The Nonuniform Transmission Line as a Broadband Termination

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The problem of obtaining a broadband microwave termination is considered from the point of view of nonuniform transmission line theory. Attention is restricted to lines in which only the distributed shunt admittance may be varied. An optimization argument is presented which leads to the consideration of a line in which the fractional increase in admittance per wavelength in the line is constant. The nonuniform transmission line equations are solved exactly for this case, and the results are expressed in terms of readily interpretable elementary functions. It is shown that a fixed geometrical length of line can lead to an arbitrarily large effective length without destroying the match at the input. The introduction of a small loss term makes the line almost totally absorbing regardless of its termination.

The line has a long-wavelength cutoff given by 4π times the actual length of the line. If the line is short-circuited at its far end, a return loss of greater than 11.4 db is obtained at all frequencies above twice the cutoff frequency. The effect of certain practical limitations on the performance of this line is also discussed.

I. INTRODUCTION

Classical transmission line analysis leads to the propagation of a wave in which neither the electric nor magnetic fields have components in the direction of propagation.[1] These transverse electromagnetic (TEM) waves are characteristic not only of the usual transmission line structures such as parallel wires and coaxial cable; they are also characteristic of plane-wave propagation in isotropic media.

It is often convenient, when dealing with TEM-wave propagation, to make use of results of classical transmission line analysis. Some care must be exercised, however, in applying these results at microwave frequencies. Consider, for example, the problem of terminating a lossless line. Classical analysis tells us that, if the line is terminated in its characteristic impedance (which is a pure resistance for a lossless line), the termination will be reflectionless. Two difficulties arise at microwave frequencies, where the physical dimensions are no longer small compared to the wavelength. First, as the frequency increases, the concept of a lumped circuit element becomes less meaningful. For example, a resistive disc termination for a coaxial line will have an effective impedance which is strongly influenced by the geometry and has very little correlation with the dc resistance of the disc.[2] The second difficulty is that, even if the appropriate effective lumped impedance is obtained, the analysis assumes that this impedance is connected across an open circuit. An open-ended line at high frequencies is by no means an electrical open circuit. There is a true open circuit, however, one-quarter wavelength in front of a short circuit. Therefore, if the line is short-circuited by a metallic surface, and the appropriate characteristic impedance is placed one-quarter wavelength in front of the short, the termination will be reflectionless. However, if the frequency is changed the quarter-wave condition is destroyed, so that the termination is not broadband.

The purpose of this paper is to consider analytically the use of a nonuniform transmission line as a broadband termination. If this structure is to be finite, it too must be terminated. To make the results independent of what is beyond the nonuniform line and still maintain physical realizability, it will be assumed that the nonuniform line is terminated by a short circuit. The problem, then, is to match from a given characteristic impedance, Z_0, to a short circuit by means of an appropriate nonuniform transmission line.

There is extensive literature[3] on the use of nonuniform transmission lines for impedance matching. Optimum matching procedures have been discussed[4] for matching two uniform lossless lines to one another by means of a lossless nonuniform line. In addition to the assumption of no loss, it is assumed that the magnitude of the reflection coefficient is much less than unity at all points along the line. Both of these assumptions, however, are not applicable for matching to a short circuit. Since the object here is to absorb all incident energy, the line cannot be lossless. Also, the magnitude of the reflection coefficient is unity at the short circuit. Thus, the matched-termination problem must be considered apart from the usual matching problem.

In order to define the problem more precisely, it is necessary to consider the transmission line equations which determine the voltage V and current I along the line:

$$\frac{dV}{dx} = -Z(x)I, \tag{1}$$

$$\frac{dI}{dx} = -Y(x)V, \tag{2}$$

where $Z(x)$ is the distributed series impedance per unit length and $Y(x)$ is the distributed shunt admittance per unit length. The line is said to be uniform if both these quantities are constant.

