Gain and Noise Figure of a Variable-Capacitance Up-Converter*

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The performance of a p-n junction nonlinear-capacitance diode as a low-noise, amplifying frequency converter is analyzed for the case in which the output signal frequency is many times greater than the input signal frequency. The diode is represented by an equivalent circuit consisting of a time varying capacitance and a constant series resistance. Formulae for the maximum available gain and the noise figure of the circuit are obtained, and representative numerical values are given for the noise figures of systems incorporating such diodes as preamplifiers.

I. INTRODUCTION

Since microwave receivers frequently must operate at a very low input signal level it is important to obtain the smallest possible noise figure. One way of accomplishing this is to use a low-noise amplifier at the input of the receiver. In this article the role of a nonlinear-capacitance p-n junction diode in performing such a function is analyzed.

The performance of a variable capacitance as a frequency converter has been extensively discussed in the literature.[1,2,3] Theory indicates that when the variable capacitance is used for conversion between a low frequency and an upper sideband (a noninverting frequency), the maximum available gain is f_2/f_1, f_1 being the input frequency and f_2 being the output frequency. Hence in up-conversion (modulation) a power gain results, with the power added to the input signal being supplied by the beat oscillator. In down-conversion a power loss results. (These statements apply when no signal power is transferred at the lower sideband or at any of the harmonic images). The situation is different when one of the two frequencies involved is a lower sideband (an inverting frequency). A negative resistance may then appear at both sets of terminals,

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resulting in regenerative gain or stable oscillations.[4] Large gains may be obtained when going either up or down in frequency. The present paper, however, is restricted to a consideration of upper sideband frequency conversion. Analysis of lower sideband operation is now in progress.

In this article the frequency conversion performance of a nonlinear capacitance (N-C) diode with parasitics is analyzed. In Section V the maximum available gain (MAG) is calculated and is compared with the MAG predicted for an ideal device. In Section VI, an equation for the noise figure of an N-C diode is obtained, taking into account only the thermal noise of the series resistance. Over-all noise figures are given in Table I for three types of receivers with diode preamplifiers, operating at specified input and output frequencies.

II. THE EQUIVALENT CIRCUIT

For the calculations to be made here, the p-n junction might be represented by the lumped-parameter equivalent circuit shown in Fig. 1. Here R_s and C_T are as defined and the "variable admittance", shown as lumped parameters, accounts for the current carried across the junction by the motion of carriers through the space charge region. A calculation of the mixing action of the diode, using Fig. 1, would involve a rather complicated formulation.[3] However, the silicon and germanium retarding-field diffused diodes now under investigation are fairly accurately represented by the equivalent circuit shown in Fig. 2. (The design and properties of these devices are discussed in other papers).[5,6] In this diagram, $C_{\min}$ represents the smallest capacitance attainable, determined by the limitation on reverse bias voltage imposed by breakdown, and $C_{\max}$ is the largest value, determined either by physical limitations or by choice.

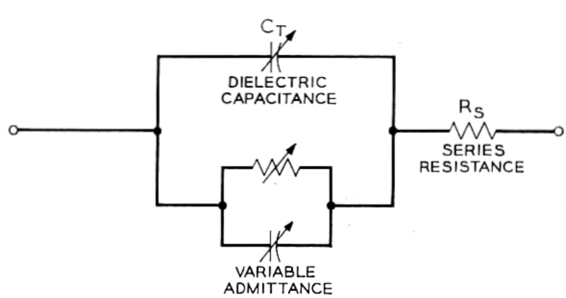

Fig. 1 — Equivalent circuit for nonlinear-capacitance diode.

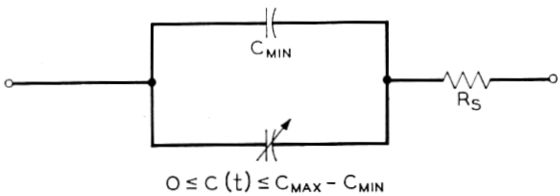

Fig. 2 — Simplified diode equivalent circuit.

III. ASSUMPTIONS FOR CALCULATION OF GAIN AND NOISE FIGURE

In this article the gain and noise figure of an N-C diode frequency converter are calculated according to the following assumptions: (1) Upper sideband operation is assumed; that is, an input frequency f_1 is converted to an output frequency f_2 using a beating oscillator at the frequency $f_2 - f_1$. (2) The lower sideband, of frequency $f_2 - 2f_1$, and all the harmonic images are assumed to be open circuited, implying that no power is delivered by the frequency converter at these frequencies. (3) The equivalent circuit of Fig. 3 is used. Here, filters are indicated which restrict transmission in the left-hand branch to a narrow band about the frequency f_1 and transmission in the right-hand branch to a narrow band about f_2; $\mathcal{E}_1$ and $\mathcal{E}_2$ are signal voltages and $\mathcal{E}_{N1}$ and $\mathcal{E}_{N2}$ are noise voltages at frequencies f_1 and f_2 respectively. The equivalent circuit of the diode which was discussed in connection with Fig. 2 is assumed applicable over the operating range of pump voltage and frequency.

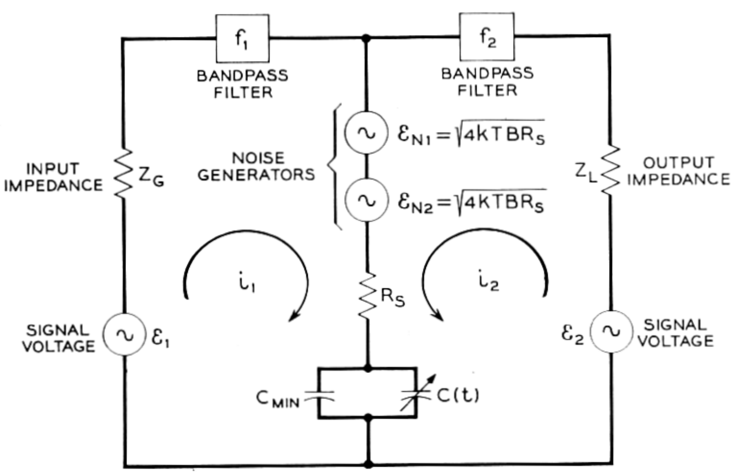

Fig. 3 — Equivalent circuit for the calculation of gain and noise figure.

