Frequency Shifts in Cavities with Longitudinally Magnetized Small Ferrite Discs

By H. SEIDEL and H. BOYET

(Manuscript received August 27, 1957)

Values of the tensor permeability components of a magnetized ferrite at microwave frequencies may be determined from the frequency shift that a sample produces in a resonant cavity.

In this paper mathematical expressions are obtained relating this frequency shift to the diagonal and off-diagonal permeability tensor values and to the cavity geometry and ferrite geometry for any TE, TM or TEM mode natural to the empty cavity. The cavities considered are axial and have generalized cross section. The expressions are valid for ferrite discs whose volume is small compared with the cavity volume; it is assumed that the ferrite sample does not perturb the fields outside the ferrite from their empty cavity values. The cross section of the ferrite disc is arbitrary in shape and is perpendicular to the axis of the cavity.

Examples are given in which the frequency shifts in a circular coaxial cavity, circular cylindrical cavity and rectangular cavity containing thin ferrite discs are derived for general TE and TM modes. The TEM-type cavity is also considered.

I. INTRODUCTION

A possible experimental determination of the elements of the permeability tensor of a magnetized ferrite at any frequency and magnetic field rests on the measurement of the frequency shift that the sample produces in a resonant cavity.[1] From this measurement and from the theoretical relation among frequency shift, ferrite and cavity geometry and ferrite properties, we may deduce the diagonal and off-diagonal components, μ and κ, of the Polder tensor.

The theoretical relation referred to above is the result of the assumption usually made that the ferrite has small volume compared to the cavity volume and produces a small perturbation of the fields in the cavity:[1,2]

$$\frac{2\,d\omega}{\omega_0} = \frac{W_s^{(M)} + W_s^{(E)}}{W_c^{(E)}}. \tag{1}$$

Here W_c is the total energy stored in the empty cavity at resonance, $W_s^{(M)}$ the additional magnetic energy stored in the sample, $W_s^{(E)}$ the additional electric energy stored in the sample, $d\omega$ the shift of resonance frequency upon introduction of the sample and ω_0 the resonant frequency of the empty cavity. The quantities $W_c^{(E)}$, $W_s^{(M)}$ and $W_s^{(E)}$ are given by

$$\begin{aligned} W_c^{(E)} &= \frac{\epsilon_0}{2}\int_{\text{cavity}} \mathbf{E}_0\cdot\mathbf{E}_0^* d\tau; \\ W_s^{(M)} &= \frac{\mu_0}{2}\int_{\text{sample}} \mathbf{M}\cdot\mathbf{H}_0^* d\tau; \qquad \mathbf{M} = \chi\mathbf{H} \\ W_s^{(E)} &= \frac{\epsilon_0}{2}\int_{\text{sample}} \mathbf{P}\cdot\mathbf{E}_0^* d\tau; \qquad \mathbf{P} = \chi_e\mathbf{E} \end{aligned} \tag{2}$$

where $\mathbf{E}_0$ and $\mathbf{H}_0$ are the electric and magnetic fields in the empty cavity, $\mathbf{E}$ and $\mathbf{H}$ are the corresponding quantities in the perturbed cavity; $\mathbf{M}$ and $\mathbf{P}$ are the magnetic and electric polarizations in the sample, and χ and χ_e are the magnetic and electric susceptibilities of the sample.

In any particular geometrical and modal situation, the right side of (1) must be evaluated and the result is then a relation between $(d\omega/\omega_0)$ on the one hand and μ, κ and cavity and sample geometry on the other. The assumption in the perturbation theory is that electric and magnetic fields just outside the sample are their (known) empty cavity values. From this and the requirement of continuity of tangential E and H and normal B and D at air-sample interfaces, we can obtain the fields inside the sample and so calculate the numerator of (1). This is the approach used by the authors in Refs. 1 and 2 and we shall continue to follow this.

Instead of specializing to a particular cavity operating in a particular mode, as is done by the various authors in Ref. 1, we feel it would be quite useful to assume the ferrite sample placed in an axial cavity of generalized cross section operating in any TE, TM or TEM mode and find the frequency shift produced by this sample. We shall assume: the cavity has a z axis which coincides with the z axis of the sample; the sample is a disc of arbitrary cross section; the sample volume is small compared to the cavity volume; the sample is magnetized along the z axis; the cavity has perfectly conducting walls. The result of our endeavor is to reduce the right side of (1) to an expression containing a series of contour integrals only, in which μ and κ are explicit coefficients of these integrals. The contour integrals are integrals of empty cavity fields (or potentials generating these fields) taken around the contour of the cavity and the contour of the sample in the transverse plane. For any particular geometry and modal distribution, these contour integrals are easily evaluated, and we give examples of their evaluation in the circular coaxial cavity, circular cylinder cavity and rectangular waveguide cavity for the TE_{pqN}, TM_{pqN}, TM_{pq0} modes. In these examples, we consider thin ferrite discs. Thus the frequency shift, within the confines of a perturbation theory, is obtained for a quite generalized cavity operating in any of its natural modes of oscillation; the frequency shift for any given cavity operating in any given mode can then be obtained from the general result.

An interesting fact that emerges from the general result is that the expression for $(d\omega/\omega_0)$ is independent of κ whenever the explicit time independent part of the field, or potential, is real. This reality of the potential corresponds to a linear polarization at any point in the transverse plane and since a linear polarization is equivalent to two equal and opposite circular polarizations (corresponding to $+\kappa$ and $-\kappa$ values of the off-diagonal element) the net effect is for κ to cancel out of the interaction of the linear wave with the ferrite at each point of the ferrite. This is the situation in the rectangular waveguide cavity with a rectangular slab sample, as we shall see later (Section 2.2.6). It is also the case in the TEM-type cavity (Section 2.2.5) since here, too, the fields are real and therefore linearly polarized at any given point in the ferrite. These cases are thus not suited for determining the off-diagonal component of the Polder tensor.

On the other hand, we shall see that when we deal with a circular coaxial cavity or circular cylinder cavity (Sections 2.2.1 to 2.2.4) and choose a circularly polarized mode, the time independent part of the potential or field is complex and goes as $e^{ip\theta}$ where θ is angle and p is angular mode number. In this case the circular polarization $e^{i(p\theta+\omega t)}$ interacts in an unbalanced way with the precessing spins and leads to a coupling to κ. The expression for $d\omega/\omega_0$ thus involves both μ and κ and is ideally suited for use in experimental determinations of Polder tensor elements.

The existence of complex potentials is associated with a degenerate state of the system. Since degeneracy is usually related to a symmetric structure (e.g. square or circular guide), we state that in almost all cases the only structures suitable for measuring κ in thin ferrite samples are symmetric ones. There are, however, asymmetric structures which also have degenerate states.

