A Network Containing a Periodically Operated Switch Solved by Successive Approximations

By C. A. DESOER

(Manuscript received June 15, 1956)

This paper concerns itself with the analysis of a type of periodically switched network that might be used in time multiplex systems. The economics of the situation require that the ratio of the switch closure time τ to the switching period T be small. Using this assumption, the analysis is performed by successive approximations. More precisely the zeroth approximation to the transmission is obtained from a block diagram analogous to those used in sampled servomechanisms. From the convergence proof of the successive approximation scheme, it follows that when τ/T is small, the zeroth approximation is very close to the exact transmission. A discussion of some examples is included.

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I. INTRODUCTION

One main contributor to the cost of transmission circuits is the transmission medium itself. Thus it is important to share the transmission medium among as many messages as possible. One possible method is the frequency multiplex where each message utilizes a different frequency band of the whole band available in the medium. An alternate method is the time multiplex where each message is assigned a time slot of duration τ and has access to that time slot once every T seconds. It is obvious that the economics of the situation requires that τ be as small as possible and T as large as possible so that the largest possible number of messages are transmitted over the medium. For this very reason the analysis of periodically switched networks is of special interest in the case where τ/T is small.

W. R. Bennett[4] has published an exact analysis of this problem without any restrictions either on the network or on the ratio τ/T. It is believed, however, that the analysis presented in this paper will, in most practical cases, give the desired answer with a considerable reduction in the amount of calculations. The simplification of the analysis is mainly a result of the assumption that τ/T is small.

First the successive approximation method of solution will be discussed in general terms. Next it will be shown that the zeroth approximation to the transmission through the network can be obtained from the gain of a block diagram analogous to those used in the analysis of sampled servomechanisms. The nature of the zeroth approximation is further clarified by some general discussion and some examples. Next it is shown that the successive approximations converge. The convergence proof then suggests some slight modifications of the block diagram to obtain a more accurate solution.

II. DESCRIPTION OF THE SYSTEM

The system under consideration is shown on Fig. 1. It consists of two reactive networks N_1 and N_2 connected through a switch S which is itself in series with an inductance ℓ. N_1 is driven at its terminal pair (1)

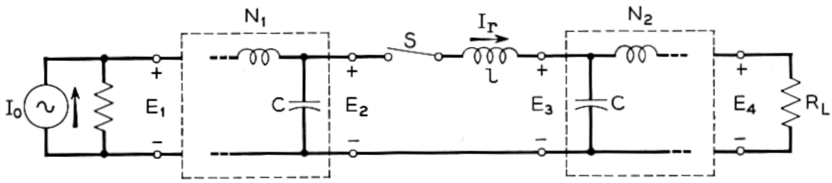
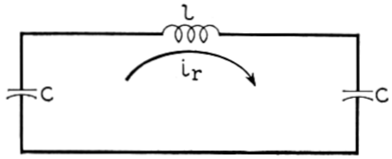


Fig. 1 — System under consideration.



$$f_r = \frac{1}{2\pi\sqrt{l\frac{C}{2}}} = \frac{\omega_0}{2\pi} \qquad \tau = \frac{1}{2f_r} = \frac{\pi}{\omega_0} = \pi\sqrt{l\frac{C}{2}}$$

Fig. 2 — Resonant circuit.

by a current source I_0 which is shunted by a one ohm resistor. N_2 is also terminated at its terminal pair (1) by a one ohm resistor R_L which is the load resistor of the system. The switch S is periodically closed for a duration τ. The switching period is T. Thus if the switch is closed during the interval $(0, \tau)$ it will be closed during the intervals $(nT, nT + \tau)$ for $n = 1, 2, 3, \cdots$. The inductance ℓ is selected so that the series circuit shown on Fig. 2 has a resonant frequency $f_r = 1/2\tau$; i.e., the time τ during which the switch is closed is exactly one-half period of the circuit of Fig. 2.

The switch S acts as a sampler and, as a result of the well-known modulating properties of sampled systems, the sampling period T must be chosen such that the frequency $1/2T$ is larger than any of the frequencies present in the signals generated by I_0. Furthermore, in order to eliminate all the sidebands generated by the switching, N_2 must have a high insertion loss for all frequencies above $1/2T$ cps.

