A Criterion to Limit Inspection Effort in Continuous Sampling Plans

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In continuous sampling plans of the type known as CSP-1, the amount of screening has an important bearing on the total inspection effort. To limit this effort an inspector may be required to take special action if the number of inspected units in one screening sequence exceeds some specified value or "critical length". The aim of the special action is to bring about improvement in the production process. This effect is possible also when the producing shop is required to do any screening called for by the inspection plan.

A procedure for calculating critical lengths may be based on simple approximations derived from the theory of runs.

I. INTRODUCTION

1.1 *Continuous Sampling Plans*

The CSP-1 continuous sampling plans introduced by H. F. Dodge[1] are designed for continuous or "belt" production of discrete units of product. To apply such a plan, inspected units must be classified as either "defective" or "nondefective". The inspector begins by inspecting each unit made in succession until a specified number, i, of consecutive units are found nondefective. A sequence of units so inspected is called a screening sequence and the number i the clearing number. After the initial screening sequence has ended, the inspector samples a fraction f of the units presented to him. He continues to sample until he finds a defective unit. At this point he again resorts to screening, following the same procedure as before, so that he alternates between screening and sampling inspection. The inspector rejects (or sets aside for correction) any inspected unit found to be defective and accepts all others.

Two refinements of this plan, CSP-2 and CSP-3, have appeared[2] as well as generalizations of CSP-1[3, 4, 5] entailing two or more levels of sampling inspection. In addition, various sequential continuous inspection plans have been proposed.[6]

The characteristics of these different sampling plans — such as AOQL, fraction inspected, or characteristic curves — have been explored under a variety of assumptions. Of these assumptions, the statistical behavior of the production process has the greatest effect on the results. There are three alternatives which have been used (but which may not cover all plausible situations):

(I) The production process is Bernoullian: each unit has the same probability of being defective independent of any other unit; the proportion of defective units converges almost certainly to this value as the number of units produced increases. It is therefore known also as the process average.

(II) The production process represents a stationary Markov chain; each unit has a probability of being defective which depends only on the defectiveness or non-defectiveness of the previous $k(\geqq 1)$ units produced and is otherwise independent of time.

(III) The production process represents a discrete stochastic process of an arbitrary nature.

Not all the continuous sampling plans introduced have been examined under each assumption.

Assumption (I) leads to the simplest mathematics and will be adopted here. Its use does not imply that the CSP-1 plans — with or without the criterion proposed below — are invalid if the production process goes out of control. These plans were designed with this condition in mind. The effect of lack of control is to alter the stated characteristics of such plans, but the author has no evidence from actual production processes that such deviations are wide.

Another factor that influences the characteristics of continuous sampling plans is the kind of sampling used when sampling is required. Again there are three alternatives commonly used:

(i) The sampling is Bernoullian; each unit bears a probability f of being sampled independent of any other unit; in this case and in (iii) below screening is usually required to begin with the next unit after a known defective.

(ii) One unit in each (disjoint) set of $1/f$ consecutive units produced is randomly chosen from the set for inspection. Screening, when required, may begin within the same set in which a defective unit is found, or it may begin with the first unit of the next set. One or the other method of starting to screen is usually specified.

(iii) Every $1/f$th unit is inspected.

In most characteristics of CSP-1 it makes little difference whether (i), (ii), or (iii) is used provided (I) is assumed. Again the mathematics is simpler with (i), and accordingly we shall follow it.

A third assumption is sometimes made about the operation of CSP-1: each defective unit inspected is replaced by a nondefective one. This assumption affects only the character of outgoing quality. It will have no bearing on the criterion for inspection effort discussed below.

1.2 *Inspector's Risk*

With this brief background we may take up the main subject of this paper. In using any inspection plan there are three areas of risk: one area pertains to the consumer's operations, another to the producer's operations, and the third to the inspector's operations. One risk in the third area is that the inspector may be called upon to perform an excessive amount of inspection for the amount of protection he furnishes. The CSP-1 plans, although admitting the necessity of high inspection rates on occasion, are not really intended to be used when inspection will continue indefinitely at a high rate. In general such a high rate would not lead to economical and effective inspection nor to economical manufacture: screening alone does not guarantee that the level of incoming quality will improve enough to diminish the amount of screening significantly in the future. Neither is there so much confidence in the outgoing quality, which poor incoming quality may affect adversely in spite of intensified screening. Indeed, the existence of such a situation may imply some basic difficulty in the process of design or manufacture that cannot be properly handled by inspection methods alone. Not only the inspector but the customer may be undergoing a special risk. Furthermore, the producer often does any screening required (as Dodge originally recommended[1]). He too might find an appropriate special action economically preferable to a great deal of screening.

Thus the inspector needs a special alarm signal to indicate that unless he takes special action a high rate of screening may continue. The following sections show how such a special alarm signal for CSP-1 plans may be devised on the basis of the number of units inspected in any one screening sequence. If this number exceeds a "critical length", n^*, chosen in advance, the inspector is to take an appointed special action.