II. ANALYSIS

2.1 *Frequency Shift Produced by Thin Ferrite Disc Sample in Cavity of Generalized Cross Section.*

2.1.1 *TE_{pqN} and TM_{pqN} type modes.* We consider a cavity of arbitrary cross section with an axis in the z direction containing a thin disc of ferrite of small cross section. The ferrite sample is magnetized in the z direction (see Fig. 1). We assume the cavity walls are perfect, lossless conductors. We shall consider standing waves in this cavity, which are the result of superposing the traveling waves $\pm f(u, v)e^{\pm j\beta z}$, where u, v are transverse coordinates and β is the longitudinal propagation number.

The electric field in the empty cavity which satisfies the boundary condition that transverse $\mathbf{E}_0$ vanish at $z = 0$ and the condition div $\mathbf{E}_0 = 0$, is obtained as a superposition of $e^{\pm j\beta z}$ waves and is given by

$$\mathbf{E}_0 = \mathbf{E}_t \sin \beta z + E_z \cos \beta z \, \mathbf{z}_0 \tag{3}$$

where $\mathbf{E}_t$ and $\mathbf{E}_z$ are functions of transverse coordinates only. The condi-

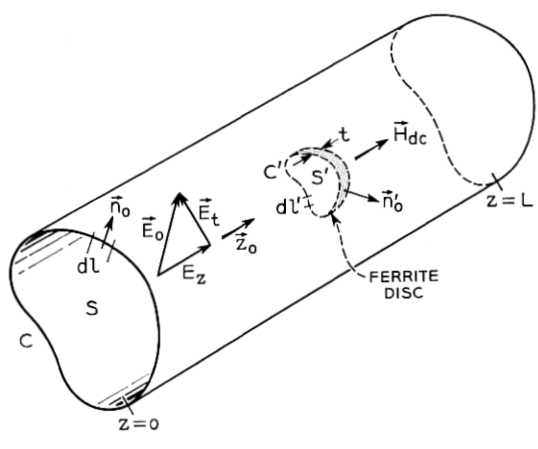

Fig. 1 — Geometry of cavity and sample. C, C', S and S' are contours and cross sections of transverse section of cavity and ferrite sample, respectively; t is thickness of ferrite disc; $\mathbf{z}_0$ is unit vector along axis of cavity; $\mathbf{H}_{dc}$ is the steady applied magnetic field; $\mathbf{E}_t$ is the transverse component of electric field $\mathbf{E}_0$; $\mathbf{n}_0$ and $\mathbf{n}_0'$ are outward unit normals to C and C' and L is the length of the cavity.

tion that transverse $\mathbf{E}_0$ vanish at $z = L$ leads to

$$\beta L = N\pi, \qquad N = 1, 2, \cdots, \tag{4}$$

where N is the longitudinal mode number.

We shall attempt to write the transverse electric field in terms of two potentials φ and ψ which are functions of the transverse coordinates:

$$\mathbf{E}_t = \nabla_t\varphi + \mathbf{z}_0 \times \nabla_t\psi \tag{5}$$

where ∇_t is the transverse gradient operator. With $\nabla = \nabla_t + \mathbf{z}_0(\partial/\partial z)$ and $\nabla\cdot\mathbf{E}_0 = 0$, we find from (3) and (5)

$$\nabla_t^2\varphi - \beta E_z = 0. \tag{6}$$

Since the z component of $\mathbf{E}_0$, $E_z \cos \beta z$, satisfies the wave equation, we find with (6) that φ satisfies

$$\begin{aligned}
&(\nabla_t^2 + k_c^2)\varphi = 0, \qquad E_z = -\frac{k_c^2}{\beta}\varphi, \\
&k_c^2 = k^2 - \beta^2, \\
&k = \frac{2\pi}{\lambda_0} = \omega\sqrt{\mu_0\epsilon_0}, \\
&\beta = \frac{2\pi}{\lambda_g},
\end{aligned} \tag{7}$$

where λ_0 and λ_g are free space wavelength and cavity wavelength, respectively.

In like manner, the transverse component of $\mathbf{E}_0$, namely

$$(\nabla_t\varphi + \mathbf{z}_0 \times \nabla_t\psi) \sin \beta z,$$

satisfies the wave equation and this leads to

$$(\nabla_t^2 + k_c^2)\psi = 0. \tag{8}$$

The complete electric and magnetic fields in the empty cavity are thus given by

$$\begin{aligned}
\mathbf{E}_0 &= (\nabla_t\varphi + \mathbf{z}_0 \times \nabla_t\psi) \sin \beta z - \frac{k_c^2}{\beta}\varphi \cos \beta z\mathbf{z}_0, \\
\mathbf{H}_0 &= \frac{j}{\omega\mu_0}\left(\nabla_t + \mathbf{z}_0\frac{\partial}{\partial z}\right) \times \mathbf{E}_0 \\
&= \frac{j\beta}{\omega\mu_0}\left[-\nabla_t\psi \cos \beta z + \frac{k^2}{\beta^2}\mathbf{z}_0 \times \nabla_t\varphi \cos \beta z - \frac{kc^2}{\beta}\psi \sin \beta z\mathbf{z}_0\right].
\end{aligned} \tag{9}$$

The existence of a φ wave alone can be interpreted as a TM wave as (9) shows. The boundary condition on φ is determined by $E_z = 0$ on C, and from (7) this is equivalent to $\varphi_c = 0$. If $\varphi_c = 0$, then inspection of (9) shows that H_{normal} also vanishes on C. Transverse **E** is zero on the end plates of the cavity provided $\beta L = N\pi$.

The existence of a ψ wave alone can be interpreted as a TE wave. The boundary condition that $H_{\text{normal}} = 0$ on all conducting surfaces is satisfied if $\beta L = N\pi$ and if $(\partial\psi/\partial n) = 0$ on C, where **n** is the unit normal directed outward from C. If $(\partial\psi/\partial n) = 0$ on C, we see from (9) that tangential **E** is also zero on C, as required.

The problem now is to find contour integral expressions for the quantities $W_s^{(M)}$, $W_s^{(E)}$ and $W_c^{(E)}$ defined in (1) and (2). For this purpose, we shall make use of the following three relations:

$$\int_S |\nabla_t\xi|^2\, dS = k_c^2 \int_S |\xi|^2\, dS + \oint_C (\xi\nabla_t\xi^*)\cdot\mathbf{n}_0\, dl, \tag{10}$$

$$2k_c^2 \int_S |\xi|^2\, dS = \oint_C [(\nabla_{k_c}\xi\cdot\mathbf{k}_c)(\nabla_t\xi^*\cdot\mathbf{n}_0) - \xi^*\nabla_t(\nabla_{k_c}\xi\cdot\mathbf{k}_c)\cdot\mathbf{n}_0]\, dl, \tag{11}$$

$$\int_S \nabla_t\xi\cdot(\mathbf{z}_0 \times \nabla_t\xi^*)\, dS = \oint_C (\xi\nabla_t\xi^*)\cdot d\mathbf{l}. \tag{12}$$

Here ∇_{k_c} is the gradient operator in $\mathbf{k}_c$ space and ξ stands for either φ or ψ. Equations (10), (11) and (12) apply to either C and S or to C' and S'. The above relations are derived in Appendices I, II and III, respectively.

