which includes geometric constants of the core and winding, and on the resistivity of the copper winding. It is instructive to analyze (8) on the basis of eliminating f_0, using the values from (7). Thus, the optimum Q can be expressed alternatively as

$$Q_0 = \frac{\pi}{\sqrt{\frac{\zeta e}{s}} + \frac{\mu(aB_m + c)}{2}}. \tag{10}$$

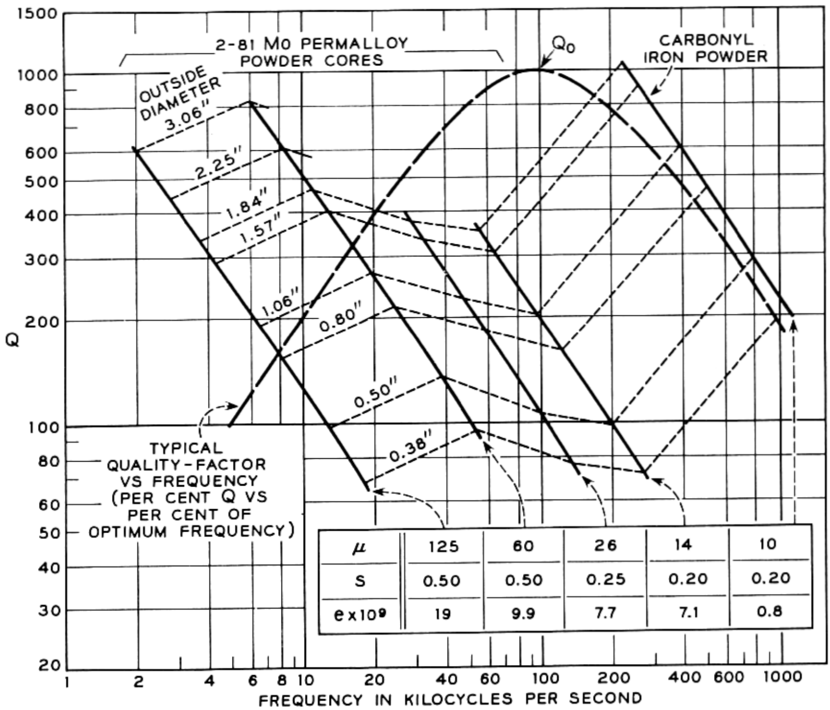

Fig. 2 — Maximum Q and optimum frequency for annular cores of conventional diameters and materials. Hysteresis and residual losses are assumed to be negligible. Dimension rations: $w' = \frac{1}{3}$, $h' = \frac{3}{8}$, $d' = \frac{1}{6}$. Commercial cores that differ from these ratios require specific calculation to determine possible further impairment of Q.

Equation (10) resembles (8), except that the copper resistance contribution to Q has become conspicuous, through its implied presence in f_0 from (7). In revealing the frequency at which Q reaches a maximum, (7) must hold, with either expression for Q_0. If hysteresis and residual core losses can be neglected in comparison with $\sqrt{\zeta e/s}$, (10) reduces to

$$Q_0 = \pi \sqrt{\frac{2s\pi dwh(D - w - h)}{\rho De(w + h + 2d) \times 10^9}}. \tag{11}$$

Equation (11) contains specific geometric dimensions of the core and winding. For clarification, it is reasonable to assume that the *relative* dimensions of any coil will be constant, regardless of the coil size. For example, the dimensions of the "archetype" coil may all be assumed proportional to the mean diameter of the core, D; i.e., $w = w'D$, $h = h'D$ and $d = d'D$, where the primed letters indicate fractions. The winding resistance coefficient, from (5a), is

$$\zeta = \frac{\rho(w' + h' + 2d') \times 10^9}{2\pi D^2 d'w'h'(1 - w' - d')}. \tag{11a}$$

Thus (10) becomes

$$Q_0 = \frac{\pi D}{\sqrt{\dfrac{\rho e(w' + h' + 2d') \times 10^9}{2s\pi d'w'h'(1 - w' - d')} + \dfrac{\mu D(aB_m + c)}{2}}}. \tag{12}$$

It is apparent that the maximum Q of an annular coil is directly proportional to the diameter of the core, to a first approximation, when hysteresis and residual losses are small. Core manufacturers have met the need for Q versus diameter adjustments by standardizing on a line of core diameters ranging from heavy cores down to a size so small that special winding machine shuttles are required to pass through the hole in the winding. The choice of dimension ratios depends on wise apportionment of space to magnetic core, copper, wire insulation, mechanical strength of core, dimensions and shape of winding shuttle, etc. The relative dimensions, width, height and winding depth of commercial cores have been adjusted to fulfill specific design needs. It has been found practicable to maintain for the winding shuttle a hole size no less than $D/3$.