A similar type of criterion could be evolved for other types of continuous sampling plans. The effectiveness of this type of criterion alone might be lessened if it were applied to other types of plans in which screening is not so promptly reinstated after a defective is found as in the CSP-1 plans. It seems certain that the "most sensitive criterion" for any of these plans, including CSP-1, would take account somehow of the observed per cent defective. On the other hand simplicity and convenience would have to be sacrificed to some extent to do so. For the

CSP-1 plans it is hoped that the proposed criterion of critical length is a satisfactory compromise between theoretical and practical requirements.

The special action to be taken when required by this criterion should depend upon the situation. It might be to notify the customer's purchasing or contracting department; it might go so far as to cause the inspector to stop inspection, effectively halting purchase of product. If such a severe action is specified, the manufacturing unit may rightly feel entitled to be informed in advance whenever such action appears imminent so that it may begin to adjust the process and to screen product ahead of the inspector. Using a different criterion from the one proposed here, an existing government inspection plan[7] does, in fact, require the inspector to stop inspecting. It is not our purpose, however, to discuss in detail any particular special action since its wisdom could be confirmed only by reference to the nature of the application. It is intended only to point out that such actions have already been devised and used.

There is no reason to adjust published AOQL figures for CSP-1 plans because of the addition of this special action criterion to their operation. If the special actions are suitable, there is no reason to expect anything but an improvement in the outgoing quality level.

II. THE CRITICAL LENGTH OF A SCREENING SEQUENCE

2.1 *The Basis for Choosing Critical Lengths*

It is generally possible for an inspection agency to state what it considers a reasonable upper limit to the amount of inspection it should be required to perform under a given CSP-1 plan. Let us call this limit F^*. Under our assumptions Dodge[1] has shown that, when the probability of a defective unit is p, the average amount of inspection (i.e., limiting fraction of units inspected) is

$$F = \frac{f}{f + (1 - f)q^i}, \qquad (q = 1 - p), \tag{1}$$

if an inspector uses a CSP-1 plan with clearing number i and sampling frequency f. It is clear that placing an upper limit F^* on F is equivalent to placing an upper limit p^* on p. Indeed, if

$$(1 - p^*)^i = \frac{f}{1 - f} \cdot \frac{1 - F^*}{F^*} = K, \tag{1'}$$

the inequality $F \leqq F^*$ implies and is implied by $p \leqq p^*$ according to (1).†

Having specified F^* as the upper limit to the amount of inspection, we need a measure of the price the inspection agency should be willing to pay to enforce it. For our purposes it will be convenient to choose as a measure the maximum probability α^* of taking special action when $F \leqq F^*$. It is equivalent to say that α^* shall be the fraction of all screening sequences in which the inspector takes special action when $F = F^*$ or when $p = p^*$. In practice the choice of F^* or α^* or both may be somewhat arbitrary. In the author's experience, the choice of $F^* = 0.5$ and $\alpha^* = 0.10$ has proved reasonable.

We may now choose a critical length n^* so that special action is taken in accordance with the risk specified above. First, the inspector is to take special action whenever a screening sequence has not terminated after the n^*th consecutive unit in the sequence has been inspected. Second, n^* is to be chosen so that when $F = F^*$ the fraction α^* of all screening sequences have not terminated after n^* units.

This second condition cannot in general be fulfilled exactly. Instead, if we call the probability that a screening sequence has not terminated after n units $T_n(p, i)$, we shall find n^* satisfying

$$T_{n^*}(p^*, i) \leqq \alpha^* < T_{n^*-1}(p^*, i). \tag{2}$$

It can be easily demonstrated that for any α^* and p^* satisfying $0 < \alpha^* < 1$ and $0 < p^* < 1$, there is a solution, n^*, to (2).

It is sensible to desire that the higher the "true" limiting fraction F of units inspected, the more likely it is that a screening sequence will exceed its critical length. The truth of this statement can be easily shown also. This guarantees that α^* is a *maximum* probability of taking the special action incorrectly.

The mathematical problem, as we have stated it, is covered by the theory of runs. Its solution has long been known[8] and will be discussed in the following section, as well as in the Appendixes. Briefly, in terms

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† Certain variations of CSP-1 lead to different expressions from those given here. For instance, if the producer does *all* screening, the inspector will often inspect a fixed proportion, f, of all units — including those already screened. It is then more sensible to apply F^* as an upper limit only to the average amount of *screening*, which is the product of $1 - q^i$ and the right side of (1). Solving for p^* or K then requires that f be added to the denominator $(1 - f)F^*$ in (1′). Another variation arises when defective units inspected are repaired and reinspected. If it were assumed that the proportion defective among repaired units is again p, it would be necessary only to divide the right side of (1) by $1 - p$ and to replace F^* by $F^*(1 - p^*)$ in (1′). Then p^* or K would be found by iteration. The effects of these two variations may be combined but not without further assumptions.

of this theory, we may restate our problem as follows: Given a Bernoullian process with "success" probability $q^* = 1 - p^*$, to find the least number of trials, n^*, in which the probability of having had no runs of i (or more) "successes" is less than or equal to α^*.