Using these assumptions, results have been obtained in a simple form which can serve as a guide to diode development and application. It is not supposed that the model used here constitutes a very accurate characterization of the frequency converter operation. In particular, assumption (2) is theoretically possible, but there would be considerable practical difficulty in satisfying these conditions in a microwave circuit.

IV. GENERAL EQUATIONS FOR FREQUENCY CONVERSION

In subsequent sections we shall need to calculate, for the circuit shown in Fig. 3, the power delivered to the load at frequency f_2 when specified signal voltages at frequencies f_1 and f_2 are applied. Hence, what is desired is a means of calculating the current i_2 for given signal voltages. For the specific applications to be made later on, we shall obtain in this section a solution in general form for the currents i_1 and i_2 resulting when signal (or noise) voltages $\mathcal{E}_1$ and $\mathcal{E}_2$, at frequencies f_1 and f_2 respectively, are applied to an N-C diode.

It is possible to express the relation between currents and voltages of a frequency converter by means of an admittance matrix; this will be done here for a nonlinear-capacitance frequency converter. Let the beat oscillator voltage be represented by V_0, and the combined signal voltages by δV. If noise is of any consequence, one may surely assume $\delta V \ll V_0$. Hence we may write (1), representing the charge on the capacitor as a function of voltage, in the form of a Taylor expansion,

$$Q(V) = Q_{V_0} + \left(\frac{\partial Q}{\partial V}\right)_{V_0} \delta V + \cdots \tag{1}$$

The subscript V_0 signifies that Q or its derivatives have the value determined by V_0 alone, i.e. δV equals 0. It is assumed that terms of higher order in δV are negligible, so that the relations to be derived will be linear in the signal quantities. (For a more complete theory of linear frequency converters, see Ref. 6).

To evaluate (1) we expand $(\partial Q/\partial V)_{V_0}$ in a Fourier series,

$$\frac{\partial Q}{\partial V} = C(t) + C_{\min} = \sum_{n=-\infty}^{\infty} C_n e^{jnbt}, \tag{2}$$

with the beat oscillator voltage assumed to be periodic, with pulsatance (angular frequency) b. The signal voltage δV may involve all the harmonics of the beat oscillator pulsatance mixed with the signal pulsatance s. Hence it is written

$$\delta V = \sum_{m=-\infty}^{\infty} \sum_{s\pm} v_{mb+s} e^{j(mb+s)t}. \tag{3}$$

Referring to (1), it is seen that the first term on the right-hand side contains only pulsatances which are harmonics of b. The second term, which we designate δQ, contains the various signal pulsatances. It is given by

$$\delta Q = \left(\frac{\partial Q}{\partial V}\right)_{V_0} \delta V = \sum_{n,m} \sum_{s\pm} C_n v_{mb+s} e^{j[(m+n)b+s]t}. \tag{4}$$

Since we shall want to select individual pulsatance terms from the right-hand side of (4), we set $m + n = l$. The result is

$$\delta Q = \sum_{l,m} \sum_{s\pm} C_{l-m} v_{mb+s} e^{j(lb+s)t}. \tag{5}$$

Also, being interested in the current components, we use the relation $i_\omega = j\omega q_\omega$, obtaining finally

$$i_{lb+s} = \sum_m j(lb + s) C_{l-m} v_{mb+s} \tag{6}$$

(we have here stopped considering the terms containing $-s$, because $i_{lb-s} = i^*_{-lb+s}$). Equation (6) is equivalent to the matrix equation

$$i_{lb+s} = \sum_m Y_{lm} v_{mb+s}, \tag{7}$$

where the elements Y_{lm} are given by

$$Y_{lm} = j(lb + s) C_{l-m}. \tag{8}$$

The admittance representation is convenient to use when only a few signal voltages have specified values and the rest are zero (i.e. short-circuited). In the case of the equivalent circuit used here, none of the voltages can be short-circuited, because of the diode series resistance. The possibility remains, however, of presenting open circuits to all the image pulsatances (all the $mb + s$) except for the two involved in frequency conversion, s and $b + s$. This means that all the currents except i_s and i_{b+s} will be assumed to vanish. The relation between voltage and current can then be simplified when it is expressed in the impedance matrix form. Hence we must obtain the matrix inverse of $\| Y \|$, which is $\| Z \| \equiv \| Y \|^{-1}$.

