Gray Codes and Paths on the n-Cube

By E. N. GILBERT

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Certain problems in coding and in switching theory require a list of distinct binary n-tuples such that each differs from the one preceding it in just one coordinate. Geometrically, such a list corresponds to a path which follows edges of an n-dimensional cube. This paper finds all types of closed paths on cubes with $n \leqq 4$. *For larger n, a process given here will produce large numbers of paths.*

I. INTRODUCTION

A Gray code is a means of quantizing an angle and representing it in a binary alphabet. The encoding is such that angles in adjacent quantum intervals are encoded into n-tuples of binary digits which differ in just one place. For example, taking $n = 3$, as the angle increases from 0° to 360°, the binary code for the angle might go through the succession 000, 001, 011, 010, 110, 111, 101, 100 and back to 000. Gray codes are used when the encoding is performed by a code wheel. At angles close to the boundary between two quantum intervals, any of the digits which change at the boundary are likely to be in error. In a Gray code there is only one such questionable digit, and a mistake in this digit only gives to the angle the code for the adjacent quantum interval.

Although the Gray code example given for $n = 3$ is easily generalized to obtain the well known Gray (reflected binary) code for any n, there are, in general, a large number of other codes which also change one digit at a time. Our problem is to find these other codes. In special applications, some of the others may be preferable to the conventional Gray code. For instance, it may be desirable to use other numbers of quantum intervals beside powers of 2; powers of 10 might be a natural choice. If the quantity being encoded is a length rather than an angle, one can drop the requirement that the first and last quantum interval have codes differing in just one position. Then there is a still larger variety of encodings from which to choose.

For $n \leqq 4$, we will exhibit all possible codes which recycle (i.e., are suitable for angles). As n grows beyond 4, the number of possibilities soon becomes enormous; we will give a procedure by which a large number of codes may be constructed. The purpose of this paper is to construct, classify and catalogue codes, but no attempt will be made to single out those codes which are particularly useful.

In searching for codes the following geometrical picture is helpful. The set of edges of a unit n-dimensional cube forms a linear graph which we will call the *n-cube graph* Q_n. Q_n has 2^n vertices, each labeled by an n-tuple $(x_1, \cdots, x_n)$, $x_i = 0$ or 1. Two vertices are joined by a line of Q_n if their coordinates agree in all but one of the n places. We are interested in the paths and cycles of Q_n. By a *path of length L* is meant a set of lines $(V_0V_1, V_1V_2, \cdots, V_{L-1}V_L)$ of Q_n where the vertices $V_0, \cdots, V_L$ are distinct. *Cycle of length L* is similarly defined but with $V_1, \cdots, V_L$ distinct and $V_0 = V_L$. If we follow a path or cycle of Q_n and interpret the n-tuples which label the vertices as codes for successive quantum intervals, we obtain an encoding in which only one digit changes from interval to interval. Thus, our problem is to find all paths and cycles of Q_n. The cycles are the encodings suitable for angles.

A permutation of digits in the n-tuples of a code produces a new code which is not significantly different from the original one. Its code wheel is obtainable from the original wheel merely by permuting tracks. Similarly, a complementation (interchange of 0 and 1 in certain coordinate positions of the n-tuples) is a minor change. These operations correspond to rotation and reflection symmetries of the cube. The symmetry group of Q_n is the hyperoctahedral group 0_n of order $2^n n!$. The typical symmetry operation of 0_n consists of one of the $n!$ possible permutations of the coordinates of the n-tuple, followed by one of the 2^n possible complementations. Two paths or cycles will be called *equivalent* or of the same *type* if one can be changed into the other by applying to Q_n one of the symmetry operations in 0_n. Although there are tremendous numbers of paths and cycles, it suffices to give just one of each type. Even our list of distinct types of cycles becomes rather long at $n = 4$. The exact number of types of cycles is not known for larger values of n. We give below a procedure for constructing large numbers of paths and cycles. If $L = 2^n - 1$, we can specify in advance the numbers $N_1, \cdots, N_n$, where N_k is the number of times x_k changes as one follows the path from V_0 to V_L. A similar specification is possible for cycles of length 2^n. The cycles of this length contain every vertex and thus are the Hamilton lines of Q_n (Ref. 2, Ch. 2). Even the number of types of Hamilton lines is found to grow rapidly with n; there are nine types of Hamilton lines for $n = 4$.