A type of core well suited to Formex-insulated solid-copper wire windings maintains the dimension ratios approximately $w' = \frac{1}{3}$, $h' = \frac{3}{8}$ and $d' = \frac{1}{6}$. It is instructive to compute f_0 and Q_0 for a series of cores of conventional materials based on these dimension ratios. Copper resistivity may be taken as 1.75×10^{-6} ohm-cm, and the winding packing factor as $s = 0.50$. The inefficiency of packing is due partly to the space occupied by insulation, but largely to the failure of circular cross-sectioned wire to occupy all of the available space. Upon inserting these values in (12) and neglecting hysteresis and residual losses, the maximum Q becomes

$$Q_0 = \frac{420D}{\sqrt{e \times 10^9}}. \tag{13}$$

The same substitutions in (7) yield the optimum frequency at which the maximum Q is obtained:

$$f_0 = \frac{7400}{\mu D\sqrt{e \times 10^9}} \quad \text{kc}. \tag{14}$$

Equations (13) and (14) apply to typical cores wound with Formex-

TABLE I — OPTIMUM FREQUENCY AND MAXIMUM Q OF IDEAL FORMEX WIRE-WOUND CORE RINGS

Dimension ratios: $w' = \frac{1}{3}$, $h' = \frac{3}{8}$, $d' = \frac{1}{6}$. Note: Core rings listed below differ more or less from these assumed dimension ratios; Q_0 and f_0 values will require specific calculation for accuracy.

Piece part number	O.D., inches	D, cm	f_0, kc	Q_0
Permalloy powder with $\mu = 125$, $e = 19 \times 10^9$, $a = 1.6 \times 10^6$, $c = 3.0 \times 10^5$				
P-11D719*	0.38	0.72	18.8	68
A-050056-2	0.50	1.01	13.3	97
A-206068-2	0.80	1.65	8.3	155
A-903157-2	1.06	2.06	6.8	190
A-254168-2	1.57	3.07	4.4	290
A-438281-2	1.84	3.53	3.8	335
A-109156-2	2.25	4.63	2.9	445
A-866142-2	3.06	6.35	2.1	600
Permalloy powder with $\mu = 60$, $e = 9.9 \times 10^9$, $a = 2.5 \times 10^6$, $c = 5.0 \times 10^5$				
P-11D719*	0.38	0.72	54.3	96
A-051027-2	0.50	1.01	38.8	134
A-848032-2	0.80	1.65	23.7	216
A-894075-2	1.06	2.06	19.0	270
A-083081-2	1.57	3.07	12.7	410
A-795135-2	1.84	3.53	11.1	460
A-488075-2	2.25	4.63	8.4	620
A-123068-2	3.06	6.35	6.2	830

* New size (Western Electric Company)

All "A" core designations follow Arnold Engineering Co. general catalog. These cores may be commercially available also from several other manufacturers. Cores of these dimensions were originally standardized by the Western Electric Company for telephone system applications.

insulated wire; they would be altered somewhat for cores of different geometrical ratios. They neglect hysteresis and residual losses, so that the actual quality factor will be correspondingly smaller than Q_0. In Table I computations of Q_0 and f_0 are shown for a series of commercial cores of various diameters, using permalloy powder cores of permeabilities 125 and 60. For any size of core, it will be noted from the table that Q_0 and f_0 are higher for 60-permeability cores.* This trend would be observed with other grades of cores having still lower permeability, but computations would not be rigorous enough to be profitable, based only on the analysis thus far presented. The reasons for this will be explained in the next section.

Thus far we have examined the coil quality factor Q in the relationships between core loss resistance and copper winding dc resistance. It

* Q_0 for 60-permeability cores is disproportionately higher, due to the decreased eddy current coefficient, as compared with 125-permeability cores.

has been found that Q reaches a maximum at a frequency f_0, defined by (7), and that the maximum value of Q_0 can be expressed in terms of core properties, (8), or of copper winding and core properties, (10). If core and winding dimensions are assumed to be proportional to core diameter, D, (12) gives a basic expression for Q_0, from which the effects of core diameter and permeability can be derived.

III. WINDING COMPLICATIONS

The analysis thus far is adequate for inductor designs for operating frequencies below about 30 kc. Greater rigor in analysis of coil design leads to a closer scrutiny of the windings. It is obvious that ac magnetic induction permeating a layer of copper wires will set up eddy currents, which result in energy loss. This correspondingly increases effective resistance of a coil, and decreases quality factor. Similarly, capacitance and leakance between the coil terminals increase losses and decrease Q. These winding complications are somewhat involved, and they will now be taken up one at a time.

3.1 *Copper Eddy Currents*

It has already been noted that eddy current losses in a magnetic core result in effective resistance contribution, $R_e = e\mu Lf^2$. The magnetic permeability of copper wire is unity, so that we may write a corresponding expression for copper eddy current resistance, $R_{ce} = R_c mf^2$, where the constant m is a function of the degree of subdivision of the copper. Calculation of m is given in the Appendix.

Use of stranded instead of solid wire involves a sacrifice of at least a further 25 to 30 per cent in winding-space efficiency, due to the large amount of insulation and the inefficiency of packing many strands in each turn of the winding. Thus, s will be reduced to 0.25 or less, in comparison with about 0.50 for nonstranded Formex-insulated wire. It is therefore evident that stranding of wire is not profitable at low frequencies, nor until the savings in eddy current loss at least offset the increase in dc resistance due to inefficient packing.