There is at least one case in which this matrix inversion can be readily performed. If $\partial Q/\partial V$ equals $C_0 + 2C_1 \cos bt$ then only the coefficients C_0 and C_1 in the expansion (2) do not vanish. The matrix of (8) is now limited to elements on the diagonal and once removed from the diagonal. As a product of infinite matrices (in these arrays only the nonzero elements are indicated), it is written

$$\| Y \| = \left\| \begin{matrix} \cdots & & & & \\ & j(-b+s) & & & \\ & & js & & \\ & & & j(b+s) & \\ & & & & j(2b+s) \\ & & & & \cdots \end{matrix} \right\| \left\| \begin{matrix} \cdots & & & & & \\ C_1 & C_0 & C_1 & & & \\ & C_1 & C_0 & C_1 & & \\ & & C_1 & C_0 & C_1 & \\ & & & & & \cdots \end{matrix} \right\| . \qquad (9)$$

The second matrix on the right-hand side of (9) may be factored to obtain (10),

$$\| C_{lm} \| = C_0 \left\| \begin{matrix} \cdots & & & & \\ \zeta & 1 & \zeta & & \\ & \zeta & 1 & \zeta & \\ & & \zeta & 1 & \zeta \\ & & & \cdots & \end{matrix} \right\| , \qquad (10)$$

where $\zeta = C_1/C_0$. Equation (9) can be written symbolically as

$$\| Y \| = \| j(lb + s) \| \, \| C_{lm} \|,$$

and, by a rule of matrix algebra, the impedance matrix is given by

$$\| Z \| = \| Y \|^{-1} = \| C_{lm} \|^{-1} \, \| j(lb + s) \|^{-1}. \qquad (11)$$

The first matrix on the right-hand side of (9) is diagonal, hence its inverse is easily obtained. The inverse of the second matrix is also known.[6] If a matrix has the form

$$\begin{aligned} M_{ij} &= 1; && \text{if} \quad i = j \\ &= \frac{\xi}{1 + \xi^2}; && \text{if} \quad i = j \pm 1 \\ &= 0; && \text{otherwise,} \end{aligned} \qquad (12a)$$

the inverse matrix is given by

$$(M^{-1})_{ij} = \frac{1 + \xi^2}{1 - \xi^2} (-\xi)^{|i-j|}. \qquad (12b)$$

This result can be used to obtain $\| C_{lm} \|^{-1}$, if we set $\zeta \equiv \xi/(1 + \xi^2)$ [see (10)]. If it is now assumed that all pulsatances in the spectrum, other than $b + s$ and s, are open-circuited, the impedance matrix reduces to

$$Z_c = \frac{1}{C_0}\frac{1+\xi^2}{1-\xi^2}\begin{Vmatrix} 1 & -\xi \\ -\xi & 1 \end{Vmatrix}\begin{Vmatrix} \dfrac{1}{js} & 0 \\ 0 & \dfrac{1}{j(b+s)} \end{Vmatrix}. \tag{13}$$

To simplify future work, we shall use the abbreviated representation

$$Z_c = \begin{Vmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{Vmatrix}. \tag{14}$$

The matrix Z_c represents the capacitative impedance of the diode. The total impedance Z must include the series resistance R; it is given by

$$Z = \begin{Vmatrix} Z_{11}+R & Z_{12} \\ Z_{21} & Z_{22}+R \end{Vmatrix}. \tag{15}$$

By the definition of an impedance matrix we have

$$\begin{Vmatrix} v_1 \\ v_2 \end{Vmatrix} = \begin{Vmatrix} Z_{11}+R & Z_{12} \\ Z_{21} & Z_{22}+R \end{Vmatrix}\begin{Vmatrix} i_1 \\ i_2 \end{Vmatrix}. \tag{16}$$

Here v_1 and v_2 are the voltages across the diode terminals at the input and output frequencies, respectively, and i_1 and i_2 are the corresponding currents. It follows from Fig. 3 that these two voltages are given by

$$\begin{aligned} v_1 &= \mathcal{E}_1 - i_1 Z_G \\ v_2 &= \mathcal{E}_2 - i_2 Z_L \end{aligned} \tag{17}$$

Substituting (17) into (16), and rearranging, we obtain

$$\begin{Vmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{Vmatrix} = \begin{Vmatrix} Z_{11}+R+Z_G & Z_{12} \\ Z_{21} & Z_{22}+R+Z_L \end{Vmatrix}\begin{Vmatrix} i_1 \\ i_2 \end{Vmatrix}. \tag{18}$$

The solution for the currents is written in matrix form

$$\begin{Vmatrix} i_1 \\ i_2 \end{Vmatrix} = \begin{Vmatrix} Z_{11}+R+Z_G & Z_{12} \\ Z_{21} & Z_{22}+R+Z_L \end{Vmatrix}^{-1}\begin{Vmatrix} \mathcal{E}_1 \\ \mathcal{E}_2 \end{Vmatrix}, \tag{19}$$

where the reciprocal matrix is given by

$$\begin{Vmatrix} Z_{11} + R + Z_G & Z_{12} \\ Z_{21} & Z_{22} + R + Z_L \end{Vmatrix}^{-1} = \frac{1}{D} \begin{Vmatrix} Z_{22} + R + Z_L & -Z_{12} \\ -Z_{21} & Z_{11} + R + Z_G \end{Vmatrix}, \tag{20}$$

and D is the determinant of the matrix:

$$D = (Z_{11} + R + Z_G)(Z_{22} + R + Z_L) - Z_{12}Z_{21}. \tag{21}$$

The output current, and hence the output power, may be determined from (19) when the voltages $\mathcal{E}_1$ and $\mathcal{E}_2$ are specified. This equation will be used as a starting point for calculations of gain and noise figure.

V. GENERAL EQUATIONS FOR MAXIMUM AVAILABLE GAIN

We shall calculate the gain of the frequency converter by using the condition

$$\begin{aligned} \mathcal{E}_1 &= \mathcal{E}, \\ \mathcal{E}_2 &= 0, \end{aligned} \tag{22}$$

where $\mathcal{E}$ is the signal voltage. The gain G is defined by

$$G = \frac{\text{power absorbed in } Z_L}{\text{power available at input}} = \frac{|\,i_2\,|^2 \operatorname{Re} Z_L}{|\,\mathcal{E}\,|^2/4 \operatorname{Re} Z_G}. \tag{23}$$

We shall calculate G with the aid of (19), (20) and (22), and then determine the values of Z_G and Z_L leading to a maximum G, designated by $G_{\max}$.