In the analysis that follows networks N_1 and N_2 will be assumed to be identical: it should, however, be stressed that this assumption is not necessary for the proposed method of analysis.* This assumption is made because in the practical situation which motivated this analysis N_1 and N_2 were identical since transmission in both directions was required.

In order for the system under consideration to achieve the maximum degree of multiplexing, the closure time τ of the switch will be taken as small as practically possible and the switching period T as large as possible (consistent with the bandwidth of the signals to be transmitted). As a result the ratio τ/T is very small, of the order of 10^{-2} or less in practical cases. Consequently the resonant frequency f_r of the series resonant circuit shown on Fig. 2, is many times larger than any of the natural frequencies of N_1 and N_2.

* The modifications required for the case where N_1 is not identical to N_2 are given in Appendix IV.

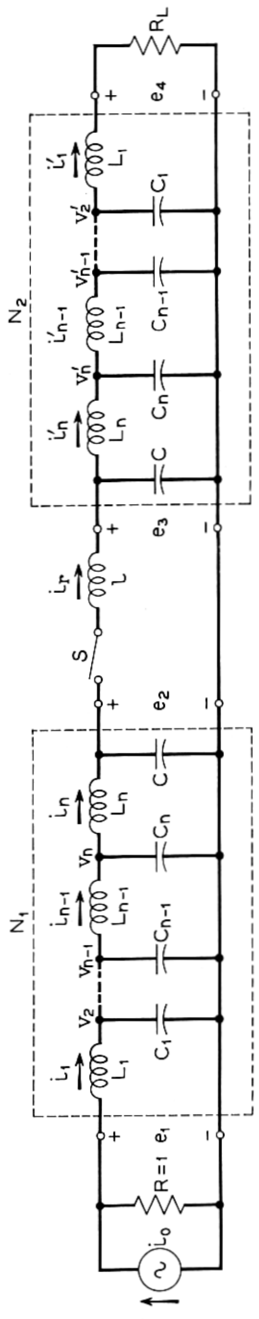


Fig. 3 — System under consideration when N_1 and N_2 are lossless ladders.

The problem is to determine the relation between E_4, the voltage across R_L, and I_0.

III. METHOD OF SOLUTION

Let us first write the equations of the system. Obviously the equations will depend on the exact configuration of the networks N_1 and N_2. For simplicity we shall write them for the case where N_1 and N_2 are dissipationless low-pass ladder networks. As will become apparent later this assumption is not essential to the argument. What is essential, however, is the fact that both N_1 and N_2 should start (looking in from the switch) with a shunt capacitor C and a series inductance L_n, the element value of L_n being much larger than ℓ. Using a method of analysis advocated by T. R. Bashkow,[5] we obtain, for the network of Fig. 3, the equations:

$$\left.\begin{aligned}
L_1 \frac{di_1}{dt} &= -Ri_1 - v_2 + Ri_0 \\
C_1 \frac{dv_2}{dt} &= i_1 - i_2 \\
&\vdots \\
C_n \frac{dv_n}{dt} &= i_{n-1} - i_n \\
L_n \frac{di_n}{dt} &= v_n - e_2 \qquad (1.\mathrm{a})
\end{aligned}\right\} I_a$$

$$\left.\begin{aligned}
C \frac{de_2}{dt} &= i_n - i_r\Delta(t) \qquad (1.\mathrm{b}) \\
\ell \frac{di_r}{dt} &= [e_2 - e_3]\Delta(t) \qquad (1.\mathrm{c}) \\
C \frac{de_3}{dt} &= i_r\Delta(t) - i_n' \qquad (1.\mathrm{d})
\end{aligned}\right\} R \qquad (1)$$

$$\left.\begin{aligned}
L_n \frac{di_n'}{dt} &= e_3 - v_n' \\
C_n \frac{dv_n'}{dt} &= i_n' - i_n{'}_{-1} \\
&\vdots \\
C_1 \frac{dv_2'}{dt} &= i_2' - i_1' \\
L_1 \frac{di_1'}{dt} &= v_2' - R_L i_1'
\end{aligned}\right\} I_b$$

where

$$\Delta(t) = \sum_{k=-\infty}^{+\infty} [u(t - kT) - u(t - kT - \tau)], \tag{2}$$

with $u(t) = 1$ for $t > 0$, and $u(t) = 0$ for $t < 0$.