Calculating first the electric energy stored in the cavity we have

$$W_c^{(E)} = \frac{\epsilon_0}{2}\int \mathbf{E}_0\cdot\mathbf{E}_0^*\, d\tau = \frac{\epsilon_0 L}{4}\int_S \left[|\nabla_t\varphi|^2 + |\nabla_t\psi|^2 + \left(\frac{k_c^2}{\beta}\right)|\varphi|^2 \right] dS, \tag{13}$$

where we are assuming that a φ and ψ wave do not exist simultaneously. Making use of (10) and (11) with the condition that $\varphi = 0$ on C and $(\partial\psi^*/\partial n) = 0$ on C, we find

$$\begin{aligned} W_c^{(E)} &= \frac{\epsilon_0 L}{8}\frac{k^2}{\beta^2}\oint_C (\nabla_{k_c}\varphi\cdot\mathbf{k}_c)(\nabla_t\varphi^*\cdot\mathbf{n}_0)\, dl \qquad \text{for } \varphi \text{ waves;} \\ W_c^{(E)} &= -\frac{\epsilon_0 L}{8}\oint_C \psi^*\nabla_t(\nabla_{k_c}\psi\cdot\mathbf{k}_c)\cdot\mathbf{n}_0\, dl \qquad \text{for } \psi \text{ waves.} \end{aligned} \tag{14}$$

We next calculate the magnetic energy stored in the sample:

$$W_s^{(M)} = \frac{\mu_0}{2} \int \mathbf{M} \cdot \mathbf{H}_0^* d\tau',$$

$$\mathbf{M} = \chi \mathbf{H}; \qquad \chi = \begin{pmatrix} \mu - 1 & j\kappa & 0 \\ -j\kappa & \mu - 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

According to the perturbation theory, the $\mathbf{H}_0$ and $\mathbf{H}$ appearing in (15) are the RF magnetic fields in the empty cavity and ferrite sample, respectively. From (15), $M_z = 0$; therefore $(H_z)_0^*$ is not needed in the expression for $W_s^{(M)}$. Continuity of tangential H from air to sample shows that we may use $\mathbf{H}_0$ in place of $\mathbf{H}$ in (15). The field to be used throughout (15) is, therefore, that given by (9) without the $(H_z)_0$ component:

$$\mathbf{H}_0 = \frac{j\beta}{\omega\mu_0}\left[-\nabla_t\psi \cos\beta z + \frac{k^2}{\beta^2}\mathbf{z}_0 \times \nabla_t\varphi \cos\beta z\right]. \tag{16}$$

When (16) is used in (15) we find (Appendix IV)

$$W_S^{(M)} = \frac{f\beta^2 \cos^2 \beta z_0}{2\omega^2\mu_0} t \int_{S'} [(\mu - 1) \mid \nabla_t\xi \mid^2 + j\kappa \nabla_t\xi \cdot \mathbf{z}_0 \times \nabla_t\xi^*]\, dS' \tag{17}$$

where z_0 (not to be confused with $\mathbf{z}_0$) is the position of the sample along the axis of the cavity and $f = 1$ or (k^2/β^2) for $\xi = \psi$ and $\xi = \varphi$, respectively. It is assumed $t \ll L$. Transforming (17) to contour integrals through (10), (11) and (12), we have

$$\begin{aligned} W_S^{(M)} = \frac{f\beta^2 \cos^2 \beta z_0}{2\omega^2\mu_0} t \Big\{ &(\mu - 1) \oint_{C'} (\xi \nabla_t \xi^*) \cdot \mathbf{n}_0'\, dl' \\ &+ \frac{\mu - 1}{2} \oint_{C'} [(\nabla_{k_c}\xi \cdot \mathbf{k}_c)(\nabla_t\xi^* \cdot \mathbf{n}_0') - \xi^* \nabla_t(\nabla_{k_c}\xi \cdot \mathbf{k}_c) \cdot \mathbf{n}_0']\, dl' \\ &+ j\kappa \oint_{C'} (\xi \nabla_t \xi^*) \cdot d\mathbf{l}' \Big\}. \end{aligned} \tag{18}$$

The final step is the calculation of the electric energy stored in the sample:

$$W_S^{(E)} = \frac{\epsilon_0}{2} \int \mathbf{P} \cdot \mathbf{E}_0^*\, d\tau', \qquad \mathbf{P} = \chi_e \mathbf{E}, \qquad \chi_e = \epsilon - 1, \tag{19}$$

ϵ being the dielectric constant of the sample and $\mathbf{E}_0$ and $\mathbf{E}$ the electric fields in empty cavity and inside ferrite, respectively. Since $\mathbf{E}_t = \mathbf{E}_{t_0}$

and $\epsilon_0 E_{z_0} = \epsilon_0 \epsilon E_z$, we have

$$W_S^{(E)} = \frac{\epsilon_0(\epsilon - 1)}{2} \int \left[| E_{t_0} |^2 + \frac{1}{\epsilon} | E_{z_0} |^2 \right] d\tau'$$
$$= \frac{\epsilon_0(\epsilon - 1)t}{2} \left[\sin^2 \beta z_0 \int_{S'} (| \nabla_t \varphi |^2 + | \nabla_t \psi |^2) \, dS' \right. \tag{20}$$
$$\left. + \frac{\cos^2 \beta z_0}{\epsilon} \left(\frac{k_c^2}{\beta} \right)^2 \int_{S'} | \varphi |^2 \, dS' \right].$$

Making use of (10) and (11), we find

$$W_S^{(E)} = \frac{\epsilon_0(\epsilon - 1)}{2} \, t \left\{ \sin^2 \beta z_0 \left[\oint_{C'} (\xi \nabla_t \xi^*) \cdot \mathbf{n}_0' \, dl' \right. \right.$$
$$\left. + \frac{1}{2} \oint_{C'} [(\nabla_{k_c} \xi \cdot \mathbf{k}_c)(\nabla_t \xi^* \cdot \mathbf{n}_0') - \xi^* \nabla_t (\nabla_{k_c} \xi \cdot \mathbf{k}_c) \cdot \mathbf{n}_0'] \, dl' \right] \tag{21}$$
$$\left. + \frac{\cos^2 \beta z_0}{2\epsilon} \left(\frac{k_c}{\beta} \right)^2 \oint_{C'} [(\nabla_{k_c} \xi \cdot \mathbf{k}_c)(\nabla_t \xi^* \cdot \mathbf{n}_0') - \xi^* \nabla_t (\nabla_{k_c} \xi \cdot \mathbf{k}_c) \cdot \mathbf{n}_0'] \, dl' \right\}$$

where for TM modes (φ waves) both the $\sin^2 \beta z_0$ and $\cos^2 \beta z_0$ terms are needed, and for TE waves (ψ waves) only the $\sin^2 \beta z_0$ term is needed.

