Timing in a Long Chain of Regenerative Binary Repeaters

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The present paper studies some of the statistical properties of the random timing deviations, or position modulation of the signal pulses, in a long chain of regenerative binary repeaters. Random timing deviations of the output signal pulses result from input noise, tuning error, random timing deviations of the input signal pulses (introduced by preceding repeaters) and other sources at each repeater.

The power spectra and the total powers (mean square values) of the timing noise, spacing noise (random deviations in spacing of two consecutive pulses from an integral number of pulse periods) and alignment noise (random deviations in alignment between an input signal pulse and its corresponding timing pulse) caused by the input noise at each repeater are determined for a long chain of regenerative repeaters using either tuned circuit or locked oscillator timing filters. The effects of tuning error are studied for a chain of repeaters employing locked oscillator timing circuits; however, the present analysis does not treat the effects of tuning error in a chain of repeaters using tuned-circuit timing filters.

I. INTRODUCTION

Regenerative binary repeaters have recently been proposed for both baseband and carrier pulse code modulation systems,[1, 2, 3, 4] in which the signal is represented by a binary pulse train. This type of repeater attempts to remove noise and other types of distortion from the incoming pulse train and to transmit a new signal which resembles the original as closely as possible. Noise and other system imperfections have two unwanted effects: (1) a certain number of errors occur at each repeater, i.e., a received pulse is transmitted as a space, or *vice versa*; (2) the signal pulses are no longer centered in equally spaced time slots but have a random position modulation, called timing noise. The present paper is concerned with some of the statistical properties of the random timing deviations in a system containing a long chain of regenerative repeaters.

A portion of a typical pulse train under ideal conditions is shown in Fig. 1, where, for purposes of illustration, the individual signal pulses have been made short enough so that they do not overlap; the curve represents either the amplitude of a baseband pulse train or the envelope of a carrier pulse train. The pulses are either "on" or "off", denoted by "1" and "0" respectively; all signal pulses present have a standard amplitude and identical shape, and are centered in equally spaced time slots.

An ideal regenerative repeater would sample the pulse train of Fig. 1 at the instants nT ($n = \cdots, -1, 0, 1, \cdots$). If the amplitude or envelope at each sample point is greater than the slicing level a new standard signal pulse is transmitted; if the amplitude or envelope is less than the slicing level no pulse is transmitted. If an additive gaussian noise is now present at the input to the regenerator there will be a certain number of errors, so that the output pulse train will no longer be identical to the input pulse train. To minimize the number of errors the slicing level should be set at one-half the peak pulse amplitude in a baseband system, a little greater than one-half the peak pulse envelope in a carrier system.[5] The error rate is then determined by the signal-to-noise ratio at the input to the regenerator.[5] In this way the effects of noise are completely eliminated, except in the relatively rare cases where the noise is large enough to cause an error.

These sampling and level-selecting operations may be performed by an idealized regenerator having the characteristics shown in Fig. 2, which approximate those of practical regenerators. In addition to the signal input and output, which may be either baseband or carrier pulses, the regenerator has an additional input for the timing or sampling pulses, which are baseband pulses. The input and output plotted in Fig. 2 repre-

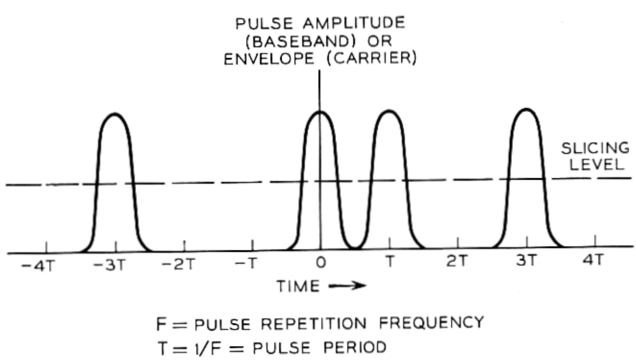

Fig. 1 — Binary pulse train.

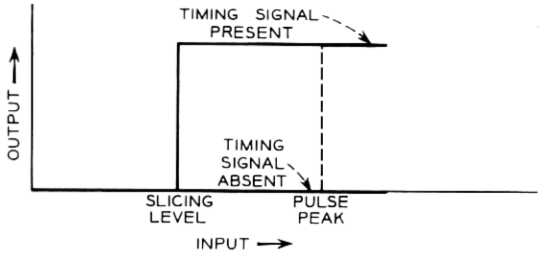

Fig. 2 — Ideal regenerator characteristic.

sent amplitude or envelope for the baseband and carrier cases respectively. If the timing signal is absent, the regenerator output is zero for all input signal levels; if a timing pulse is present, the regenerator operates as an ideal slicer, with zero output for inputs less than the slicing level and a constant output for inputs greater than the slicing level. If the timing pulses are much shorter than the signal pulses, the regenerator output will consist of identical short baseband or carrier pulses, which may be transmitted through an appropriate filter to yield standard output signal pulses. Thus, the system will produce a standard output pulse each time a timing pulse occurs when the input signal has an amplitude or envelope greater than the slicing level. The position of this output pulse is determined only by the position of the timing pulse and not by the position of the input signal pulse; this type of response has been called "complete retiming".[3, 6]

In an ideal regenerative repeater with an input signal as shown in Fig. 1, the regenerator of Fig. 2 must be supplied with timing impulses occurring at the pulse repetition frequency and centered exactly at the sample points nT. However, in the self-timed repeaters studied here the timing pulses must be derived by the repeater itself from the signal, which will no longer be the ideal signal of Fig. 1 but will have added noise and random position modulation introduced by the preceding repeaters.

In systems employing complete retiming (such as the one discussed above), in which the timing pulse alone determines the position of the corresponding output signal pulse, the timing pulses may be derived only from the input signal. However, in systems employing partial retiming, in which the position of each output signal pulse depends on the position of both the corresponding timing and input signal pulses, the timing pulses may be derived from either the input or the output signal pulses.[3, 6] Repeaters with essentially complete retiming appear to be of greatest interest for microwave systems and so, for the present, we con-

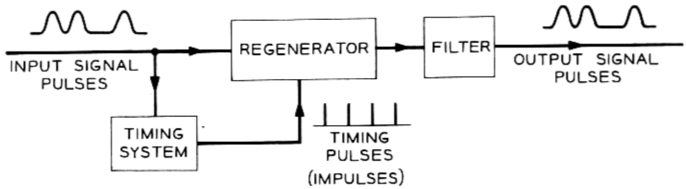

Fig. 3 — Regenerative binary repeater with timing from the input.

sider only the case of complete retiming. The analysis is easily extended to include partial retiming, with the timing pulses derived from either the input or the output.

The mathematical model chosen to represent a regenerative repeater is shown in Fig. 3. The regenerator characteristics are assumed to be those given in Fig. 2. The input and output signals are either baseband or carrier pulses; the timing signal is a train of baseband impulses, derived from the input pulse train. The filter converts the short pulses at the output of the regenerator into standard signal pulses for transmission to the next repeater.

The timing system in Fig. 3 contains a narrow band-pass filter tuned as close as possible to the pulse repetition frequency. This timing circuit is excited by baseband pulses derived from the input signal pulses. Its output, called the timing wave, is approximately a sine wave at the pulse repetition frequency, and may be considered to be a sine wave at this frequency with both random amplitude and phase modulation. The amplitude modulation results from the statistical nature of the signal pulse pattern and from the random noise introduced at the input to the repeater; the phase modulation is produced by the random noise at the input, the random variations in pulse position introduced by preceding repeaters in the chain, and the tuning error of the timing circuit. The timing circuit is followed by an ideal limiter which removes the amplitude modulation; the resulting waveform is then used to generate the timing pulses applied to the regenerator, for example, by producing a timing pulse at the instant of each negative- (or positive-) going zero crossing of this waveform, so that the phase of the timing wave determines the position of the timing pulses.

The baseband driving pulses for the timing circuit may be obtained in various ways. The simplest method is to drive the timing circuit with the signal pulses themselves in a baseband system or with their rectified envelope (obtained by a linear envelope detector) in a carrier system. However, it may be advantageous to first pass the baseband signal pulses through a nonlinear device such as a peak amplifier or a square-law rectifier to suppress the low-level noise in the absence of signal pulses;[2] similarly, a square-law rectifier might be used for the same purpose in a carrier system. Alternately, a standard baseband driving pulse for the timing circuit might be generated whenever the amplitude or envelope of the baseband or carrier signals passes through a critical level (approximately half the peak) in the ascending direction.[7] The noise at the input of each repeater will affect the timing performance of these different systems in different ways.

In order to simplify the analysis of the timing behavior of a long chain of repeaters and to permit the application of the results to systems employing different types of repeaters, such as those discussed above, the following general assumptions will be made:

1. The reception of each signal pulse is assumed to initiate an independent transient in the timing circuit. This requirement is obviously satisfied in any baseband or carrier system in which signal pulses in adjacent time slots do not overlap. It will remain satisfied if adjacent signal pulses do overlap only under special conditions, e.g., (a) in a baseband system, if the received signal pulses drive the timing circuit directly, (b) in a carrier system with a linear envelope detector and coherent carrier phase between adjacent signal pulses.

2. The timing filter is assumed to be a simple resonant circuit, characterized by its Q. Its natural resonant frequency would ideally be made equal to the pulse repetition frequency, but in practice there will be a small tuning error. A related problem in which an idealized locked oscillator is used to generate the timing wave will also be considered.

3. The input noise at each repeater causes random timing deviations of the output signal pulses, in addition to those present from other sources. This real input noise may be replaced by adding an equivalent fictitious position modulation or timing noise to the input signal pulses, such that the random timing deviations at the output of the repeater remain the same. This permits the effects of the input noise to be treated in the same manner as the effects of the random position modulation introduced by the preceding repeaters. For simplicity, we assume that the equivalent timing deviations added to the different input signal pulses are statistically independent, the mean square value of the added timing noise being given. This assumption is plausible, since the real input noise will have a bandwidth comparable to that of the signal. A detailed analysis of a particular repeater is required, of course, to establish this equivalence rigorously; the rms value of the equivalent timing noise may depend on the average number of pulses present as well as on

the noise level at the repeater input. While no attempt will be made to give detailed consideration to any of the various types of repeaters discussed above, the dependence of the equivalent added timing noise on the pulse pattern will be discussed briefly for several cases of interest.

4. In order to permit the treatment of the timing behavior of a chain of repeaters in a simple way, approximations are made which linearize the analysis. It is then strictly valid only when the equivalent timing noise added at the input of each repeater is vanishingly small. Although the effects of these approximations are not known in detail, it seems plausible that the analysis remains valid in the practical case where the input noise at each repeater is small.

5. The input signal-to-noise ratio must be moderately high in a satisfactory system so that very few errors are made in recognizing pulses and spaces; these errors are neglected in the present timing analysis.

Consider first the case where the timing filters are simple tuned circuits. Even with the above approximations a rigorous solution for the timing behavior of a chain of repeaters has been obtained only under special conditions. In particular, the effects of the input noise may be determined if the tuning error of every repeater is zero, for only the following signal pulse patterns: (1) all pulses present; (2) every Mth pulse present; (3) any general periodic pulse pattern. In the first two cases the analysis is straightforward, but in the third case the complexity increases with the complexity of the pulse pattern and becomes somewhat prohibitive for all but the simpler periodic pulse patterns. The most interesting case — a random pulse pattern — has been treated in a simple way only by making a further approximation, in which the variation of the timing wave amplitude is neglected at an appropriate point in the analysis.[7] In general, no accurate estimate is avialable for the error introduced by this approximation. It gives accurate results in two cases which can be solved by other methods. These are: (1) a chain of repeaters with a periodic pulse pattern containing two pulses located in arbitrary positions, as discussed above; (2) a single repeater with a random pulse pattern, which has been treated by W. R. Bennett in a different way.[7]

Other problems concerning a chain of repeaters employing tuned circuits as timing filters, which may well be of greater practical importance than the effects of input noise, have not been treated in the present analysis. The first of these is the effect of random tuning errors at the different repeaters. This problem has been treated for a single repeater by W. R. Bennett.[7] The second problem is the effect of the finite width of the pulses used to drive the timing circuit, which will be shown to add an identical timing noise at the output of each repeater and to be quite similar to the effects of amplitude-to-phase conversion in the limiter in the timing system.[2,8] Since the added timing noise is identical at each repeater, this type of disturbance must be treated differently from independent random disturbances added at the different repeaters, and is potentially more serious.

As stated above, in order to obtain an approximate analysis for a chain of repeaters with zero tuning error and a random pulse pattern we modify the timing response of each repeater by neglecting the variation in timing wave amplitude. Alternately, we may ask what kind of system is exactly described by the modified equations. It turns out that a somewhat idealized locked oscillator, synchronized by the signal pulses or their envelope, corresponds exactly to this analysis, within the other approximations described above. For a chain of repeaters using locked oscillators as timing circuits, both the effects of input noise and the effects of tuning error can be treated for arbitrary or random pulse patterns without further approximations. Also, finite pulse width introduces no additional timing noise in this case. Thus, the analysis for locked oscillators is considerably more tractable than is that for tuned circuits.

The statistical properties of three different quantities are of interest in studying the timing behavior of a repeater chain. These are:

1. *Timing noise*, or random deviations of the signal pulses from equally spaced time slots, will cause a random delay modulation of the original signal, unless special precautions are taken in the final decoding of the PCM signal. This random delay modulation will, for example, produce crosstalk in a frequency-division multiplex signal,[2,7] and will degrade other types of signals in different ways.[2]

2. *Spacing noise*, or the random deviations in the spacing of two consecutive signal pulses (which do not necessarily occupy adjacent time slots) from an integral number of pulse periods could cause pulses occupying adjacent time slots to interfere with each other. This would degrade the performance of the regenerator and cause an increased number of errors, i.e., a received pulse transmitted as a space, or *vice versa*. If too large, it could also cause some received pulses to be assigned to incorrect time slots, with resulting errors in the decoding of the PCM signal.

3. *Alignment noise*, or the random deviations in alignment between an input signal pulse and the corresponding timing pulse, can also degrade the performance of the regenerator and increase the number of errors, since for optimum margin against noise the timing or sampling pulse should fall exactly at the center of the corresponding signal pulse.

These three quantities are defined in Fig. 4, which will be discussed in greater detail below.

The present analysis determines the power spectra and the total powers (mean square values) for these three types of deviations for a long chain of repeaters, under the conditions discussed above. The way in which these disturbances vary along a repeater chain is obviously of great importance. It is found that, while the low-frequency timing noise does grow along the chain, the spacing and alignment noise spectra approach limiting values rather quickly, as might be expected on physical grounds. Since the timing, spacing and alignment deviations are discrete functions, i.e., defined only for integral values of their agrument, their power spectra are defined somewhat differently than in the usual case of continuous functions; a brief description of the Fourier analysis of such discrete functions is included.

Related studies of the timing deviations in a repeater chain have been given by DeLange[2] and Sunde,[3] using different methods than those employed here. The present analysis follows closely the approach first used by J. R. Pierce in determining the timing deviations in a single repeater,[6] extending the analysis to a chain of repeaters and to include the spacing and alignment deviations.

II. THE TIMING RESPONSE OF A SINGLE REPEATER

In this section we relate the timing deviations of the output pulses to the timing deviations of the input pulses and to the pulse pattern for a single repeater. As discussed above, we consider both tuned-circuit and locked-oscillator timing circuits. In order to study the response of a repeater to the timing deviations of the input pulses it is necessary to consider only the simplest case of impulse excitation of the timing circuit. Finite pulse width may introduce additional output timing deviations for tuned circuit timing filters but not for idealized locked oscillators; excitation of a tuned circuit by raised cosine pulses is therefore also considered.

2.1 *Tuned Circuit Excited by Impulses*[6]

Assume the timing circuit to be a parallel resonant inductance L, capacitance C and resistance R. The impulse response of this circuit is given by the real part of the complex impulse response $H(t)$:

$$H(t) = \frac{1}{C}\left(1 + \frac{j}{2Q}\right) e^{-(\pi/Q)f_0t} e^{+j2\pi f_0 t}; \quad t > 0, \tag{1}$$

where

$$\omega_0 = 2\pi f_0 = \sqrt{\frac{1}{LC} - \frac{1}{(2RC)^2}}, \qquad Q = \omega_0 RC, \qquad \frac{\omega_0}{2Q} = \frac{1}{2RC}. \tag{2}$$

In (1) and in all subsequent complex expressions the real part is implied.