This system of linear time varying equations may be broken up into three sub-systems I_a, R and I_b. It is this subdivision that suggests a successive approximation scheme that will be shown to converge to the exact solution.

The zeroth approximation is obtained as follows: when the switch is closed, i.e., $\Delta(t) = 1$, the resonant current i_r is much larger than the currents i_n and i_n'. Thus, during the switch closure time, i_n and i_n' are neglected with respect to i_r in (1.b) and (1.d). Hence when $\Delta(t) = 1$ the system R may be solved for $i_r(t)$, $e_2(t)$ and $e_3(t)$ in terms of the initial conditions. The resulting function $e_2(t)$ and given function $i_0(t)$ are then the forcing functions of the system I_a. The other function $e_3(t)$ is the forcing function of the system I_b. Under these assumptions, the periodic steady-state solution corresponding to an applied current $i_0(t) = I_0 e^{i\omega t}$ is easily obtained.

The zeroth approximation will be distinguished by a subscript "0". Thus $i_{r0}(t)$ is the (steady state) zeroth approximation to the exact solution $i_r(t)$.

The first approximation will be the solution of the system (1), provided that during the switch closure time the functions $i_n(t)$ and $i_n'(t)$ in (1.b) and (1.d) are respectively replaced by the known functions $i_{n0}(t)$ and $i_{n0}'(t)$. And, more generally, the $(k + 1)$th approximation will be the solution of (1) provided that during the switch closure time, the functions $i_n(t)$ and $i_n'(t)$, in (1.b) and (1.d), are respectively replaced by the known solutions for $i_n(t)$, and $i_n'(t)$ given by the kth approximation It will be shown later that this successive approximation scheme converges. Let us first describe a simple method for obtaining the zeroth approximation.

IV. THE ZEROTH APPROXIMATION

4.1 *Introduction*

The problem is to obtain the steady-state solution of (1) under the excitation $i_0(t) = I_0 e^{i\omega t}$. Using the approximations indicated above, during the switch closure time (that is when $\Delta(t) = 1$) the system R becomes

$$C \frac{de_2}{dt} = -i_r(t)\Delta(t), \tag{3}$$

$$\ell \frac{di_r}{dt} = [e_2 - e_3]\Delta(t), \tag{4}$$

$$C \frac{de_3}{dt} = i_r(t)\Delta(t). \tag{5}$$

Differentiating the middle equation and eliminating de_2/dt and de_3/dt we get for $0 \leqq t < \tau$:

$$\frac{d^2 i_r}{dt^2} = -\frac{2}{\ell C} i_r \Delta(t) + \frac{1}{\ell} [e_2(t) - e_3(t)]\delta(t) \tag{6}$$

in which we used the notation $\delta(t)$ for the Dirac function and the knowledge that

$$\frac{d\Delta(t)}{dt} = \delta(t) - \delta(t - \tau). \tag{7}$$

Equation (6) represents the behavior of the resonant circuit of Fig. 2 for the following initial conditions:

$$i_r(0+) = 0, \tag{8}$$

$$\frac{di_r(0+)}{dt} = \frac{e_2(0) - e_3(0)}{\ell}. \tag{9}$$

In Appendix I it is shown that the resulting current $i_r(t)$ is, for the interval $0 \leqq t < \tau$,

$$i_r(t) = C[e_2(0) - e_3(0)]s_1(t), \tag{10}$$

where

$$s_1(\,) = \begin{cases} \dfrac{\pi}{2\tau} \sin \dfrac{\pi t}{\tau} = \dfrac{1}{2} \omega_0 \sin \omega_0 t & \text{for} \quad 0 \leqq t < \tau \\ 0 \text{ elsewhere} \end{cases} \tag{11}$$

with

$$\omega_0 = \frac{\pi}{\tau} = \sqrt{\frac{2}{\ell C}}. \tag{12}$$

Thus the zeroth approximation to the exact $i_r(t)$ is given for the interval $0 \leqq t \leqq T$ by

$$i_{r0}(t) = C[e_2(0) - e_3(0)]s_1(t). \tag{13}$$

We shall now show that the zeroth approximation may be conveniently obtained from the block diagram of Fig. 4.