2.1.2 *TM_{pq0} type modes.* The TM_{pq0} type modes are obtained by considering $\beta = 0$ in all $e^{\pm j\beta z}$ field dependences, i.e. the cavity is cut off in the z direction and there is to be no field dependence on z. In this case we have

$$\begin{aligned} (\nabla_t^2 + k^2) E_z &= 0, \\ k_c^2 = k^2 - \beta^2 = k^2, &\qquad \mathbf{k}_c = \mathbf{k}. \end{aligned} \tag{22}$$

Since transverse $\mathbf{E}$ must be zero at $z = 0$ and $z = L$, and since there can be no z dependence in this case, transverse E must vanish everywhere in the cavity. There can only be an E_z component. Thus a TE_{pq0} mode can not exist, for then $\mathbf{E}_0 \equiv 0$ everywhere and $\mathbf{H}_0 \propto$ curl $\mathbf{E}_0$ gives $\mathbf{H}_0 \equiv 0$ also. A TM_{pq0} mode is the only possibility and we have

$$\begin{aligned} \mathbf{E}_0 &= E_z \mathbf{z}_0, \\ \mathbf{H}_0 &= \frac{j}{\omega\mu_0} \nabla \times \mathbf{E}_0 = \frac{-j}{\omega\mu_0} \mathbf{z}_0 \times \nabla_t E_z, \end{aligned} \tag{23}$$

where E_z is a function of transverse coordinates only. E_z plays the same role now as φ did in the TM_{pqN} case before. The boundary condition $E_z = 0$ on C causes H_{normal} to vanish automatically on C, as required, as is seen from (23).

Proceeding exactly as in the steps leading to (14), (18) and (21), we find

$$W_C^{(E)} = \frac{\epsilon_0 L}{4k^2} \oint_C (\nabla_k E_z \cdot \mathbf{k})(\nabla_t E_z^* \cdot \mathbf{n}_0)\, dl, \tag{24}$$

$$\begin{aligned} W_S^{(M)} = \frac{t}{2\omega^2 \mu_0} \Big\{ (\mu - 1) \oint_{C'} (E_z \nabla_t E_z^*) \cdot \mathbf{n}_0'\, dl' \\ + \frac{\mu - 1}{2} \oint_{C'} [(\nabla_k E_z \cdot \mathbf{k})(\nabla_t E_z^* \cdot \mathbf{n}_0') - E_z^* \nabla_t (\nabla_k E_z \cdot \mathbf{k}) \cdot \mathbf{n}_0']\, dl' \\ + j\kappa \oint_{C'} (E_z \nabla_t E_z^*) \cdot \mathbf{dl}' \Big\}, \end{aligned} \tag{25}$$

$$\begin{aligned} W_s^{(E)} = \frac{\epsilon_0(\epsilon - 1)t}{4\epsilon k^2} \oint_{C'} [(\nabla_k E_z \cdot \mathbf{k})(\nabla_t E_z^* \cdot \mathbf{n}_0') \\ - E_z^* \nabla_t (\nabla_k E_z \cdot \mathbf{k}) \cdot \mathbf{n}_0']\, dl'. \end{aligned} \tag{26}$$

2.1.3 *TEM type cavity*. In this case $\lambda_g = \lambda_0$, $\beta = k$, and $k_c^2 = 0$. We can write the fields in terms of a single potential φ:

$$\begin{aligned} \mathbf{E}_0 &= (\nabla_t \varphi) \sin \beta z, \\ \mathbf{H}_0 &= \frac{j}{\omega \mu_0} \nabla \times \mathbf{E}_0 = \frac{j\beta}{\omega \mu_0} (\mathbf{z}_0 \times \nabla_t \varphi) \cos \beta z, \end{aligned} \tag{27}$$

where φ satisfies div $\mathbf{E}_0 = 0$ or

$$\nabla_t^2 \varphi = 0. \tag{28}$$

The boundary condition $E_{\text{tangential}} = 0$ on cavity walls gives $(\partial\varphi/\partial l) = 0$ or $\varphi =$ constant on C. This automatically makes $H_{\text{normal}} = 0$ on cavity walls as we see from (27). The end plate condition on transverse $\mathbf{E}$ again gives $\beta L = N\pi$.

The various stored energies are found as before, with use made of (10), (11) and (12):

$$\begin{aligned} W_C^{(E)} = \frac{\epsilon_0}{2} \int \mathbf{E}_0 \cdot \mathbf{E}_0^* \, d\tau = \frac{\epsilon_0 L}{4} \int_S |\nabla_t \varphi|^2 \, dS \\ = \frac{\epsilon_0 L}{4} \oint_C (\varphi \nabla_t \varphi^*) \cdot \mathbf{n}_0 \, dl, \end{aligned} \tag{29}$$

$$\begin{aligned} W_S^{(M)} = \frac{\beta^2 \cos^2 \beta z_0}{2\omega^2 \mu_0} t \Big[(\mu - 1) \oint_{C'} (\varphi \nabla_t \varphi^*) \cdot \mathbf{n}_0'\, dl' \\ + j\kappa \oint_{C'} (\varphi \nabla_t \varphi^*) \cdot d\mathbf{l}' \Big], \end{aligned} \tag{30}$$

$$W_S^{(E)} = \frac{\epsilon_0}{2}\int \mathbf{P}\cdot\mathbf{E}_0{}^* \, d\tau' = \frac{\epsilon_0(\epsilon - 1)}{2}\int |\nabla_t\varphi|^2 \sin^2\beta z' \, dS' \, dz'$$

$$= \frac{\epsilon_0(\epsilon - 1)}{2} t \sin^2 \beta z_0 \int_{S'} |\nabla_t\varphi|^2 \, dS' \tag{31}$$

$$= \frac{\epsilon_0(\epsilon - 1)}{2} t \sin^2 \beta z_0 \oint_{C'} (\varphi\nabla_t\varphi^*)\cdot\mathbf{n}_0' \, dl',$$

where we have, in (31), used the fact that $\mathbf{E}$ is purely transverse and must be continuous from air to sample ($\mathbf{E} = \mathbf{E}_0$).

2.2 *Circular Coaxial Cavity (Higher Modes); Circular Cylindrical Cavity; TEM Type Cavity; Rectangular Cavity.*

We are now in a position to calculate the frequency shift in various cases from (1).

2.2.1 *Circular coaxial cavity, ferrite ring, TE_{pqN} modes.* The geometry we are considering is shown in Fig. 2.

We are here dealing with a ψ wave only. Consider the sample placed at $\sin \beta z_0 = 0$ (node of the electric field) so that from (21) there is no electric energy stored in the sample.