Common sense demands that the special action never be taken until there has been some chance to complete the screening sequence. That is, the critical length of a screening sequence must be larger than the clearing number. It is shown in Appendix A that it is equivalent to require

$$\alpha^* < \frac{F^* - f}{(1 - f)F^*}. \tag{3}$$

This restriction is usually minor. For instance, if, as above, $F^* = 0.5$ and $\alpha^* = 0.10$, then according to (3) $f < 9/19 = 0.474^-$. For $\alpha^* \leqq 0.5$ it is more convenient generally to use the inequality

$$F^* \geqq f + 0.70\alpha^*. \tag{3'}$$

As is shown in Appendix A, (3) is satisfied whenever (3′) is.

In any case the value computed for n^* will depend upon the assumptions discussed in Section I. If these are inexact, the probability statements outlined above will generally be inexact also. Nevertheless, the same value of n^* may still be used with good prospect of limiting screening effort without added penalty to the manufacturer.

2.2 *Computation of* n^*

As noted above, the exact relation between n^*, i, p^* and α^* is known, but it is difficult to use. To simplify computation it has been found advisable to resort to an approximation for n^*, which assumes the form

$$n^* \doteq a_1 i + a_0 , \tag{4}$$

where a_0 and a_1 depend only on K, defined in (1′), and α^*. A derivation of this approximation is given in Appendix B. It is based on asymptotic results for large n^* and i.

It is interesting to note that K, when defined in terms of p^* and i, is the probability of terminating a screening sequence in exactly i trials. For the purposes of this paper we shall determine K in terms of f and F^*.

For convenience the coefficients a_0 and a_1 are presented in graphical form in Fig. 1 with values of K on the abscissa and with separate curves for $\alpha^* = 0.01, 0.05, 0.10$. The requirement (3) is observed by plotting these curves only over the interval of values of K satisfying $\alpha^* < 1 - K < 1$. While the immediate field of interest is inspection, the

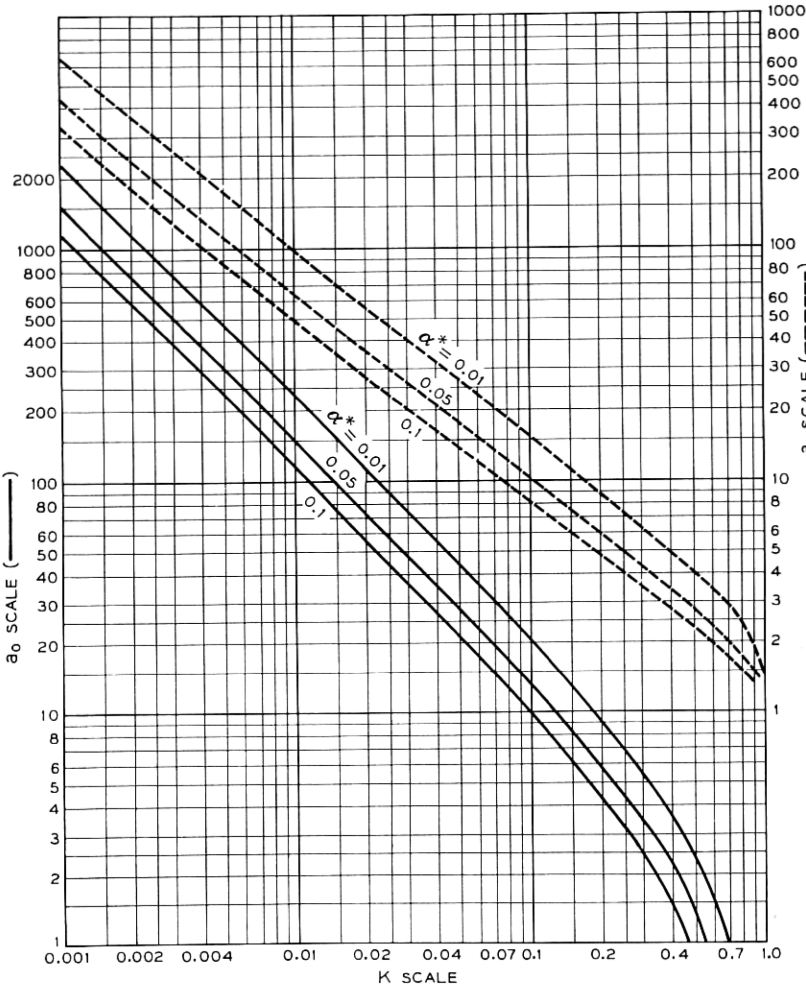

Fig. 1 — The coefficients a_0 and a_1 as functions of $K = f(1 - F^*)/F^*(1 - f)$ for $\alpha^* = 0.01, 0.05, 0.10$. The critical length n^* is approximated by $a_1 i + a_0$.

values of a_0 and a_1 read from this chart obtain equally well in other uses of the theory of runs. Therefore, the range of K in Fig. 1 is considerably larger than would be necessary to handle this particular problem alone. Given the values f, i, F^* and α^*, the value K may be computed from (1′). If $\alpha^* = 0.01, 0.05$, or 0.10, we may choose the proper curve for a_0 , read off its ordinate at the computed value of K, follow a similar procedure to find a_1 , and compute n^* from (4).