Another application of paths on Q_n may be found in switching theory. Here the labels on the vertices represent the states of a relay network with n relays. Coordinate x_i is 0 or 1 according to whether the ith relay magnet is turned off or on. Since it is physically impossible to change the state of two relay magnets precisely simultaneously, the state of the entire network can change only by following lines of Q_n. If the network is a counting circuit it is made so that its state follows a path or a cycle according to whether the network is intended to lock or recycle at the end of the count (Ref. 1, Ch. 11).

II. COORDINATE SEQUENCES

A path is specified completely by listing the $L + 1$ vertices V_0, $\cdots$, V_L in order. For example, on the 3-cube, a list might be

$$000, 001, 011, 010, 110, 100, 101, 111.$$

Ignoring the starting vertex, a more compact notation is to list in order only the coordinate places in which the change occurs. In the example cited, one would obtain (3231232). This L-tuple of coordinate places will be called the *coordinate sequence* for the path.

The vertices of a path might equally well have been written down in the reverse order V_L, $\cdots$, V_0. Hence the list of coordinate places written in reverse order [i.e., (2321323)] will be regarded as another notation for the same coordinate sequence.

Each coordinate sequence represents not only the given path but also every other path obtainable from it by one of the 2^n complementations. All of these paths are of the same type; it suffices here to study coordinate sequences rather than the paths themselves.

Two paths, P and P', are of the same type if and only if one of the $n!$ permutations of coordinates changes the coordinate sequence of P into the coordinate sequence of P'. Our problem thus becomes one of classifying coordinate sequences into symmetry types with respect to the symmetric group on the n coordinates.

In a similar way, a cycle (say 001, 011, 111, 101, 001) may be represented by a coordinate sequence (2121). Now, however, in addition to the two orders in which the coordinates may be written, there are L different vertices at which the list of changing coordinates can begin. Thus, for any cycle there may be as many as $2L$ distinct L-tuples of digits, all of which are considered to be the same coordinate sequence.

Not every list of digits is the coordinate sequence of a path or a cycle. A typical list, such as (121323131), represents a way of wandering along lines of the cube graph, possibly visiting some vertices more than once. In the example cited, the vertices visited after the first and the sixth steps are the same. In the intermediate steps (213231) all digits $k = 1$, 2, 3 appear an even number of times; for all k the net change in x_k during these steps is even, i.e., zero. This observation leads simply to the following result.

Theorem I. An L-tuple $A = (a_1, a_2, \cdots, a_L)$ $a_i = 1, \cdots, n$ *is the coordinate sequence of a path of* Q_n *if and only if every one of the* $L(L + 1)/2$ *blocks of consecutive digits* $(a_i, a_{i+1}, \cdots, a_j)$ *contains at least one of the* n *digits an odd number of times. A is the coordinate sequence of a cycle of* Q_n *if and only if every one of the blocks of length* $1, \cdots$, *or* $L - 1$ *contains some digit an odd number of times while A itself contains every digit an even number of times.*

As an illustration of Theorem I, we construct a simple cycle of length 2^n (Hamilton line) on Q_n. For $k = 1, 2, \cdots, 2^n$ define the kth digit of the 2^n-tuple to be the number

$$a_k = \underset{1 \leqq d \leqq n}{\text{Max}} \; (d \text{ such that } 2^{d-1} \text{ divides } k).$$

That one obtains a cycle thereby is easily proved by induction on n. When $n = 2$, the construction yields (1212), the coordinate sequence of a cycle on the square. For larger n, the construction yields a 2^n-tuple of the form

$$A_n = (B_n, n, B_n, n), \tag{1}$$

where B_n is the $(2^{n-1} - 1)$-tuple obtained from A_{n-1} by removing the $n - 1$ in the 2^{n-1} place. Consider any block C of $q < 2^n$ consecutive digits of A_n. If C does not contain one of the two n's, it is a block of the path B_n. If C contains only one n, n appears an odd number of times. If C contains both n's, the digits of A_n not in C form a block of B_n and contain some digit k an odd number of times; since k appears an even number of times in A_n it appears an odd number of times in C. In any case, some digit appears an odd number of times in C. The Hamilton line in question corresponds to the conventional Gray code.