The natural resonant frequency f_0 (as distinguished from the steady-state resonant frequency $f_r = 1/2\pi\sqrt{LC}$) will be made as close as possible to the pulse repetition frequency F. Therefore we set

$$f_0 = F + \delta f = F\left(1 + \frac{\delta f}{F}\right), \tag{3}$$

where

F = pulse repetition frequency,

δf = tuning error of the timing circuit,

$F = \frac{1}{T}$, T being the pulse period (Fig. 1),

$\frac{\delta f}{F} \ll 1.$

Substituting into (1), the response to a unit impulse at the time t_0 is equal to the real part of

$$H(t - t_0) = \frac{1}{C}\left(1 + \frac{j}{2Q}\right) e^{-(\pi/Q)F(1+\delta f/F)(t-t_0)} e^{j2\pi F(1+\delta f/F)(t-t_0)}; \qquad t - t_0 > 0. \tag{4}$$

The pulse train driving the tuned circuit is given by

$$g(t) = \sum_{n=-\infty}^{\infty} a_n \delta(t - t_n), \tag{5}$$

where $\delta(t - t_n)$ is a unit impulse occurring at t_n, the time of arrival of the signal pulse corresponding to the nth time slot. If the nth time slot contains a pulse $a_n = 1$; if this time slot is vacant $a_n = 0$. In the absence of timing noise the signal pulses would be centered in their corresponding time slots, so that for the pulse corresponding to the nth time slot

$$t_n = nT. \tag{6}$$

If the deviation in the position of this pulse is δt_n, then the time of arrival becomes

$$t_n = nT + \delta t_n . \tag{7}$$

It is convenient to normalize the timing deviation with respect to the pulse period T. Thus, let

$$\epsilon^i(n) = \frac{\delta t_n}{T}, \tag{8}$$

where $\epsilon^i(n)$ is the normalized timing deviation of the signal pulse in the nth time slot at the input to the repeater, referred to simply as the timing deviation when no confusion will arise. Then the time of arrival of this pulse in (7) becomes

$$\begin{aligned} t_n &= nT + \epsilon^i(n)T, \\ Ft_n &= \quad n + \epsilon^i(n). \end{aligned} \tag{9}$$

From (4), (5) and (9) the response of the timing circuit to all of the signal pulses up to and including the pulse in the mth time slot is equal to the real part of

$$G(t) = \frac{1}{C}\left(1 + \frac{j}{2Q}\right) \sum_{n=-\infty}^{m} a_n e^{-(\pi/Q)(1+\delta f/F)[Ft-n-\epsilon^i(n)]} \cdot e^{j2\pi(1+\delta f/F)[Ft-n-\epsilon^i(n)]}. \tag{10}$$

This expression gives the output of the timing circuit for values of t lying between the arrival times of the pulse in the mth time slot and the next pulse that is present in the signal. Formally,

$$\begin{aligned} t_m &< t < t_k , \\ m + \epsilon^i(m) &< Ft < k + \epsilon^i(k), \\ a_n &= \begin{cases} 1; & n = m, k \\ 0; & m < n < k. \end{cases} \end{aligned} \tag{11}$$

The response of the timing circuit given in (10) may be written in the form of a carrier at the pulse repetition frequency F with both amplitude and phase modulation:

$$G(t) = \frac{1}{C}\left(1 + \frac{j}{2Q}\right) A(t)e^{j2\pi Ft}; \qquad A(t) = a(t)e^{j\varphi(t)}. \tag{12}$$

In this equation, $A(t)$ is a complex function of time whose magnitude $| A(t) | = a(t)$ equals the normalized amplitude of the timing wave and whose angle $\angle A(t) = \varphi(t)$ equals the phase deviation of the timing wave. For convenience, we have chosen to remove the constant factor $(1/C)\ (1 + j/2Q)$ in defining $A(t)$. From (10),

$$A(t) = \sum_{n=-\infty}^{\infty} a_n e^{-(\pi/Q)(1+\delta f/F)[Ft-n-\epsilon^i(n)]} e^{j2\pi\{(1+\delta f/F)[Ft-n-\epsilon^i(n)]-Ft\}}, \quad (13)$$

keeping in mind the restrictions of (11).

Equation (13) shows the way in which the amplitude and phase of the timing wave vary with time when timing noise is present. The amplitude decreases exponentially between signal pulses, increasing abruptly at the instant a pulse is received. In the absence of tuning error the phase is constant between signal pulses, changing abruptly when a pulse is received. If the tuning error is not zero, the phase has, in addition, a small constant linear variation between signal pulses. If the tuning error δf and the input timing deviations $\epsilon^i(n)$ are both equal to zero, we have from (13)

$$\angle A(t) = \varphi(t) = 0. \quad (14)$$

It is convenient to assume that under these conditions the delays in the repeater of Fig. 3 have been adjusted so that the timing pulses supplied to the regenerator are properly aligned with the input signal pulses, i.e., occur at the instants nT in Fig. 1. The quantity $\varphi(t)$ then gives a true measure of the timing deviations of the timing pulses.

We now assume that the phase of the timing wave at the instant immediately following the reception of a signal pulse determines the timing deviation of the corresponding timing pulse. This, of course, may not be strictly true. For example, the timing pulse might be generated at the next negative-going zero crossing of the timing wave, as discussed in the introduction. This will occur approximately $T/4$ seconds after the arrival of the signal pulse because, as shown in the subsequent analysis, in any satisfactory system the alignment error (defined in the introduction) remains small, and consequently the maxima of the timing wave occur close to the driving pulses for the timing circuit. However, for zero tuning error the timing wave phase $\varphi(t)$ is constant between signal pulses and consequently may be evaluated equally well at any time during the pulse period T following the arrival of a signal pulse. Even if the tuning error is not zero $\varphi(t)$ changes very slowly during the interval T, so that a small error in the time at which it is evaluated becomes unimportant; evaluating the phase just after the arrival of a signal pulse rather than $T/4$ seconds later will cause a very small error in the dc value of the output timing deviation, but will have no other effect on the analysis.

Consequently, evaluating (13) at the time

$$Ft = Ft_m = m + \epsilon^i(m) \tag{15}$$

we obtain

$$A(t_m) = e^{-j2\pi\epsilon^i(m)} \sum_{n=-\infty}^{m} a_n e^{-(\pi/Q)(1+\delta f/F)[(m-n)+\epsilon^i(m)-\epsilon^i(n)]} \cdot e^{j2\pi\{\epsilon^i(m)-\epsilon^i(n)+(\delta f/F)[(m-n)+\epsilon^i(m)-\epsilon^i(n)]\}}. \tag{16}$$

Denoting the normalized output timing deviation of the repeater by $\epsilon^o(m)$, defined as in (8), we have for the case of complete retiming, where the timing deviation of each output signal pulse is identical to the timing deviation of the corresponding timing pulse,

$$\epsilon^o(m) = -\frac{\varphi(t_m)}{2\pi}; \qquad \varphi(t_m) = \angle A(t_m), \tag{17}$$

where $A(t_m)$ is given by (16). The minus sign in (17) occurs because the timing pulse will occur too soon for $\varphi(t)$ positive, corresponding to a negative timing deviation according to (7) and (8).

Equations (16) and (17) determine the general relation between the input and output timing deviations for a single repeater. The output timing deviation is seen to be a rather complicated function of the past input timing deviations. In order to make possible a reasonably simple analysis that can readily be extended to a chain of repeaters, certain restrictions will be imposed that linearize the relations between input and output timing deviations.

Equation (16) gives $A(t_m)$ as the sum of an infinite number of vectors, each associated with one of the past input signal pulses. The amplitudes of these vectors become exponentially smaller the farther back in time the corresponding signal pulse occurred. If the angles between those vectors which give the essential contribution to the sum are small enough so that the sine of an angle is approximately equal to the angle, then the vector summation is quite easily accomplished. Taking the vector corresponding to the present input pulse ($n = m$) as a reference, the in-phase component is approximately the sum of the magnitudes of all of the vectors, the quadrature component is the sum of the magnitudes times the relative angles. In order for these approximations to be valid we must have

$$\left.\begin{array}{l} \left| \dfrac{\delta f}{F}(m - n) \right| \ll 1 \\ \left| \epsilon^i(m) - \epsilon^i(n) \right| \ll 1 \end{array}\right\} \text{for all significant } n. \tag{18}$$

The significant terms in the summation are for values of n bounded by

$$n_0 < n < m,$$

where

$$(m - n_0) \gg \frac{Q}{\pi}. \tag{19}$$

The first condition of (18) thus becomes

$$\left| \frac{\delta f}{F} \right| \ll \frac{\pi}{Q}. \tag{20}$$

There are two different ways in which the second condition of (18) may be satisfied. Either the input timing deviation must be small compared to one or it must change so slowly that the difference in the deviations of pulses separated by Q/π pulse periods will be small compared to one. Thus,

$$| \epsilon^i(n) | \ll 1$$

or

$$| \Delta\epsilon^i(n) | = | \epsilon^i(n + 1) - \epsilon^i(n) | \ll \frac{\pi}{Q}, \tag{21}$$

where $\Delta\epsilon^i(n)$ represents the first forward difference of $\epsilon^i(n)$.

Subject to (20) and one of the conditions of (21) we have, from (16) and (17), the normalized timing wave amplitude A_m and the normalized output timing deviation $\epsilon^o(m)$ (with complete retiming):

$$A_m = | A(t_m) | = \sum_{n=-\infty}^{m} a_n e^{-(\pi/Q)(m-n)}, \tag{22}$$

$$\epsilon^o(m) = \frac{\sum_{n=-\infty}^{m} a_n e^{-(\pi/Q)(m-n)} \left[\epsilon^i(n) - \frac{\delta f}{F}(m - n) \right]}{A_m}. \tag{23}$$

Alternately, with the summations rewritten, (22) and (23) become

$$A_n = \sum_{k=0}^{\infty} a_{n-k} e^{-(\pi/Q)k}, \tag{24}$$

$$\epsilon^o(n) = \frac{\sum_{k=0}^{\infty} a_{n-k} e^{-(\pi/Q)k} \left[\epsilon^i(n - k) - \frac{\delta f}{F} k \right]}{A_n}. \tag{25}$$

For zero tuning error, (22) and (23) or (24) and (25) give the output timing deviation as a linear function of past input timing deviations for a repeater with complete retiming, subject to the restriction of (20) and (21). It is usually implied that $a_m = 1$ in (23) or $a_n = 1$ in (25), since the output timing deviation has meaning only for values of its argument corresponding to time slots that contain signal pulses. The timing wave phase corresponding to vacant time slots will not normally be of interest (except insofar as there may be a preferred time to examine a vacant time slot to determine that no signal pulse is present).

These equations show that, for $\delta f = 0$, each repeater may be considered a linear transducer to the timing deviations. In general, the equivalent transducer will be time-varying, since the timing wave amplitude A_n will vary from pulse to pulse; the pulse pattern enters explicitly into the analysis, both in determining A_n and through the a's in the numerator of (23) or (25). However, if all pulses are present or if the pulse pattern is periodic with every Mth pulse present (all pulses present correspond to the special case $M = 1$), A_n is constant at every signal pulse. This is the only case in which the repeater acts as a strictly invariant linear transducer, permitting a simple analysis of a chain of repeaters.

In dealing with general pulse patterns it is convenient to define a new "primed" independent variable that numbers the signal pulses consecutively, rather than the time slots. Fig. 4(a) shows a portion of a typical pulse train; consecutive time slots are denoted by the variable n, consecutive pulses by n'. A quantity regarded as a function of n' or any other primed independent variable will be distinguished by the symbol ~; functions of n or any other *un*primed independent variable will be written as before. For any given pulse pattern n is a function of n', which of course differs for each different pulse pattern, and *vice versa*. As shown in Fig. 4, the number of vacant time slots between the $(n' - 1)$th and the n'th (consecutive) pulses is defined as $b_{n'}$. Referring to Fig. 4, the following examples illustrate this notation:

$\epsilon(n)$ = timing deviation of pulse corresponding to the nth time slot,

$\tilde{\epsilon}(n')$ = timing deviation of the n'th pulse,

A_n = timing wave amplitude corresponding to the nth time slot,

$\tilde{A}_{n'}$ = timing wave amplitude corresponding to the n'th pulse,

$$\begin{gathered} \tilde{\epsilon}(n' - 1) = \epsilon(n - b_{n'}), \qquad \tilde{\epsilon}(n') = \epsilon(n), \\ \tilde{A}_{n'-1} = A_{n-b_{n'}}, \qquad \tilde{A}_{n'} = A_n . \end{gathered} \tag{26}$$

Thus, for the particular pulse pattern of Fig. 4(a) we have, for example:

$$\tilde{\epsilon}(0) = \epsilon(0); \qquad \tilde{\epsilon}(2) = \epsilon(6); \qquad \tilde{A}_3 = A_9 .$$

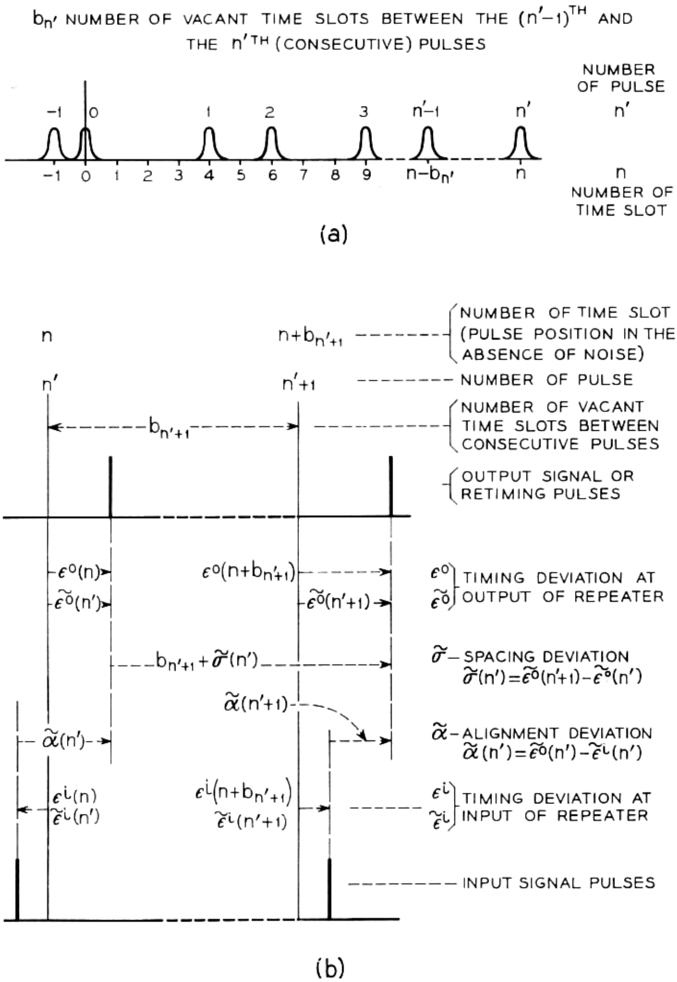

Fig. 4 — Definition of timing, spacing and alignment deviations for a repeater with complete retiming: (a) general pulse pattern, in absence of timing noise; (b) timing, spacing and alignment deviations.

For all pulses present $n' = n$ and $b_{n'} = 1$, and the two notations coincide.

The timing, spacing and alignment deviations defined in Section I are illustrated in Fig. 4(b). The timing deviation of the input and the timing and output pulses have been defined above. The spacing deviation is given by

$$\tilde{\sigma}(n') = \tilde{\epsilon}^o(n' + 1) - \tilde{\epsilon}^o(n') = \Delta\tilde{\epsilon}^o(n'), \tag{27}$$

where Δ indicates the first forward difference, and the alignment deviation for a repeater with complete retiming is given by

$$\tilde{\alpha}(n') = \tilde{\epsilon}^o(n') - \tilde{\epsilon}^i(n'), \tag{28}$$

recalling that, with complete retiming, the timing deviations of the timing and the output signal pulses are equal.