The solution of $(\nabla_t^2 + k_c^2)\psi = 0$ for the empty cavity potential in this geometry is

$$\begin{aligned} \psi &= e^{jp\theta}[J_p(k_c r) + CN_p(k_c r)], \\ &= e^{jp\theta}Z_p(k_c r), \qquad a_1 < r < a_2, \end{aligned} \tag{32}$$

where J_p and N_p are Bessel functions of the first and second kind of

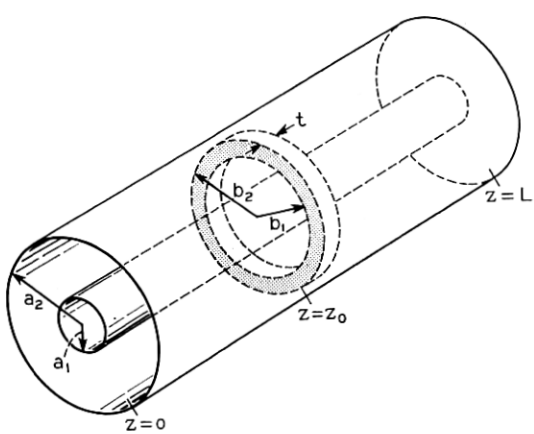

Fig. 2 — Circular coaxial cavity with ferrite ring of thickness t placed at $z = z_0$; a_1 and a_2 are inner and outer radii of cylinders; b_1 and b_2 are inner and outer radii of ferrite ring.

order p, Z_p is the general cylinder function and C is an arbitrary constant to be determined by the boundary conditions. Z_p satisfies[3]

$$Z_p''(x) + \frac{1}{x} Z_p'(x) + \left(1 - \frac{p^2}{x^2}\right) Z_p(x) = 0, \qquad (33)$$
$$Z_p' = Z_{p-1} - \frac{p}{x} Z_p .$$

The boundary condition $(\partial\psi/\partial n) = 0$ on C $(r = a_1, a_2)$ gives

$$\begin{aligned} Z_p'(k_c a_1) &= Z_p'(k_c a_2) = 0, \\ -C &= \frac{J_p'(k_c a_1)}{N_p'(k_c a_1)} = \frac{J_p'(k_c a_2)}{N_p'(k_c a_2)}, \\ k_c a_2 &= \sigma_{pq}, \qquad p, q = \text{integers}, \\ \sigma_{pq} &= q\text{th root of } J_p'(k_c a_1) N_p'(k_c a_2) - J_p'(k_c a_2) N_p'(k_c a_1) = 0, \\ \beta L &= N\pi. \end{aligned} \qquad (34)$$

When (32) and (33) are used in (14) and (18) (remembering that C consists of the contours $r = a_1$ and $r = a_2$ taken in opposite senses and C' consists of the contours $r = b_1$, $r = b_2$ also taken oppositely) we find

$$\frac{d\omega}{\omega_0} = \frac{\frac{\beta^2}{k^2}\frac{t}{L}\left[(\mu - 1)\left\{Z_p^2\left(1 - \frac{2p}{k_c^2 b^2}\right) + \frac{2}{k_c b} Z_p Z_{p-1}(1 - p) + Z_{p-1}^2\right\} \pm \frac{2\kappa p}{k_c^2 b^2} Z_p^2\right] k_c^2 b^2\Big\}_{b_1}^{b_2}}{\left\{k_c^2 a^2\left(1 - \frac{p^2}{k_c^2 a^2}\right) Z_p^2(k_c a)\right\}_{a_1}^{a_2}} \qquad (35)$$

with $(\beta^2/k^2) = (\lambda_0/\lambda_g)^2$, $\beta L = N\pi$, $k_c a_2 = \sigma_{pq}$ = roots of (34). The $\pm\kappa$ term arises from each of the two types of circular polarizations which are possible, i.e. $e^{\pm jp\theta}$. The functions Z_p in the numerator are defined by $Z_p = Z_p(k_c b)$.

2.2.2 *Circular coaxial cavity, ferrite ring, TM_{pqN} modes.* Here we are considering a φ wave with the sample again placed at $\sin \beta z_0 = 0$. The solution of $(\nabla_t^2 + k_c^2)\varphi = 0$ in the geometry of Fig. 2 is again given by (32), but the boundary condition $\varphi = 0$ on $r = a_1, a_2$ now gives

$$\begin{aligned} Z_p(k_c a_1) &= Z_p(k_c a_2) = 0, \\ -C &= \frac{J_p(k_c a_1)}{N_p(k_c a_1)} = \frac{J_p(k_c a_2)}{N_p(k_c a_2)}, \\ k_c a_2 &= \tau_{pq}, \qquad p, q = \text{integers}, \\ \tau_{pq} &= q\text{th root of } J_p(k_c a_1) N_p(k_c a_2) - J_p(k_c a_2) N_p(k_c a_1) = 0, \\ \beta L &= N\pi. \end{aligned} \qquad (36)$$

The evaluation of the cavity energy, magnetic energy in a sample, and electric energy in a sample [(14), (18) and (21)] is straightforward and leads to the following result for the frequency shift:

$$\frac{d\omega}{\omega_0} = \frac{\frac{t}{L}\left[(\mu - 1)\left\{ Z_p^2\left(1 - \frac{2p}{k_c^2b^2}\right) + \frac{2}{k_cb} Z_pZ_{p-1}(1 - p) + Z_{p-1}^2 \right\} \pm \frac{2\kappa p}{k_c^2b^2} Z_p^2 + \frac{\epsilon - 1}{\epsilon}\frac{k_c^2}{k^2}\left\{ Z_p^2 - \frac{2p}{k_cb} Z_pZ_{p-1} + Z_{p-1}^2 \right\} \right] k_c^2b^2 \Big\}_{b_1}^{b_2}}{\left\{ k_c^2a^2Z_{p-1}^2(k_ca) \right\}_{a_1}^{a_2}} \tag{37}$$

with $(k_c^2/k^2) = 1 - (\lambda_0/\lambda_g)^2$, $\lambda_g = (2L/N)$, $k_ca_2 = \tau_{pq} =$ roots of (36). The argument of the cylinder functions in the numerator of (37) is implicitly k_cb.

2.2.3 *Circular coaxial cavity, ferrite ring, TM_{pq0} modes.* The solution of $(\nabla_t^2 + k^2)E_z = 0$ in this geometry is

$$\begin{aligned} E_z &= e^{jp\theta}Z_p(kr), \\ Z_p &= J_p + CN_p . \end{aligned} \tag{38}$$

The boundary condition $E_z = 0$ on $r = a_1$, a_2 gives the same set of equations as in (36) with k_c replaced by k, except that $\beta L = N\pi$ is not now applicable. When (38), (36) and (33) are used in (24), (25) and (26) we find for the frequency shift

$$\frac{d\omega}{\omega_0} = \frac{\frac{t}{L}\left[(\mu - 1)\left\{ Z_p^2\left(1 - \frac{2p}{k^2b^2}\right) + \frac{2}{kb} Z_pZ_{p-1}(1 - p) + Z_{p-1}^2 \right\} \pm \frac{2\kappa p}{k^2b^2} Z_p^2 + \frac{\epsilon - 1}{\epsilon}\left\{ Z_p^2 - \frac{2p}{kb} Z_pZ_{p-1} + Z_{p-1}^2 \right\} \right] b^2 \Big\}_{b_1}^{b_2}}{2\left\{ a^2Z_{p-1}^2(ka) \right\}_{a_1}^{a_2}} \tag{39}$$

with $ka_2 = \tau_{pq}$, $\tau_{pq} =$ roots of (36). The numerator argument is kb.