We first solve for i_2 by substituting the conditions of (22) into (19) and obtain

$$i_2 = -\frac{1}{D} Z_{21}\mathcal{E}. \tag{24}$$

Hence, for the gain we have

$$\frac{|\,Z_{21}\,|^2\,|\,\mathcal{E}\,|^2\, R_L/|\,D\,|^2}{|\,\mathcal{E}\,|^2/4R_G} = 4R_GR_L\frac{|\,Z_{21}\,|^2}{|\,D\,|^2} \tag{25}$$

which, with the aid of (21) becomes

$$G = \frac{4R_GR_L\,|\,Z_{21}\,|^2}{|(Z_{11} + R + Z_G)(Z_{22} + R + Z_L) - Z_{12}Z_{21}\,|^2}. \tag{26}$$

To maximize G, we first impose the tuning conditions

$$j \operatorname{Im} Z_G = -Z_{11}, \qquad j \operatorname{Im} Z_L = -Z_{22}.$$

[Note that, according to (13), the quantities Z_{11} and Z_{22} are pure imaginary numbers]. As a result, we obtain

$$G = \frac{4R_G R_L \mid Z_{21} \mid^2}{[(R + R_G)(R + R_L) - Z_{12}Z_{21}]^2}. \quad (27)$$

Here R_G is Re Z_G, R_L is Re Z_L, and $Z_{12}Z_{21}$ is a real number (since it is the product of two imaginary numbers), and we may dispense with the absolute magnitude sign in the denominator.

To obtain G_{max} we must maximize the right hand side of (27) with respect to R_G and R_L. The values of R_G and R_L satisfying this condition are the matching values, and will be designated by R_G' and R_L'. They may be obtained by solving the pair of equations

$$\frac{\partial G}{\partial R_G} = 0, \qquad \frac{\partial G}{\partial R_L} = 0. \quad (28)$$

On solving these we obtain

$$R_G = \frac{R^2K^2 + RR_L}{R + R_L}, \qquad R_L = \frac{R^2K^2 + RR_G}{R + R_G}, \quad (29)$$

where $K = \sqrt{1 - Z_{12}Z_{21}/R^2}$. These equations give the value of R_G which maximizes G for any selected value of R_L, and correspondingly for R_L. The matching values R_G' and R_L' satisfy (29) simultaneously. They are

$$R_G' = R_L' = KR. \quad (30)$$

The equality of the input and output matching resistors could have been predicted from the form of (27) for G, which is completely symmetrical in R_G and R_L; i.e. an interchange of R_G and R_L, whatever their values, does not alter the value of G. Thus, let us suppose that we determine R_G' and R_L' experimentally by trial and error, and assume that they are a unique pair. Interchanging them must leave G a maximum, hence R_G' equals R_L'.

An expression for G_{max} may now be obtained by substituting (30) into (27). We find, noting that $Z_{12}Z_{21}$ equals $-R^2(K^2 - 1)$,

$$G_{max} = \frac{4K^2R^2 \mid Z_{21} \mid^2}{[R^2(K + 1)^2 - Z_{12}Z_{21}]^2} = \frac{\mid Z_{21} \mid^2}{R^2(K + 1)^2}. \quad (31)$$

Equation (31) expresses the maximum available gain as a function of the diode parameters R and $C_{\min}$, and of the time variation of the diode capacitance, through the quantities C_0 and C_1. For a given diode the first two quantities are assumed fixed, while the last two may be varied within limits set by their status as Fourier coefficients of a real, positive, periodic function of time. In the Appendix, $G_{\max}$ is evaluated explicitly in terms of R and $C_{\min}$ for the case that Z_c is given by (13). The result is

$$G_{\max} = \frac{f_2}{f_1} \frac{1}{(x + \sqrt{1 + x^2})^2}, \tag{32}$$

where $\sqrt{1 + 1/x^2} = K$ as introduced in (29), and

$$x = \lambda \bar{\omega} C_{\min} R,$$

with λ a numerical factor depending upon C_0 and C_1, and

$$\bar{\omega} = \sqrt{\omega_1 \omega_2} = \sqrt{s(b + s)}.$$

To simplify future analysis, we write

$$x = \lambda \frac{\bar{f}}{f_c}, \tag{33}$$

where

$$\bar{f} = \sqrt{f_1 f_2}$$

and

$$f_c = \frac{1}{2\pi R_s C_{\min}} = \text{a figure of merit for the diode.}$$

The expression for x in (33) is convenient for evaluating $G_{\max}$ from (32). We shall first consider low-frequency operation, where $\bar{f}$ is very much less than f_c/λ. Then x is very much less than 1, and

$$G_{\max} \approx f_2/f_1 .$$

As x increases, $G_{\max}/(f_2/f_1)$ monotonically decreases from its asymptotic value of 1. This is shown graphically in Fig. 4. Since $x = \lambda \bar{f} 2\pi R_s C_{\min}$, the best operation is obtained for minimum values of R_s, $C_{\min}$ and λ, or a maximum value of f_c/λ. In the Appendix it is shown that, for the type of operation under consideration, the minimum attainable value of λ is 5.83.

We next consider high-frequency operation, characterized by

$\bar{f} \gg f_c/\lambda$ or $x \gg 1$. It is then convenient to rewrite (32) in the following form:

$$G_{\max} = \frac{f_2}{f_1 x^2} \frac{1}{\left(1 + \sqrt{1 + \frac{1}{x^2}}\right)^2} = \frac{f_2}{f_1 x^2} \frac{1}{(1 + K)^2}. \tag{34}$$

As x approaches ∞, (34) becomes

$$\lim_{x \to \infty} G_{\max} \to G_{\text{lim}} = \frac{f_2}{4 f_1 x^2} = \left(\frac{f_c}{2\lambda f_1}\right)^2. \tag{35}$$

Since we would like to have G_{lim} as large as possible, it is evident that f_c/λ should be large, or, correspondingly, that R_s, $C_{\min}$ and λ should be small, as was true for the low-frequency limit of operation.