III. THE 4-CUBE

Since cycles are of greatest interest, we have constructed a list of all types of cycles on Q_4. This list also includes all the cycles on Q_2 and Q_3, since such cubes are contained as subgraphs (faces) of Q_4. There are no cycles of odd length, since every coordinate must change an even number of times. The numbers of types of cycles of lengths $4, 6, \cdots, 16$

are 1, 2, 7, 10, 23, 20 and 9. Of these, the cycle of length 4 is found on Q_2 ; both cycles of length 6 and one cycle of length 8 are found on Q_3 .

The computation consisted of the enumeration of a large number of cases using whatever *ad hoc* simplifications could be found. For example, consider any cycle of length L which contains no pair of points which differ in all four coordinates. Complementing all four coordinate places changes this cycle into a new disjoint one. Then $2L \leqq 16$. It follows that every cycle of length 10 contains such a pair of "diametrically opposite" points. The cycle can be cut into two paths joining these points, one of length 4, the other of length 6. Hence, coordinate sequences for types of cycles of length 10 can all be written in the form (1234 . . . . . .), and only the ways of filling the six empty places must be enumerated.

It is also helpful to draw a picture of the 4-cube in such a way that certain equivalences between cycles becomes geometrically obvious. The diagram described by Keister, Ritchie, and Washburn (Ref. 1, Appendix to Chap. 8) is convenient. Their cube has the appearance of a piece of graph paper; it is agreed that any two points which can be connected by a path made up of horizontal and vertical line segments of length 4 represent the same point on the cube. The graph paper itself has symmetries, each of which is also a symmetry of the cube. Then any two cycles which, when drawn on the graph paper cube, can be transformed into one another by a sequence of translation, reflection or rotation symmetries of the graph paper must be of the same type. Unfortunately, these symmetries account for only 128 of the 384 symmetries of the 4-cube. Cycles which are equivalent with respect to the 4-cube group 0_4 may fall into as many as three distinct types with respect to the subgroup of graph paper symmetries. This phenomenon is illustrated in Fig. 1, which shows three cycles which are of the same type in spite of the differences in their graph paper pictures.

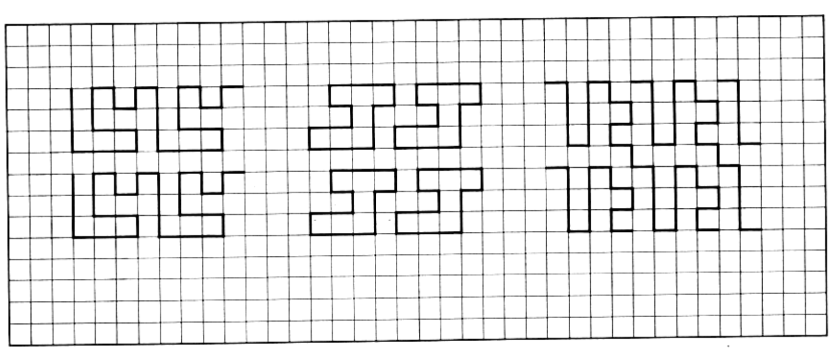

Fig. 1. — Graph paper representation of cycles on Q_4 .