Analysis of the interaction of copper eddy current losses with other coil properties can be made by reference to (6). The contribution of copper eddy current loss will affect the coil Q both by the factor $(1 + mf^2)$ and by substitution of an inferior packing factor s. Thus, (6) becomes

$$Q = \frac{2\pi}{\dfrac{\zeta(1 + mf^2)}{s\mu f} + \mu(aB_m + c + ef)}. \tag{15}$$

Solving for conditions for maximum Q yields the necessary new optimum frequency,

$$f_{00} = \frac{1}{\mu} \sqrt{\frac{\zeta}{e\left(s + \frac{m\zeta}{\mu^2 e}\right)}}. \tag{16}$$

Comparing this with (7), it is evident that copper eddy currents shift the optimum frequency downward by the relation

$$f_{00} = \frac{f_0}{\sqrt{1 + \frac{m\zeta}{\mu^2 es}}}. \tag{17}$$

We can now proceed to calculate the new optimum Q, following a procedure similar to that of Section 2.3. Solving (16) for the copper resistance coefficient gives

$$\zeta = \frac{se\mu^2 f_{00}^2}{1 - mf_{00}^2}. \tag{18}$$

Substituting in (15) gives the new maximum quality factor,

$$Q_{00} = \frac{\pi}{\frac{\mu e f_{00}}{1 - mf_{00}^2} + \frac{\mu(aB_m + c)}{2}}. \tag{19}$$

This corresponds to (8), but shows the penalty exacted by eddy currents in the coil windings. If hysteresis and residual losses may be neglected, (19) can be rewritten

$$Q_{00} = \frac{\pi(1 - mf_{00}^2)}{\mu e f_{00}}. \tag{20}$$

The reduction in Q due to copper winding eddy current losses is represented by the term mf_{00}^2, which should be as small as possible in relation to the first term, 1. Decision as to where stranding will be required rests on a further analysis of the term mf_{00}^2. From (50) of the Appendix, the critical term can be written

$$mf_{00}^2 = \frac{932 \times 10^{-6} f_{00}^2}{nN} [Dsd'(1 - w' - d')]^3. \tag{21}$$

For the cores considered in Table I, which have $w' = \frac{1}{3}$ and $d' = \frac{1}{6}$, (21) becomes

$$mf_{00}^2 = \frac{0.539 \times 10^{-6} f_{00}^2 D^3 s^3}{nN}. \tag{21a}$$

The value of the packing factor, s, is linked to the number of strands, n, changing from about 0.50 when $n = 1$ to 0.25 and progressively lower for higher values of n. Designs of coils are usually arranged to make the winding turns, N, large enough to adjust the specified inductance to a precision of better than 2 per cent per turn; i.e., N must be 100 turns or more.

With these assumptions, Table II has been calculated. It is apparent that nonstranded wire ($n = 1$) is already very disadvantageous at a frequency of 50 kc. Multiple stranding of very high n is required at higher frequencies, and for large-size cores. For example, wire composed of 30 strands entails a decrease in Q of 9.8 per cent at 50 kc, on a 3-centimeter core. The table should be recognized as pessimistic, in that most windings will have more than the assumed 100 turns, and correspondingly will have less than the indicated value of eddy current loss coefficient mf_{00}^2.

For purpose of inductor computation, it may now be convenient to express Q_{00} in terms giving explicit recognition to the copper resistance coefficient ζ, instead of (20). Thus, inserting the value of f_{00} from (16) in (15) gives an alternative expression for the new maximum quality factor:

$$Q_{00} = \frac{\pi}{\sqrt{\frac{\zeta e}{s}\left(1 + \frac{m\zeta}{es\mu^2}\right) + \frac{\mu(aB_m + c)}{2}}}. \tag{22}$$

TABLE II — VALUES OF mf_{00}^2 = FRACTION DECREASE IN TOROIDAL COIL Q DUE TO STRANDED WIRE WINDING (100 TURNS)

Frequency f_{00}, kc	Core diameter, cm	Number of strands and winding efficiency			
		$n = 1$, $s = 0.5$	$n = 7$, $s = 0.25$	$n = 30$, $s = 0.20$	$n = 81$, $s = 0.20$
50	1	1.7	0.030	0.0036	0.00133
	3	—	0.81	0.098	0.029
	6	—	—	0.78	0.283
100	1	—	0.12	0.0144	0.0053
	3	—	—	0.39	0.0144
	6	—	—	—	0.116
200	1	—	0.48	0.058	0.021
	3	—	—	0.157	0.058
	6	—	—	—	0.467
400	1	—	—	0.232	0.086
	3	—	—	—	0.23
	6	—	—	—	—

This corresponds to (10), but shows the penalty exacted by copper eddy current losses. If hysteresis and residual losses may be neglected, (22) reduces to the form

$$Q_{00} = \frac{\pi}{\sqrt{\frac{\zeta e}{s}\left(1 + \frac{m\zeta}{es\mu^2}\right)}}. \tag{23}$$

This expression can be converted by inserting the values of ζ and m into an equation resembling (13):

$$Q_{00} = \frac{1.88 \times 10^{-2} \sqrt{\frac{s}{e}}\, D}{\sqrt{1 + \frac{0.015Ds^2}{Ne\mu^2}}}. \tag{24}$$

This equation is inconvenient for general use, since it requires knowledge as to the specific values of winding turns and strand number involved in the term introduced by copper eddy current losses. Assumptions as to a tolerable value for this term may be made — for example, that it decrease the coil Q by, say, 10 per cent. The necessary stranding can then be calculated for any core material, diameter and number of winding turns. It should be remarked that these computations apply at the optimum frequency f_{00}, defined by (16).