Equation (35) indicates that, for a given input frequency f_1, the gain should approach a limiting value as f_2 approaches ∞. This behavior is shown graphically in Fig. 5. It is noteworthy that G_{lim} decreases with increasing f_1, and in particular that, for $f_1 \geqq f_c/2\lambda$, $G_{\text{lim}} \leqq 1$. This places a fundamental upper limit on the input frequency, if power gain is to be obtained.

The above results show how the maximum available gain of a diode amplifier is affected by parasitics. Of greater significance in characteriz-

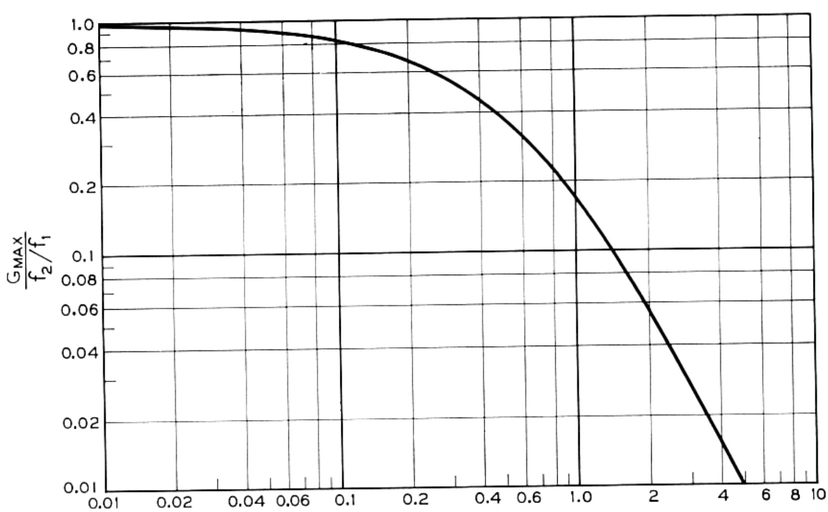

Fig. 4 — Ratio of MAG of a diode with parasites to the MAG of an ideal variable capacitance, plotted against the parameter x.

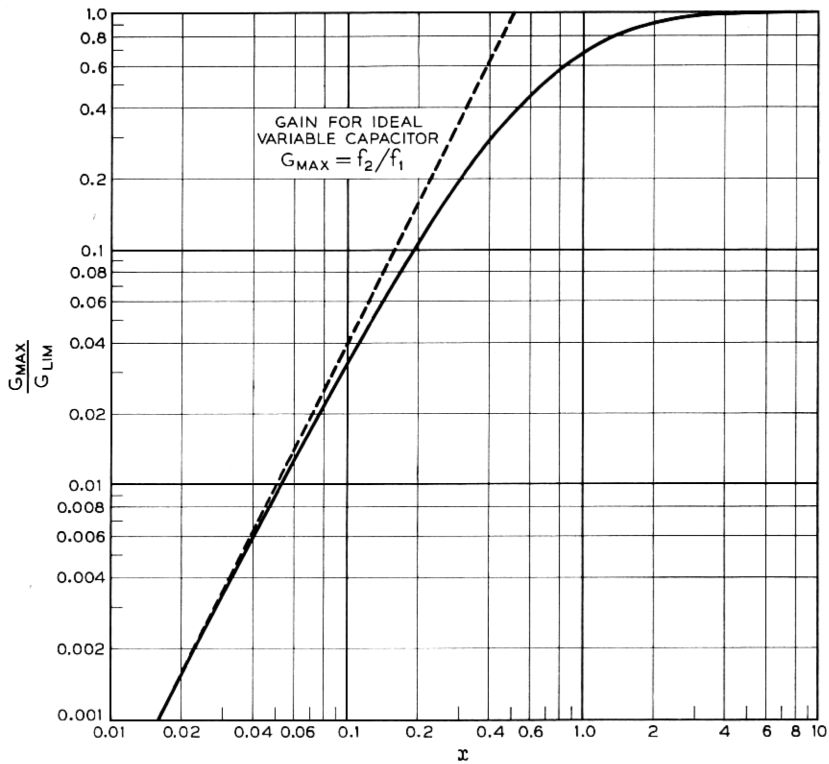

Fig. 5 — The ratio G_{max}/G_{lim} versus x, the input frequency f_1 being kept constant while the output frequency f_2 varies.

ing its performance is the noise figure, which is discussed in the next section.

VI. GENERAL EQUATIONS FOR NOISE FIGURE

A p-n junction diode is thought to contain two sources of noise power, one being shot noise at the p-n junction, the other being thermal noise in the series resistance R of the bulk semiconductor. It has been shown[6] that the shot noise of the p-n junction of a nonlinear-capacitance diode approaches zero so long as the frequency is high enough for recombination to be negligible, and not so high that the p-n junction no longer exhibits purely capacitative behavior. Hence in the following calculation of noise figure only the contribution of the thermal noise of R is considered.

The noise figure of an N-C diode frequency converter is calculated using the equivalent circuit of Fig. 3 and the assumptions of Section III, together with the additional assumption that the noise generators at the frequencies f_1 and f_2 are uncorrelated.