TABLE I — COORDINATE SEQUENCES OF CYCLES OF DISTINCT TYPE ON Q_4

Length 2	(1 1)
Length 4	(1 2 1 2)
Length 6	(1 2 1 3 2 3)
	(1 2 3 1 2 3)
Length 8	(1 2 1 3 1 2 1 3)
	(1 2 1 3 2 4 3 4)
	(1 2 3 1 4 2 3 4)
	(1 2 3 1 4 2 4 3)
	(1 2 3 1 4 3 2 4)
	(1 2 3 4 1 2 3 4)
	(1 2 3 4 1 4 3 2)
Length 10	(1 2 1 3 4 3 1 2 1 4)
	(1 2 1 3 4 1 2 1 3 4)
	(1 2 1 3 4 1 2 1 4 3)
	(1 2 1 3 4 1 2 3 1 4)
	(1 2 1 3 4 1 2 4 1 3)
	(1 2 1 3 4 1 4 2 1 3)
	(1 2 1 3 4 1 3 1 2 4)
	(1 2 1 3 4 2 1 3 1 4)
	(1 2 1 3 4 2 1 4 1 3)
	(1 2 1 3 1 4 3 1 2 4)
Length 12	(1 2 1 3 4 3 1 2 1 3 4 3)
	(1 2 1 3 4 3 1 2 3 4 1 3)
	(1 2 1 3 4 3 1 2 3 1 4 3)
	(1 2 1 3 4 3 1 2 3 1 3 4)
	(1 2 1 3 4 3 1 3 4 2 1 3)
	(1 2 1 3 4 3 1 3 2 1 3 4)
	(1 2 1 3 4 3 1 3 2 4 1 3)
	(1 2 1 3 4 3 1 3 2 3 1 4)
	(1 2 1 3 4 1 4 2 4 1 3 4)
	(1 2 1 3 4 1 3 2 3 1 4 3)
	(1 2 1 3 4 1 4 2 4 1 4 3)
	(1 2 1 3 4 1 4 2 4 3 1 4)
	(1 2 1 3 4 2 3 1 3 4 1 3)
	(1 2 1 3 4 2 4 1 4 3 1 4)
	(1 2 1 3 4 2 1 2 3 1 2 4)
	(1 2 1 3 4 2 1 3 2 1 2 4)
	(1 2 1 3 1 2 1 4 1 3 1 4)
	(1 2 1 3 1 4 1 2 1 3 1 4)
	(1 2 3 4 1 2 4 2 3 4 2 4)
	(1 2 3 4 1 3 1 2 3 1 4 3)
	(1 2 3 4 1 3 1 2 3 4 1 3)
	(1 2 3 4 2 1 4 2 4 3 2 4)
	(1 2 3 1 4 3 1 2 3 1 4 3)
Length 14	(1 2 1 3 4 3 1 2 1 3 1 4 1 3)
	(1 2 1 3 4 3 1 2 3 4 1 4 3 4)
	(1 2 1 3 4 3 1 2 3 2 4 2 1 3)
	(1 2 1 3 4 3 1 3 4 2 4 1 3 4)
	(1 2 1 3 4 3 1 3 2 4 1 4 3 4)
	(1 2 1 3 4 1 4 2 1 4 1 3 1 4)
	(1 2 1 3 4 1 4 2 4 3 2 1 2 4)
	(1 2 1 3 4 2 1 4 2 4 3 2 1 4)
	(1 2 1 3 4 2 1 4 2 4 3 1 2 4)
	(1 2 1 3 4 2 1 3 2 3 4 2 1 3)
	(1 2 1 3 4 2 4 1 2 4 3 2 1 4)
	(1 2 1 3 4 2 4 1 2 4 3 1 2 4)
	(1 2 1 3 4 2 4 1 4 3 2 1 2 4)
	(1 2 1 3 4 2 3 2 1 2 3 4 1 3)

TABLE I — *Concluded*

	(1 2 1 3 4 2 3 1 2 3 4 2 1 3)
	(1 2 1 3 1 2 1 4 1 2 1 3 2 4)
	(1 2 1 3 1 2 1 4 1 2 3 1 2 4)
	(1 2 1 3 1 2 1 4 2 1 3 1 2 4)
	(1 2 1 3 1 2 4 1 2 1 3 1 2 4)
	(1 2 1 3 1 2 4 1 2 1 3 2 1 4)
	(1 2 1 3 2 1 2 4 1 2 1 3 1 4)
Length 16	(1 2 1 3 1 2 1 4 1 2 1 3 1 2 1 4)
	(1 2 1 3 1 2 1 4 2 1 2 3 2 1 2 4)
	(1 2 1 3 2 1 2 4 1 2 1 3 2 1 2 4)
	(1 2 1 3 2 1 2 4 2 3 2 1 3 2 3 4)
	(1 2 1 3 2 1 2 4 3 2 3 1 2 3 2 4)
	(1 2 3 2 1 2 3 4 3 2 1 2 3 2 1 4)
	(1 2 3 2 1 2 3 4 1 2 3 2 1 2 3 4)
	(1 2 3 2 1 2 3 4 1 3 1 2 1 3 1 4)
	(1 2 1 3 4 1 4 2 4 3 2 1 2 3 4 3)

A complete list of coordinate sequences for distinct types of cycles on Q_4 is given in Table I.