3.2 *Winding Capacitance*

The ends of the coil winding have unavoidably some small capacitance and associated conductance with respect to each other. Similarly, adjacent turns have mutual capacitance, and they have capacitance to the magnetic core. These several capacitances and conductances may be taken effectively as lumps, C and G, shunted across the terminals of the coil, so as to be in parallel with the inductance L and resistance R of the coil, which are taken to be in series with each other. Computations of such a network at frequency corresponding to $\omega = 2\pi f$ show that it will be observed to have inductance and resistance

$$L_{\text{obs}} = \frac{L(1 - \omega^2 LC) - CR^2}{(1 - \omega^2 LC)^2 + 2GR + G^2(R^2 + \omega^2 L^2) + \omega^2 C^2 R^2} \tag{25}$$

and

$$R_{\text{obs}} = \frac{R + G(R^2 + \omega^2 L^2)}{(1 - \omega^2 LC)^2 + 2GR + G^2(R^2 + \omega^2 L^2) + \omega^2 C^2 R^2}. \tag{26}$$

These expressions simplify at frequencies well below resonance to the approximations

$$L_{\text{obs}} \doteq L(1 + \omega^2 LC), \tag{27}$$

$$R_{\text{obs}} \doteq (R + G\omega^2 L^2)(1 + 2\omega^2 LC). \tag{28}$$

It is evident that both inductance and resistance of a coil effectively increase above their initial values, with the addition term proportional to the square of the frequency. If a precise inductance is required at some operating frequency, the contribution due to distributed capacitance must be taken into consideration. Furthermore, this contribution is unfortunate in that it robs the coil of that constancy of inductance which may be desired for circuit performance over a broad band of frequencies.

Since distributed capacitance is undesirable, means are sought for reducing its value. Such means include spacing apart the end turns of the toroidal winding, bank-winding the entire coil and spacing the winding away from the magnetic core by means of material having low dielectric constant and low conductance (high dielectric Q_c). Such spacing is desirable in that it provides a controlled low capacitance, in series with the capacitance due to the core material, composed of insulated metallic particles, or perhaps ferrite, having high dielectric constant.

The dielectric quality factor, Q_c, for such composite systems of capacitance tends to be much lower than for the usual insulating materials — values around $Q_c = 30$ are not uncommon. This factor is very much dependent upon the moisture content of the dielectrics, and it can be improved or multiplied several fold by thorough drying of the coil and core structure. Proper design requires provision of permanent means for exclusion of moisture from a coil.

Spacing of windings away from core is costly, since this absorbs possible winding space and compels the use of smaller gage wire than was assumed in the computations of Section 2.3 and Table I. As already noted in Section 3.1, the impairment of copper winding space due to stranding shows up in a drastic decrease in the winding packing factor s and a corresponding increase in the dc resistance. The exact apportionment of available space to insulating spacing and copper tends to be a compromise, which is often resolved by the availability of insulating materials in convenient thicknesses, or of stranded wires.

The ultimate criterion of coil design is generally the coil quality factor Q achieved at the desired operating frequency. Since both inductance and resistance have been seen to depend upon the distributed capacitance, the effective quality factor will reflect such dependence. The observed coil quality factor derived from (25) and (26) is

$$Q_{\text{obs}} = \frac{\omega L_{\text{obs}}}{R_{\text{obs}}} = \frac{\omega L \left(1 - \omega^2 LC - \dfrac{CR^2}{L}\right)}{R\left[1 + GR\left(1 + \dfrac{\omega^2 L^2}{R^2}\right)\right]}. \tag{29}$$

This expression can be simplified by inserting the intrinsic coil quality factor $Q = \omega L/R$ and the capacitance quality factor $Q_c = \omega C/G$. Making these substitutions and rearranging terms yields

$$Q_{\text{obs}} = Q \frac{1 - \omega^2 LC \left(1 + \dfrac{1}{Q^2}\right)}{1 + \omega^2 LC \left(1 + \dfrac{1}{Q^2}\right) \dfrac{Q}{Q_c}}. \tag{30}$$

Since intrinsic coil Q is generally much larger than one, the equation above can be rewritten to close approximation as

$$Q_{\text{obs}} = Q \frac{1 - \omega^2 LC}{1 + \omega^2 LC \dfrac{Q}{Q_c}}. \tag{31}$$

The effect of distributed capacitance on coil Q can be more clearly seen by expressing (31) in series form. Thus, performing the indicated division yields

$$Q_{\text{obs}} = Q\left[1 - \omega^2 LC\left(1 + \frac{Q}{Q_c}\right)\left(1 - \omega^2 LC \frac{Q}{Q_c} + \omega^4 L^2 C^2 \frac{Q^2}{Q_c^2} - \cdots\right)\right], \tag{32}$$

which is convergent for $\omega^2 LCQ/Q_c < 1$. For small values of $\omega^2 LCQ/Q_c$ the series reduces practically to its first two terms.