4.2 *Description of the Block Diagram*

All the blocks of the block diagram are unilateral and their corresponding transfer functions are defined in the following. Capital symbols represent $\mathfrak{L}$-transform of the corresponding time functions, thus $I_0(p)$ is the $\mathfrak{L}$-transform of $i_0(t)$.

Referring to Fig. 1,

$$Z_{12}(p) = \left.\frac{E_2(p)}{I_0(p)}\right|_{I_r=0}.$$

Thus $Z_{12}(p)$ represents the transfer impedance of N_1 when its output is open-circuited (i.e., $I_r = 0$). Since N_1 and N_2 are identical we also have, from $R_L = 1$ and reciprocity, $Z_{12}(p) = E_4/I_r$, where I_r is the current entering N_2.

The impulse modulator is periodically operated every T seconds, and has the property that if its input is a continuous function $f(t)$ its output is a sequence of impulses:

$$\sum_{-\infty}^{+\infty} f(t)\,\delta(t - kT).$$

The transfer function $S_1(p)$ is defined by

$$S_1(p) = \mathfrak{L}[s_1(t)] = \frac{\omega_0^2}{p^2 + \omega_0^2} \cosh \frac{p\tau}{2}\, e^{-p\tau/2}. \tag{14}$$

Let $Z(p)$ be the driving point impedance at the terminal pair (2) of N_1. It is also that of N_2 since N_1 and N_2 are assumed to be identical.

Let $V(p)$ be the output of the first block, then, by definition, $V(p) = Z_{12}(p)I_0$. Let $v(t)$ be the corresponding time function. The voltage $v(t)$

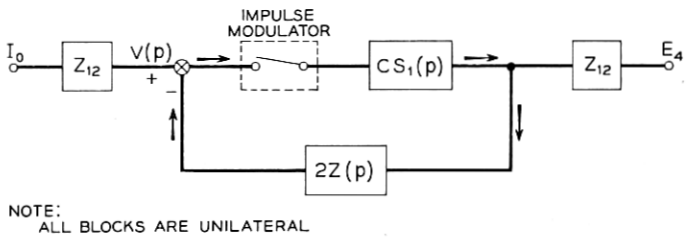


Fig. 4 — Zeroth-approximation block diagram.

is the output voltage of N_1, when N_1 is excited by the current source I_0 and the switch S remains open at all times.

4.3 *Analysis of the Block Diagram*

For simplicity, suppose that the system starts from a relaxed condition (i.e., no energy stored) at $t = 0$. Let $z(t) = \mathfrak{L}^{-1}[Z(p)]$. Considering the network N_1 as driven by i_0 and i_{r0}, it follows that the voltage $e_2(t)$ shown on Fig. 3 is given by

$$e_{20}(t) = v(t) - \int_0^t i_{r0}(t')z(t - t')\,dt'. \tag{15}$$

Similarly

$$e_{30}(t) = \int_0^t i_{r0}(t')z(t - t')\,dt'. \tag{16}$$

Thus

$$e_{20}(t) - e_{30}(t) = v(t) - 2\int_0^t i_{r0}(t')z(t - t')\,dt'. \tag{17}$$

These equations have been derived by considering Fig. 1. They could have been also derived from the block diagram of Fig. 4 as follows: let $I_{r0}(p)$ be the output of $CS_1(p)$. As a result, the output of the block $2Z(p)$ is $2Z(p)I_{r0}(p)$. When this latter quantity is subtracted from $V(p)$ one gets $V(p) - 2Z(p)I_{r0}(p)$, which is the $\mathfrak{L}$-transform of the right-hand side of (17). Referring to the block diagram it is also seen that this quantity is the input to the impulse modulator.

Thus we see that if $I_{r0}(p)$ is the output of $CS_1(p)$, then the input of the impulse modulator is $e_{20}(t) - e_{30}(t)$ by virtue of (17). If this is the case the output of $CS_1(p)$ is given by $C[e_{20}(0) - e_{30}(0)]s_1(t)$, for $0 \leqq t < T$, which, according to (9), is $i_{r0}(t)$.