A uniform line may be used to match from an impedance of Z_0 to a short circuit if Z and Y are both large complex constants (so that the wavelength in the line is small and the losses are large) and if, in addition, $\sqrt{Z/Y}$ equals Z_0. Thus, in principle, a fixed length of uniform line can be made electrically long and extremely lossy, and yet it may have the same characteristic impedance Z_0 as does the line it is desired to terminate. In practice, however, this may be difficult to accomplish.

Consider, for example, the problem of finding a microwave-absorbing material for anechoic chambers. Using the above principle it would be necessary to find a material with large complex relative permeability and relative dielectric constant, but such that the ratio of these two quantities was unity. It is not too difficult to obtain high dielectric constants at microwave frequencies,[5] but it is difficult in general to obtain equally large permeabilities.[6] It is of interest, then, to consider a medium with a permeability equal to that of free space, and to attempt to match this medium to a short circuit by increasing the dielectric constant as the termination is approached. In the analogous transmission line problem, the distributed series impedance, $Z(x)$, is a fixed constant, and only the distributed shunt admittance, $Y(x)$, is at our disposal.

The following problem is thus suggested. A uniform lossless line, characterized by a distributed series impedance per unit length of $j\omega L_1$ and a distributed shunt admittance per unit length of $j\omega C_1$, is to be terminated by a nonuniform line of length s (see Fig. 1). The nonuniform line has a constant distributed series impedance per unit length of $j\omega L_1$, but the distributed shunt admittance per unit length, $Y(x)$, is as yet unspecified. This line is, in turn, terminated in a short circuit at $x = s$ It is desired to find that function $Y(x)$ which minimizes the reflection coefficient at $x = 0$.

The general variational problem is outside the scope of the present work. In the following sections a somewhat intuitive argument will be presented which leads to a particular form for $Y(x)$. An exact solution of the transmission line equations will be obtained for this particular nonuniform line, and it will be shown that the line indeed has the properties indicated by the simpler intuitive arguments.

II. DETERMINATION OF THE ADMITTANCE VARIATION

Approximation techniques[7] indicate that, if the fractional change in the properties of the line per "local wavelength over 2π"* is small, then reflections may be assumed to be negligible. The exact solutions of particular nonuniform transmission lines, such as the exponential line,[8] indicate that reflections become important when this condition is violated.

Since we are attempting to match from an admittance level $j\omega C_1$ to an infinite admittance, it is natural to ask the following question: How large can the admittance be made at $x = s$, subject to the conditions

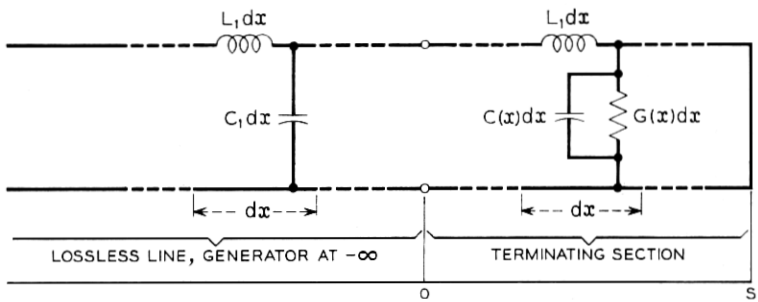

Fig. 1 — Nonuniform transmission line termination.

that it be equal to $j\omega C_1$ at $x = 0$ and that the fractional change in admittance per local wavelength in the line be small?