The analysis of inductor design again has lost generality, in that impairment to Q at any frequency is dependent upon the specific values of L, C, Q and Q_c. Useful design advance can be made by setting some limit on the value of $\omega^2 LC$, such as, say, 0.02. Reference to (27) indicates that the inductance under this condition would appear to be 2 per cent higher than its low frequency value, L. Regarding the observed Q, (32) indicates that Q is decreased by the same percentage for an inductor having very high Q_c, and by a greater percentage for inductors in which Q_c is low in relationship to Q. Thus, for $Q/Q_c = 5$ and $\omega^2 LC =$ 0.02, (32) indicates that the observed Q will be approximately 10 per cent less than for the same inductor with very high Q_c. Similar calculation with $Q/Q_c = 1$ yields a loss of Q of some 4 per cent, corresponding to the assumed 2 per cent increase of inductance. Such illustrative calculations show the importance of maximizing Q_c — by use of low-loss dielectric materials in wire insulation, spacing and core insulation; by rigorous drying of coil and core structure before use; and by preservation of dryness throughout the lifetime of the inductor by means of hermetic seals.

The relationship of specific values of inductance and distributed capacitance can be derived, illustratively, from (27) on the above-assumed basis that $\omega^2 LC = 0.02$. The following table has been computed for the inductance which satisfies the assumption (2 per cent increase) at various frequencies and several values of the distributed capacitance C:

	Inductance limit (maximum), in millihenries			
C, in μμf:	5	10	20	50
Frequency, in kc				
30	113.0	56.3	28.2	11.3
50	40.6	20.3	10.15	4.06
100	10.1	5.06	2.58	1.12
150	4.54	2.27	1.13	0.454
200	2.54	1.27	0.63	0.254

This table show the limitations on inductance necessary to maintain less than 2 per cent apparent increase at any frequency. It is evident that low distributed capacitance is desirable, if large inductance values are to be employed. Experience shows that capacitances as low as those of the first two columns can be realized only at the expense of great pains in spacing of winding from the core, and in careful bank winding of wire.

Measurement of distributed capacitance may be conveniently made with the use of (27) and ac bridge measurements of inductance over a frequency range wide enough to make the term $\omega^2 LC$ vary from approximately zero up to say 0.02 or more. Q-meter measurements, in which the resonating capacitance is observed at a low frequency f_1 and again at a frequency twice f_1, are also convenient. The distributed capacitance is then obtained as $(C_1 - 4C_2)/3$, where C_1 and C_2 are the resonating capacitances at f_1 and $2f_1$, respectively. Both these methods assume constancy of intrinsic inductance over the frequency range, i.e., freedom from appreciable eddy current shielding in the core material.

IV. PRACTICAL DESIGN PROCEDURE

It was found in Section 2.1 that copper winding resistance and core loss resistance can be adjusted to yield the largest possible coil quality factor Q_0 at some frequency f_0, with these parameters being functions of core size, permeability and loss characteristics. Complications enter at higher frequencies, due to eddy current losses in the copper wire winding and to capacitance and dielectric losses in the insulating materials adjacent to the winding. Copper eddy current losses impair the quality factor by the factor mf_{00}^2, as shown in (19) and more specifically in (21a), where the interplay of core size, winding turns and strand number is shown. For practical coils, the turns number is generally more than 100, in which case Table II gives the values of mf_{00}^2 for common stranding numbers. From this table it is possible to select the stranding number needed to avoid excessive impairment of Q at any frequency.

It is to be noted that the winding packing factor s declines markedly upon the adoption of stranded wire. Similarly, the steps taken to minimize distributed capacitance in the winding entail sacrifices in the possible s. The minimum winding hole size that preserves space for a shuttle is $D/3$. Experience shows that copper packing factor s in the available winding space ranges from about 0.50 for Formex-insulated solid wire to 0.25, 0.20, or lower for stranded wire windings with bank winding and anticapacitance spacing from the core. A further impairment to packing factor is introduced by the intervals between commercial wire sizes, which may prevent maximum filling of winding space with the number of turns needed to obtain the desired inductance. Theoretically, this impairment means that the packing factor will range downward by as much as 20 per cent of its highest attainable value before the next regular wire size may be used for producing the desired inductance in a regular series of inductances on a given core. Reference to (24) shows that a reduction of 20 per cent in packing factor is reflected in a reduction of Q by more than 10 per cent. This range of impairment must be considered in any practical design based upon the foregoing calculations, which have assumed step-free change of wire sizes.