2.2.4 *Circular cylindrical cavity, circular ferrite disc, TE_{pqN}, TM_{pqN}, TM_{pq0} modes.* The geometry of this situation is obtained from Fig. 2 by letting a_1, b_1 approach 0, so that the ferrite ring becomes a disc. The frequency shifts in the various modal cases can then be obtained from (35), (37) and (39) by letting a_1, b_1 approach 0 and putting $C = 0$ in (32), (34), (36) and (38), since only the Bessel function which is regular at

$r = 0$ (J_p) is now allowed. Under these conditions the following quantities approach zero for $p \geqq 0$:

$$r^2\left(1 - \frac{2p}{r^2}\right) J_p^2(r), \qquad 2rJ_p(r)J_{p-1}(r)(1 - p),$$

$$r^2J_{p-1}^2(r), \qquad pJ_p^2(r), \qquad r^2J_p^2(r), \qquad 2prJ_p(r)J_{p-1}(r)$$

where $r = k_c a_1$ or $k_c b_1$ approaches 0. In other words, all quantities at the lower limits vanish in (35), (37) and (39) for this geometry. Equation (35) becomes for TE_{pqN} modes:

$$\frac{d\omega}{\omega_0} = \frac{\dfrac{\beta^2}{k^2}\dfrac{t}{L}\dfrac{b_2^2}{a_2^2}\left[(\mu - 1)\left\{J_p^2\left(1 - \dfrac{2p}{k_c^2b^2}\right) + \dfrac{2}{k_cb}J_pJ_{p-1}(1 - p) + J_{p-1}^2\right\} \pm \dfrac{2\kappa p}{k_c^2b^2}\right]_{b=b_2}}{\left[\left(1 - \dfrac{p^2}{k_c^2a^2}\right)J_p^2(k_ca)\right]_{a=a_2}} \tag{40}$$

with $(\beta^2/k^2) = (\lambda_0/\lambda_g)^2$, $\beta L = N\pi$, $k_ca_2 = s_{pq} =$ roots of $J_p'(k_ca_2) = 0$, $p, q =$ integers. The numerator argument is k_cb.

For TM_{pqN} modes, (37) becomes

$$\frac{d\omega}{\omega_0} = \frac{\dfrac{t}{L}\dfrac{b_2^2}{a_2^2}\left[(\mu - 1)\left\{J_p^2\left(1 - \dfrac{2p}{k_c^2b^2}\right) + \dfrac{2}{k_cb}J_pJ_{p-1}(1 - p) + J_{p-1}^2\right\} \pm \dfrac{2\kappa p}{k_c^2b^2}J_p^2 + \dfrac{(\epsilon - 1)}{\epsilon}\dfrac{k_c^2}{k^2}\left\{J_p^2 - \dfrac{2p}{k_cb}J_pJ_{p-1} + J_{p-1}^2\right\}\right]_{b=b_2}}{J_{p-1}^2(k_ca_2)} \tag{41}$$

$$\frac{k_c^2}{k^2} = 1 - \left(\frac{\lambda_0}{\lambda_g}\right)^2, \qquad \lambda_g = \frac{2L}{N},$$

with $k_ca_2 = t_{pq} =$ roots of $J_p(k_ca_2) = 0$, $p, q =$ integers. The numerator argument is k_cb.

For TM_{pq0} modes, (39) becomes

$$\frac{d\omega}{\omega_0} = \frac{\frac{t}{L}\frac{b_2^2}{a_2^2}\left[(\mu - 1)\left\{J_p^2\left(1 - \frac{2p}{k^2b^2}\right) + \frac{2}{kb}J_pJ_{p-1}(1 - p) + J_{p-1}^2\right\} \pm \frac{2\kappa p}{k^2b^2}J_p^2 + \frac{\epsilon - 1}{\epsilon}\left\{J_p^2 - \frac{2p}{kb}J_pJ_{p-1} + J_{p-1}^2\right\}\right]_{b=b_2}}{2J_{p-1}^2(ka_2)} \quad (42)$$

$ka_2 = t_{pq}$ = roots of $J_p(ka_2) = 0$, $k = (2\pi/\lambda_0)$, p, q = integers. Again the sample is considered placed at $\sin \beta z_0 = 0$ in (40) and (41). The numerator argument is kb.

2.2.5 *TEM type cavity.* In this case the quantity φ appearing in (29), (30) and (31) is real so that

$$\oint_{C'} \varphi \nabla_t \varphi^* \cdot \mathbf{dl}' = \frac{1}{2}\oint_{C'} \nabla_t \varphi^2 \cdot \mathbf{dl}' = \frac{1}{2}\oint_{C'} \frac{\partial(\varphi^2)}{\partial l'}\, dl' \equiv 0.$$

Consequently, there is no coupling to κ and this type of cavity is not suited for measurement of the off-diagonal component of the Polder tensor. If the sample is placed at $\sin \beta z_0 = 0$, the frequency shift is given by

$$\frac{d\omega}{\omega_0} = \frac{t}{L}(\mu - 1)\frac{\oint_{C'} \frac{\partial(\varphi^2)}{\partial n'}\, dl'}{\oint_C \frac{\partial(\varphi^2)}{\partial n}\, dl} \quad (43)$$

where $\partial/\partial n'$ is the derivative along the normal to C or C'.

2.2.6 *Rectangular cavity, thin rectangular ferrite slab, TE_{pqN}, TM_{pqN} modes.* The geometry is shown in Fig. 3.

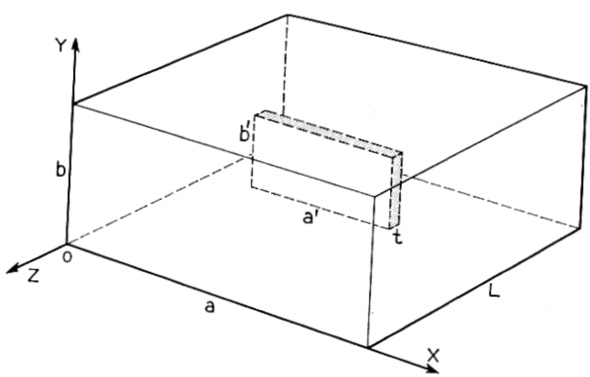

Fig. 3 — Rectangular cavity containing rectangular ferrite slab.

For TE_{pqN} modes, $\psi = \cos k_x x \cos k_y y$. The boundary condition $\partial\psi/\partial n = 0$ on C ($x = 0, x = a, y = 0, y = b$) gives $k_x a = p\pi$, $k_y b = q\pi$ where p, q = integers. The wave equation $(\nabla_t^2 + k_c^2)\psi = 0$ then gives $k_c^2 = k_x^2 + k_y^2 = k^2 - \beta^2$. Again $\beta L = N\pi$.

For TM_{pqN} modes, $\varphi = \sin k_x x \sin k_y y$ and this satisfies the boundary condition $\varphi = 0$ on C if $k_x a = p\pi$, $k_y b = q\pi$. Again $k_c^2 = k_x^2 + k_y^2$ and $\beta L = N\pi$.