The choice of n' as the independent variable in the analysis rather than the original independent variable n is more natural in several ways. As mentioned above, we are usually interested in the timing wave phase only for time slots that contain pulses. The definition of spacing deviation in (27) is much easier to work with than it would be if it were in terms of the original variable n. As will appear below, this change of independent variable facilitates the approximate treatment of a chain of repeaters with random or general periodic pulse patterns. While it would present some complications in dealing with quantities that must be expressed in real time, no such difficulties arise in the present treatment of a repeater chain.

Returning to the special case of a periodic pulse pattern with every Mth pulse present, the results of (21) to (25) may be written very conveniently in terms of the new independent variable n'. We have

$$a_n = \begin{cases} 1; & n = Mn' \\ 0; & \text{otherwise} \end{cases} \tag{29}$$

and

$$n = Mn'; \qquad k = Mk'. \tag{30}$$

The timing wave amplitude takes on the constant value A at every signal pulse; from (24),

$$A = A_{Mn'} = \tilde{A}_{n'} = \sum_{k'=0}^{\infty} e^{-(\pi M/Q)k'}, \qquad A = \frac{1}{1 - e^{-\pi M/Q}}. \tag{31}$$

The output timing deviation in (25) becomes

$$\begin{aligned} \tilde{\epsilon}^o(n') &= (1 - e^{-\pi M/Q})\left[\sum_{k'=0}^{\infty} \tilde{\epsilon}^i(n' - k')e^{-(\pi M/Q)k'} - \frac{\delta f}{F} M \sum_{k'=0}^{\infty} k' e^{-(\pi M/Q)k'}\right] \\ &= (1 - e^{-\pi M/Q}) \sum_{k'=0}^{\infty} \tilde{\epsilon}^i(n' - k')e^{-(\pi M/Q)k'} - \frac{\delta f}{F}\frac{Me^{-\pi M/Q}}{1 - e^{-\pi M/Q}}. \end{aligned} \tag{32}$$

For the high-Q case we may set

$$\frac{1}{A} = (1 - e^{-\pi M/Q}) \approx \frac{\pi M}{Q}; \qquad Q \gg \pi M \tag{33}$$

in (31) and (32), so that (32) becomes

$$\bar{\epsilon}^o(n') = \frac{\pi M}{Q} \sum_{k'=0}^{\infty} \bar{\epsilon}^i(n' - k')e^{-(\pi M/Q)k'} - \frac{\delta f}{F}\frac{Q}{\pi}; \qquad Q \gg \pi M. \tag{34}$$

The second term of (32) and (34) represents a constant timing deviation introduced by the tuning error; the first term shows that the repeater is a strictly invariant linear transducer to the input timing deviations for this special case.

In a chain of repeaters the timing wave amplitude A_n is identical at each repeater, since the pulse pattern is the same at each repeater. Thus, for random or general periodic pulse patterns we must treat the repeaters as identical linear time-varying transducers to the timing deviations. Now, if Q is large, the variation in A_n will be small;[7] we can make use of this fact to treat the repeaters as approximately invariant transducers. Equations (23) and (25) may be written in an alternate form, with n' as the independent variable, that has a simple physical interpretation and proves useful in making the further approximation that permits the analysis of a chain of repeaters with random or general periodic pulse patterns, for zero tuning error.

We now examine the change in the output timing deviation that occurs during the interval between two consecutive signal pulses. Making use of (26), we have, from (24) and (25), the first backward difference of the output timing deviation as a function of n':

$$\bar{\epsilon}^o(n') - \bar{\epsilon}^o(n' - 1) = \frac{1}{\tilde{A}_{n'}}\left[\bar{\epsilon}^i(n') - \bar{\epsilon}^o(n' - 1) + \frac{\delta f}{F} b_{n'}\right] - \frac{\delta f}{F} b_{n'}. \tag{35}$$

The timing wave amplitude is given by (24), as before. Referring to Fig. 5, we may interpret this relation as follows: Immediately after the reception of the $(n' - 1)$th pulse the timing wave amplitude is $\tilde{A}_{n'-1}$ and the output timing deviation is $\bar{\epsilon}^o(n' - 1)$, with corresponding timing wave phase $-2\pi\bar{\epsilon}^o(n' - 1)$. During the interval of $b_{n'}$ pulse periods from the arrival of the $(n' - 1)$th pulse until just before the arrival of the n'th pulse the timing wave amplitude will decay exponentially to the value $e^{-(\pi/Q)b_{n'}}\tilde{A}_{n'-1}$ and the timing wave phase will advance linearly by an angle $2\pi(\delta f/F)b_{n'}$, so that the resultant phase is

$$-2\pi[\bar{\epsilon}^o(n' - 1) - (\delta f/F)b_{n'}],$$

as illustrated by the top vector coming from the origin in Fig. 5. The n'th pulse, with timing deviation $\tilde{\epsilon}^i(n')$, initiates a corresponding unit vector of phase $-2\pi\tilde{\epsilon}^i(n')$, which adds to the vector representing the previous value of the timing wave to give a resultant timing wave vector of amplitude $\tilde{A}_{n'}$, phase $-2\pi\tilde{\epsilon}^o(n')$. Since the relevant angles are small, the result given in (35) may be readily obtained.

Equations (35) and (25) are completely equivalent statements describing the timing response of a single repeater using a resonant circuit to generate the timing wave. They are derivable from each other without any approximation and are both subject to the restrictions given in (20) and (21). They provide somewhat different representations of the repeater as a linear time-varying transducer to the timing deviation. For the special case of a periodic pulse pattern with every Mth pulse present and $\tilde{A}_{n'}$ constant, given by (31), in which the repeater becomes a linear invariant transducer, (35) is equivalent to (32). We shall show in Section 2.2 that (35) is in the appropriate form for the approximate treatment of a chain of repeaters with zero tuning error for random and general periodic pulse patterns, by neglecting the variation of the timing wave amplitude.

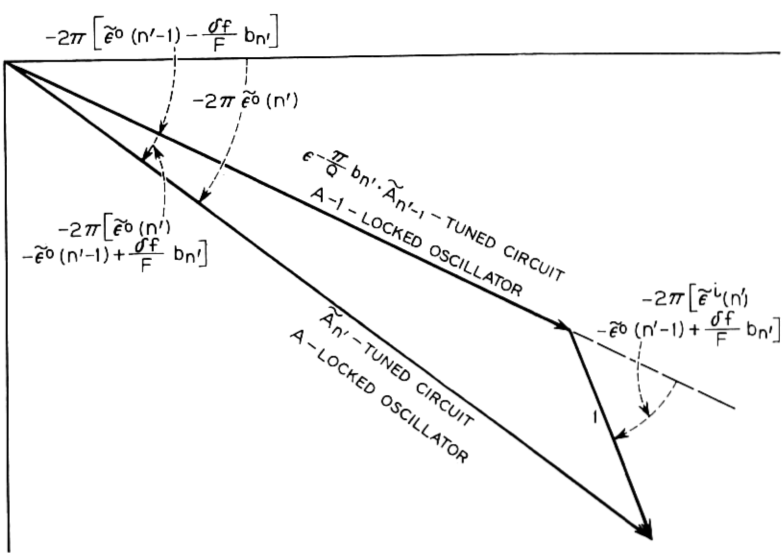

Fig. 5 — Change in output timing deviation during the interval between two consecutive signal pulses.

2.2 *Approximate Timing Response for Random and General Periodic Pulse Patterns — Tuned Circuit Timing Filter, Zero Tuning Error.*

W. R. Bennett has pointed out that, for a high enough Q, the variation of timing wave amplitude in a tuned circuit will be very small, and has used this fact in his treatment of a single repeater.[7] Making use of this fact, we show how a repeater using a tuned circuit timing filter with zero tuning error may be treated as an approximately invariant linear transducer to the timing deviations.

We might first think that we could set $\tilde{A}_{n'}$ equal to its average value in (25), but this does not yield a satisfactory approximation to the behavior of a repeater. As pointed out in the discussion following (17), the numerator of (25) is simply the quadrature component and the denominator is the in-phase component of the timing wave, referred to the transient started by the present input pulse. For zero tuning error, both numerator and denominator decay exponentially between signal pulses, so that the timing wave phase remains constant; this is apparent either on physical grounds or on examination of (24) and (25). If we arbitrarily set $\tilde{A}_{n'}$ constant in (25), we should obtain an exponentially decreasing phase deviation between signal pulses, rather than a constant deviation. Further, for zero tuning error a constant dc input timing deviation produces an equal constant output timing deviation for any arbitrary pulse pattern. If, however, we set $\tilde{A}_{n'}$ constant in (25) the output timing deviation fluctuates about its average value, so that it would now contain false ac components. Since it turns out that the timing deviation in a repeater chain consists principally of low-frequency components, an approximate treatment of a single repeater should yield accurate results for dc and slowly varying input timing deviations.

The correct approach to this problem has been indicated by J. L. Kelly, Jr. If the tuning error is zero, a useful approximation to the behavior of a repeater is obtained by setting the timing wave amplitude equal to its average value in (35). Thus, setting $\delta f = 0$, we have

$$\tilde{\epsilon}^o(n') - \tilde{\epsilon}^o(n' - 1) = \frac{1}{A}\,[\tilde{\epsilon}^i(n') - \tilde{\epsilon}^o(n' - 1)], \qquad A = \langle \tilde{A}_{n'} \rangle, \quad (36)$$

where the brackets $\langle\ \rangle$ indicate an ensemble average for a random pulse pattern, and A_n is given by (24). Referring to Fig. 5 and to the discussion following (35), in this approximation the phase deviation of the timing wave remains constant between signal pulses, as it should. A dc input timing deviation produces an equal dc output timing deviation, with no false ac components; the error in this approximation should be small for the low-frequency components that form the major part of the timing deviation.

Equation (36) describes a repeater with a tuned circuit timing filter with zero tuning error as a linear invariant transducer to the timing deviation, greatly simplifying the analysis of a chain of repeaters. For random and general periodic pulse patterns the variation in the timing wave amplitude will modify the timing behavior somewhat; for periodic pulse patterns with every Mth pulse present (including all pulses present) (36) is exact. The average timing wave amplitude A and its standard deviation are determined in Section 2.4; A is proportional to the average number of pulses present and to the tuned-circuit Q. We shall show in Section 2.3 that (36) gives an exact description of a locked oscillator with zero tuning error for any arbitrary pulse pattern.

2.3 *Locked Oscillator Timing Circuit*

In the preceding section the approximate timing response of a repeater with zero tuning error was found by setting $\tilde{A}_{n'}$ equal to its average value in (35). It is natural to ask what kind of system is described exactly by (35) with $\tilde{A}_{n'}$ equal to a constant A (δf is now not necessarily equal to zero).

Such a system may be regarded as an idealized locked oscillator. The discussion of Fig. 5 immediately following (35) remains applicable except that the amplitude of the timing wave no longer decays between signal pulses but has a constant length $(A - 1)$. The unit vector initiated by the n'th pulse adds to the vector representing the previous value of the timing wave, causing an abrupt change in its phase as before and increasing its length momentarily to A. However, we now assume that the timing wave amplitude returns to its constant value $(A - 1)$ before the next signal pulse arrives. In a practical locked oscillator with a weak synchronizing signal, the transients started by the signal pulses will be so much smaller than the steady-state oscillator wave that the timing wave amplitude will be essentially constant, independent of the pulse pattern.

Setting $\tilde{A}_{n'} = A$ in (35), we have an exact expression for the timing behavior of a locked oscillator with tuning error (subject of course to the usual small angle restrictions) for any arbitrary pulse pattern:

$$\tilde{\epsilon}^o(n') - \tilde{\epsilon}^o(n' - 1) = \frac{1}{A}\left[\tilde{\epsilon}^i(n') - \tilde{\epsilon}^o(n' - 1) + \frac{\delta f}{F} b_{n'}\right] - \frac{\delta f}{F} b_{n'} . \quad (37)$$

The parameter A is now determined by the ratio of the amplitudes of the oscillator wave and the synchronizing signal. Equation (37) shows that a locked oscillator timing circuit with tuning error may be treated rigorously as a linear transducer to the timing deviation for any arbitrary pulse pattern; the equivalent linear transducer has two inputs, $\tilde{\epsilon}^i(n')$ and $b_{n'}$.

Comparison with the results of the preceding section shows that for zero tuning error (37) also describes the approximate timing response of a tuned circuit timing filter, where A is now the average timing wave amplitude, as in (36). Unfortunately, (37) does *not* provide a valid approximation to the behavior of a tuned circuit when tuning error is present. This is best shown by comparing a tuned circuit and the corresponding locked oscillator in a simple case.

Consider a random pulse pattern where the probability that any time slot contains a pulse is very close to one, independently for each time slot. Only rarely will vacant time slots lie close together in this pulse pattern, and so it suffices to study the effect of a single missing pulse, as shown in Fig. 6(a). For $n' \leqq 0$ both the output timing deviation and the timing wave amplitude are constant for both the locked oscillator and the tuned circuit. Making A for the locked oscillator equal to the

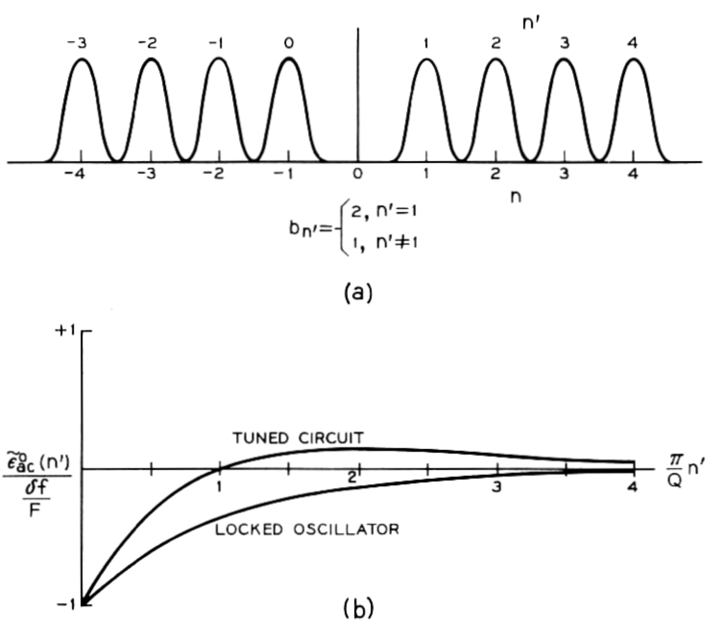

Fig. 6 — Output timing deviation for locked oscillator and tuned-circuit timing filters with tuning error, for single missing pulse: (a) pulse pattern; (b) output timing deviations.

steady-state timing wave amplitude for the tuned circuit, we have for both cases, from (24) and (35) to (37):

$$\tilde{A}_{n'} = A = \sum_{k=0}^{\infty} e^{-(\pi/Q)k} = \frac{1}{1 - e^{-\pi/Q}}; \qquad n' \leqq 0, \tag{38}$$

$$\tilde{\epsilon}^{o}(n') = \tilde{\epsilon}_{dc}{}^{o} = -\frac{\delta f}{F} A \left(1 - \frac{1}{A}\right) = -\frac{\delta f}{F} \frac{e^{-\pi/Q}}{1 - e^{-\pi/Q}}; \qquad n' \leqq 0. \tag{39}$$

The transient or ac component of the output timing deviation, $\tilde{\epsilon}_{ac}{}^{o}(n')$, is of principal interest, where

$$\tilde{\epsilon}^{o}(n') = \tilde{\epsilon}_{dc}{}^{o} + \tilde{\epsilon}_{ac}{}^{o}(n'). \tag{40}$$

Then, from (35), we have

$$\begin{aligned} \tilde{\epsilon}_{ac}{}^{o}(n') &- \tilde{\epsilon}_{ac}{}^{o}(n' - 1) \\ &= -\frac{1}{\tilde{A}_{n'}} \tilde{\epsilon}_{ac}{}^{o}(n' - 1) + \frac{\delta f}{F} \frac{A - \tilde{A}_{n'}}{\tilde{A}_{n'}}; \qquad n' > 1, \\ \tilde{A}_{n'} &= \begin{cases} A & ; \quad \text{locked oscillator} \\ A - e^{-(\pi/Q)n'}; & \quad \text{tuned circuit.} \end{cases} \end{aligned} \tag{41}$$

The initial condition for the difference equation (41) is given by

$$\tilde{\epsilon}_{ac}{}^{o}(1) = -\frac{\delta f}{F}\left[2 - \frac{1}{\tilde{A}_1}(A + 1)\right]. \tag{42}$$

For the locked oscillator, the second term on the right-hand side of (41) vanishes and, subject to (42), we find

$$\tilde{\epsilon}_{ac}{}^{o}(n') = -\frac{\delta f}{F} e^{-(\pi/Q)n'}; \qquad \text{locked oscillator.} \tag{43}$$

For the tuned circuit, the second term of (41) no longer vanishes, but adds an inhomogeneous term to the solution. From (41), $A - \tilde{A}_{n'} = e^{-(\pi/Q)n'}$ and, approximating the $\tilde{A}_{n'}$ occurring in the denominators of (41) and (42) by A, we find

$$\tilde{\epsilon}_{ac}{}^{o}(n') = -\frac{\delta f}{F} e^{-(\pi/Q)n'}[(2 - e^{-\pi/Q}) - (1 - e^{-\pi/Q})n']; \qquad \text{tuned circuit.} \tag{44}$$

For the high-Q case, this may be further simplified:

$$\tilde{\epsilon}_{ac}{}^{o}(n') = -\frac{\delta f}{F} e^{-(\pi/Q)n'}\left[1 - \frac{\pi}{Q} n'\right]; \qquad \text{tuned circuit, } Q \gg 1. \tag{45}$$

These results may alternately be obtained by inspection from the vector diagram of Fig. 5.