The wavelength in the uniform lossless line characterized by L_1 and C_1 is

$$\lambda = \frac{2\pi}{\omega\sqrt{L_1 C_1}}. \tag{3}$$

The local wavelength in the nonuniform line is given by $\lambda/\sqrt{\epsilon(x)}$, where

$$\epsilon(x) = \frac{Y(x)}{j\omega C_1} = \frac{C(x)}{C_1} - \frac{jG(x)}{\omega C_1}, \tag{4}$$

in which $C(x)$ is the distributed shunt capacitance per unit length and $G(x)$ is the distributed shunt conductance per unit length. For the pur-

* "Local wavelength" is defined as the wavelength in a uniform line which has the same distributed constants as the line in question at the point in question.

pose of the present section, it will be assumed that the nonuniform line is lossless, so that $G(x)$ is zero and $\epsilon(x)$ is real. The condition that the fractional change in admittance per local wavelength over 2π be small can then be written

$$\frac{\lambda}{2\pi} \frac{1}{\epsilon^{3/2}} \frac{d\epsilon}{dx} \leqq a, \tag{5}$$

where a is an as yet unspecified small constant.

The problem to be considered is thus: Given s greater than zero, find that function $\epsilon(x)$ which maximizes $\epsilon(s)$, subject to the conditions that $\epsilon(0)$ equals 1 and that the inequality (5) is satisfied over the interval $[0, s]$. The solution to this problem is obtained by replacing the inequality in (5) by an equality, which leads to the result

$$\epsilon(x) = \left(1 - \frac{\pi a}{\lambda} x\right)^{-2}. \tag{6}$$

Thus, as x approaches $\lambda/\pi a$, $\epsilon(x)$ becomes infinite. It would then appear that, in a fixed length of line, an arbitrarily large change in admittance can be utilized, and consequently a large effective length obtained, without violating the slowly varying condition. The introduction of a small imaginary component (shunt conductance) to $\epsilon(x)$ should then make the line totally absorbing regardless of termination.

In order to verify the above conjectures analytically it is necessary to solve the transmission line equations. This will be done in the following section.

III. SOLUTION OF THE TRANSMISSION LINE EQUATIONS

For a line with uniformly distributed series impedance per unit length $j\omega L_1$ and distributed shunt admittance per unit length $j\omega C_1\epsilon(x)$, the transmission line equations may be rewritten in the form

$$\frac{d^2V}{dx^2} + \left(\frac{2\pi}{\lambda}\right)^2 \epsilon(x)V = 0, \tag{7}$$

$$\frac{d^2I}{dx^2} - \frac{1}{\epsilon}\frac{d\epsilon}{dx}\frac{dI}{dx} + \left(\frac{2\pi}{\lambda}\right)^2 \epsilon(x)I = 0, \tag{8}$$

where λ is given by (3).

If $\epsilon(x)$ is of the form $(A + Bx)^n$, (7) may be transformed[9] into Bessel's equation of order $1/(n + 2)$. The fact that considerable simplifications result when n equals -2 has been noted previously,[10] but the physical implications do not appear to have been discussed.

The expression for $\epsilon(x)$, given by (6), is real. Since we are interested in absorption, the shunt admittance must contain a conductance term in addition to the capacitance, and consequently ϵ must be complex. In order to maintain the same functional dependance for ϵ, it will be assumed that the ratio of shunt conductance to shunt capacitance is constant,

$$\frac{G(x)}{C(x)} = \sigma, \tag{9}$$

so that $\epsilon(x)$ is given by

$$\epsilon(x) = \frac{1 - j\sigma/\omega}{\left(1 - \frac{\pi a}{\lambda} x\right)^2}. \tag{10}$$

If the change of variables

$$r = -\ln\left(1 - \frac{\pi a}{\lambda} x\right) \tag{11}$$

is made, (7) and (8) can be rewritten

$$\frac{d^2V}{dr^2} + \frac{dV}{dr} + \frac{4}{a^2}(1 - j\sigma/\omega)V = 0, \tag{12}$$

$$\frac{d^2I}{dr^2} - \frac{dI}{dr} + \frac{4}{a^2}(1 - j\sigma/\omega)I = 0, \tag{13}$$

where the expression for ϵ as given by (10) has been used. The solutions of these equations are

$$V = e^{-r/2}[V_1 e^{-\gamma r} + V_2 e^{\gamma r}] \tag{14}$$

and

$$I = j\frac{a}{2}\sqrt{C_1/L_1}\, e^{r/2}\left[-\left(\gamma + \frac{1}{2}\right)V_1 e^{-\gamma r} + \left(\gamma - \frac{1}{2}\right)V_2 e^{\gamma r}\right], \tag{15}$$

where

$$\gamma = \sqrt{\frac{1}{4} - \frac{4}{a^2}(1 - j\sigma/\omega)} \tag{16}$$

may be interpreted as an effective propagation constant and r may be interpreted as an effective length.