The noise figure is calculated in accordance with

$$F - 1 = \frac{\text{noise output originating in thermal noise of } R}{\text{noise output due to thermal noise at 290°K in source resistance}}. \tag{36}$$

Thus, as a "standard signal", one uses the thermal noise output of the source resistance, assumed to be at a temperature of 290°K. The noise output of the diode is assumed to arise from two components of the thermal noise of R. One of these, the thermal noise emf $\mathcal{E}_{N1}$, sets up a current of frequency f_1 in the left-hand loop of Fig. 3, which, by the frequency-converting action of the diode, is partly converted into a current i_2, of frequency f_2; this then passes into the right-hand loop and introduces noise power into the load. The other component, $\mathcal{E}_{N2}$, also causes a current i_2, of frequency f_2, to flow through the load. The noise power received by the load is determined by these two currents, which are assumed to be uncorrelated.

The noise power can be readily calculated from (19) and (20). The current i_2 due to $\mathcal{E}_{N1}$ is given by solving

$$\left\| \begin{matrix} i_1 \\ \\ i_2 \end{matrix} \right\| = \frac{1}{D} \left\| \begin{matrix} Z_{22} + R + Z_L & -Z_{12} \\ \\ -Z_{21} & Z_{11} + R + Z_G \end{matrix} \right\| \left\| \begin{matrix} \mathcal{E}_{N1} \\ \\ 0 \end{matrix} \right\|. \tag{37a}$$

Similarly, the current i_2 due to $\mathcal{E}_{N2}$ is obtained from

$$\left\| \begin{matrix} i_1 \\ \\ i_2 \end{matrix} \right\| = \frac{1}{D} \left\| \begin{matrix} Z_{22} + R + Z_L & -Z_{12} \\ \\ -Z_{21} & Z_{11} + R + Z_G \end{matrix} \right\| \left\| \begin{matrix} 0 \\ \\ \mathcal{E}_{N2} \end{matrix} \right\|. \tag{37b}$$

Solving these, we obtain

$$\mathcal{E}_{N1} \neq 0: \quad i_2 = -\frac{1}{D} Z_{21}\mathcal{E}_{N1}, \tag{38a}$$

$$\mathcal{E}_{N2} \neq 0: \quad i_2 = \frac{1}{D} (Z_{11} + R + Z_G)\mathcal{E}_{N2}. \tag{38b}$$

The noise power delivered to the load is then:

$$\langle i_2^2 \rangle R_L = \frac{4kTBRR_L}{| D |^2} (| Z_{11} + R + Z_G |^2 + | Z_{21} |^2), \tag{39}$$

in accordance with the assumption that the currents in (38) are uncorrelated. Following (36), the noise figure is given by

$$F - 1 = \frac{\langle i_2^2 \rangle R_L}{290kBG}, \tag{40}$$

where $290kB$ is the available thermal noise of the source resistance and G is the gain of the network under the same conditions, defined by (23). The gain was calculated in Section V, with the result in (25), given here by

$$G = 4R_G R_L \frac{| Z_{21} |^2}{| D |^2}. \tag{41}$$

Combining this with (39) and (40), we obtain

$$F - 1 = \frac{R}{R_G} \frac{| Z_{11} + R + Z_G |^2 + | Z_{21} |^2}{| Z_{21} |^2} \frac{T}{290}. \tag{42}$$

The noise figure here depends upon the diode structure, through R, Z_{11} and Z_{21}; upon the beat frequency oscillator waveform and the operating frequencies, through Z_{11} and Z_{21}; and upon the input termination, through Z_G. (Note that F is independent of the load impedance.) Even if all the other quantities are fixed, considerable variation in performance can be obtained by varying Z_G. In this presentation, the noise figure is calculated for the value of Z_G which gives maximum available gain, and for the value giving minimum noise figure.

The noise figure corresponding to the condition of maximum available gain is obtained by inserting the appropriate value of Z_G into (42). In Section V this was shown to be $Z_G = KR - Z_{11}$. Equation (42) then gives

$$F - 1 = \frac{1}{K} \frac{R^2(K + 1)^2 + | Z_{21} |^2}{| Z_{21} |^2} \frac{T}{290}. \tag{43}$$

On the other hand, from (31) we have

$$G_{\max} = \frac{| Z_{21} |^2}{R^2(K + 1)^2}.$$

Hence, (43) may be written

$$F - 1 = \frac{1}{K}\left(1 + \frac{1}{G_{\max}}\right)\frac{T}{290}. \tag{44}$$

Before proceeding further, it is interesting to note that (44) can be obtained by a simple intuitive analysis, making use of (36) directly. We assume that the diode series resistance R generates noise emfs equal to $\sqrt{4kTBR}$ at the two frequencies f_1 and f_2. Due to the emf at frequency f_2, the load receives noise power proportional to R. The noise power generated at frequency f_1 is also proportional to R; it is converted to frequency f_2 with a gain G, so that the load receives additional noise power proportional to GR, making the total noise power from the diode proportional to $(G + 1)RT$. This is to be compared with the noise received at the load due to a matching resistance at the input. This matching resistance $R_G = KR$, and the corresponding noise power received is proportional to $290GKR$. Hence, according to (36),

$$F - 1 = \frac{(G + 1)RT}{290GKR} = \frac{1}{K}\left(1 + \frac{1}{G_{\max}}\right)\frac{T}{290}.$$

Equation (44) can be expressed in a more useful form by substituting an equivalent expression for K. From (34) and (35) it follows that

$$G_{\max} = G_{\lim} \frac{4}{(1 + K)^2}. \tag{45}$$

Hence

$$K = 2\left(\frac{G_{\lim}}{G_{\max}}\right)^{1/2} - 1. \tag{46}$$

Substituting this into (44) gives

$$F = 1 + \frac{1}{2\left(\frac{G_{\lim}}{G_{\max}}\right)^{1/2} - 1}\left(1 + \frac{1}{G_{\max}}\right)\frac{T}{290}. \tag{47}$$