Once the numerous sources of impairment to quality factor Q are recognized, it is necessary to embark on a practical design by suppressing the impairments as much as possible. Thus, (24) yields the maximum Q for any diameter of core, packing factor and core eddy current coefficient. Assuming that wire stranding is such as to eliminate copper eddy current losses, and that bank winding and spacing eliminate capacitance impairments, calculations of Q_0 can be made for other types of cores than those given in Table I. Such calculations are reported in Table III, for permalloy powder cores of lower permeabilities, and for carbonyl iron powder cores. A summary of core data from both Table I and Table III is shown in Fig. 2. This graph has been prepared to show maxi-

TABLE III — OPTIMUM FREQUENCY AND MAXIMUM Q OF STRANDED WIRE-WOUND CORE RINGS

Dimension ratios: $w' = \frac{1}{3}, h' = \frac{3}{8}, d' = \frac{1}{6}$; hysteresis and residual losses neglected.

Material	O.D., inches	D, cm	f_0, kc	Q_0
Permalloy powder with $\mu = 26$,	0.38	0.72	142	77
$e = 7.7 \times 11^9$,	0.50	1.01	102	109
$a = 6.9 \times 10^6$	0.80	1.65	62	178
$c = 9.6 \times 10^5$	1.06	2.06	49	221
$s = 0.25$	1.57	3.07	33	330
	1.84	3.53	29	380
Permalloy powder with $\mu = 14$,	0.38	0.72	275	72
$e = 7.1 \times 11^9$,	0.50	1.01	197	101
$a = 11.4 \times 10^6$,	0.80	1.65	121	165
$c = 14.3 \times 10^5$,	1.06	2.06	97	206
$s = 0.20$	1.57	3.07	65	307
	1.84	3.53	56	353
Carbonyl iron powder with $\mu = 10$,	0.38	0.72	1160	213
$e = 0.8 \times 11^9$,	0.50	1.01	830	300
$a = 5.0 \times 10^6$,	0.80	1.65	500	460
$c = 6.0 \times 10^5$,	1.06	2.06	400	620
$s = 0.20$	1.57	3.07	270	910
	1.84	3.53	235	1020

mum Q and optimum frequency for permalloy powder cores and carbonyl iron powder cores of typical dimensions. It is useful in selecting core size and permeability to fulfill any desired Q and frequency requirement. The graph must be recognized as optimistic, in that it does not show the reductions in Q due to copper eddy current losses, distributed capacitance losses and failure to fill the winding space to its maximum capability. These reductions in Q can be estimated by means of the formulae provided in the text.

APPENDIX

Copper Eddy Current Losses in Windings on Annular Cores

Theoretical analysis of copper eddy current loss in solenoidal windings has been attacked by Wien.[4] We now proceed to a similar calculation of the more complicated shape of an annular winding.

Thus a turn of copper wire t centimeters in diameter in a coil experiences a transverse alternating flux $B \sin \omega t$, which induces eddy currents in the copper very similar to those in a magnetic lamination. In the diagram (Fig. 3) flux transverse to the wire induces emf in the "ribbon" elements dy thick in the wire, which drives eddy currents as indicated. Assuming a length of l centimeters of wire, the emf in a circuit of area $l \times 2y$ is

$$e = 2ly \frac{dB}{dt} = 2lyB_m\omega \cos \omega t,$$

and (33)

$$E = 4\pi lyf \frac{B_m}{\sqrt{2}} \quad \text{abvolt, rms.}$$

The resistance around the periphery of this area is

$$dR = \frac{\rho l}{\sqrt{\frac{t^2}{4} - y^2}\, dy} \quad \text{abohm},$$

when ρ is expressed in abohm-centimeters.

The corresponding power consumption is

$$dP = \frac{E^2}{dR} = \frac{8l\pi^2 f^2 B_m^{\ 2}}{\rho} y^2 \sqrt{\frac{t^2}{4} - y^2}\, dy.$$

Integrating from $y = 0$ to $y = t/2$ gives the total power consumption in the length l of a copper wire as

$$P_1 = \frac{l\pi^3 f^2 B_m^{\ 2} t^4}{32\rho} \quad \text{erg}. \tag{34}$$

It is now appropriate to compute the value of the induction B_m, which causes the copper eddy current loss. This value will vary from point to point throughout the winding depending upon the integrated magnetizing forces due to current in the nearby turns of the winding.

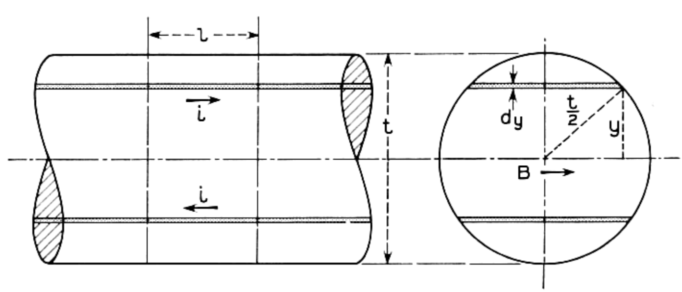

Fig. 3 — Sectional diagram of copper wire, showing eddy current sheets in sections of thickness dy. Magnetic induction B is in direction shown.