The lossless line ($\sigma = 0$) is cut off when the effective propagation constant becomes real; that is, when $a \geqq 4$. If the shunt capacitance, $C(x)$, is to be frequency-independent, it follows from the form of $\epsilon(x)$ that a

must be proportional to wavelength. A cutoff wavelength λ_0 and cutoff frequency ω_0 may then be defined such that

$$a = \frac{4\lambda}{\lambda_0} = \frac{4\omega_0}{\omega}. \tag{17}$$

Thus, γ can be written as

$$\gamma = \left(\frac{1}{2}\right)\sqrt{1 - \frac{\omega^2}{\omega_0^2} + j\sigma\frac{\omega}{\omega_0^2}}. \tag{18}$$

It is interesting to note that these results are identical in form to the results that would be obtained for an exponential line,[8] in which the admittance and impedance vary as e^{ax} and e^{-ax} respectively, and where r is defined as ax. However, in order to obtain the same effective length, R, the actual length of the exponential line must be longer by a factor $R/(1 - e^{-R})$, which increases linearly with R for large effective lengths. Thus, the equivalent exponential line will, in general, be considerably longer than the line considered here.

IV. REFLECTION COEFFICIENT

If the boundary condition, $V = 0$ at $x = s$, is substituted into (14), the following relation is obtained for the input admittance:

$$Y_i = \frac{I(x=0)}{V(x=0)} = -2j\frac{\omega_0}{\omega}\sqrt{\frac{C_1}{L_1}}\left(\frac{1}{2} + \gamma\coth\gamma R\right), \tag{19}$$

where R is the value of r corresponding to $x = s$; that is,

$$R = -\ln\left(1 - \frac{4\pi}{\lambda_0}s\right). \tag{20}$$

The complex voltage reflection coefficient, ρ, is given by

$$\rho = \frac{\sqrt{C_1/L_1} - Y_i}{\sqrt{C_1/L_1} + Y_i}. \tag{21}$$

Substitution of (19) into (21) gives

$$\rho = \frac{1 + j\frac{\omega_0}{\omega} + 2j\gamma\frac{\omega_0}{\omega}\coth\gamma R}{1 - j\frac{\omega_0}{\omega} - 2j\gamma\frac{\omega_0}{\omega}\coth\gamma R}. \tag{22}$$

For the reflection coefficient to be small over a broad frequency band, it is necessary that the transmission loss be large; that is, the real part of γR must be large. For any nonzero loss, $\sigma > 0$, one can, in principle,