In order to compute the coefficients a_0 and a_1 for any values of α^* and F^* satisfying (3), it is necessary to compute

$$w = -\ln K = \ln \frac{F^*}{1 - F^*} - \ln \frac{f}{1 - f} \tag{5}$$

and to solve

$$we^{-w} = ve^{-v}, \qquad (v \neq w \text{ for } w \neq 1), \tag{6}$$

for v, the letters "ln" indicating the logarithm to the base e. It is usually easiest to solve (6) by the following convergent iterative procedure: If $w \geqq 1$, put

$$v_0 = we^{-w}, \qquad v_{m+1} = v_0 e^{v_m}; \tag{7}$$

if $w \leqq 1$, put

$$v_0 = w - \ln w, \qquad v_{m+1} = v_0 + \ln v_m . \tag{7'}$$

In either case v may be obtained with as much accuracy as desired by simple iteration with formulas (7) or (7′).

The coefficients a_0 and a_1 then may be expressed as

$$a_1 = \frac{1}{v}\left[\ln \frac{w - v}{2(1 - v)} - \ln \frac{w\alpha^*}{2}\right], \tag{8}$$

$$a_0 = a_1 \cdot \frac{w - v}{2(1 - v)} - \frac{v + w - 2}{2(1 - v)^2} - 1. \tag{9}$$

The limiting values of a_1 and a_0 as w and v approach unity are given by (B9), (B11), and (B12) in Appendix B.

The accuracy of the approximation (4) has been investigated and found to be adequate. For small i and f and large F^* slightly greater precision is possible with the Uspensky approximation[9], computation of which is simplified by Feller's iterative procedure[10] (see Appendix B). Both approximations lose accuracy as α^* increases.

For $F^* = 0.5$ and $\alpha^* = 0.1$, Table I presents a comparison of the exact integral value satisfying (2) with the two approximations, in which the value of n^* satisfies the equation concerned as precisely as possible and is therefore not integral. For this table the exact recursion formulas (A3) and (C3) were used. The latter was found by Miss M. N. Torrey and the writer and appears to be new.

2.3 *Some Properties of the Criterion of Critical Length*

According to the previous discussion, it is proposed to take special action whenever a screening sequence exceeds its critical length. Since

TABLE I — TABLE OF CRITICAL NUMBERS n^* FOR UPPER LIMIT OF FRACTION INSPECTED $F^* = 0.5$ AND MAXIMUM PROBABILITY OF ERROR $\alpha^* = 0.1$†

Clearing Number, i	Sampling Frequency, f								
	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45
5	88 87.8 85.7	47 46.6 45.9	32 31.5 31.2	24 23.4 23.3	19 18.2	15 14.5	12 11.7	10 9.4	8 7.4
10	153 152.2 151.2	84 83.3 83.0−	58 57.5 57.4	44 43.5	35 34.4	28 27.8	23 22.8	19 18.6	16 14.8
20	283 282.5 282.0−	158 157.3 157.1	110 109.8	84 83.7	67 66.7	55 54.4	45 44.8	38 36.9	31 29.8
50	675 674.7 674.5	380 379.6	267 266.8	205 204.5	164 163.6	135 134.1	111 111.0+	94 91.9	75 74.6
100	1329 1329 1329	751 750.3 750.2	529 528.6	406 405.8	326 325.2	268 266.8 267.0−	221 221.4	187 183.5	150 149.3
300	3946 3947 3946	2233 2234 2233	1576 1577 1576	1212 1212 1212	973 970.9 971.9	800 797.0− 798.4	661 662.7 662.9	560 550.1	450 448.0+

† The triad of numbers appearing for f and i combinations are, reading down, the exact value, the Uspensky approximation, and the approximation (4) to (9). The last is omitted if it agrees with the second to 0.1. Approximate values less than 1,000 should be rounded to the next higher integral value to obtain the result corresponding to the exact value of n^*. This method was followed in rounding approximate values greater than 1,000.

the aim of such action would be to bring about improvement in the process, it might be justifiable to resume inspection with sampling after the special action has been taken. There is a question in any case whether the screening sequence, once interrupted by special action, should be resumed at the point it was stopped. A cautious procedure would be to resume inspection with a new screening sequence not involving any previously inspected units. This course would lead to a lower AOQL but a higher fraction inspected, F^C, than the original plan.‡ Resumption with sampling would have the opposite effect, but the changes in either case should be slight in practice.

We shall consider in detail only the effect of increasing the total amount of inspection when inspection is resumed with a new screening sequence after special action has been taken. With this alteration in the CSP-1 inspection plan, the limiting fraction inspected, F^C, according to our

‡ There is no change from the original values if the inspector takes special action as soon as he finds a defective unit after $n^* - i$ units in a single screening sequence and before that sequence is ended.

mathematical model, is given by (D24) in Appendix D. Furthermore, the upper limit to this fraction is according to (D34)

$$F^{C*} = \frac{F^*}{F^* + (1 - F^*)\beta^*}, \tag{10}$$

where

$$\beta^* = (1 - \alpha^*)/(1 - \alpha^* K - T_{n^*+i}(p^*, i)) \tag{11}$$

and all other quantities are as defined previously. The comparatively minor change from F^* to F^{C*} in the range of interest can be illustrated by noting that if $F^* = 0.5$, $\alpha^* = 0.1$, and $f \geqq 0.05$, we will have $0.5 < F^{C*} < 0.51$.