Equations (43) and (45) are plotted in Fig. 6(b). For the tuned circuit, $\tilde{\epsilon}_{ac}{}^{o}(n')$ initially approaches zero twice as fast as for the locked oscillator. Consequently, a tuned circuit and the corresponding locked oscillator are approximately equivalent only for zero tuning error; when tuning error is present their behavior is no longer similar. In the present paper we are able to treat the effect of tuning error in a chain of repeaters only for locked-oscillator timing circuits.

2.4 *Average Value and Fluctuation of the Timing Wave Amplitude*

We now determine the average value and standard deviation of the timing wave amplitude $\tilde{A}_{n'}$, denoted by $A = \langle \tilde{A}_{n'} \rangle$ and σ_A respectively, for a tuned circuit. The average value is needed in (36), which gives the approximate timing response for a tuned circuit timing filter with zero tuning error. The standard deviation indicates the magnitude of the departure of $\tilde{A}_{n'}$ from its average value A, which is neglected in this approximate analysis.

Consider a random pulse pattern with the probability p that any time slot contains a pulse, independently for each time slot. Equation (24) gives A_n; for the a_{n-k}'s we have

$$a_n = 1,$$

$$\left.\begin{aligned} \Pr[a_{n-k} = 1] &= p \\ \Pr[a_{n-k} = 0] &= 1 - p \end{aligned}\right\} k > 1. \tag{46}$$

The different a's are independent. Then, from (24), we have

$$A = \langle \tilde{A}_{n'} \rangle = \sum_{k=0}^{\infty} \langle a_{n-k} \rangle e^{-(\pi/Q)k} = 1 + \frac{pe^{-\pi/Q}}{1 - e^{-\pi/Q}}, \tag{47}$$

$$\begin{aligned} \langle \tilde{A}_{n'}{}^2 \rangle &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \langle a_{n-k} a_{n-l} \rangle e^{-(\pi/Q)(k+l)} \\ &= 1 + 2p \frac{e^{-\pi/Q}}{1 - e^{-\pi/Q}} + \left(\frac{pe^{-\pi/Q}}{1 - e^{-\pi/Q}} \right)^2 + p(1-p) \frac{e^{-2\pi/Q}}{1 - e^{-2\pi/Q}}, \end{aligned} \tag{48}$$

$$\sigma_A{}^2 = \langle \tilde{A}_{n'}{}^2 \rangle - \langle \tilde{A}_{n'} \rangle^2 = p(1-p) \frac{e^{-2\pi/Q}}{1 - e^{-2\pi/Q}}. \tag{49}$$

For the high-Q case

$$A = \frac{pQ}{\pi}, \quad {\sigma_A}^2 = p(1 - p)\frac{Q}{2\pi}, \quad \left(\frac{\sigma_A}{A}\right)^2 = \frac{1 - p}{p}\frac{\pi}{2Q}; \qquad Q \gg 1. \quad (50)$$

This agrees with the results of a similar calculation made by W. R. Bennett.[7]

The timing-wave amplitude will be close to its average value with high probability if

$$\frac{\sigma_A}{A} \ll 1, \qquad Q \gg \frac{1 - p}{p}. \qquad (51)$$

The smaller the probability that a time slot contains a pulse, the larger Q must be in order for the variation in the timing wave amplitude to be small.

For any periodic pulse pattern, $\tilde{A}_{n'}$ is easily determined as a periodic function of n' from (24). For a high enough Q the average timing wave amplitude A is again given by (50), where p is now the average number of time slots containing pulses. In the special case of a periodic pattern with every Mth pulse present, $\tilde{A}_{n'}$ is of course constant, given by (31) or (33).

2.5 *Tuned Circuit Excited by Raised Cosine Pulses*

Up to now we have considered only the case of a timing filter excited by impulses. However, in the simplest repeaters the signal pulses themselves or their rectified envelopes (in baseband or carrier systems, respectively) would be used to drive the timing circuit. The finite width of these driving pulses may add an additional component of timing noise at the output of each repeater. We now consider the behavior of a tuned circuit excited by raised cosine pulses, which approximate the pulse shape that might be used in a practical system.

A driving pulse centered at $t = 0$ is given by

$$p(sFt) = \frac{1}{2}(1 + \cos 2\pi sFt); \qquad |Ft| < \frac{1}{2s}, \qquad (52)$$

where F is the pulse repetition frequency and s is a parameter determining the pulse width. For $s = 1$ adjacent pulses are just resolved; for $s = \frac{2}{3}$ adjacent pulses overlap such that the amplitude half way between sample points is equal to one-half the peak pulse amplitude. These two cases are illustrated in Fig. 7. The complex response $P(t)$ of a tuned cir-

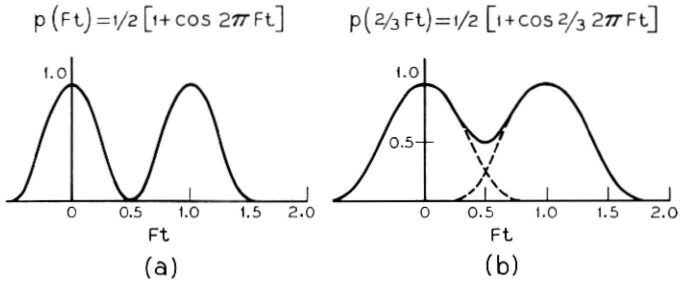

Fig. 7 — Driving pulses for timing circuit: (a) pulses resolved; (b) pulses overlapping.

cuit to this pulse is

$$P(t) = \int_{-\infty}^{t} p(sF\tau)H(t - \tau)\, d\tau, \tag{53}$$

where $H(t)$, given in (1) and (4), is the complex impulse response of the tuned circuit. Thus,

$$P(t) = \frac{1}{2F} I_s(Ft)H(t), \tag{54}$$

where

$$I_s(Ft) = \begin{cases} 0; & Ft < -\dfrac{1}{2s} \\ \displaystyle\int_{-1/2s}^{Ft} (1 + \cos 2\pi s\tau)e^{+(\pi/Q)(1+\delta f/F)\tau} e^{-j2\pi(1+\delta f/F)\tau}\, d\tau; & \\ & -\dfrac{1}{2s} < Ft < +\dfrac{1}{2s} \\ I_s\left(\dfrac{1}{2s}\right); & +\dfrac{1}{2s} < Ft. \end{cases} \tag{55}$$

The pulse train applied to the tuned circuit now becomes [from (5) through (9)]:

$$g(t) = \sum_{n=-\infty}^{\infty} a_n p\{s[Ft - n - \epsilon^i(n)]\}. \tag{56}$$

The tuned-circuit response $G(t)$ to all pulses up to and including the pulse in the mth time slot is given by

$$G(t) = \sum_{n=-\infty}^{m} a_n P[t - nT - \epsilon^i(n)T], \tag{57}$$

corresponding to (10), and $G(t)$ may again be written in a form similar to (12) and (13):

$$G(t) = \frac{I_s\left(\frac{1}{2s}\right)}{2FC}\left(1 + \frac{j}{2Q}\right) A(t)e^{j2\pi Ft}, \tag{58}$$

$$A(t) = a(t)e^{j\varphi(t)},$$

where, as before, $|A(t)| = a(t)$ is the normalized amplitude and $\angle A(t) = \varphi(t)$ is the phase deviation of the timing wave. Corresponding to (13), we now have for $A(t)$

$$\begin{aligned} A(t) = \sum_{n=-\infty}^{m} & a_n e^{-(\pi/Q)(1+\delta f/F)[Ft-n-\epsilon^i(n)]} \cdot e^{j2\pi\{(1+\delta f/F)[Ft-n-\epsilon^i(n)]-Ft\}} \\ & + a_m \left[\frac{I_s[Ft - m - \epsilon^i(m)]}{I_s\left(\frac{1}{2s}\right)} - 1\right] e^{-(\pi/Q)(1+\delta f/F)[Ft-m-\epsilon^i(m)]} \\ & \cdot e^{j2\pi\{(1+\delta f/F)[Ft-m-\epsilon^i(m)]-Ft\}}; \end{aligned} \tag{59}$$

$$(m - 1) + \epsilon^i(m - 1) + \frac{1}{2s} < Ft < (m + 1) + \epsilon^i(m + 1) - \frac{1}{2s},$$

where t is restricted such that only the pulse in the mth time slot differs from 0.

We now assume that the phase of the timing wave $\varphi(t)$ at the negative-going zero crossing following the peak of each received pulse determines the timing deviation of the corresponding timing pulse. In the analysis for impulse excitation in Section 2.1 the timing wave phase was evaluated just after the reception of a driving impulse, since a different choice would introduce only a small dc error in the output timing deviation. In contrast, with driving pulses of finite width the choice of time at which $\varphi(t)$ is evaluated has an appreciable effect on the last term of (59), which gives the additional output timing deviation due to finite pulse width. Thus, evaluating (59) at

$$Ft = F\left(t_m + \frac{T}{4}\right) = m + \epsilon^i(m) + \frac{1}{4}, \tag{60}$$

where t_n is given in (9), we obtain

$$A\left(t_m + \frac{T}{4}\right) = [e^{-(\pi/Q)(1+\delta f/F)1/4} e^{j(\delta f/F)\pi/2}]$$
$$\cdot e^{-j2\pi\epsilon^i(m)} \left\{ \sum_{n=-\infty}^{m} a_n e^{-(\pi/Q)(1+\delta f/F)[(m-n)+\epsilon^i(m)-\epsilon^i(n)]} \right. \tag{61}$$
$$\left. \cdot e^{j2\pi\{\epsilon^i(m)-\epsilon^i(n)+(\delta f/F)[(m-n)+\epsilon^i(m)-\epsilon^i(n)]\}} + a_m \left[\frac{I_s\left(\frac{1}{4}\right)}{I_s\left(\frac{1}{2s}\right)} - 1\right]\right\},$$

corresponding to (16). In order for the restriction of (59) to be satisfied s must be greater than $\frac{2}{3}$. The terms in the first bracket arise from the fact that we have determined the phase one-quarter of a pulse period after the peak of the received pulse, rather than just after the peak as in Section 2.1; they represent a small decrease in amplitude and a small dc phase shift, and may be neglected. Then, making use of the discussion in Section 2.1 and assuming as before that the alignment deviation $\alpha(n) = \epsilon^o(n) - \epsilon^i(n)$ remains small, we obtain from (17) and (61) the normalized timing wave amplitude and output timing deviation (with complete retiming):

$$A_n = \sum_{k=0}^{\infty} a_{n-k} e^{-(\pi/Q)k}, \tag{62}$$

$$\epsilon^o(n) = \frac{\sum_{k=0}^{\infty} a_{n-k} e^{-(\pi/Q)k} \left[\epsilon^i(n-k) - \frac{\delta f}{F} k\right]}{A_n} - w_s \frac{a_n}{A_n}, \tag{63}$$

where w_s depends on the pulse width and is given by

$$w_s = \frac{1}{2\pi} \operatorname{Im}\left[\frac{I_s\left(\frac{1}{4}\right)}{I_s\left(\frac{1}{2s}\right)} - 1\right]. \tag{64}$$

These results are again subject to the restrictions of (20) and (21). The following approximate results for the high-Q case may be obtained from

(55):

$s = 1$,	$s = \frac{2}{3}$,
$w_1 = 0.0253 + \frac{0.0325}{Q}$,	$w_{2/3} = 0.1038 + \frac{0.349}{Q}$,
$I_1\left(\frac{1}{4}\right) = 0.534 + \frac{0.185}{Q} + j\left[0.0795 - \frac{0.298}{Q}\right]$,	$I_{2/3}\left(\frac{1}{4}\right) = 0.430 + \frac{0.235}{Q} + j\left[0.166 - \frac{0.429}{Q}\right]$,
$I_1\left(\frac{1}{2}\right) = 0.5\left[1 - j\frac{0.75}{Q}\right]$.	$I_{2/3}\left(\frac{3}{4}\right) = 0.255\left[1 - j\frac{2.3}{Q}\right]$.

Except for the last term, (63) is identical to (25) for impulse excitation; the last term represents the additional output timing deviation caused by the finite pulse width. This timing deviation is inversely proportional to the instantaneous timing wave amplitude, and is thus quite similar to the effect of amplitude-to-phase conversion in the limiter of the timing system.[2, 8] In both cases an identical timing deviation will be added at the output of every repeater of the chain, since the pulse pattern is the same for every repeater. This additional timing deviation is caused by the fact that the driving pulse is not zero when the timing wave goes through zero. However, the zero crossings corresponding to vacant time slots are not disturbed; W. M. Goodall has pointed out that inverting the pulse pattern in each repeater will therefore eliminate this source of timing noise. If the driving pulse is short enough so that it falls to zero before the zero crossing of the timing wave occurs, no additional timing deviation will be produced; in particular, the driving pulses may be impulses as above or square pulses of length $T/2$.[7] Finally, a weakly coupled locked oscillator will not be affected by the finite width of the synchronizing pulses.

III. FOURIER ANALYSIS OF DISCRETE FUNCTIONS

3.1 *Transforms of Discrete Functions*

The timing deviation $\epsilon(n)$ or $\bar{\epsilon}(n')$ and the various other quantities discussed above may be called discrete real functions, since they are defined only for integral values of their argument. The Fourier analysis of discrete functions is quite similar to that for continuous functions and is summarized below.

The transform $X(f)$ of a discrete function $x(n)$ is defined by[9, 10]

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn}. \tag{65}$$

The inverse transform is then

$$x(n) = \int_{-1/2}^{+1/2} X(f)e^{+j2\pi fn}\, df. \tag{66}$$

This transformation is closely related to the z-transform[11, 12] or generating function. Equations (65) and (66) may be regarded as a statement of the Fourier series theorem, in which the usual roles of function and transform have been reversed.

The following table [Equation (67)] summarizes some of the properties of discrete transforms that will be of use; $x_1(n)$ and $x_2(n)$ are any two different discrete functions with corresponding transforms $X_1(f)$ and $X_2(f)$:

Function	*Transform*	
$x_1(n)$	$X_1(f)$	(67a)
$x_2(n)$	$X_2(f)$	(67b)
$x_1(n+k)$	$e^{j2\pi fk}X_1(f)$	(67c)
$\Delta x_1(n) = x_1(n+1) - x_1(n)$	$(e^{j2\pi f} - 1)\, X_1(f)$ $= 2je^{j\pi f} \sin \pi f X_1(f)$	(67d)
$x_1(n) \otimes x_2(n) = \sum_{k=-\infty}^{\infty} x_1(n-k)x_2(k)$ $= \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$	$X_1(f)X_2(f)$	(67e)
$x_1(n)x_2(n)$	$X_1(f) \otimes X_2(f)$ $= \int_{-1/2}^{+1/2} X_1(f-\tau)X_2(\tau)\, d\tau$ $= \int_{-1/2}^{+1/2} X_1(\tau)X_2(f-\tau)\, d\tau$	(67f)
$\varphi_1(k) = \sum_{n=-\infty}^{\infty} x_1(n)x_1(n+k)$	$E_1(f) = X_1(f)X_1^*(f)$ $= \lvert X_1(f) \rvert^2$.†	(67g)

† The * denotes the complex conjugate.