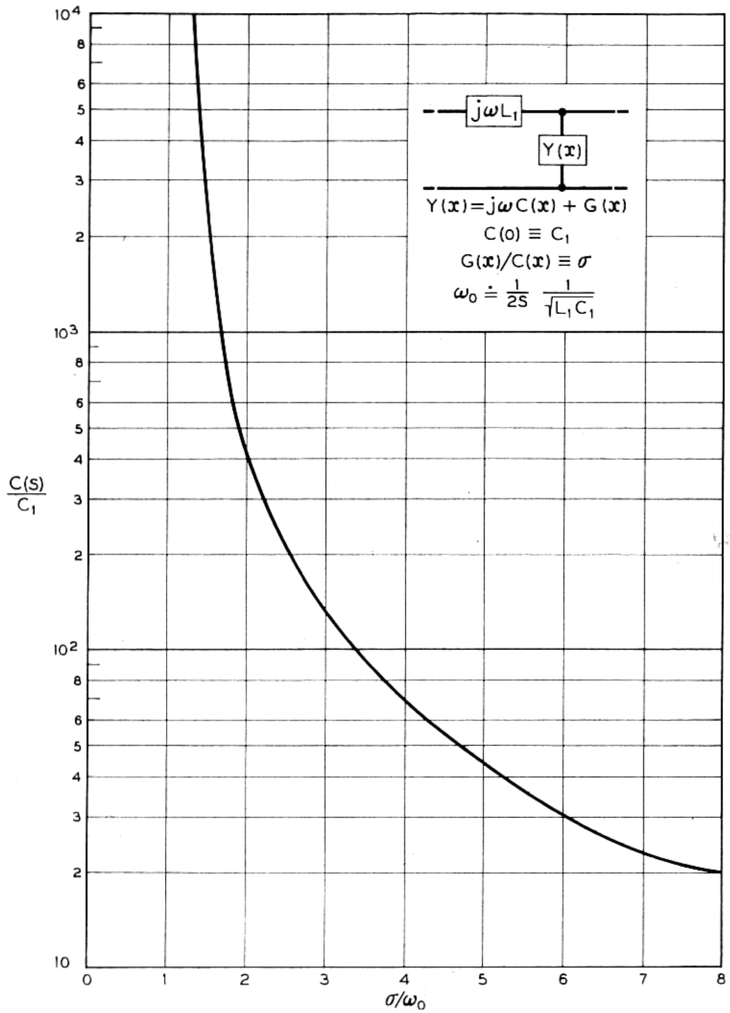

Fig. 2 — Admittance ratio required for 13-db transmission loss.

make the effective length of the line, R, sufficiently large so that the transmission loss is as large as desired. It follows from (20) that this can be accomplished in a line whose length is

$$s = \frac{\lambda_0}{4\pi}(1 - e^{-R}). \tag{23}$$

Thus, if the physical length and the desired effective length of the line are specified, the cutoff wavelength can be determined from (23). For large R, the cutoff wavelength is essentially given by $4\pi s$.

A large value of R is obtained by approaching the singularity in $\epsilon(x)$. The real part of $\epsilon(x)$, at $x = s$, is given by

$$\frac{C(s)}{C_1} = \left(1 - \frac{4\pi}{\lambda_0} s\right)^{-2} = e^{2R}, \tag{24}$$

so that exceedingly large values of distributed shunt capacitance are required to obtain moderately large effective lengths.

The above formalism may be used to calculate the distributed shunt capacitance at $x = s$ required to ensure a given transmission loss. This result will, of course, depend on the value of the loss parameter, σ/ω_0. As an example, in Fig. 2 the value of $C(s)/C_1$ required to ensure a minimum transmission loss of 13 db for all $\omega \geqq \omega_0$ is shown as a function of σ/ω_0.* If σ/ω_0 is very much less than one, an astronomically large value of $C(s)/C_1$ is required. Although this may not be realizable in practice, it gives an upper limit to the ideal behavior of the line.

If the transmission loss is sufficiently large,

$$\coth \gamma R \doteq 1. \tag{25}$$

If, in addition, σ/ω_0 is neglected in comparison to unity, it follows from (18) and (22) that

$$|\rho|^2 = \frac{1 - \sqrt{1 - \omega_0^2/\omega^2}}{1 + \sqrt{1 - \omega_0^2/\omega^2}}. \tag{26}$$

Equation (26) gives the intensity reflection coefficient of the ideal line. The return loss ($-$ 10 $\log_{10} |\rho|^2$) is plotted as a function of frequency in Fig. 3. The return loss is zero at the cutoff frequency, but increases rapidly as the frequency is increased.†