IV. COMPOSITION

The example (1) suggests the following result.

Theorem II. Let the L-tuple $A = (a_1, \cdots, a_L)$ *and the M-tuple* $B = (b_1, \cdots, b_M)$ *be coordinate sequences of two paths in* Q_{n-1}. *Then the* $(L + M + 1)$*-tuple* $(A, n, B) = (a_1, \cdots, a_L, n, b_1, \cdots, b_M)$ *is the coordinate sequence of a path in* Q_n. *If each of the digits* $1, \cdots, n - 1$ *appears an even number of times among* $a_1, \cdots, a_L, b_1, \cdots, b_M$, *then* (A, n, B, n) *is the coordinate sequence of a cycle in* Q_n.

A formal proof is easily given along the lines of the example. However, we prefer to note only that the theorem is obvious geometrically. We may envision Q_n as composed of two $(n - 1)$-cubes R and S. An extra coordinate $x_n = 0$ is added to the labels on the points of R and, similarly, $x_n = 1$ for S. Then 2^{n-1} lines are added joining corresponding points of R and S in order to construct Q_n. The path obtained by following the A path in R, then stepping over into S and following the B path is represented by (A, n, B). If all digits appear an even number of times in $a_1, \cdots, a_L, b_1, \cdots, b_M$ then the two-end points of (A, n, B) differ only in their x_n coordinate; an additional step back to R returns one to the starting point and (A, n, B, n) represents a cycle.

The n-dimensional path and cycle (if it exists) so constructed will be called the *composite path* and *composite cycle* of the paths with coordinate sequences A and B. Of the nine Hamilton lines listed, only the last is not composite.

A large family U_n of paths and cycles on the n-cube can now be constructed inductively. For $n = 2$, the family consists of all paths and cycles. When U_{n-1} has been constructed, U_n will contain all members of U_{n-1} plus all paths and cycles obtainable by making a composite from a pair of paths of U_{n-1}. It will be convenient to admit single vertices as paths (having the null coordinate sequence). Then such coordinate sequences as (n, A) with A in U_{n-1}, or even (n), represent paths in U_n. Next, we complete the construction of U_n by adding all paths and cycles equivalent to the ones just constructed. Paths and cycles belonging to U_n will be called *ultracomposite*.

The Hamilton line given as an example following Theorem I is ultracomposite, as may be seen using (1) and induction. The first five of the nine Hamilton lines in Table I are ultracomposite.

V. CHANGE NUMBERS

Let A be a coordinate sequence and let N_k, $k = 1, \cdots, n$, be the number of appearances of the digit k in A. N_k will be called the kth *change number* of A. The change numbers of any other sequence A' of the same type as A are just a rearrangment of $N_1, \cdots, N_n$. Hence a comparison of change numbers often suffices to prove two coordinate sequences to be of different type. There are, however, many examples of coordinate sequences of different type but having the same set of change numbers (the list of Hamilton lines contains two with change numbers 6, 6, 2, 2 and five with change numbers 6, 4, 4, 2).

If A and B are coordinate sequences of paths on the $(n - 1)$-cube and have change numbers $N_1, \cdots, N_{n-1}$ and $M_1, \cdots, M_{n-1}$, then the composite (A, n, B) has the change numbers

$$N_1 + M_1, \cdots, N_{n-1} + M_{n-1}, 1.$$

This observation suggests the possibility of an arithmetic test to decide whether a given set of numbers $N_1, \cdots, N_n$ are the change numbers of some ultracomposite.

In what follows we call an n-tuple $(N_1, \cdots, N_n)$ a *word* if $N_1, \cdots, N_n$ are the change numbers of an ultracomposite path. We ignore ultracomposite cycles because they may be found merely by adding single steps to those ultracomposite paths which have words in which all but one change number is even.

Following the inductive definition of U_n given above, an inductive scheme for computing all words may be given:

(i) (0,0), (0,1), (1,1) *and* (1,2) *are words.*

(ii) *If* $(N_1, \cdots, N_n)$ *and* $(N_1', \cdots, N_n')$ *are words; so are* $(N_1, \cdots, N_n, 0)$ *and* $(N_1 + N_1', \cdots, N_n + N_n', 1)$.