In (67g) the autocorrelation function $\varphi(k)$ and the energy spectrum $E(f)$ of a discrete function $x(n)$ have been defined; the fact that they are transforms of each other follows from the convolution theorem (67e) and the fact that $x(n)$ is real. The total energy E of a discrete function $x(n)$ is defined by

$$E = \sum_{n=-\infty}^{\infty} x^2(n). \tag{68}$$

Then, from (67g),

$$E = \varphi(0) = \int_{-1/2}^{1/2} E(f)\, df = \int_{-1/2}^{1/2} | X(f) |^2\, df. \tag{69}$$

The results of the present section apply only to discrete functions having a finite energy E as defined in (68), so that their transforms exist. Discrete random functions, which have infinite energy but a finite average power, will be discussed in Section 3.3.

3.2 *Discrete Transducers*

A discrete linear invariant transducer may be characterized by its impulse response. A discrete unit impulse at $n = n_0$ is defined by

$$\delta(n - n_0) = \begin{cases} 1; & n = n_0 \\ 0; & n \neq n_0, \end{cases} \tag{70}$$

corresponding to the delta function in the continuous case. Let $h(n)$ be the output of a discrete linear invariant transducer for a unit impulse input at the origin; for stable systems

$$h(n) = 0; \qquad n < 0. \tag{71}$$

Then the output $x^o(n)$ for an arbitrary input $x^i(n)$ is given by the convolution of the input and the impulse response $h(n)$ of the transducer:

$$\begin{aligned} x^o(n) &= \sum_{k=-\infty}^{n} x^i(k)h(n-k) \\ &= \sum_{k=0}^{\infty} x^i(n-k)h(k). \end{aligned} \tag{72}$$

Let the transforms according to (65) and (66) of $h(n)$, $x^i(n)$ and $x^o(n)$ be $H(f)$, $X^i(f)$ and $X^o(f)$ respectively; $H(f)$ is the frequency response or transfer function of the transducer. Then, from (67e), we have

$$X^o(f) = H(f)X^i(f). \tag{73}$$

If the input consists of a single exponential of frequency f,

$$x^i(n) = A_i e^{j2\pi f n}, \tag{74}$$

then the output is an exponential of the same frequency,

$$\begin{aligned} x^o(n) &= A_o e^{j2\pi f n}, \\ A_o &= H(f) A_i, \end{aligned} \tag{75}$$

as in the continuous case.

Finally, consider two cascaded discrete transducers with individual impulse responses $h_1(n)$ and $h_2(n)$ and corresponding transfer functions $H_1(f)$ and $H_2(f)$. The over-all impulse response $h(n)$ is equal to the convolution of $h_1(n)$ and $h_2(n)$, and the over-all transfer function $H(f)$ is equal to the product of $H_1(f)$ and $H_2(f)$, as in the continuous case:

$$h(n) = \sum_{k=0}^{n} h_1(n-k) h_2(k), \tag{76}$$

$$H(f) = H_1(f) H_2(f). \tag{77}$$

3.3 *Discrete Random Functions*

A discrete stationary random function $x(n)$ will have infinite energy but a finite average power P given by

$$P = \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x^2(n). \tag{78}$$

The covariance $\rho(k)$ is defined as

$$\rho(k) = \langle x(n) x(n+k) \rangle, \tag{79}$$

where $\langle \ \rangle$ denotes an average over the ensemble. For a stationary ergodic ensemble the power spectrum $P(f)$ and the covariance $\rho(k)$ are transforms of each other, according to (65) and (66):

$$P(f) = \sum_{k=-\infty}^{\infty} \rho(k) e^{-j2\pi f k}, \tag{80}$$

$$\rho(k) = \int_{-1/2}^{1/2} P(f) e^{+j2\pi f k} \, df. \tag{81}$$

The average power P is the integral of the power spectrum

$$P = \int_{-1/2}^{1/2} P(f) \, df. \tag{82}$$

A white noise of power N_0 has a covariance

$$\rho(k) = N_0\delta(k) = \begin{cases} N_0 ; & k = 0 \\ 0; & k \neq 0. \end{cases} \tag{83}$$

Thus the power spectrum is

$$P(f) = N_0 . \tag{84}$$

This process is called a white noise because the power spectrum is constant with frequency. The values of this noise for different values of the independent variable n are uncorrelated; this random process is used to describe the equivalent timing noise added at the input of each repeater, as discussed in Section I.

IV. TIMING IN A CHAIN OF REPEATERS WITH ZERO TUNING ERROR AND COMPLETE RETIMING

4.1 *Introduction*

In this section we determine the timing, spacing and alignment noise power spectra and total powers (mean square values) caused by the input noise at each repeater for a chain of repeaters with zero tuning error and complete retiming. The repeaters may employ either tuned-circuit or locked-oscillator timing circuits.

From (36) or (37) the response of each repeater is given by

$$\begin{gathered} \tilde{\epsilon}^o(n') - \tilde{\epsilon}^o(n' - 1) = \frac{1}{A} [\tilde{\epsilon}^i(n') - \tilde{\epsilon}^o(n' - 1)], \\ A = \langle \tilde{A}_{n'} \rangle. \end{gathered} \tag{85}$$

For locked oscillators, (85) is exact. The normalized timing wave amplitude A is equal to the ratio of the amplitude of the oscillator wave to the amplitude of the transient started by a single signal pulse.

For tuned circuits we assume that the driving pulses for the timing circuit are short enough so that no additional output timing noise results from the finite pulse width, as discussed in Section 2.5, and we study only the effects of input noise. Equation (85) will be exact only for periodic pulse patterns with every Mth pulse present (including all pulses present); for random or general periodic pulse patterns it gives a good approximation for the response of a repeater. The normalized timing wave amplitude $\tilde{A}_{n'}$ is given by (24); A is given by (31) or (50).

Let the discrete transforms of the input and output timing deviations

$\tilde{\epsilon}^i(n')$ and $\tilde{\epsilon}^o(n')$ be $\tilde{C}^i(f)$ and $\tilde{C}^o(f)$ respectively, where the symbol ~ indicates that the transforms have been taken with respect to the independent variable n' [Fig. 4(a)]. Then, transforming (85) by the use of (67), we obtain for the transfer function $\tilde{H}(f)$ of a single repeater

$$\tilde{H}(f) = \frac{\tilde{C}^o(f)}{\tilde{C}^i(f)} = \frac{1/A}{1 - (1 - 1/A)e^{-j2\pi f}}. \tag{86}$$

It is convenient to define $\tilde{Q}$ as the effective "Q" of the timing circuit to the timing deviations, as a function of the average timing wave amplitude A:

$$A = \frac{1}{1 - e^{-\pi/\tilde{Q}}}. \tag{87}$$

In the high-$\tilde{Q}$ case,

$$\tilde{Q} = \pi A, \qquad \tilde{Q} \gg 1. \tag{88}$$

For tuned-circuit timing filters $\tilde{Q}$ may be conveniently expressed in terms of the tuned circuit Q. For a periodic pulse pattern with every Mth pulse present, from (31),

$$\tilde{Q} = \frac{Q}{M}. \tag{89}$$

For all pulses present, $M = 1$ and $\tilde{Q} = Q$. For random or general periodic pulse patterns (50) yields, for the high-Q case,

$$\tilde{Q} = pQ, \qquad \tilde{Q} \gg 1, \tag{90}$$

where p is either the probability that a time slot contains a pulse or the average number of time slots containing pulses.

In terms of $\tilde{Q}$, the frequency response of a repeater becomes

$$\tilde{H}(f) = \frac{1 - e^{-\pi/\tilde{Q}}}{1 - e^{-\pi/\tilde{Q}}e^{-j2\pi f}}. \tag{91}$$

Taking the inverse transform of $\tilde{H}(f)$, the impulse response $\tilde{h}(n')$ of a single repeater is

$$\tilde{h}(n') = (1 - e^{-\pi/\tilde{Q}})e^{-(\pi/\tilde{Q})n'}; \qquad n' \geq 0. \tag{92}$$

Finally, the square of the absolute magnitude of the frequency response of a single repeater may be written

$$|\tilde{H}(f)|^2 = \frac{1}{1 + \operatorname{csch}^2 \dfrac{\pi}{2\tilde{Q}} \sin^2 \pi f}. \tag{93}$$

Equation (93) is plotted in Fig. 8(a) for the high-$\tilde{Q}$ case. The low-frequency components of the timing deviation are transmitted with little loss in amplitude, and the high-frequency components are substantially suppressed in a single repeater. The 3-db bandwidth is given by

$$f_{3\text{db}} = \frac{1}{2\tilde{Q}}; \qquad \tilde{Q} \gg 1. \tag{94}$$

4.2 *Impulse and Frequency Response of a Chain of Repeaters*

The transfer function of K repeaters in cascade, $\tilde{H}_K(f)$, is equal to the Kth power of the transfer function of a single repeater. From (77)

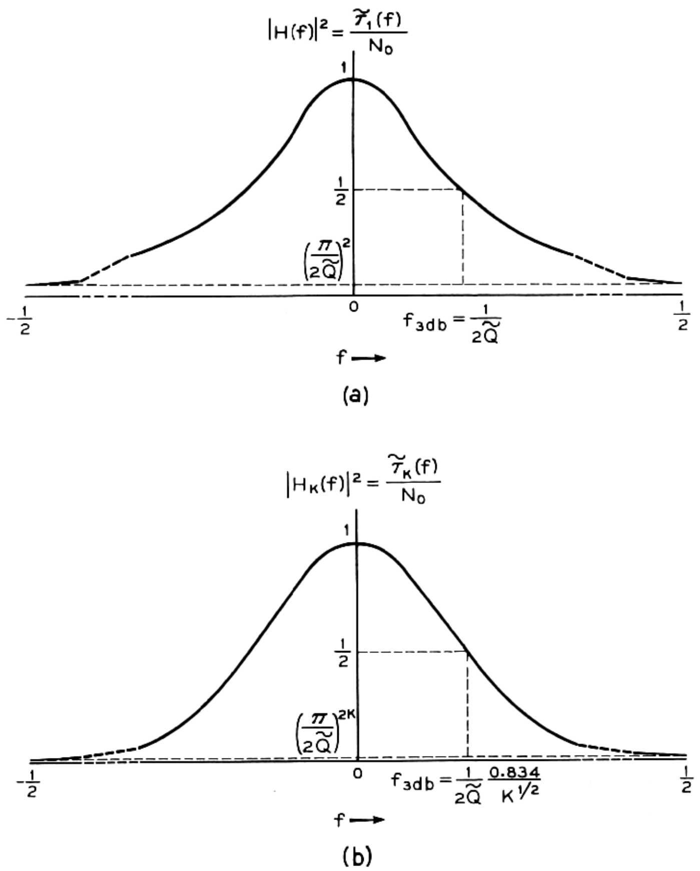

Fig. 8 — Transfer function for repeaters with zero tuning error and complete retiming, with $\tilde{Q} \gg 1$: (a) single repeater; (b) k cascaded repeaters, $k \gg 1$.

and (91),

$$\tilde{H}_K(f) = \tilde{H}^K(f) = \left[\frac{1 - e^{-\pi/\tilde{Q}}}{1 - e^{-\pi/\tilde{Q}}e^{-j2\pi f}}\right]^K. \tag{95}$$

From (93),

$$|\tilde{H}_K(f)|^2 = |\tilde{H}(f)|^{2K} = \left[\frac{1}{1 + \operatorname{csch}^2 \frac{\pi}{2\tilde{Q}} \sin^2 \pi f}\right]^K. \tag{96}$$

Equation (96) is plotted in Fig. 8(b); the 3-db bandwidth for the high-$\tilde{Q}$ case is

$$f_{3\text{db}} = \frac{1}{2\tilde{Q}} \sqrt{2^{1/K} - 1}; \qquad \tilde{Q} \gg 1. \tag{97}$$

For a long chain of repeaters (97) becomes

$$f_{3\text{db}} = \frac{1}{2\tilde{Q}} \frac{0.834}{K^{1/2}}; \qquad \tilde{Q} \gg 1, \qquad K \gg 1. \tag{98}$$

The bandwidth of a chain of repeaters to the timing deviation varies inversely as the square root of the number of repeaters.

The impulse response for K cascaded repeaters, $\tilde{h}_K(n')$, may be found either by taking the inverse transform of the frequency response, given in (95), or by finding the $(K - 1)$-fold convolution of the impulse response for a single repeater, given in (92). Using the latter method,

$$\tilde{h}_K(n') = (1 - e^{-\pi/\tilde{Q}})^K e^{-(\pi/\tilde{Q})n'} \underbrace{\sum_{n_{K-1}=0}^{n'} \sum_{n_{K-2}=0}^{n_{K-1}} \cdots \sum_{n_2=0}^{n_3} \sum_{n_1=0}^{n_2}}_{K-1 \text{ summations}} 1. \tag{99}$$

This expression is easily evaluated using functions occurring in the theory of difference equations.[13] The function $k^{[m]}$ is defined as follows:

$$k^{[m]} = k(k - 1)(k - 2) \cdots (k - m + 1) = \frac{\Gamma(k + 1)}{\Gamma(k - m + 1)}. \tag{100}$$

The brackets around the superscript m indicate that it is not to be regarded as an exponent. Note that

$$k^{[0]} = 1. \tag{101}$$

This function behaves with respect to difference and summation operators in a similar manner to the power function x^m with respect to differential and integral operators. In particular,

$$\Delta k^{[m]} = (k + 1)^{[m]} - k^{[m]} = mk^{[m-1]},$$
$$\sum_{k=M}^{N} k^{[m]} = \left[\frac{k^{[m+1]}}{m+1}\right]_M^{N+1}. \tag{102}$$

Substituting these relations into (99), we obtain

$$\tilde{h}_K(n') = (1 - e^{-\pi/\tilde{Q}})^K e^{-(\pi/\tilde{Q})n'} \frac{\Gamma(n' + K)}{\Gamma(K)\Gamma(n' + 1)} \tag{103}$$

for the impulse response of K repeaters in cascade. For $K = 1$ this expression becomes identical to (92). For a single repeater the impulse response decreases exponentially to zero; for a chain of repeaters the impulse response starts from zero, increases to a maximum value, and decreases again to zero.

4.3 *Timing, Spacing and Alignment Noise*

Consider a chain of N repeaters with an independent white timing noise of average power N_0 [(83) and (84)] added at the input of every repeater. The output timing noise power spectrum is equal to the sum of the power spectra produced by the individual noise sources, since the system is linear and the various noise sources are independent.

Let $\tilde{\tau}_K(f)$ be the power spectrum of the timing noise at the output of a chain of K repeaters with a white timing noise of power N_0 introduced at the input of only the first repeater. Then, from (73) and (96),

$$\tilde{\tau}_K(f) = N_0 \mid \tilde{H}(f) \mid^{2K}. \tag{104}$$

This noise spectrum has the same shape as the transfer function of (93) or (96), shown on Fig. 8, so these curves also show $\tilde{\tau}_K(f)$. The timing noise power spectrum $\tilde{T}_N(f)$ at the output of a chain of N repeaters with an independent white timing noise of power N_0 added at the input of each repeater is now

$$\tilde{T}_N(f) = \sum_{K=1}^{N} \tilde{\tau}_K(f) = N_0 \frac{1 - \mid \tilde{H}(f) \mid^{2N}}{\mid \tilde{H}(f) \mid^{-2} - 1}, \tag{105}$$

where $\mid \tilde{H}(f) \mid^2$ is given in (93).

Fig. 9(a) shows $\tilde{T}_N(f)$ for a high $\tilde{Q}$ and a fairly long chain of repeaters. At zero frequency

$$\tilde{T}_N(0) = N_0 N. \tag{106}$$

The low-frequency noise components from each noise source are unattenuated, and their powers add at the output of the repeater chain. The high-frequency components are strongly attenuated in each repeater. The 3-db bandwidth in the high-$\tilde{Q}$ case is

$$f_{3\text{db}} = \frac{1}{2\tilde{Q}} \frac{1.265}{N^{1/2}}; \qquad \tilde{Q} \gg 1, \qquad N \gg 1. \tag{107}$$

As the length of the repeater chain approaches infinity, the timing-noise power spectrum approaches a limiting form

$$\lim_{N\to\infty} \tilde{T}_N(f) = N_0 \frac{\sinh^2 \dfrac{\pi}{2\tilde{Q}}}{\sin^2 \pi f}, \tag{108}$$

as indicated in Fig. 9(a). In this limiting case, for each narrow band the decrease in noise power in passing through each repeater is equal to the noise power added at the input of the next repeater.