Under these same assumptions two other characteristics of the modified CSP-1 plan can be readily computed: The average number of special actions per 10,000 units produced and the average number of special actions per 10,000 units inspected. These may be computed by multiplying C in (D27) and C^I in (D32) respectively by 10^4. The first of these two averages may be the more useful to the practitioner, who can use the value of this average at $p = p^*$ as an added measure of the price paid for using the criterion of critical length. In some cases he may prefer it to α^*. For $F^* = 0.5$, $\alpha^* = 0.1$, and $p = p^*$ this number varies from about 0.4 for $f = 0.05$ and $i = 5$ to about 15 for $f = 0.45$ and $i = 100$.

Another more theoretical use may be made of these two averages. We may wish to compare the operation of the criterion of critical length with that of any other criterion adopted for the same purpose. The parameters of the criterion to be compared to the present one could be adjusted so that one or both of these two averages agree for the two schemes when $p = p^*$. Then the average number of special actions per 10,000 units produced could be plotted against p or F in both cases. On the other hand one may wish the fraction inspected to be the same at $p = p^*$ for the two schemes. However, criteria calling for special action at certain times when the last inspected unit was defective lead to the same fraction inspected as found in the original CSP-1 plan. In such cases it is not possible to obtain equal fractions inspected, since $F^C > F$. It appears better in general to deal with the two averages C and C^I for the purpose of comparing criteria. At any event, as has been mentioned above, such formal comparisons are not complete measures of practical value in themselves.

III ACKNOWLEDGMENTS

The author is indebted to Miss M. N. Torrey for her assistance, particularly in the work summarized in Appendix C and Table I. Many thanks are due also to Miss J. Zagrodnick, who did most of the computations, to Miss B. A. Leetch, who computed the functions given in Fig. 1, and to S. W. Roberts, who assisted in programming many of the computations. The comments of H. F. Dodge and J. W. Tukey on earlier versions of this paper have been most helpful to the writer.

APPENDIX A

Some Properties of the Run Probability $T_n(p, i)$

As before let $T_n(p, i)$ be the probability that a screening sequence with clearing number i and process average p has not terminated after the nth consecutive unit has been inspected. This is the same as the probability of no run of i or more "successes" each having probability $q = 1 - p$ in n independent trials. Except when necessary for clarity to do otherwise, we shall abbreviate $T_n(p, i)$ by T_n.

It is easy to see that

$$T_n = 1, \qquad n = 0, 1, \cdots, i - 1, \tag{A1}$$

$$T_i = 1 - q^i, \ (q = 1 - p), \tag{A2}$$

$$T_{n-1} - T_n = pq^i T_{n-i-1}, \qquad (n > i). \tag{A3}$$

From these relations it appears that the generating function of T_n,

$$T(x) = \textstyle\sum_{n=0}^{\infty} x^n T_n,$$

satisfies

$$T(x) = \frac{1 - q^i x^i}{1 - x + pq^i x^{i+1}}, \tag{A4}$$

in which both numerator and denominator have the common factor $1 - qx$.

If it is required that $T_i > \alpha^*$, we have directly from (A2)

$$1 - q^{*i} > \alpha^*. \tag{A5}$$

From (1′) and (A5) the inequality (3) follows. In turn $F^* - f$ is seen from (3) to exceed $\alpha^* f(1 - f)/[1 - \alpha^*(1 - f)]$. Maximizing this quantity with respect to f, we have $2(1 - \sqrt{1 - \alpha^*})/\alpha^* - 1$, which is less than α^* for $0 < \alpha^* < 1$. This maximum and its derivative with respect to α^* are increasing in this interval, and the former assumes the value $3 - 4/2^{-1/2} < 0.35$ at $\alpha^* = 0.5$. The inequality (3′) follows immediately.

APPENDIX B

Derivation of an Approximation for Critical Length

From the expansion of (A4) in partial fractions, Uspensky has shown that as n approaches infinity,

$$T_n \sim \frac{1 - q\xi}{p(i + 1 - i\xi)} \cdot \frac{1}{\xi n^{+1}} \tag{B1}$$

with ξ the unique positive root of

$$\sum_{s=0}^{i-1} (qx)^s = \frac{1}{px}. \tag{B2}$$

The Uspensky approximation for T_n leads to the approximation (4) for n^* satisfying (2) for any given α^* $(0 < \alpha^* < 1)$, i, and $p^*(0 < p^* < 1)$. It can, in fact, be shown that

$$n^* \sim a_1 i + a_0 + a_{-1} i^{-1} + a_{-2} i^{-2} + \cdots .$$

If in (B1) and (B2), we put $p = p^*$, $q = q^*$, and $n = n^*$, it follows from (1′) and (B2) that

$$T_{n*} \sim \frac{1 - K\xi^i}{(i + 1 - i\xi)} \cdot \xi^{-n*} . \tag{B3}$$

Likewise, making the same substitutions in (B2) it follows that

$$K(1 - K^{1/i})x^{i+1} - x + 1 = 0 \tag{B4}$$

has two and only two positive real roots, $x_1 = 1/q^*$ and $x_2 = \xi$.