As a more practical example, the return loss will also be calculated for $\sigma/\omega_0 = 2$. It follows from Fig. 2 that this requires $C(s)/C_1$ to be 400 to ensure a 13-db transmission loss. This choice, in addition to being physically reasonable, leads to some computational simplifications. It follows from (18) that, if σ/ω_0 is 2, then

$$\gamma = \frac{1}{2}\left(1 + j\frac{\omega}{\omega_0}\right), \tag{27}$$

* The 13-db transmission loss requirement is equivalent to $|e^{-2\gamma R}| = 0.05$. It can be shown from (18) that the real part of γ has a minimum, α_m, given by the smaller of $\sqrt{\sigma/8\omega_0}$ and $\sigma/4\omega_0$. These two results, together with (24), were used to obtain Fig. 2.

† In Section II the parameter a was introduced, and it was assumed that reflections would be small if a were small. It has since been shown [in (17)] that $a = 4/(\omega/\omega_0)$. Fig. 3 then gives a quantitative demonstration of the initial supposition. For $a = 4$, there is total reflection. As a decreases (ω/ω_0 increases), the reflection is seen to decrease rapidly.

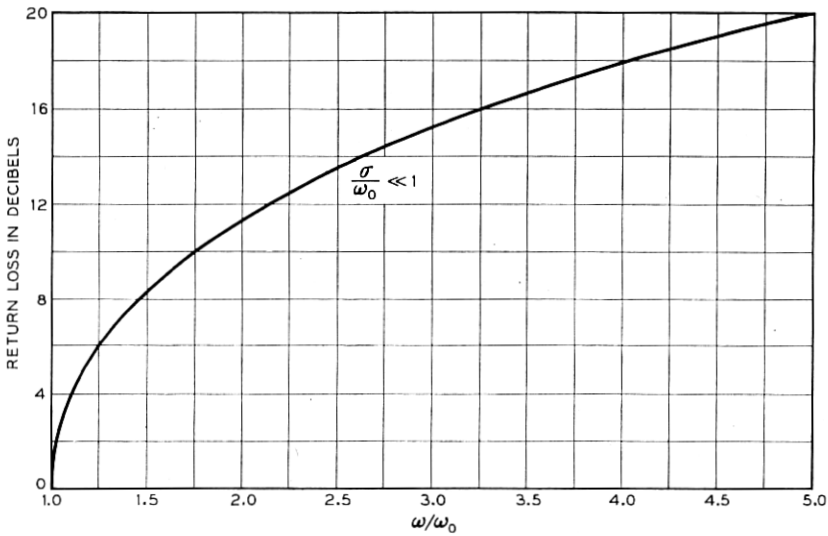

Fig. 3 — Return loss of the "ideal line".

so that the effective attenuation constant (real part of γ) is frequency-independent. The effective length of this line is $R = 3$. For these values of γ and R, coth γR varies between 0.9 and 1.1 as the frequency is changed. If, as before, it is assumed that coth $\gamma R = 1$ (thus assuming $R \gg 1$, which is equivalent to neglecting interference effects due to multiple reflections), the reflection coefficient of the line is given by

$$|\rho|^2 = \frac{1}{1 + (\omega/\omega_0)^2}. \tag{28}$$

If this approximation is not made, the exact expression for $|\rho|^2$ is more complicated but still easily amenable to numerical evaluation. The solid curve in Fig. 4 gives the return loss as a function of frequency, as evaluated from the exact expression. The dashed curve gives the return loss, neglecting interference effects, as determined from (28). In the frequency range depicted, the exact return loss is seen to oscillate about the value obtained when interference effects are neglected. However, in the high-frequency limit ($\omega \gg \omega_0$) the two curves diverge. The return loss increases without limit if the effective length of the line is infinite. However, for $R = 3$, the return loss at high frequencies approaches 26 db, which is just the two-way transmission loss of the line.