This is the noise figure corresponding to a matching input impedance. In using (47), $G_{\max}$ and $G_{\lim}$ may be determined for any set of operating conditions, with the help of (34) and (35). Two limiting expressions obtained from (47) are of interest:

$$G_{\max} \ll G_{\lim}: \quad F \cong 1 + \frac{1}{2}\left(\frac{G_{\max}}{G_{\lim}}\right)^{1/2}\left(1 + \frac{1}{G_{\max}}\right)\frac{T}{290}, \tag{48a}$$

$$G_{\max} \cong G_{\lim}: \quad F \cong 1 + \left(1 + \frac{1}{G_{\max}}\right)\frac{T}{290}. \tag{48b}$$

Hence, when $G_{\max}$ is very much less than $G_{\lim}$, F is not much greater than 1. When $G_{\max}$ approaches $G_{\lim}$, F becomes approximately equal to to 2. Thus, even under the most unfavorable conditions, the theoretical noise figure will not be much more than 3 db if there is a significant amount of gain.

The minimum noise figure may also be found from (42), and applying first the tuning condition $jX_G = -Z_{11}$, we obtain

$$F - 1 = \frac{R}{R_G} \frac{(R + R_G)^2 + |Z_{21}|^2}{|Z_{21}|^2} \frac{T}{290}. \tag{49}$$

The value of R_G making F a minimum can be found from (49) by the standard procedures of differential calculus. The appropriate value of R_G, represented by R_G'', is

$$R_G'' = (|Z_{21}|^2 + R^2)^{1/2}. \tag{50}$$

Here $G_{\max}$ is, as before, the maximum available gain, obtained with the matching value of R_G. On comparing (31) and (45), one determines that

$$|Z_{21}|^2 = 4R^2 G_{\text{lim}}. \tag{51}$$

Substituting (50) and (51) into (49) gives

$$\begin{aligned} F_{\min} &= 1 + \frac{(4G_{\text{lim}} + 1)^{1/2} + 1}{2G_{\text{lim}}} \frac{T}{290} \\ &\cong 1 + \frac{T}{290}\left(\frac{1}{\sqrt{G_{\text{lim}}}} + \frac{1}{2G_{\text{lim}}} + \cdots\right). \end{aligned} \tag{52}$$

The approximate form is valid when $G_{\text{lim}} \gg 1$. Equation (52) indicates that a noise figure of close to zero db is theoretically attainable.

It is also important to determine the maximum gain G_N attainable under the conditions leading to minimum noise figure, as G_N may be needed to calculate the over-all noise figure of a system using a diode preamplifier. To obtain G_N, we substitute into (27) the value $R_G = R_G''$ given by (50), together with the value of R_L that makes G a maximum. This quantity, represented by R_L'', is found from (29). In the present case it is:

$$R_L'' = \frac{R^2K^2 + RR_G''}{R + R_G''}. \tag{53}$$

Substituting (53) and (50) into (27) gives G_N. Making use of the definition $K = \sqrt{1 - Z_{12}Z_{21}/R^2}$ and $|Z_{21}|^2 = 4R^2G_{\text{lim}}$ given in (51), we find

$$G_N = \frac{4}{1 + R/R_G''} \frac{G_{\text{lim}}}{R_G''/R + K^2}. \tag{54}$$

Using the relation $K = 2(G_{1\text{im}}/G_{\max})^{1/2} - 1$ given in (46), equation (50) and the assumption $G_{1\text{im}} \gg 1$, we obtain the approximate expression:

$$G_N \cong \left\{ \frac{4G_{1\text{im}}}{\left[2\sqrt{G_{1\text{im}}} + \left[2\left(\frac{G_{1\text{im}}}{G_{\max}}\right)^{1/2} - 1\right]\right]^2} \right\} \left(\frac{1}{1 + \frac{1}{2\sqrt{G_{1\text{im}}}}} \right). \quad (55)$$

We now consider two special cases. For $G_{\max}$ sufficiently small,

$$\left(\frac{G_{1\text{im}}}{G_{\max}}\right)^{1/2} \gg 1, \quad \text{and} \quad \frac{G_{1\text{im}}}{G_{\max}} \gg \sqrt{G_{1\text{im}}}\,.$$

Therefore

$$G_N \cong G_{\max}\,. \quad (56a)$$

For $G_{\max} \cong G_{1\text{im}}$

$$G_N \cong 2\sqrt{G_{1\text{im}}}\,. \quad (56b)$$

The equations obtained for the noise figure have been evaluated for certain receiver systems, and the results are given in Table I. The examples include, respectively, an N-C diode up-converter followed by (1) a travelling-wave tube amplifier operating at 3 kmc with a noise figure of 4.5 db; (2) a point-contact superheterodyne converter plus IF amplifier stage operating at an input frequency of 10 kmc with a noise figure of 7.0 db; (3) a system similar to (2) operating at an input frequency of 55 kmc with a noise figure of 10 db (a value that may seem optimistic but is probably attainable by combining the best point-contact diodes with the best IF amplifier). Values for over-all noise figures of each system are obtained from equations (47) and (52), and compared; these corre-

TABLE I—NOISE FIGURES FOR CERTAIN RECEIVER SYSTEMS

f_c = 160 kmc λ = 6.0 T = 290°K

Receiver System	Output Frequency, f_2 (kmc)	Noise Figure, F_2 (db)	Over-all Noise Figure of System, F_{system} (db)			
			Input Frequency, f_1 = 500 mc		Input Frequency, f_1 = 1000 mc	
			F_A	F_B	F_A	F_B
(1)	3	4.5	1.40	1.40	2.51	2.48
(2)	10	7.0	1.24	1.20	2.15	2.20
(3)	55	10.0	1.20	1.14	1.94	2.10

F_A = noise figure of system corresponding to maximum available gain for diode.
F_B = noise figure of system corresponding to minimum noise figure for diode.

spond to the conditions of maximum available gain and minimum noise figure, respectively. The difference between them is not significant in most cases. The calculations were made assuming $\lambda = 6.0$, $T = 290°K$ and $f_c = 160$ kmc, the latter figure being obtainable with recently produced diffused silicon diodes.