We shall consider a system of equations in five variables equivalent to the system (B3) and (B4) in the five variables K, i, ξ, n^*, and T_{n*}. We shall call the new variables w, z, v, φ, and α^*. The new system of equations is

$$\alpha^* = \frac{1 - e^{-w}(1 - vz)^{-1/z}}{1 - v - vz} \cdot (1 - vz)^{\varphi/z}, \tag{B5}$$

and

$$e^{-w}(1 - e^{-wz})(1 - vz)^{-1/z} = vz, \tag{B6}$$

where only finite positive values for all variables are going to be considered with $vz < 1$ and $0 < \alpha^* < 1$. If in (B5) and (B6) we put $w = -\ln K$, $z = i^{-1}$, $v = i(\xi - 1)/\xi$, $\varphi = i^{-1}(n^* + 1)$, and $\alpha^* = T_{n*}$, the result is (B3) and (B4) with the symbol "$\sim$" replaced by "$=$" in (B3) and with $x = \xi$ in (B4).

Like (B4), (B6) has one "extraneous" root, $v_1(w, z) = (1 - e^{-wz})/z$. Either positive root, $v_1(w, z)$ or $v_2(w, z)$, is such that there exists a finite function $v_{s0}(w) = \lim_{z \to 0} v_s(w, z)$, $(s = 1, 2)$. Indeed, $v_{10}(w) = w$ and $v_{20}(w) = v_0(w)$ satisfy the limiting form of (B6) given by (6). Clearly either $0 < w \leqq 1 \leqq v_0(w) < \infty$ or $0 < v_0(w) \leqq 1 \leqq w < \infty$, the equality signs holding simultaneously.

Taking logarithms of both sides of (B5), we have

$$\varphi = \left(\ln \alpha^* + \ln \left[\frac{1 - v - vz}{1 - e^{-w}(1 - vz)^{-1/z}} \right] \right) \Big/ \frac{1}{z} \ln(1 - vz).$$

As z approaches zero, φ approaches

$$\varphi_0(w, \alpha^*) = -\frac{1}{v_0(w)} \left(\ln \alpha^* + \ln \left[\frac{w(1 - v_0(w))}{w - v_0(w)} \right] \right). \tag{B7}$$

We may differentiate (B6) and (B5) to obtain respectively

$$\left. \frac{\partial v(w, z)}{\partial z} \right|_{z=0} = \frac{v_0(w)}{2} \cdot \frac{(v_0^2(w) - w)}{(1 - v_0(w))}$$

and

$$\begin{aligned} \varphi_0'(w, \alpha^*) &= \left. \frac{\partial \varphi}{\partial z} \right|_{z=0} \\ &= \frac{w - v_0(w)}{2(1 - v_0(w))} \cdot \varphi_0(w, \alpha^*) - \frac{v_0(w) + w - 2}{2(1 - v_0(w))^2}. \end{aligned} \tag{B8}$$

Substituting the values $w = -\ln K$, $z = i^{-1}$, and $\varphi = i^{-1}(n^* + 1)$ in $\varphi = \varphi_0(w, \alpha^*) + z\varphi_0'(w, \alpha^*) + \cdots$ and putting

$$v_0(w) = v, \qquad a_1 = \varphi_0(w, \alpha^*), \qquad a_0 = \varphi_0'(w, \alpha^*) - 1, \tag{B9}$$

we have the approximation given by equations (4) through (9).

If both sides of equation (6) are divided by $d = v_0(w) - w$, we get

$$w = d/(e^d - 1). \tag{B10}$$

From (B7), (B8), and (B10) we find

$$\varphi_0(w, \alpha^*) = \ln 2 - \ln \alpha^* + 0_1(d), \tag{B11}$$

$$\varphi_0'(w, \alpha^*) = \varphi_0(w, \alpha^*) - \tfrac{1}{3} + 0_2(d). \tag{B12}$$

The related approximation for T_n, as n and i approach infinity, is of the form

$$T_n \sim A e^{-B(n+C)/(i+D)}, \tag{B13}$$

where A, B, C, and D depend only on q^i.

APPENDIX C

A Recursion Formula for $T_n(p, i)$

In order to investigate the error in n^* computed from the Uspensky approximation (B1) or the approximation (4), a convenient form of the exact value of $T_n(p, i)$ was needed. Such an expression is

$$T_n = \sum_{s=0}^{\min(k,r)} (-1)^s \binom{r}{s} (pq^i)^s T_{(k-s)i}, \tag{C1}$$

where $n = ki + r$ and $k \geqq 1$ and $0 \leqq r \leqq i$. This may be established easily by an inductive argument. Indeed, if $k = 1$ and $r = 0$, (C1) yields an identity. By adding successive expressions of the form of (A3), we obtain

$$T_{ki+r} = T_{ki} - pq^i \textstyle\sum_{s=0}^{r-1} T_{ki-i+s} \, . \tag{C2}$$

For $k = 1$, (C2) yields with the aid of (A1)

$$T_{i+r} = T_i - rpq^i,$$

substantiating (C1) for $k = 1$ and $1 \leqq r \leqq i$. Next we assume (C1) to be true for some $k \geqq 1$ and some r, $(0 \leqq r \leqq i)$. We wish to show that (C1) is true for $n = (k + 1)i + r$. From (C2) and the induction assumption

$$T_{(k+1)i+r} = T_{(k+1)i} - pq^i \sum_{s=0}^{r-1} \sum_{t=0}^{\min(k,s)} (-1)^t \binom{s}{t} (pq^i)^t T_{(k-s)i} \, .$$