(iii) *Any permutation of the change numbers of a word produces another word.*

Thus, the only words with $n = 3$ are (000), (001), (011), (012), (111), (112), (113), (122), (123), (124), (133), and their permutations. The following theorem gives an arithmetic property of words.

Theorem III. Let the change numbers of a word be written in numerical order, $a_1 \leqq a_2 \leqq \cdots \leqq a_n$. *Then for all* $k - 1, \cdots n$,

$$\sum_{i=k}^{n} a_i \leqq 2^n - 2^{k-1}. \tag{2}$$

Proof: (Induction on n)

When $n = 2$, all words satisfy (2) (see (i) above). Suppose (2) is true for words of lengths $\leqq n - 1$ and consider one of length n. If $a_1 = 0$, the ultracomposite path described by the given word lies entirely on a lower-dimensional face of Q_n. Then $(a_2, \cdots, a_n)$ is a word of length $n - 1$ and thus satisfies (2). If $a_1 = 1$, then the given word describes a composite path made from two paths B and C. For $i = 2, \cdots, n$, a_i is a sum $B_i + C_i$ of certain change numbers of B and C. Then for $k = 2, \cdots, n$,

$$\sum_{i=k}^{n} a_i = \sum_{i=k}^{n} B_i + \sum_{i=k}^{n} C_i \leqq \sum_{i=k-1}^{n-1} b_i + \sum_{i=k-1}^{n-1} c_i, \tag{3}$$

where $b_1, \cdots, b_{n-1}$ are the numbers $B_2, \cdots, B_n$ arranged in numerical order and c_i are similarly defined. Since the b_i and c_i satisfy (2), the sums on the right of (3) total $\leqq 2^n - 2^{k-1}$. Finally, when $k = 1$, (2) is necessary because there are only 2^n vertices in Q_n.

Not every n-tuple satisfying (2) is a word. An example is (0002). It is immediately recognized as not a word because every word either must contain some 1's or contain only 0's. Even ruling out such obvious examples, one still finds others, such as (114). In the case of paths of length $2^n - 1$ the following stronger result is obtained.

Theorem IV. A set of necessary and sufficient conditions that an n-tuple composed of numbers a_i, $a_1 \leqq \cdots \leqq a_n$ *shall be the word for an ultracomposite path of length* $2^n - 1$ *is that* (2) *holds for* $k = 2, \cdots, n$; *that* $a_1 = 1$ *and, in addition, that*

$$\sum_{i=1}^{n} a_i = 2^n - 1. \tag{4}$$

Proof:

Given an ultracomposite path, (2) is necessary for $k = 2, \cdots, n$ by Theorem III. Also $a_1 = 1$, since any composite has some coordinate changing only once. Equation (4) is the requirement that the length of the path be $2^n - 1$.

Conversely, suppose the given a_i satisfy the stated arithmetic conditions. If $n = 2$, the only possibility is $a_1 = 1$, $a_2 = 2$ and the coordinate sequence in question is (212). If $n \geqq 3$, we are able to use a lemma of C. E. Shannon (Ref. 4, p. 84). Given a set of numbers $b_1 \leqq b_2 \leqq \cdots \leqq b_n$ containing a pair b_r, b_s with $1 < b_r < b_s$, a *flow* operation is defined which replaces b_r by $b_r + 1$ and b_s by $b_s - 1$. Shannon shows that those sets of numbers which are obtainable by repeated flow operations starting from the initial set $1, 2, \cdots, 2^{n-1}$ are exactly those sets $a_1 \leqq \cdots \leqq a_n$ which satisfy our (2), (4) and $a_1 = 1$. The lemma in question states that if $1, a_2, \cdots, a_n$ is such a flow pattern then for $k = 2, \cdots, n$

$$a_k = B_k + C_k,$$

where $B_2, \cdots, B_n$ are a set of numbers (not necessarily in increasing order) obtainable from $1, 2, \cdots, 2^{n-2}$ by means of flows, and similarly for $C_2, \cdots, C_n$. By induction, $(B_2, \cdots, B_n)$ and $(C_2, \cdots, C_n)$ are themselves words for some paths B and C. Then $(B, 1, C)$ is a path having the word $(a_1, \cdots, a_n)$ and the theorem is proved.