The total timing noise power is given simply by the integral of the power spectrum, according to (82). Thus,

$$\begin{aligned} \tilde{\tau}_K &= \int_{-1/2}^{1/2} \tilde{\tau}_K(f)\, df, \\ \tilde{T}_N &= \int_{-1/2}^{1/2} \tilde{T}_N(f)\, df, \end{aligned} \tag{109}$$

where $\tilde{\tau}_K$ is the total timing noise power at the output of a chain of repeaters with noise added at the input of only the first repeater, $\tilde{T}_N$ is the total timing noise power at the output of a chain of repeaters with noise added at the input of every repeater and the power spectra $\tilde{\tau}_K(f)$ and $\tilde{T}_N(f)$ are given by (104) and (105). While exact expressions for $\tilde{\tau}_K$ and $\tilde{T}_N$ can be found, they are so complicated as to be of little use except for short repeater chains, and are not suitable for considering the high-$\tilde{Q}$ case. An asymptotic expression for $\tilde{T}_N$ has been found by S. O. Rice and is derived in the Appendix; a similar analysis has been performed for $\tilde{\tau}_N$, but only the final result is given here. For the high-$\tilde{Q}$ case:

$$\tilde{\tau}_K = N_0 \frac{1}{\pi} \sinh \frac{\pi}{2\tilde{Q}} \frac{\Gamma(\frac{1}{2})\, \Gamma(K - \frac{1}{2})}{\Gamma(K)}; \qquad K \geqq 2, \qquad \tilde{Q} \gg 1, \tag{110}$$

$$\tilde{T}_N = N_0 \frac{2N}{\pi} \sinh \frac{\pi}{2\tilde{Q}} \frac{\Gamma(\frac{1}{2})\, \Gamma(N + \frac{1}{2})}{\Gamma(N + 1)}; \qquad \tilde{Q} \gg 1. \tag{111}$$

For a long chain of repeaters Stirling's approximation yields:

$$\tilde{\tau}_K = N_0 \frac{1}{2\tilde{Q}} \sqrt{\frac{\pi}{K - 1}}; \qquad \tilde{Q} \gg 1, \qquad K \gg 1, \tag{112}$$

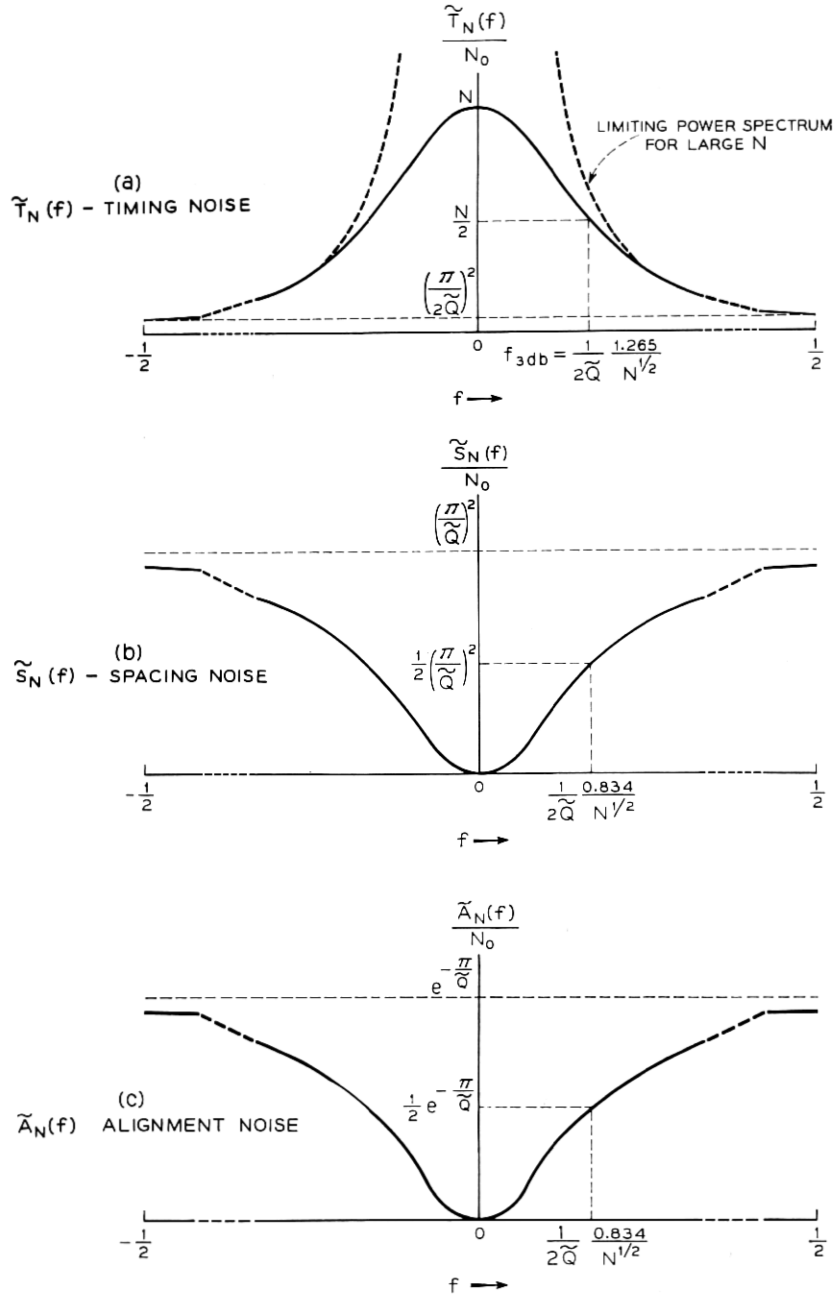

Fig. 9 — Timing (a), spacing (b) and alignment (c) noise power spectra for a chain of N repeaters with zero tuning error and complete retiming, with a white timing noise of power N_0 added at the input of each repeater; $Q \gg 1$, $N \gg 1$.

$$\tilde{T}_N = N_0 \frac{\sqrt{\pi N}}{\tilde{Q}}; \qquad \tilde{Q} \gg 1, \qquad N \gg 1. \tag{113}$$

Finally, for a short repeater chain the exact results are reasonably simple:

$$\begin{aligned} \tilde{T}_1 = \tilde{\tau}_1 &= N_0 \tanh \frac{\pi}{2\tilde{Q}}, \\ \tilde{\tau}_2 &= N_0 \tanh^2 \frac{\pi}{2\tilde{Q}} \operatorname{ctnh} \frac{\pi}{\tilde{Q}}. \end{aligned} \tag{114}$$

For the high-$\tilde{Q}$ case

$$\begin{aligned} \tilde{T}_1 = \tilde{\tau}_1 &= N_0 \frac{\pi}{2\tilde{Q}}; \qquad \tilde{Q} \gg 1, \\ \tilde{\tau}_2 &= N_0 \frac{\pi}{4\tilde{Q}}; \qquad \tilde{Q} \gg 1. \end{aligned} \tag{115}$$

The spacing and alignment noise are easily found in terms of the above results. Let $\tilde{S}_N(f)$ and $\tilde{A}_N(f)$ be the power spectra of the spacing and alignment noise and $\tilde{S}_N$ and $\tilde{A}_N$ be the total spacing and alignment noise powers at the output of a chain of N repeaters with an independent white timing noise of power N_0 added at the input of each repeater. From (27) and (67d),

$$\tilde{S}_N(f) = 4 \sin^2 \pi f \, \tilde{T}_N(f). \tag{116}$$

Substituting (105) and (93),

$$\tilde{S}_N(f) = N_0 \left\{ 4 \sinh^2 \frac{\pi}{2\tilde{Q}} \left[1 - | \tilde{H}(f) |^{2N}\right] \right\}. \tag{117}$$

From (112),

$$\tilde{S}_N = N_0 \left(\frac{\pi}{\tilde{Q}}\right)^2 \left(1 - \frac{1}{2\tilde{Q}} \sqrt{\frac{\pi}{N-1}}\right); \qquad \tilde{Q} \gg 1, \qquad N \gg 1. \tag{118}$$

The second term of (118) is negligible.

From (28),

$$\tilde{A}_N(f) = | \tilde{H}(f) - 1 |^2 [\tilde{T}_{N-1}(f) + N_0]. \tag{119}$$

From (91) and (93),

$$\operatorname{Re} \tilde{H}(f) = \frac{1 + e^{-\pi/\tilde{Q}}}{2} | \tilde{H}(f) |^2 + \frac{1 - e^{-\pi/\tilde{Q}}}{2}, \tag{120}$$

so that

$$| \tilde{H}(f) - 1 |^2 = e^{-\pi/\tilde{Q}}[1 - | \tilde{H}(f) |^2]. \tag{121}$$

Substituting (121) and (105) into (119),

$$\tilde{A}_N(f) = N_0 e^{-\pi/\tilde{Q}}[1 - | \tilde{H}(f) |^{2N}], \tag{122}$$

and from (112)

$$\tilde{A}_N = N_0 e^{-\pi/\tilde{Q}}\left[1 - \frac{1}{2\tilde{Q}}\sqrt{\frac{\pi}{N-1}}\right]; \qquad \tilde{Q} \gg 1, \qquad N \gg 1. \tag{123}$$

The second term of (123) is again negligible.

Figs. 9(b) and 9(c) show $\tilde{S}_N(f)$ and $\tilde{A}_N(f)$. They have the same general shape, but the alignment noise has considerably greater magnitude. For high $\tilde{Q}$ they have an essentially white spectrum, except for very low frequencies; their spectra and total power change very little along the repeater chain. The main contribution to the spacing and alignment deviations comes from the high-frequency components of the input timing deviation, so only the input noise at the Nth repeater gives a significant contribution to these quantities.

The small angle restrictions of (21) were assumed to hold throughout the analysis. Equation (118) shows that if N_0 is small enough so that the first equation of (21) is satisfied for a single repeater, then the second equation of (21) will remain satisfied for the entire chain of repeaters.

4.4 *Discussion*

Each repeater acts as a discrete low-pass filter to the timing deviation, transmitting the low frequencies with little loss and attenuating the high-frequency components. The resulting timing noise at the output of a repeater chain, with noise added at every repeater, contains primarily low frequencies; the total timing noise power grows as the square root of the number of repeaters. The timing deviations of consecutive pulses are strongly correlated. Although the total timing deviation may become large, it changes so slowly that the spacing and alignment deviations remain small. The spacing and alignment noise have an almost white spectrum and remain almost constant along the repeater chain; successive spacing and alignment deviations are almost uncorrelated.

The results of this section are exact for locked-oscillator timing circuits and for tuned circuits with periodic pulse patterns containing every Mth pulse; for tuned circuits with random or general periodic pulse patterns they are only approximate. The exact analysis for a repeater chain with a general periodic pulse pattern is discussed briefly in the following section; the approximate results of the present section are in good agreement with the exact results for this case. The present results also agree with those given by W. R. Bennett for the total output timing noise of a single repeater with a random pulse pattern.[7]

In order to apply these results to specific cases, a detailed analysis of the particular type of repeater is necessary to determine the power N_0 of the equivalent timing noise at the input of each repeater as a function of the real input noise, the pulse pattern, and other parameters of the system. Of course, N_0 will be proportional to the input noise power; its general dependence on the pulse pattern can be described in a qualitative way without a detailed analysis.[2]

Consider first tuned-circuit timing filters. From (115) and (90), the total output timing noise in a single repeater with an input white timing noise of power N_0 is approximately

$$\bar{\tau}_1 = N_0 \frac{\pi}{2pQ}; \qquad pQ \gg 1, \tag{124}$$

where Q is the tuned-circuit "Q" and p is the average number of time slots containing pulses. We consider four cases, discussed in the first section; in all of them from (50) the power of the timing wave is proportional to p^2.

1. Baseband repeater, signal pulses drive tuned circuit directly. The real output noise power remains constant for different pulse patterns; since $\bar{\tau}_1$ is proportional to the output noise to signal ratio, $\bar{\tau}_1 \propto 1/p^2$, and, from (124), $N_0 \propto 1/p$.

2. Baseband repeater, signal pulses passed through square-law or similar nonlinear device before driving tuned circuit. The real noise in the vacant time slots is now substantially suppressed; the real noise power at the output might be expected to vary approximately as p. Consequently $\bar{\tau}_1 \propto 1/p$, and N_0 is approximately independent of p.

3. Carrier frequency repeater, linear envelope detector. This case is somewhat different from case 1 above. During a time slot containing a pulse, only the in-phase component of the real input noise contributes to the random component of the envelope of the input wave. In the absence of a signal pulse, both the in-phase and quadrature components of the real input noise contribute equally to the output random phase modulation of the tuned circuit. Consequently, as p decreases, $\bar{\tau}_1$ will increase somewhat faster than $1/p^2$, and N_0 will increase somewhat faster than $1/p$.

4. Carrier frequency repeater, square-law or similar nonlinear detector. This type of repeater behaves essentially the same as a baseband repeater with a square-law device, case 2 above; N_0 is approximately independent of p. This has been verified in detail for the special case in which the timing circuit is an ideal flat filter, rather than a tuned circuit.

For locked oscillator timing circuits, (115) gives

$$\bar{\tau}_1 = N_0 \frac{\pi}{2\tilde{Q}}; \qquad \tilde{Q} \gg 1, \tag{125}$$

with $\tilde{Q}$, given by (87) or (88), now independent of the pulse pattern; the timing-wave amplitude is also constant. A similar discussion to the one above shows that N_0 for a locked oscillator has approximately the same functional dependence on p as it would for a tuned circuit for the corresponding type of repeater (i.e., baseband or carrier, linear or square-law detector). Since $\tilde{Q}$ is now independent of p, the variation of the output timing noise power $\bar{\tau}_1$ for a locked oscillator is p times that for a tuned circuit, with the same type of repeater.

Improved performance is thus attained by using a square-law or similar nonlinear detector in both baseband and carrier-frequency repeaters, using either tuned circuit or locked oscillator timing circuits, as pointed out by De Lange.[2] In this case the equivalent input timing noise power N_0 will be approximately independent of the pulse pattern, and may be treated as a constant in the analysis. Qualitatively, the principal effect of the input noise is simply to add a random displacement to the input signal pulses, as assumed in the analysis.

The above calculation of the alignment deviation must be modified for linear baseband repeaters (without a square-law or other nonlinear element) and for carrier frequency repeaters using a linear envelope detector. Here for $p < 1$ the equivalent position modulation required in the analysis will exceed the actual effective position modulation of the input signal pulses by the real input noise. Instead of (122) and (123) we would have approximately for the linear baseband repeater:

$$\tilde{A}_N(f) = N_0\{pe^{-\pi/\tilde{Q}}[1 - |\tilde{H}(f)|^{2N}] + (1 - p)|\tilde{H}(f)|^2\}, \tag{126}$$

$$\tilde{A}_N = N_0\left\{pe^{-\pi/\tilde{Q}}\left(1 - \frac{1}{2\tilde{Q}}\sqrt{\frac{\pi}{N-1}}\right) + (1 - p)\frac{\pi}{2\tilde{Q}}\right\}; \qquad \tilde{Q} \gg 1, \qquad N \gg 1, \tag{127}$$

with a similar result for the carrier repeater with a linear envelope detector. Since in this case $N_0 \propto 1/p$, the net effect as p decreases is an increase in the low frequency portion of the alignment noise power spectrum, with only a small increase in the total alignment noise power. The over-all behavior of the alignment noise is thus very similar both for linear repeaters and for repeaters using a square-law or similar nonlinear element, for which (122) and (123) apply with N_0 a constant independent of p. The above results for timing and spacing noise, of course, remain valid for all cases.