If the order of summation is reversed, the double sum becomes

$$- pq^i \sum_{t=0}^{\min(k,r-1)} (-1)^t (pq^i)^t T_{(k-t)i} \sum_{s=t}^{r-1} \binom{s}{t}$$

$$= \sum_{t=0}^{\min(k+1,r)-1} (-1)^{t+1} (pq^i)^{t+1} T_{(k-t)i} \binom{r}{t+1},$$

so that

$$T_{(k+1)i+r} = \sum_{s=0}^{\min(k+1,r)} (-1)^s \binom{r}{s} (pq^i)^s T_{(k+1-s)i} \, .$$

The special form of (C1) used in checking accuracy was that with $r = i$ and $1 \leqq k \leqq i$:

$$T_{(k+1)i} = \sum_{s=0}^{k} (-1)^s \binom{i}{s} (pq^i)^s T_{(k-s)i} \, . \tag{C3}$$

APPENDIX D

Some Characteristics of the CSP-1 Plan With and Without the Criterion of Critical Length

The use of generating functions is helpful in characterizing the original CSP-1 plans. The theory of Markov chains, applied somewhat as in Reference 4, also leads to some of the results found here. While this theory is convenient to show the validity of the strong law of large numbers as applied to fraction inspected and other ratios, the task of computing (to which we restrict ourselves here) appears generally simpler with the generating function technique. Let P_r be the probability that the rth unit produced is the first one in some sampling period, and let Q_r be the probability of being in a sampling period on that unit. Then

$$P_r = 0, (0 \leqq r \leqq i), \qquad P_{i+1} = q^i, \qquad\qquad \text{(D1)}$$
$$P_r = q^i p[1 - (1 - f)Q_{r-i-1}], \quad (r > i + 1),$$

and

$$Q_r = \textstyle\sum_{s=0}^{r} P_s(1 - fp)^{r-s}, \qquad (r \geqq 0). \quad \text{(D2)}$$

If $P(x)$ and $Q(x)$ are the corresponding generating functions, we have from (D1) and (D2) respectively

$$P(x) = \frac{x^{i+1}q^i}{1 - x}[1 - qx - p(1 - f)(1 - x)Q(x)] \qquad \text{(D3)}$$

and

$$Q(x) = P(x)/[1 - (1 - fp)x], \qquad \text{(D4)}$$

whence

$$Q(x) = \frac{q^i x^{i+1}}{1 - x}\left[1 - p(1 - f)x \sum_{m=0}^{i-1} (qx)^m\right]^{-1}. \qquad \text{(D5)}$$

With some manipulation of partial fractions we obtain

$$Q(x) = q^i x^{i+1}\left[\frac{(1 - x)^{-1}}{f + (1 - f)q^i} + \sum_{r=0}^{\infty} e_r x^r\right], \qquad \text{(D6)}$$

where it can be shown that e_r approaches zero as r approaches infinity. It follows that

$$Q = \lim_{r\to\infty} Q_r = \frac{q^i}{f + (1 - f)q^i}, \qquad \text{(D7)}$$

and, therefore, the limit of P_r exists as does the limit of

$$F_r = 1 - (1 - f)Q_r, \tag{D8}$$

the probability that the rth unit produced will be inspected. From (D7) and (D8) we may obtain (1).

There are similar results in terms of units inspected. Let P_r^I and Q_r^I correspond to P_r and Q_r with r in the former pair indicating the ordinal number of the unit inspected. Thus

$$\begin{aligned} P_r^I &= 0, (0 \leqq r \leqq i), \qquad & P_{i+1}^I &= q^i, \\ P_r^I &= q^i p, & &(r > i + 1), \end{aligned} \tag{D9}$$

and

$$\begin{aligned} Q_r^I &= 0, (0 \leqq r \leqq i), \\ Q_r^I &= \sum_{s=0}^{r} P_s^I (1 - p)^{r-s} = q^i, \quad (r \geqq i + 1). \end{aligned} \tag{D10}$$

Before passing to the modified CSP-1 plan, we observe that we may write the expression for P_r in (D1) and (D3) in a different way. First, let R_r be the probability that screening is stopped for the first time after the rth unit produced (i.e., a run of i nondefectives has been completed for the first time with that unit). The generating function of R_r, $R(x)$, then satisfies

$$R(x) = 1 - (1 - x)T(x) = \frac{q^i x^i (1 - qx)}{1 - x + pq^i x^{i+1}}, \tag{D11}$$

where $T(x)$ is given by (A4). Next we may put

$$P_r = R_{r-1} + fp(Q_0 R_{r-1} + \cdots + Q_{r-1} R_0), \tag{D12}$$

or

$$P(x) = xR(x)[1 + fpQ(x)], \tag{D13}$$

and

$$Q(x) = xR(x)/[1 - x + fpx(1 - R(x))]. \tag{D14}$$

We now take up the case in which a criterion for the length of screening sequences is applied in the operation of the CSP-1 plan. To simplify notation a little, we will hereafter call the (fixed) critical length n rather than n^*. Also the probabilities corresponding to P_r and Q_r will be denoted by $P_r^{\ C}$ and $Q_r^{\ C}$. We shall treat the occurrence of an incomplete screening sequence of length n as a recurrent event; as soon as such a sequence occurs, the whole inspection procedure begins anew with the next unit produced.