In each of these cases it is clear from (32) that the ferrite sample does not couple to κ since $\xi = \varphi$ or ψ is real. As a result,

$$\kappa \oint_{C'} \xi \nabla_t \xi^* \cdot \mathbf{dl}' = \frac{\kappa}{2} \oint_{C'} \nabla_t(\xi^2) \cdot \mathbf{dl}' = 0,$$

unlike the situation in the circular coaxial cavity [(35), (37) and (39)]. Thus the situation described here, as with the TEM cavity, is not suited to a measurement of κ. We shall, however, write down the frequency shift in the TE_{pqN} case.

We shall again consider the sample placed at $\sin \beta z_0 = 0$, so that there is no electric energy stored in the sample (21). After evaluating the various quantities needed in (14) and (18) and carrying out the integrations, we find for the frequency shift of a transversely centered slab.

$$\frac{d\omega}{\omega_0} = \frac{t}{L}\frac{\beta^2}{k^2}\frac{a'b'}{ab}(\mu - 1)\left[1 + (-1)^{p+q+1}\frac{\sin\theta_1}{\theta_1}\frac{\sin\theta_2}{\theta_2} + (-1)^p \frac{k_y^2 - k_x^2}{k_y^2 + k_x^2}\frac{\sin\theta_1}{\theta_1} + (-1)^q \frac{k_x^2 - k_y^2}{k_x^2 + k_y^2}\frac{\sin\theta_2}{\theta_2}\right], \quad (44)$$

$$\frac{\beta^2}{k^2} = \left(\frac{\lambda_0}{\lambda_g}\right)^2, \quad \beta L = N\pi, \quad k_x = \frac{p\pi}{a}, \quad k_y = \frac{q\pi}{b}, \quad \theta_1 = k_x a', \quad \theta_2 = k_y b',$$

p, q = integers.

III. REMARKS

In the case of TM_{pqN} modes in the circular coaxial cavity or circular cylindrical cavity, we see from (37) and (41) that the frequency shift depends on the values of the three quantities μ, $|\kappa|$ and ϵ. In these cases we placed the sample at $\sin \beta z_0 = 0$. If now we place the sample at $\cos \beta z_0 = 0$, (18) shows that the stored magnetic energy in the sample would be zero while (21) would give a stored electric energy in the sample proportional to $\epsilon - 1$. The frequency shift corresponding to (37) and (41) for this situation would thus depend only on ϵ. From these two situations we could infer all three quantities μ, κ and ϵ. A similar argument holds in the TE_{pqN} case in circular coaxial or circular cylindrical cavities [(35) and (40)]. If we placed the sample at $\cos \beta z_0 = 0$ we would again find zero magnetic energy in the sample and now $d\omega/\omega_0$ would depend only on $\epsilon - 1$ [see (18) and (21)]. Thus, we could again infer the values of μ, κ and ϵ separately. In the TM_{pq0} case, on the other hand, we can never separate the effects of μ and κ on the one hand and ϵ on the other, since the stored magnetic and electric energies in the sample are independent of longitudinal position of sample, as (25) and (26) show.

An interesting mathematical point is the manner in which μ, κ and ϵ appear in the equations. From (18), (21), (25), (26), (30) and (31) we see that the contour integrals which are the coefficients of μ and ϵ involve normal components at the ferrite periphery of vector functions of the field quantities, while the contour integral coefficient of κ involves tangential components of vector functions of field quantities.

Another point observed from (18) is that cavities in which the field potential (aside from the time dependence) is real (e.g. rectangular or TEM type cavities) are not suited for determination of κ since the basic κ integral, $\oint_{c'} \xi \nabla_t \xi^* \cdot d\mathbf{l}'$, vanishes. A polarization that is natural to the spin precession is needed and this is provided in the circular or square type cavities. In these cases the spatial field potential is complex and the κ integral does not vanish.

All results presented here apply also to the situation when the ferrite is lossy. In that case, μ, ϵ, κ and ω become complex in all formulas. A discussion of the Q of the cavity in this case is given in the various papers of Ref. 1.

APPENDIX I

Derivation of Equation (10)

Gauss' Theorem applied to the disc of Fig. 1 is

$$\int \operatorname{div}\,(\xi \nabla_t \xi^*)\, d\tau' = \int (\xi \nabla_t \xi^*) \cdot \mathbf{n}_0'\, dA'. \tag{45}$$

Since $\xi \nabla_t \xi^*$ is parallel to the surface S', there is no flux of this vector through S'. Thus, with $dA' = t dl'$ and $d\tau' = dS' \cdot t$ we have

$$\int \operatorname{div}\,(\xi \nabla_t \xi^*)\, dS' = \oint_{c'} (\xi \nabla_t \xi^*) \cdot \mathbf{n}_0'\, dl' = \int (\xi \nabla_t^2 \xi^* + |\nabla_t \xi|^2)\, dS'.$$

With $(\nabla_t^2 + k_c^2)\xi = 0$, this becomes

$$\int_{S'} |\nabla_t \xi|^2 \, dS' = k_c^2 \int_{S'} |\xi|^2 \, dS' + \oint_{C'} (\xi \nabla_t \xi^*) \cdot \mathbf{n}_0' \, dl', \qquad (46)$$

which is also applicable to C and S.

APPENDIX II

Derivation of Equation (11)

Consider two functions ξ_1 and ξ_0 which differ slightly, both satisfying the wave equation, and corresponding to the wave vectors $\mathbf{k}_{c_1}$ and $\mathbf{k}_c$, respectively:

$$\begin{aligned} (\nabla_t^2 + k_c^2)\xi_0 &= 0, \\ (\nabla_t^2 + k_{c_1}^2)\xi_1 &= 0, \\ \mathbf{k}_{c_1} &= \mathbf{k}_c + d\mathbf{k}_c, \\ k_{c_1}^2 &= k_c^2 + 2\mathbf{k}_c \cdot d\mathbf{k}_c, \\ \xi_1 &= \xi_0 + (\nabla_{k_c} \xi_0) \cdot d\mathbf{k}_c, \end{aligned} \qquad (47)$$

where ∇_{k_c} is the gradient operator in $\mathbf{k}_c$ space. We insert (47) into Green's identity which, for the geometry of Fig. 1, is

$$\int_{S'} (\xi_0^* \nabla_t^2 \xi_1 - \xi_1 \nabla_t^2 \xi_0^*) \, dS' = \oint_{C'} (\xi_0^* \nabla_t \xi_1 - \xi_1 \nabla_t \xi_0^*) \cdot \mathbf{n}_0' \, dl', \qquad (48)$$

and equate coefficients of $d\mathbf{k}_c$ terms on both sides. Then in the limit as ξ_1 approaches $\xi_0 \equiv \xi$, and noting from (48) and the wave equation that

$$\oint_{C'} (\xi_0^* \nabla_t \xi_0 - \xi_0 \nabla_t \xi_0^*) \cdot \mathbf{n}_0' \, dl' = 0,$$

this procedure gives

$$2k_c^2 \int_{S'} |\xi|^2 \, dS' = \oint_{C'} [(\nabla_{k_c} \xi \cdot \mathbf{k}_c)(\nabla_t \xi^* \cdot \mathbf{n}_0') - \xi^* \nabla_t (\nabla_{k_c} \xi \cdot \mathbf{k}_c) \cdot \mathbf{n}_0'] \, dl', \qquad (49)$$

which is again applicable to C and S. A one-dimensional form of this result is given by Sommerfeld.[4]