For example, if the winding comprises several layers upon an annular core (Fig. 1) the air induction in any layer through the hole in the core will be

$$B_r = H_r = 2\sqrt{2}\,\frac{N_r I}{r}, \tag{35}$$

where r is the distance of the layer from the center, and N_r is the number of turns in the winding, inside the radius r. This number of turns is

$$N_r = \frac{\pi\,(r^2 - r_h^2)}{a}, \tag{36}$$

where r_h is the radius of the hole left through the winding and a is the area of each turn, including copper, insulating material and packing inefficiency. It will be noticed that the analysis of the present section is based on the *radius* of winding layers, core, etc., in contrast with the earlier sections, which were based upon the mean *diameter* of the core. This modification is introduced in the interest of mathematical convenience and it will be converted later into terms of D.

Combining (35) and (36) yields the air induction transverse to any layer of wire through the hole in a core:

$$B_r = 2\sqrt{2}\pi\,\frac{(r^2 - r_h^2)I}{ar}. \tag{37}$$

Referring to (34), the power expended in a length l of a wire at radius r inside the core will be

$$P_{lr} = l\pi^5 f^2 t^4\,\frac{(r^2 - r_h^2)^2 I^2}{4\rho a^2 r^2}. \tag{38}$$

The entire power consumption in the layer requires multiplication of the single-wire loss, per (38), by the number of wires at radius r. Since the turns in the winding are presumed to be stranded, the wire diameter t applies to the individual strands, for this eddy current computation. The cross-sectional area occupied by one layer of turns is $2\pi r\sqrt{a}$, assuming square wire. The number of turns in the layer is then $2\pi r/\sqrt{a}$, and the number of strands is $2\pi rn/\sqrt{a}$, where n is the number of strands per turn. The number of strands per centimeter of radius will then be $2\pi rn/a$, and the power consumption in a differential thickness dr will be

$$dP_{lr} = \frac{nl\pi^6 f^2 t^4 I^2 (r^2 - r_h^2)^2\,dr}{2\rho a^3 r}. \tag{39}$$

The total loss for axial height l of winding inside the hole of the core is obtained by integrating (39) between the limits r_i and r_h, where r_i is the radius of the hole in the core, as follows:

$$P = \frac{nl\pi^6 f^2 t^4 I^2}{2\rho a^3}\left[\frac{r_i^4 - r_h^4}{4} - r_h^2(r_i^2 - r_h^2) + r_h^4 \log \frac{r_i}{r_h}\right]. \quad (40)$$

The effective resistance due to this power loss is P/I^2, or

$$R_{ce} = \frac{nl\pi^6 f^2 t^4}{2\rho a^3}\left[\frac{r_i^4 - r_h^4}{4} - r_h^2(r_i^2 - r_h^2) + r_h^4 \log \frac{r_i}{r_h}\right]. \quad (41)$$

It is interesting to compare this resistance increase at frequency f with the dc resistance. Thus, the dc resistance of the N turns through the hole in the core, for an axial height l, is

$$R_c = \frac{4\rho l N}{n\pi t^2},$$

whence

$$\frac{R_{ce}}{R_c f^2} = \frac{n^2\pi^7 t^6}{8\rho^2 a^3 N}\left[(r_i^4 - r_h^4) - r_h^2(r_i^2 - r_h^2) + r_h^4 \log \frac{r_i}{r_h}\right]. \quad (42)$$

The winding area relationship gives

$$a = \frac{\pi(r_i^2 - r_h^2)}{N}, \quad (43)$$

whence

$$\frac{R_{ce}}{R_c f^2} = \text{“}m\text{”} = \frac{n^2 N^2 \pi^4 t^6}{8\rho^2(r_i^2 - r_h^2)^3}\left[(r_i^4 - r_h^4) - r_h^2(r_i^2 - r_h^2) + r_h^4 \log \frac{r_i}{r_h}\right]. \quad (44)$$

Noting that the cross-sectional area of copper in any turn of the winding is $n\pi t^2/4$, (44), becomes

$$m = \frac{8\pi N^2 a_c^3}{n\rho^2} F(r), \quad (45)$$

where a_c is the cross-sectional area of all the copper strands in each turn of the winding, and $F(r)$ is a complicated function of the inside radii of winding hole and core hole.

The above calculations apply to that part of the winding in the hole through the core. Similar computations for the winding on the outer periphery of the core show that (44) becomes

$$\frac{R_{ce}}{R_c f^2} = m$$
$$= \frac{n^2 N^2 \pi^4 t^6}{8\rho^2 (r_w^2 - r_0^2)^3} \left[(r_0^4 - r_w^4) - r_w^2 (r_0^2 - r_w^2) + r_w^4 \log \frac{r_0}{r_w} \right], \tag{44'}$$

where the hole and inside core radii of (44) are replaced by the outside core radius r_0 and outside winding radius r_w, respectively. Obviously, the coefficient of (44) remains unchanged, for the inside and outside parts of the winding, and only the function $F(r)$ varies, to accommodate the different radii.