It is seen from Fig. 4 that there is a 3-db return loss at the cutoff frequency of the "ideal line." However, as the frequency increases, the loss increases only gradually; at $\omega = 4\omega_0$ the return loss is 13.3, db as com-

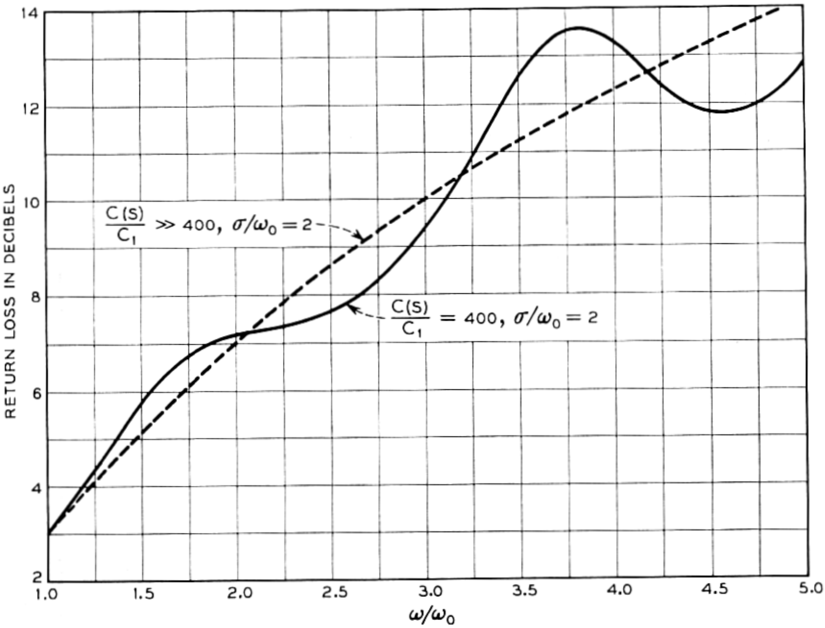

Fig. 4 — Return loss of a "practical line".

pared to 17.8 db for the ideal line. Thus, practical limitations on maximum shunt capacitance have an appreciable influence on the behavior of the line.

V. CONCLUSIONS

Properties of nonuniform transmission lines have led to the consideration of a line in which the fractional change in shunt admittance per wavelength in the line is constant. The transmission line equations have been solved exactly for this case. The solution indicates that a fixed length of line s can be made to have as large an effective length as desired. Hence, with the introduction of a small loss term, all energy matched into the line is essentially completely absorbed regardless of the line's termination.

It has been shown that the line has a long wavelength cutoff given essentially by $\lambda_0 \doteq 4\pi s$. As the frequency increases beyond cutoff frequency ω_0, the reflected intensity from the short-circuited line diminishes rapidly, being 11.4 db down at $2\omega_0$ and 17.8 db down at $4\omega_0$, as seen from Fig. 3.

If practical considerations limit the maximum shunt capacitance, it is necessary to use a larger shunt conductance to obtain the same transmission loss. This degrades the performance of the line somewhat, as shown by the example in Fig. 4. If a return loss of 10 db is required, it is seen from Fig. 4 that the line must be operated at frequencies above 3.1 ω_0. Thus, the length of the line would be $s = \lambda_0/4\pi = 0.25\lambda$, where λ is the longest wavelength for which the return loss would be equal or greater than 10 db. This length is one-third of that which would be required with the equivalent exponential line.

The "ideal line", as indicated by Fig. 3, gives a 10-db return loss at 1.75 ω_0. The length of line required for a 10-db absorption would then be 0.14λ. The differences between the ideal structure and the practical example become even more pronounced as greater absorption is required. However, practical structures may approach the performance of the ideal line if one considers a variation of the loss term σ in addition to the variation $C(x)$.

The nonuniform transmission line analyzed here may be considered to be a singularity of the general Bessel line. It is of analytic interest because the solutions are in the form of readily interpretable elementary functions. It is also of physical interest because the particular variation of line parameters is suggested by a common approximation procedure for analyzing nonuniform transmission lines, and because the exact solutions indicate that the line has desirable properties as a broadband termination.

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