APPENDIX III

Derivation of Equation (12)

The following identity holds:

$$\nabla_t \times (\xi \nabla_t \xi^*) = \nabla_t \xi \times \nabla_t \xi^* + \xi \nabla_t \times \nabla_t \xi^* = \nabla_t \xi \times \nabla_t \xi^*, \tag{50}$$

so that

$$\begin{aligned} &\int_{S'} (\nabla_t \xi \cdot \mathbf{z}_0 \times \nabla_t \xi^*)\, dS' = \int_{S'} \mathbf{z}_0 \cdot (\nabla_t \xi^* \times \nabla_t \xi)\, dS' \\ &= \int_{S'} (\nabla_t \xi^* \times \nabla_t \xi) \cdot d\mathbf{S}' = \int_{S'} \nabla_t \times (\xi \nabla_t \xi^*) \cdot d\mathbf{S}' = \oint_{C'} (\xi \nabla_t \xi^*) d\mathbf{l} \cdot' \end{aligned} \tag{51}$$

where Stokes' theorem has been employed and $d\mathbf{S}' = \mathbf{z}_0 dS'$.

APPENDIX IV

Derivation of Magnetic Energy Stored in Ferrite, Equation (17)

We decompose the RF $\mathbf{H}_0$ field in the cavity (16) into a combination of two circular polarizations of opposite senses and possibly different amplitudes H_+ and H_-,

$$\mathbf{H}_0 = H_+ \boldsymbol{\sigma}_+ + H_- \boldsymbol{\sigma}_-, \tag{52}$$

where $\boldsymbol{\sigma}_+$ and $\boldsymbol{\sigma}_-$ are two-dimensional column vectors describing circular polarizations of opposite senses and unit amplitudes:

$$\boldsymbol{\sigma}_+ = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -j \end{pmatrix}, \boldsymbol{\sigma}_- = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ j \end{pmatrix}. \tag{53}$$

Both $\boldsymbol{\sigma}_+$ and $\boldsymbol{\sigma}_-$ satisfy the following relations:

$$\begin{aligned} &\boldsymbol{\sigma}_+^* = \boldsymbol{\sigma}_-, \quad \boldsymbol{\sigma}_+ \cdot \boldsymbol{\sigma}_+^* = 1, \quad \boldsymbol{\sigma}_- \cdot \boldsymbol{\sigma}_-^* = 1, \quad \boldsymbol{\sigma}_+ \cdot \boldsymbol{\sigma}_+ = 0, \\ &\boldsymbol{\sigma}_-^* = \boldsymbol{\sigma}_+, \quad \boldsymbol{\sigma}_+ \cdot \boldsymbol{\sigma}_- = 1, \quad \boldsymbol{\sigma}_- \cdot \boldsymbol{\sigma}_+ = 1, \quad \boldsymbol{\sigma}_- \cdot \boldsymbol{\sigma}_- = 0. \end{aligned} \tag{54}$$

Further from (15) we have

$$\begin{aligned} \chi \boldsymbol{\sigma}_\pm &= \chi_\pm \boldsymbol{\sigma}_\pm, \\ \chi_\pm &= (\mu - 1) \pm \kappa. \end{aligned} \tag{55}$$

Then it is easily shown that

$$\mathbf{M} \cdot \mathbf{H}_0^* = \chi_+ \,|\, H_+ \,|^2 + \chi_- \,|\, H_- \,|^2, \tag{56}$$

where from (16) and (53)

$$H_{\pm} = \mathbf{H}_0 \cdot \boldsymbol{\sigma}_{\pm}^*$$
$$= \frac{-j\beta \cos \beta z}{\sqrt{2}\omega\mu_0}\left[\left(\frac{\partial\psi}{\partial u} \pm j\,\frac{\partial\psi}{\partial v}\right) \text{ or } \frac{-k^2}{\beta^2}\left(-\frac{\partial\varphi}{\partial v} \pm j\,\frac{\partial\varphi}{\partial u}\right)\right] \tag{57}$$

for a ψ wave and φ wave, respectively. Then (56) and (57) give

$$\mathbf{M}\cdot\mathbf{H}_0^* = f\,\frac{\beta^2 \cos^2 \beta z}{\omega^2\mu_0^2}\,[(\mu - 1) \mid \nabla_t\xi \mid^2 + j\kappa\nabla_t\xi\cdot\mathbf{z}_0 \times \nabla_t\xi^*] \tag{58}$$

where $f = 1$ or $(k^2/\beta^2)^2$ for $\xi = \psi$ and $\xi = \varphi$, respectively. Equation (17) follows from an integration of (58) throughout the sample, with $d\tau' = dS'\,dz$.

REFERENCES

1. Artman, J. O. and Tannenwald, P. E., Measurement of Permeability Tensor in Ferrites, Phys. Rev., **91,** Aug. 15, 1953, p. 1014; Measurement of Susceptibility Tensor in Ferrites, J. Appl. Phys., **26,** Sept. 1955, p. 1124.
2. Rowen, J. H. and Von Aulock, W., Measurement of the Complex Tensor Permeability of Ferrites, Phys. Rev., **96,** Nov. 1954, p. 1151; Measurement of Dielectric Magnetic Properties of Ferromagnetic Material at Microwave Frequencies, B. S. T. J., **36,** March 1957, p. 427.
   Lax, B. and Berk, A. D., Convention Record of the Meeting of the I.R.E., March 1953 (unpublished).
   Beljers, H. G., Measurements on Gyromagnetic Resonance of a Ferrite Using Cavity Resonators, Physica, **14,** Feb. 1949, p. 629.
   Spencer, E. G., LeCraw, R. C. and Reggia, F., Circularly Polarized Cavities for Measurement of Tensor Permeabilities, J. Appl. Phys., **26,** March 1955, p. 354.
   Berk, A. D. and Lengyel, B. A., Magnetic Fields in Small Ferrite Bodies with Applications to Microwave Cavities Containing Such Bodies, Proc. I.R.E., **43,** Nov. 1955, p. 1587.
   Waldron, R. A., Proc. Inst. Elect. Eng., **104B,** No. 6, Oct.-Nov. 1956, pp. 307–315.
2. Casimir, H. B. G., On the Theory of Electromagnetic Waves in Resonant Cavities, Philips Res. Rept., **6,** 1951, pp. 162–182.
   Bethe, H. A. and Schwinger, J., N.D.R.C. Rept. D1-117, March 1943.
   Muller, J., Hochfr.-Tech. und Elektr., **54,** 1939, p. 157.
3. Jahnke, E. and Emde, F., *Tables of Functions*, 4th Ed., Dover Publishers, N. Y., 1945, p. 144.
4. Sommerfeld, A., *Partial Differential Equations*, Academic Press, N. Y., 1949, p. 310.