Computations for the radiating parts of the turns on the top and bottom of the core would lead to a complicated relationship, which essentially would yield some sort of average between the values of (44) and (44′). Rather than work through such expressions, we will simply postulate that the effective resistance of the entire copper winding will be a function of the mean diameter of the core supporting the winding. Thus,

$$m = \frac{8\pi N^2 a_c^3}{n\rho^2} F(D). \tag{46}$$

The functional relationship $F(D)$ in (46) is most conveniently found empirically, by resistance measurements at high frequencies of windings on nonmagnetic cores of various sizes. Using for the resistivity of copper 1750 abohm-centimeters, the equation above becomes

$$m = \frac{R_{ce}}{R_c f^2} = \frac{8.2 \times 10^{-6} N^2 a_c^3}{n} F(D). \tag{47}$$

Proceeding with construction and measurement of coils of important sizes has involved preparing nonmagnetic wooden or phenol fiber rings, and winding them as efficiently as possible in approximately bank distribution with stranded wire. A survey of data obtained over the past 20 years is given in Table IV. The data have been "boiled down" in terms of only basic geometrical facts of the core diameter, stranding, number and diameter of individual wires, number of turns in the winding and the value of $R_{ce}/(R_c f^2) = m$ measured over a sufficiently wide frequency range. The ratio of the observed value of m to the winding factors in (47) yields values for $F(D)$.

Inspection of Table IV shows that $F(D)$ is inversely proportional to D^3, over a considerable range of diameters. Thus, for D between 2 to 6 centimeters, the value of $F(D)$ averages $3.61/D^3$. Deviations of specific windings in this range amount to -36 and $+38$ per cent maximum. Considering that the average deviation is 15.7 per cent, the analysis is of sufficient accuracy to warrant its acceptance for a useful range of core

TABLE IV — DATA ON STRANDED WIRE-WOUND AIR CORE TOROIDS

D, cm	n, strands	B&S gage	t, cm × 10^3	N, turns	a_c, cm² × 10^4	$\frac{8.2 \times 10^{-6} N^2 a_c^3}{n}$, × 10^{12}	Observed $m = \frac{R_{ce}}{R_e f^2}$ × 10^{12}	$F(D)$	$F(D) \times d^3$
1.65	60	42	6.33	100	18.9	9.22	3.20	0.347	1.56
2.10	120	46	3.99	100	15.0	2.31	0.97	0.420	3.90
2.10	90	44	5.02	100	17.8	5.14	1.40	0.272	2.52
3.14	81	38	10.1	68	64.6	148	11.0	0.0743	2.30
3.7	7	40	8.00	400	3.507	8.09	0.66	0.0815	4.13
3.7	7	40	8.00	800	3.507	32.4	2.17	0.0670	3.40
3.7	7	40	8.00	1000	3.507	50.5	3.40	0.0673	3.42
3.7	7	40	8.00	1336	3.507	90.1	6.90	0.0765	3.88
3.7	19	40	8.00	208	9.52	16.1	1.23	0.0763	3.87
3.7	30	40	8.00	100	15.03	9.31	0.76	0.0816	4.13
3.7	30	40	8.00	180	15.03	30.2	2.22	0.0735	3.73
3.7	30	40	8.00	300	15.03	83.9	6.46	0.0769	3.90
3.7	30	40	8.00	400	15.03	149	11.0	0.0737	3.74
3.7	7	34	1.60	400	14.99	631	32.0	0.0507	2.57
3.7	1	25	4.55	400	16.24	5620	336	0.0598	3.03
6.20	30	36	12.7	600	38.0	5400	97.6	0.0181	4.33
6.20	30	40	8.00	1002	15.03	935	19.5	0.0209	4.99
9.05	1	21	72.3	390	41.05	8630	1825	0.0212	15.7
9.05	81	38	10.1	485	64.6	6430	116	0.0180	13.3
Average (omitting 1.65 cm and 9.05 cm cores)									3.61

sizes. Beyond 6.2 centimeters the coefficient deviates considerably. Thus (47) can be rewritten (for core diameters 2 to 6 centimeters):

$$m = 30 \times 10^{-6} \frac{N^2 a_c^3}{n D^3}. \tag{48}$$

Equation (48) is inconvenient, since it includes N, the number of turns in the winding, and a_c, the area of copper in each turn. These parameters can be combined by noting that Na_c is the total copper area in the winding, and N_{ac}/s is the entire available winding area $A_c = \pi d(D - w - d)$. Thus, for archetype cores, as discussed above, prior to (12), these relationships reduce to

$$Na_c = s\pi D^2 d'(1 - w' - d'). \tag{49}$$

Hence, (48) becomes

$$m = 30 \times 10^{-6} \frac{D^3}{nN} s^3 \pi^3 d'^3 (1 - w' - d')^3,$$

or

$$m = \frac{932 \times 10^{-6}}{nN} [Dsd'(1 - w' - d')]^3. \tag{50}$$

It appears, finally, that the copper eddy current loss coefficient is directly proportional to the cube of the mean diameter of the core, and inversely proportional to the number of turns and number of strands per turn of the coil winding.

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