The decomposition of a_k into a sum $B_k + C_k$ can be done either by inspection or using the procedure by which Shannon's lemma is proved. Thus the problem of finding a path for a given word reduces to two such problems in lower-dimensional cubes. Continuing, one finally requires only 2^{n-2} paths on certain square faces of Q_n. Since the decomposition of the a_k can generally be done in several different ways, there may be many types of paths which can be found for a given word.

VI. HIGHER DIMENSIONS

For the two applications mentioned earlier, one might suppose that a complete list of all types of paths, or only of Hamilton lines, would be useful. Such tables must lengthen rapidly as n increases. For example, the number H_n of types of ultracomposite Hamilton lines is at least as large as the number of words, satisfying Theorem IV, for which $a_2, \cdots, a_n$ are all even. An enumeration of such words shows $H_5 \geqq 19$. A rapidly increasing lower bound on H_n is now given.

Theorem V. Let $s_0, s_1, \cdots$ be defined by the recurrence

$$s_{n+1} = (n + 1)s_n - \binom{n}{2} s_{n-2} \tag{5}$$

and the initial values $s_0 = 1$, $s_1 = 1$, $s_2 = 2$. *Then* $H_{n+2} \geqq s_n$. *For large* n, *an asymptotic formula is*

$$s_n \approx n!(n\pi)^{-(1/2)}e^{3/4}. \tag{6}$$

Proof:

By Theorem IV the $(n + 1)$-tuple $W = (2^1, 2^2, \cdots, 2^n, 1)$ is a word for some coordinate sequence T. Let a permutation P, say $k \to p(k)$, be applied just to the first n coordinate. Then T changes to PT with the word $(2^{p(1)}, \cdots, 2^{p(n)}, 1)$. The coordinate sequence $S(P) = (T, n + 2, PT, n + 2)$ describes an ultracomposite Hamilton line in which the kth coordinate change number is

$$N_k(P) = \begin{cases} 2^k + 2^{p(k)} & \text{if } k = 1, \cdots, n \\ 2 & \text{if } k = n + 1 \text{ or } n + 2. \end{cases}$$

Among the $n!$ Hamilton lines $S(P)$ there are at least as many distinct types as there are distinct sets of coordinate change numbers. It follows from the uniqueness of binary notation that $2^a + 2^b = 2^c + 2^d$ is equivalent to the statement that the (unordered) pairs (a, b) and (c, d) are the same. If one of the coordinate change numbers $2^K + 2^{p(K)}$ of $S(P)$ equals a coordinate change number of some other line $S(P')$ then P' [say $i \to p'(i)$] satisfies either $p'(K) = p(K)$ or $p'[p(K)] = K$. Then, in order for P' to have the same set of coordinate change numbers as P, every cycle of P' must either be a cycle of P or the inverse of a cycle of P. To get a lower bound on H_{n+2}, we may count the number s_n of equivalence classes of permutations in which P and P' are considered equivalent when every cycle of P' is either a cycle of P or the inverse of a cycle of P. Precisely this enumeration is also required for Cayley's problem of counting the number of terms in the expansion of a symmetric determinant. In this connection, (5) and (6) were given by I. Schur. Derivations may be found in Pólya and Szégö's book (Ref. 3, Vol. 2, Ch. 7, probs. 45 and 46).

In higher-dimensional cubes, it seems likely that the majority of the types of Hamilton lines will not be ultracomposite or even composite. To support this guess, we now construct a large class of non-composite lines. The last of the nine types listed for $n = 4$ was one such type. Using its coordinate sequence, $(1213 \cdots 3)$, we construct sequences of the form

$$A = (1, A_1, 2, A_2, 1, A_3, \cdots, 3, A_{16}),$$

where each of $A_1, \cdots, A_{16}$ represents a path of length $2^{n-4} - 1$ and contains only the digits $5, \cdots, n$. We also require that $5, \cdots, n$ each appear an even number of times. Each such A is the coordinate sequence of a non-composite Hamilton line.

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