V. EXACT ANALYSIS FOR A CHAIN OF REPEATERS USING TUNED-CIRCUIT TIMING FILTERS, FOR PERIODIC PULSE PATTERNS — ZERO TUNING ERROR, COMPLETE RETIMING

The results of Section IV are exact only in those cases where the timing wave amplitude $\tilde{A}_{n'}$ is strictly constant, i.e., for tuned circuits with a periodic pulse pattern with every Mth pulse present (including all pulses present), and for locked oscillators with any arbitrary pulse pattern. For tuned circuits with random or general periodic pulse patterns these results are only approximate. In this section the exact solution for a chain of repeaters using tuned circuit timing filters is discussed for a simple periodic pulse pattern; in this case the analysis of Section IV provides a very satisfactory approximation. As above, the repeaters are assumed to have zero tuning error and complete retiming, and the driving pulses for the tuned circuits are assumed to be short enough so that no additional timing noise results from the finite pulse width (Section 2.5). The response of a chain of repeaters to a single frequency input $e^{j2\pi fn'}$ first is determined and then used to determine the noise response of the system. Since the pulse pattern is the same, the variation of $\tilde{A}_{n'}$ is identical at every repeater; the repeaters will consequently be identical time-varying transducers to the timing deviation.

Setting $\delta f = 0$ in (35), which gives the exact response of a single repeater with a tuned-circuit timing filter, yields

$$\tilde{\epsilon}^o(n') - \tilde{\epsilon}^o(n' - 1) = \frac{1}{\tilde{A}_{n'}} [\tilde{\epsilon}^i(n') - \tilde{\epsilon}^o(n' - 1)]. \qquad (128)$$

The timing wave amplitude $\tilde{A}_{n'}$ is given by (24). For a general periodic pulse pattern $\tilde{A}_{n'}$ will be a periodic function of n'. The simplest case of interest is a periodic pulse pattern of period M containing only two pulses, located in arbitrary time slots; $\tilde{A}_{n'}$ then takes on only two values and may be written as

$$\tilde{A}_{n'} = A(1 + \delta e^{j\pi n'}), \qquad (129)$$

where A is the average timing wave amplitude and δ is the normalized deviation of $\tilde{A}_{n'}$ from its average value.

Equation (129) is the Fourier series for $\tilde{A}_{n'}$; A and δ are determined by (24). From (50),

$$A = \frac{2Q}{\pi M}; \qquad Q \gg \frac{\pi M}{2}; \tag{130}$$

δ will depend on the particular pulse pattern. The pattern giving the greatest variation in timing wave amplitude, and hence the largest δ, will consist of two adjacent pulses followed by $M - 2$ vacant time slots, as shown in Fig. 10. In this case,

$$\delta = -\frac{\pi}{4Q}(M - 2); \qquad Q \gg \frac{\pi M}{2}. \tag{131}$$

For other patterns containing two pulses, δ will vary between zero and the value given in (131). For periodic pulse patterns containing more pulses, the Fourier series for $\tilde{A}_{n'}$ corresponding to (129) will contain more terms.

We now seek the response of a repeater chain to the input $e^{j2\pi fn'}$. From (128) and (129) the input and output are related by a modulation process involving the term

$$e^{j\pi n'} = e^{j2\pi(1/2)n'}, \tag{132}$$

of frequency $\frac{1}{2}$. Since $e^{j2\pi n'} = 1$,

$$\begin{aligned} e^{j\pi n'} e^{j2\pi fn'} &= e^{j2\pi(f-1/2)n'}, \\ e^{j\pi n'} e^{j2\pi(f-1/2)n'} &= e^{j2\pi fn'}. \end{aligned} \tag{133}$$

Therefore, for $f > 0$, if the input timing error to a repeater contains a sinusoidal component at the frequency f the output will contain com-

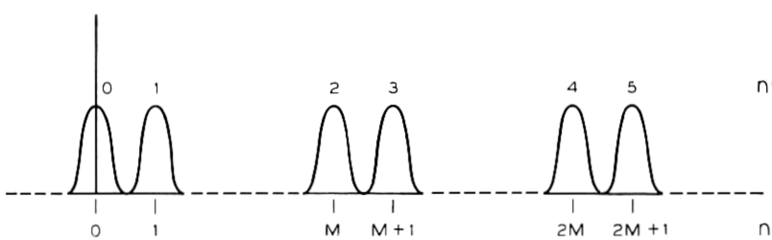

Fig. 10 — Periodic pulse pattern containing two pulses giving maximum variation in timing wave amplitude.

ponents at frequencies f and $f - \frac{1}{2}$; an input at the frequency $f - \frac{1}{2}$ will give rise to output components at frequencies $f - \frac{1}{2}$ and f. (If $f < 0$, an input at f yields outputs at f and $f + \frac{1}{2}$; an input at $f + \frac{1}{2}$ yields outputs at $f + \frac{1}{2}$ and f.) Assuming that $f > 0$, the timing error at the output of the $(K - 1)$th and the Kth repeaters may be written as follows:

$$\tilde{\epsilon}_{K-1}{}^{o}(n') = A_{K-1}(f)e^{j2\pi f n'} + B_{K-1}(f)e^{j2\pi(f-1/2)n'}, \tag{134}$$

$$\tilde{\epsilon}_{K}{}^{o}(n') = A_{K}(f)e^{j2\pi f n'} + B_{K}(f)e^{j2\pi(f-1/2)n'}. \tag{135}$$

Substituting (134) and (135) for $\tilde{\epsilon}^{i}(n')$ and $\tilde{\epsilon}^{o}(n')$ respectively in (128), and substituting (129) for $\tilde{A}_{n'}$, equating the coefficients of each of the two frequencies on each side of the resulting equation, we obtain

$$aA_K(f) + bB_K(f) = A_{K-1}(f), \tag{136}$$

$$cA_K(f) + dB_K(f) = B_{K-1}(f), \tag{137}$$

where the parameters a, b, c, and d are constants which depend on the input frequency f in a simple way. Since the input to the repeater chain is $e^{j2\pi f n'}$, the initial conditions for the linear difference equations (136) and (137) are

$$A_0 = 1, \qquad B_0 = 0. \tag{138}$$

These difference equations are easily solved by the usual methods, yielding the response of the system to a single input frequency; the results may then be used to determine the timing, spacing and alignment noise along the repeater chain with an independent white timing noise added at the input of every repeater. Although the solution is straightforward the analysis is lengthy, and the problem does not seem to be of sufficiently general interest to warrant the inclusion of the detailed results. Examination of these results shows that for quite moderate values of Q the variation in the timing wave amplitude may be safely neglected in considering the timing behavior of a chain of repeaters. The approximate analysis of Section IV, which neglects the additional modulation products caused by the variation of the timing wave amplitude, is reliable for a long chain of repeaters with tuned-circuit timing filters for this particular pulse pattern.

This analysis may be generalized to more complicated periodic pulse patterns. However, as the number of pulses in one period increases, so does the number of simultaneous difference equations corresponding to (136) and (137), increasing the complexity of the analysis.

VI. THE EFFECT OF TUNING ERROR ON THE TIMING OF A CHAIN OF REPEATERS USING LOCKED-OSCILLATOR TIMING CIRCUITS WITH COMPLETE RETIMING

We next consider the timing deviations introduced by random tuning errors of the different repeaters, in a chain of repeaters using locked-oscillator timing circuits with complete retiming. From (37) the timing response of such a repeater is given by

$$\tilde{\epsilon}^o(n') - \tilde{\epsilon}^o(n' - 1) = \frac{1}{A}\,[\tilde{\epsilon}^i(n') - \tilde{\epsilon}^o(n' - 1)] - \frac{\delta f}{F}\left(1 - \frac{1}{A}\right) b_{n'}\,, \tag{139}$$

where A is the ratio of the amplitude of the oscillator wave to the amplitude of the transient started by a single signal pulse. As pointed out in Section 2.3, this relation does *not* approximate the behavior of a tuned circuit with tuning error. Since no equivalent analysis has been devised for tuned circuits, the present treatment of the effects of tuning error on a repeater chain is confined exclusively to repeaters employing locked oscillators.

The output timing deviation of a repeater using a locked-oscillator timing circuit is linearly related to the input timing deviation, but has an additional equivalent input timing deviation related to the pulse pattern through the $b_{n'}$ and proportional to the tuning error. In the present section we consider only the timing noise caused by the tuning errors of the different repeaters, since the effects of input noise for this case have been discussed in Section IV.

We will consider only random pulse patterns with a probability p that any time slot contains a pulse, independently for each time slot. For such a pulse pattern the quantities $b_{n'}$ are independent random variables and consequently have a white spectrum, in addition to a dc component. The quantity $(b_{n'} - 1)$ has a geometric distribution,[14] so that the mean and variance of $b_{n'}$ are

$$\begin{aligned} \langle b_{n'} \rangle &= \frac{1}{p}, \\ \sigma_b^2 = \langle b_{n'}^2 \rangle - \langle b_{n'} \rangle^2 &= \frac{1 - p}{p^2}. \end{aligned} \tag{140}$$

Separating $b_{n'}$ into dc and ac components,

$$b_{n'} = B + B_{n'}\,, \qquad B = \langle b_{n'} \rangle = \frac{1}{p}, \tag{141}$$

$$\langle B_{n'} \rangle = 0, \qquad \langle B_{n'}^2 \rangle = \frac{1-p}{p^2}. \tag{142}$$

It is now convenient to separate (139) into dc and ac components, as in (40) to (45). Writing

$$\tilde{\epsilon}(n') = \tilde{\epsilon}_{\rm dc} + \tilde{\epsilon}_{\rm ac}(n'), \tag{143}$$

we have, from (139):

$$\tilde{\epsilon}_{\rm dc}{}^i - \tilde{\epsilon}_{\rm dc}{}^o = \frac{\delta f}{F}(A-1)B, \tag{144}$$

$$\tilde{\epsilon}_{\rm ac}{}^o(n') - \tilde{\epsilon}_{\rm ac}{}^o(n'-1) = \frac{1}{A}\left[\tilde{\epsilon}_{\rm ac}{}^i(n') - \tilde{\epsilon}_{\rm ac}{}^o(n'-1)\right] - \frac{\delta f}{F}\left(1 - \frac{1}{A}\right)B_{n'}. \tag{145}$$

Equation (144) represents a constant delay through the repeater due to tuning error. There will be a corresponding dc alignment deviation between the input signal and the retiming pulses, but otherwise the dc component is of no further interest in the analysis of the chain of repeaters. The ac timing deviation is governed by (145); for simplicity, the subscripts $_{\rm ac}$ will be dropped in the remainder of this section.

We now let $\tilde{C}^i(f)$, $\tilde{C}^o(f)$ and $\tilde{B}(f)$ be the transforms of $\tilde{\epsilon}_{\rm ac}{}^i(n')$, $\tilde{\epsilon}_{\rm ac}{}^o(n')$ and $B_{n'}$ respectively. Taking the transform of (145),

$$\tilde{C}^o(f) = \tilde{H}(f)\left[\tilde{C}^i(f) - \frac{\delta f}{F}\frac{1}{e^{+\pi/\tilde{Q}} - 1}\tilde{B}(f)\right], \tag{146}$$

where

$$\tilde{H}(f) = \frac{1 - e^{-\pi/\tilde{Q}}}{1 - e^{-\pi/\tilde{Q}}e^{-j2\pi f}}, \tag{147}$$

$$A = \frac{1}{1 - e^{-\pi/\tilde{Q}}}; \qquad \tilde{Q} = \pi A, \qquad \tilde{Q} \gg 1. \tag{148}$$

$\tilde{H}(f)$ and $\tilde{Q}$ are the same transfer function and effective "Q" previously defined for the repeater in (91) and (87) and (88), and $B_{n'}$ is a white noise of power $(1-p)/p^2$.

In Section IV the effects of independent white timing noise added at the input of every repeater were determined. The present analysis is somewhat different in that the added timing noises at the different repeater inputs are no longer independent, since the pulse pattern and hence $B_{n'}$ and $B(f)$ are identical at each repeater. In general, we no longer may simply add the output power spectra caused by the various noise inputs, but must keep track of the phase angles of the different components. Thus the timing noise power spectrum $\tilde{T}_N(f)$ at the output of the Nth repeater, in terms of the tuning errors $(\delta f)_K$ of the various repeaters, becomes

$$\tilde{T}_N(f) = \frac{1}{F^2}\left[\frac{1}{e^{+\pi/\tilde{Q}} - 1}\right]^2 \frac{1-p}{p^2} \sum_{K=1}^{N} \sum_{L=1}^{N} (\delta f)_{N-K+1}(\delta f)_{N-L+1} \tilde{H}^K(f)\tilde{H}^{*L}(f). \tag{149}$$

Next assume that the tuning errors $(\delta f)_K$ themselves are independent random variables with zero mean. Thus,

$$\begin{aligned} \langle(\delta f)_K(\delta f)_L\rangle &= 0; \qquad K \neq L, \\ \langle(\delta f)_K^2\rangle &= \langle(\delta f)^2\rangle; \qquad K = L. \end{aligned} \tag{150}$$

The tuning errors at the different repeaters are uncorrelated. The equivalent input timing noises at the different repeaters are thus also uncorrelated, although not independent; in this special case, the output power spectra of the different components may be added directly. Substituting (150) into (149),

$$\begin{aligned} \tilde{T}_N(f) &= \frac{\langle(\delta f)^2\rangle}{F^2}\left[\frac{1}{e^{+\pi/\tilde{Q}} - 1}\right]^2 \frac{1-p}{p^2} \frac{1 - |\tilde{H}(f)|^{2N}}{|\tilde{H}(f)|^{-2} - 1}, \\ \langle(\delta f)_K(\delta f)_L\rangle &= \begin{cases} 0 \quad ; & K \neq L \\ \langle(\delta f)^2\rangle; & K = L \end{cases}. \end{aligned} \tag{151}$$

Comparing with (105), the timing noise power spectrum caused by uncorrelated tuning errors has the same shape as that caused by independent white timing noise at every repeater input. Thus, the total output timing noise power in the present case is given by (111) or by (113) to (115), where N_0 must be replaced by

$$\frac{\langle(\delta f)^2\rangle}{F^2}\left[\frac{1}{e^{+\pi/\tilde{Q}} - 1}\right]^2 \frac{1-p}{p^2}.$$

For the high-$\tilde{Q}$ case:

$$\begin{aligned} \tilde{T}_1 &= \frac{\langle(\delta f)^2\rangle}{F^2} \frac{\tilde{Q}}{\pi} \frac{1-p}{2p^2}; \qquad \tilde{Q} \gg 1, \\ \tilde{T}_N &= \frac{\langle(\delta f)^2\rangle}{F^2} \frac{\tilde{Q}}{\pi} \frac{1-p}{p^2} \sqrt{\frac{N}{\pi}}; \qquad \tilde{Q} \gg 1, \quad N \gg 1. \end{aligned} \tag{152}$$

The spacing and alignment noise power spectra and total powers are found in the same way from the results of Section 4.3.

The results of (151) and (152) are no longer valid if the tuning errors at the different repeaters are correlated. For example, if all tuning errors are identical, making use of (121), (149) becomes

$$\tilde{T}_N(f) = \left(\frac{\delta f}{F}\right)^2 \frac{e^{+\pi/\tilde{Q}}}{(e^{+\pi/\tilde{Q}} - 1)^2} \frac{1 - p}{p^2} \frac{|1 - \tilde{H}^N(f)|^2}{|\tilde{H}(f)|^{-2} - 1}, \tag{153}$$

$$(\delta f)_K = \delta f.$$

In this case, all repeaters are tuned correctly but the pulse repetition frequency is incorrect. The last factor of (153) approaches N^2 as f approaches zero [the corresponding factor of (151) or (105) approaches N, as illustrated in Fig. 9(a)]. The low-frequency noise is much greater in this case, since an identical timing noise has been added at every repeater and the corresponding noise amplitudes, rather than powers, add at the output.

For a single repeater using a locked oscillator, the output timing noise power given in (152) is just twice as large as the corresponding quantity computed by W. R. Bennett for a tuned circuit.[7] This is in accord with the discussion of the two cases given in (38) to (45) and illustrated in Fig. 6 for p close to 1. A similar analysis for the timing noise produced by tuning error in a chain of repeaters using tuned circuit timing filters has not been found.

VII. TIMING IN A CHAIN OF REPEATERS WITH PARTIAL RETIMING

The preceding analysis has been confined to the case of complete retiming, in which the timing deviation of each output signal pulse is equal to the timing deviation of the corresponding timing pulse. All of these results are easily extended to the case of partial retiming, in which the timing deviation of each output pulse depends linearly on the timing deviations of both timing and input signal pulses.