VII. CONCLUSIONS

Equations for the maximum available gain (MAG), and the noise figure have been obtained for a nonlinear capacitance diode containing parasitics. Because of the series resistance and minimum capacitance, the MAG reaches a limiting value as the output frequency increases without limit. The thermal noise of the series resistance is also responsible for noise output of the diode. In contrast, a perfect variable capacitance would have its MAG always equal to the ratio of output frequency to input frequency, and it would generate no noise.

The results obtained depend upon the diode cutoff frequency f_c, and upon the operating conditions through the parameter λ. Values of f_c up to 200 kmc have been obtained with recently prepared diffused diodes, hence the use of the value $f_c = 160$ kmc is not overly optimistic.

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APPENDIX

Derivation of the Equation for Maximum Available Gain

In Section V the maximum available gain was obtained in the following form, from (31):

$$G_{\max} = \frac{|Z_{21}|^2}{R^2(K+1)^2}, \tag{57}$$

where $K = \sqrt{1 - Z_{12}Z_{21}/R^2}$. It is convenient to define the quantity x such that $K = \sqrt{1 + 1/x^2}$. Hence

$$x^2 = -\frac{R^2}{Z_{12}Z_{21}} \tag{58}$$

After (58) has been solved for R^2, (57) can be written

$$G_{\max} = \frac{|Z_{21}|^2}{-Z_{12}Z_{21}} \frac{1}{x^2(K+1)^2} = \frac{|Z_{21}|^2}{-Z_{12}Z_{21}} \frac{1}{(x + \sqrt{x^2+1})^2}. \tag{59}$$

The impedance matrix elements, obtained from (13), are

$$Z_{12} = j\frac{1+\xi^2}{1-\xi^2}\frac{\xi}{b+s}\frac{1}{C_0}, \qquad Z_{21} = j\frac{1+\xi^2}{1-\xi^2}\frac{\xi}{s}\frac{1}{C_0}. \tag{60}$$

Then

$$G_{\max} = \frac{b+s}{s}\frac{1}{(x+\sqrt{x^2+1})^2} = \frac{\omega_2}{\omega_1}\frac{1}{(x+\sqrt{x^2+1})^2}, \tag{61}$$

where

$$x = \frac{1-\xi^2}{\xi(1+\xi^2)}\bar{\omega}RC_0. \tag{62}$$

The quantity $G_{\max}$ in (61) is the gain obtained with matching terminations. It depends upon the diode parameters R_s and $C_{\min}$, and upon the operating conditions through the operating frequencies and the Fourier amplitudes C_0 and C_1. The diode parameters are fixed for a given device, of course, but C_0 and C_1 may be varied to further maximize G. To demonstrate this, we first express the variables C_0 and C_1 in terms of $C_{\max}$ and $C_{\min}$, where $C_{\min}$ is a constant, namely, the lowest value of capacitance obtainable. [From (69), below, it can be seen that, for any given value of $C_{\max}$, the gain increases with decreasing $C_{\min}$]. We therefore need consider only $C_{\max}$ to be variable. For convenience, in the calculation of maximum gain, below, the variable μ is used, where $\mu^2 = C_{\max}/C_{\min}$.

The process of maximizing G is equivalent to minimizing x. To accomplish this we start with (62) and express the quantities ξ and C_0 in terms of μ and $C_{\min}$. From (12a) we have

$$\zeta \equiv \frac{C_1}{C_0} = \frac{\xi}{1+\xi^2}. \tag{63}$$

Since $C(t)$ is a sine wave, it follows that

$$C_0 - C_{\min} = 2C_1. \tag{64}$$

Equations (63) and (64) may be used to eliminate C_1, obtaining

$$C_0 = \frac{1+\xi^2}{(1-\xi)^2}C_{\min}. \tag{65}$$

Next, we express ξ in terms of μ. As $C(t)$ is sinusoidal, it also follows that

$$C_{\max} = C_0 + 2C_1. \tag{66}$$

Combining (64), (65) and (66), we obtain

$$C_{\max} = \left(\frac{1+\xi}{1-\xi}\right)^2 C_{\min}, \tag{67a}$$

or

$$\mu^2 \equiv \frac{C_{\max}}{C_{\min}} = \left(\frac{1+\xi}{1-\xi}\right)^2. \tag{67b}$$

Solving for ξ, we obtain

$$\xi = \frac{\mu - 1}{\mu + 1}. \tag{68}$$

Finally, combining (62), (65) and (68), we get

$$x = \frac{\mu(\mu + 1)}{\mu - 1} \bar{\omega} R C_{\min},$$

or (69)

$$\lambda = \frac{\mu(\mu + 1)}{\mu - 1}.$$

The minimum value of λ is found to be

$$\lambda = (1 + \sqrt{2})^2 = 5.83$$

for (70)

$$\mu = 1 + \sqrt{2}.$$

The corresponding value of $C_{\max}/C_{\min}$ is

$$\frac{C_{\max}}{C_{\min}} = \mu^2 = (1 + \sqrt{2})^2 = 5.83. \tag{71}$$

Thus, the maximum gain and minimum noise figure are obtained for a variation of capacitance over a 5.8–to–1 ratio. In practice, however, the dynamic range of capacitance may be no greater than 3 to 1. This corresponds to $\mu = \sqrt{3}$, and $\lambda = 6.5$. Hence, for this limited range the performance would still not be far from optimum.

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