Now let $R_r^{\ C}$ be the probability that screening is stopped for the first time after the rth unit produced. Then

$$R_{r+1}^{\ C} = T_n^{\ [r/n]} R_{r-n[r/n]}\,, \tag{D15}$$

and the generating function is

$$R^C(x) = R^*(x)/(1 - T_n x^n), \tag{D16}$$

where

$$R^*(x) = \textstyle\sum_{r=0}^{n} R_r x^r. \tag{D17}$$

If the superscript C is attached to the symbols P, Q, and R in (D2) and (D12), we arrive at valid equations. Therefore, we may write down the generating function $Q^C(x)$ by placing the superscript C on the same letters in (D14). Using the identity

$$R^*(x) = 1 - T_n x^n - (1 - x)T^*(x), \tag{D18}$$

where

$$T^*(x) = \textstyle\sum_{r=0}^{n-1} T_r x^r, \tag{D19}$$

we have

$$Q^C(x) = \frac{xR^*(x)}{1 - x}\,[1 - T_n x^n + fpxT^*(x)]^{-1}. \tag{D20}$$

Again we find by the use of partial fractions

$$Q^C(x) = xR^*(x)\left[\frac{(1 - x)^{-1}}{1 - T_n + fpT^*(1)} + \sum_{r=0}^{\infty} e_r' x^r\right], \tag{D21}$$

where it can be shown as before the e_r' approaches zero as r approaches infinity. Hence,

$$Q^C = \lim_{r\to\infty} Q_r^{\ C} = \frac{1 - T_n}{1 - T_n + fpT^*(1)}\,, \tag{D22}$$

and the limit of the probability of inspecting the rth unit produced,

$$F_r^C = 1 - (1 - f)Q_r^C, \tag{D23}$$

is

$$F^C = \lim_{r\to\infty} F_r^C = \frac{f(1 - T_n) + fpT^*(1)}{1 - T_n + fpT^*(1)}. \tag{D24}$$

We may likewise compute the probability C_r that the criterion of critical length will be applied on the rth unit produced:

$$C_r = D_r + fp(Q_{r-n[r/n]}{}^C T_n{}^{[r/n]} + \cdots + Q_{r-2n}{}^C T_n{}^2 + Q_{r-n}{}^C T_n), \tag{D25}$$

where $D_r = T_n^k$ if $r = nk$ and is zero otherwise. The generating function is

$$C(x) = \frac{T_n x^n}{1 - T_n x^n}(1 + fpQ^C(x)), \tag{D26}$$

so that

$$C = \lim_{r\to\infty} C_r = \frac{fpT_n}{1 - T_n} Q^C = \frac{fpT_n}{1 - T_n + fpT^*(1)}. \tag{D27}$$

As with the original CSP-1 plan we may find the probabilities P_r^{IC} and Q_r^{IC} corresponding to P_r^C and Q_r^C in terms of inspected rather than produced units. We may also obtain C_r^I, the probability of applying the criterion on the rth unit inspected. The generating functions are easily seen to satisfy

$$P^{IC}(x) = xR^C(x)[1 + pQ^{IC}(x)], \tag{D28}$$

$$Q^{IC}(x) = P^{IC}(x)/(1 - qx), \tag{D29}$$

and

$$C^I(x) = \frac{T_n x^n}{1 - T_n x^n}(1 + pQ^{IC}(x)). \tag{D30}$$

We find, using the previous methods,

$$Q^{IC} = \lim_{r\to\infty} Q_r^{IC} = \frac{1 - T_n}{1 - T_n + pT^*(1)}, \tag{D31}$$

$$C^I = \lim_{r\to\infty} C_r^I = \frac{pT_n}{1 - T_n + pT^*(1)}. \tag{D32}$$

It is possible to obtain the limiting probabilities (D27) and (D32) indirectly as the reciprocals of expectations of recurrence times, although the method of generating functions allows a more complete characterization. If m is the number of units produced until the criterion of critical length is applied and m_I is the number of units inspected until the same event, it is interesting to note from (D24), (D27), and (D32) that

$$F^C = C/C^I = \mathrm{E}(m_I)/\mathrm{E}(m), \tag{D33}$$

where E is the expectation operator.

The values of these same limiting probabilities are of particular interest when $p = p^*$. Then $T_n = \alpha^*$ and

$$p^*KT^*(1) = 1 - K - T_{n^*+i},$$

assuming an exact solution to (2). From (D24), for instance, we have for $p = p^*$

$$F^C = F^{C*} = \frac{f}{f + (1 - f)K\beta^*}, \tag{D34}$$

and

$$C^I = C^{I*} = \alpha^*\beta^*p^*K/(1 - \alpha^*), \tag{D35}$$

where β^* is given by (11). The denominator of β^* can be fairly well approximated by

$$1 - \alpha^*(K + e^{-v}),$$

where v is defined by (6). Finally (D33) may be used to find C^*.

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