As stated in Section I, in systems employing complete retiming the timing pulses may be derived only from the input signal pulses. If such a system with timing from the output signal pulses could somehow be started, its subsequent timing behavior would be completely independent of the input pulse pattern. Thus, the repeater would have no way of determining when the timing pulses were properly centered on the input signal pulses. However, in systems employing partial retiming, the timing pulses may be derived from either the input or the output signal pulses.

Let $\bar{\epsilon}^i(n')$ and $\bar{\epsilon}^o(n')$ be the normalized timing deviations of the n'th input and output signal pulses with transforms $\tilde{C}^i(f)$ and $\tilde{C}^o(f)$, as before, and let $\bar{\epsilon}^t(n')$ be the timing deviation of the n'th timing pulse with transform $\tilde{C}^t(f)$. Then, following Pierce,[6] we assume that

$$\begin{aligned} \bar{\epsilon}^o(n') &= \alpha\bar{\epsilon}^i(n') + (1 - \alpha)\bar{\epsilon}^t(n'), \\ \tilde{C}^o(f) &= \alpha\tilde{C}^i(f) + (1 - \alpha)\tilde{C}^t(f). \end{aligned} \tag{154}$$

For complete retiming assumed up to now, $\alpha = 0$; for no retiming, $\alpha = 1$.

Consider first a repeater with the timing wave derived from the input signal pulses. All of the previous analysis for a single repeater (Section II, Section 4.1 and the first part of Section VI) remains valid if we replace $\bar{\epsilon}^o(n')$ and $\tilde{C}^o(f)$ by $\bar{\epsilon}^t(n')$ and $\tilde{C}^t(f)$ respectively. Thus, from (86) to (91), for a repeater with zero tuning error

$$\frac{\tilde{C}^t(f)}{\tilde{C}^i(f)} = \tilde{H}(f) = \frac{1 - e^{-\pi/\tilde{Q}}}{1 - e^{-\pi/\tilde{Q}}e^{-j2\pi f}}, \tag{155}$$

where $\tilde{H}(f)$ and $\tilde{Q}$ remain the same as before. Defining $\tilde{H}_{\alpha i}(f)$ as the transfer function of a single repeater with partial retiming with timing derived from the input, from (154) and (155)

$$\tilde{H}_{\alpha i}(f) = \alpha + (1 - \alpha)\tilde{H}(f). \tag{156}$$

For $\alpha = 0$, $\tilde{H}_{\alpha i}(f) = \tilde{H}(f)$. Making use of (120),

$$\begin{aligned} |\tilde{H}_{\alpha i}(f)|^2 &= \gamma + (1 - \gamma)|\tilde{H}(f)|^2, \\ \gamma &= \alpha[1 - e^{-\pi/\tilde{Q}} + \alpha e^{-\pi/\tilde{Q}}], \end{aligned} \tag{157}$$

where $|\tilde{H}(f)|^2$ is given in (93). Substituting $|\tilde{H}_{\alpha i}(f)|^2$ for $|\tilde{H}(f)|^2$ in (104) and (105), the timing noise power spectrum at the output of a chain of repeaters with an independent white timing noise added at the input to each repeater is determined as in Section 4.3. For moderate values of α the timing noise spectrum will be slightly larger at high frequencies than for complete retiming, $\alpha = 0$. Thus, the total output timing noise power will be slightly larger than for complete retiming, given in (110) to (115). The spacing and alignment noise power spectra may also be easily determined.

Next consider a repeater with the timing wave derived from the output signal pulses. For zero tuning error we now have

$$\frac{\tilde{C}^t(f)}{\tilde{C}^o(f)} = \tilde{H}(f) = \frac{1 - e^{-\pi/\tilde{Q}}}{1 - e^{-\pi/\tilde{Q}}e^{-j2\pi f}}. \tag{158}$$

Substituting into (154), the transfer function $\tilde{H}_{\alpha o}(f)$ of a single repeater with partial retiming with timing derived from the output is

$$\tilde{H}_{\alpha o}(f) = \frac{\alpha}{1 - (1 - \alpha)\tilde{H}(f)}. \tag{159}$$

The 3-db bandwidth of a single repeater for small α and high $\tilde{Q}$ is approximately

$$f_{3\text{db}} = \frac{\alpha}{2\tilde{Q}}; \qquad \alpha \ll 1, \qquad \tilde{Q} \gg 1. \tag{160}$$

The bandwidth is α times the bandwidth of the corresponding repeater with the timing wave derived from the input [which for small α will be only slightly larger than the value for complete retiming, given in (94)]. The output timing deviations are thus much smaller with timing from the output than with timing from the input (for zero tuning error). However, in the limit as $\alpha \to 0$ the behavior of the repeater becomes independent of the input pulses, and the effects of any noise sources in the regenerator itself become correspondingly more important.

The analysis of Section VI for a locked oscillator with tuning error is readily extended in a similar manner to a repeater with partial retiming. With timing from the input there is again little change in behavior for small α. However, with timing from the output the dc alignment deviation $\tilde{\epsilon}_{\text{dc}}{}^{i} - \tilde{\epsilon}_{\text{dc}}{}^{t}$ is proportional to $1/\alpha$. The bandwidth of the repeater decreases as above, while the equivalent input timing noise due to tuning error increases.

Both tuning error and additional noise sources in the regenerator will determine whether for a given small value of α any advantage can be obtained by deriving the timing wave from the output rather than the input. For complete retiming, $\alpha = 0$, the timing wave must be derived from the input.

VIII. DISCUSSION

To illustrate the above results we consider a chain of repeaters using resonant-circuit timing filters with zero tuning error, complete retiming (the timing wave must of course be derived from the input), driving pulses short enough so that no additional timing deviations are introduced by the finite pulse width and a random pulse pattern. We further assume that the equivalent normalized timing deviations added to the input pulses are distributed uniformly between −0.1 and +0.1; the equivalent input timing noise power is then

$$N_0 = 0.0033, \qquad \sqrt{N_0} = 0.0578.$$

Take

$$N = 100 \text{ repeaters,}$$

$$Q = 100, \text{ tuned circuit ``}Q\text{''},$$

$$p = \tfrac{1}{2}, \text{ probability that a time slot contains a pulse.}$$

From (90)

$$\tilde{Q} = 50.$$

The rms values of the timing, spacing, and alignment noise at the output of the repeater chain, from (113), (118) and (123), are:

$$\sqrt{\tilde{T}_N} = 0.0344,$$

$$\sqrt{\tilde{S}_N} = 0.00363,$$

$$\sqrt{\tilde{A}_N} = 0.056.$$

All of these normalized quantities are of course measured in units of a pulse period.

Next consider the effect of finite pulse width. An extreme case may give some indication of the importance of this effect. Consider the change in output timing deviation caused by changing the pulse pattern from all pulses present to every Mth pulse present, with raised cosine driving pulses as illustrated in Fig. 7. From (63) and (33), the change in output timing deviation will be $w_s(\pi/Q)(M - 1)$. For overlapping pulses [Fig. 7(b)] and for $Q = 100$ as in the above example, from (64) and the table following it, $w_s = 0.1073$, and the change in output timing deviation will be 0.00337 $(M - 1)$. Thus, for $Q = 100$ and overlapping pulses as shown in Fig. 7(b), changing the pulse pattern from all pulses present to every 10th pulse present causes a phase shift of 10.9° in each repeater.

The analysis for the effect of finite pulse width (or amplitude-to-phase conversion in the limiter of the timing system) in a chain of repeaters for a random pulse pattern is complicated by the fact that the timing deviation added at each repeater is directly related to the timing wave amplitude. It is not obvious that the variations in timing wave amplitude can be neglected in this case, as was done in studying the effects of input noise (which will be independent of the pulse pattern).

Finally, consider a chain of repeaters using locked-oscillator timing circuits with random tuning errors, complete retiming (with the timing wave derived from the input), and a random pulse pattern. Assume

$$\tilde{Q} = 100,$$
$$p = \tfrac{1}{2}.$$

Then, from (151), the equivalent rms input timing noise per repeater is

$$45 \frac{\sqrt{\langle(\delta f)^2\rangle}}{F}.$$

The effects of tuning error will be the same as the effects of an input timing noise uniformly distributed between -0.1 and $+0.1$ (corresponding to $\sqrt{N_0} = 0.0578$, as in the first example) for an rms fractional tuning error of

$$\frac{\sqrt{\langle(\delta f)^2\rangle}}{F} = 0.0013.$$

As discussed above, similar results are not available for the effects of tuning error in a chain of repeaters using tuned circuit timing filters.

IX. ACKNOWLEDGMENTS

The present analysis is based on unpublished work of J. R. Pierce on the timing response of a single repeater. The author would like to thank S. O. Rice for performing the analysis given in the Appendix, which gives an approximate expression for $\tilde{T}_N$, the total timing noise power at the output of a chain of repeaters. He would also like to thank W. R. Bennett, O. E. DeLange, W. M. Goodall, and J. L. Kelly, Jr., for many helpful discussions and suggestions.

APPENDIX

Approximate Solution for $\tilde{T}_N$

In this appendix we present the approximate solution obtained by S. O. Rice for $\tilde{T}_N$, the normalized timing noise power at the output of a chain of N repeaters with an independent white timing noise of power N_0 introduced at the input to each repeater. This solution is most useful when the timing circuits have a high $\tilde{Q}$. From (109), (105) and (93),

$$\tilde{T}_N(f) = N_0 \frac{1 - |\tilde{H}(f)|^{2N}}{|\tilde{H}(f)|^{-2} - 1},$$

$$|\tilde{H}(f)|^2 = \frac{1}{1 + \operatorname{csch}^2 \dfrac{\pi}{2\tilde{Q}} \sin^2 \pi f}, \qquad (161)$$

$$\tilde{T}_N = \int_{-1/2}^{1/2} \tilde{T}_N(f)\, df.$$

Substituting

$$\theta = 2\pi f \tag{162}$$

into (161), we obtain

$$\begin{aligned} \tilde{T}_N &= \frac{N_0}{\pi} \int_0^\pi \frac{1 - y^N(\theta)}{y^{-1}(\theta) - 1}\, d\theta, \\ y(\theta) &= \frac{1}{1 + a \sin^2 \dfrac{\theta}{2}}, \end{aligned} \tag{163}$$

where we have replaced $\operatorname{csch}^2 \pi/(2\tilde{Q})$ by a to simplify the analysis. Making the following change of variable,

$$x = \tan \frac{\theta}{2}, \qquad \sin^2 \frac{\theta}{2} = \frac{x^2}{1 + x^2}, \qquad d\theta = \frac{2\,dx}{1 + x^2}, \tag{164}$$

Equation (163) becomes

$$\tilde{T}_N = \frac{N_0}{\pi} \frac{2}{a} \int_0^\infty \frac{dx}{x^2} \left\{1 - \left[\frac{1 + x^2}{1 + (a + 1)x^2}\right]^N\right\}. \tag{165}$$

Integrating (165) by parts, we find

$$\tilde{T}_N = N_0 \frac{4N}{\pi} \int_0^\infty \frac{dx}{(1 + x^2)^2} \left[\frac{1 + x^2}{1 + (a + 1)x^2}\right]^{N+1}. \tag{166}$$

Making another change of variable

$$u = \frac{1 + (a + 1)x^2}{1 + x^2}, \qquad du = \frac{2ax\,dx}{(1 + x^2)^2}, \qquad x^2 = \frac{u - 1}{a + 1 - u}, \tag{167}$$

Equation (166) becomes

$$\tilde{T}_N = N_0 \frac{2N}{\pi a} \int_1^{a+1} \frac{du}{u^{N+1}} \left[\frac{(a + 1) - u}{u - 1}\right]^{1/2}. \tag{168}$$

Making one last change of variable,

$$v = u - 1, \tag{169}$$

Equation (168) becomes

$$\tilde{T}_N = N_0 \frac{2N}{\pi a} \int_0^a \frac{dv}{(1 + v)^{N+1}} \left(\frac{a - v}{v}\right)^{1/2}. \tag{170}$$

The principal contribution to the integral of (170) comes from the singularity at $v = 0$. We split off this portion of the integral and use the remaining terms to get bounds on the error. Thus,

$$\begin{aligned}\tilde{T}_N &= N_0 \frac{2N}{\pi a}\left[\int_0^a \frac{dv}{v^{1/2}(1+v)^{N+1}}\, a^{1/2} - \int_0^a \frac{dv}{v^{1/2}(1+v)^{N+1}}\,[a^{1/2}-(a-v)^{1/2}]\right] \\ &= N_0 \frac{2N}{\pi a^{1/2}}\left\{\int_0^\infty \frac{dv}{v^{1/2}(1+v)^{N+1}} - \int_a^\infty \frac{dv}{v^{1/2}(1+v)^{N+1}} - \int_0^a \frac{dv}{v^{1/2}(1+v)^{N+1}}\left[\frac{a^{1/2}-(a-v)^{1/2}}{a^{1/2}}\right]\right\} \\ &= N_0 \frac{2N}{\pi a^{1/2}}\left[\frac{\Gamma(\frac{1}{2})\Gamma(N+\frac{1}{2})}{\Gamma(N+1)} - R_1 - R_2\right],\end{aligned} \tag{171}$$

$$\begin{aligned} R_1 &= \int_a^\infty \frac{dv}{v^{1/2}(1+v)^{N+1}}, \\ R_2 &= \int_0^a \frac{dv}{v^{1/2}(1+v)^{N+1}}\left[\frac{a^{1/2}-(a-v)^{1/2}}{a^{1/2}}\right].\end{aligned} \tag{172}$$

We next find bounds on R_1 and R_2.

For R_1, we have

$$\begin{aligned} 0 < R_1 &= \int_a^\infty \frac{dv}{v^{1/2}(1+v)^{N+1}} < \int_a^\infty \frac{dv}{v^{N+3/2}} = \frac{1}{a^{N+1/2}(N+\frac{1}{2})}, \\ 0 < R_1 &< \frac{1}{a^{N+1/2}\,(N+\frac{1}{2})}.\end{aligned} \tag{173}$$

For R_2, we first state the inequality

$$a^{1/2} - (a-v)^{1/2} = \frac{v}{a^{1/2}+(a-v)^{1/2}} < \frac{v}{a^{1/2}}; \qquad 0 < v < a. \tag{174}$$

Thus,

$$\begin{aligned} 0 < R_2 &= \int_0^a \frac{dv}{v^{1/2}(1+v)^{N+1}}\left[\frac{a^{1/2}-(a-v)^{1/2}}{a^{1/2}}\right] \\ &< \frac{1}{a}\int_0^a \frac{v^{1/2}\,dv}{(1+v)^{N+1}} < \frac{1}{a}\int_0^\infty \frac{v^{1/2}\,dv}{(1+v)^{N+1}} \\ &= \frac{1}{a}\,\frac{\Gamma(\frac{3}{2})\Gamma(N-\frac{1}{2})}{\Gamma(N+1)}, \\ 0 < R_2 &< \frac{1}{a}\,\frac{\Gamma(\frac{3}{2})\Gamma(N-\frac{1}{2})}{\Gamma(N+1)}.\end{aligned} \tag{175}$$

Replacing a by $\operatorname{csch}^2 \pi/2\tilde{Q}$ in (172), (173) and (175), we obtain as the final result

$$\tilde{T}_N = N_0 \frac{2N}{\pi} \sinh \frac{\pi}{2\tilde{Q}} \left[\frac{\Gamma(\frac{1}{2})\Gamma(N + \frac{1}{2})}{\Gamma(N + 1)} - R_1 - R_2 \right],$$

$$0 < R_1 < \sinh^{2N+1} \frac{\pi}{2\tilde{Q}} \left[\frac{1}{N + \frac{1}{2}} \right], \tag{176}$$

$$0 < R_2 < \sinh^2 \frac{\pi}{2\tilde{Q}} \left[\frac{\Gamma(\frac{3}{2})\Gamma(N - \frac{1}{2})}{\Gamma(N + 1)} \right].$$

The bounds are quite close for a moderately high $\tilde{Q}$, and improve with increasing $\tilde{Q}$ or N.

If $\tilde{Q} \gg 1$, $N \gg 1$, the sinh may be replaced by its argument and the gamma functions by Stirling's approximation, yielding the result given in (113):

$$\tilde{T}_N = N_0 \frac{\sqrt{\pi N}}{\tilde{Q}}; \qquad \tilde{Q} \gg 1, \qquad N \gg 1. \tag{177}$$

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