Thus the block diagram of Fig. 4 is a convenient way of obtaining the zeroth approximation to the periodic steady-state solution.

In order to use the techniques developed for sampled data systems,[1, 2] we introduce the following notation.[2] If $f(t) = \mathfrak{L}^{-1}[F(p)]$, then we define $F^*(p)$ by the relation

$$F^*(p) = \frac{1}{T}\sum_{n=-\infty}^{+\infty} F(p + jn\omega_s), \tag{18}$$

where

$$\omega_s = \frac{2\pi}{T}. \tag{19}$$

If $f(0+)$ is defined by $\lim_{\epsilon \to 0} f(\epsilon^2)$, then, provided $f(0+) = 0$,*

$$F^*(p) = \mathfrak{L}\left[\sum_{n=0}^{\infty} f(t)\,\delta(t - nT)\right]. \tag{20}$$

Going back to the system of Fig. 4 we get[2]

$$E_{40}(p) = \frac{[Z_{12}(p)I_0(p)]^* C S_1(p) Z_{12}(p)}{1 + 2C[S_1(p)Z(p)]^*}, \tag{21}$$

and

$$I_{ro}(p) = \frac{[Z_{12}(p)I_0(p)]^* C S_1(p)}{1 + 2C[S_1(p)Z(p)]^*}, \tag{22}$$

where according to the notation defined by (18)

$$[Z_{12}(p)I_0(p)]^* = \frac{1}{T}\sum_{n=-\infty}^{\infty} Z_{12}(p + jn\omega_s)I_0(p + jn\omega_s),$$

$$[S_1(p)Z(p)]^* = \frac{1}{T}\sum_{n=-\infty}^{+\infty} S_1(p + jn\omega_s)Z(p + jn\omega_s).$$

It should be stressed that (21) and (22) are not valid when τ is made identical to zero. When $\tau = 0$, $S_1(p) = 1$ for all p's and since $Z(p) \sim 1/Cp$ as $p \to \infty$ the time function whose transform is $Z(p)S_1(p)$ is different from zero at $t = 0$. In such a case (20) does not hold. From a physical point of view, the feedback loop of Fig. 4 is unstable when τ is identically zero since an impulse generated by the impulse modulator produces instantaneously a step at the input of the impulse modulator. This step causes an instantaneous jump in the measure of the impulse at the output of the impulse modulator and so on. In short the feedback loop is unstable.

It should be pointed out that if the power density spectrum of I_0 is zero for frequencies higher than $\omega_s/2$, (21) reduces to

$$\frac{E_{40}}{I_0} = \frac{1}{T}\frac{C S_1(p) Z_{12}^{\,2}(p)}{1 + 2C[S_1(p)Z(p)]^*} \quad \text{for} \quad |p| < \frac{\omega_s}{2}. \tag{23}$$

For certain applications it is convenient to rewrite (21) in a slightly

* When $f(0)$, as defined above, is different from zero, (20) should be replaced by

$$\mathfrak{L}\left[\sum_{n=0}^{\infty} f(t)\,\delta(t - nT)\right] = F^*(p) + \frac{1}{2} f(0+).$$

different form. Advancing the time function $s_1(t)$ by $\tau/2$ seconds, one gets the function $s_0(t)$ which is even in t. As a result its transform $S_0(p)$ is purely real, that is,

$$S_0(p) = \frac{\omega_0^2}{p^2 + \omega_0^2} \cosh \frac{p\tau}{2}.$$

From an analysis carried out in detail in Appendix IV we finally obtain

$$E_{40}(p) = \frac{[Z_{12}(p)I_0(p)]^* S_0(p) Z_{12}(p)}{2[Z(p)S_0(p)]^*}. \tag{24}$$

It should be pointed out that (23) is still valid when $\tau = 0$. Equations (20) and (23) give the zeroth approximation to the gain of the system for any driving current $i_0(t)$.

In many cases it is sufficient to know only the steady-state response $E_{40}(p)$ to an input $i_0(t) = I_0 e^{j\omega_0 t}$. The response $E_{40}(p)$, as given by (23) [or (20)] includes both transient and steady-state terms. Since $I_0(p) = \dfrac{I_0}{p - j\omega_0}$ equation (24) gives

$$E_{40}(p) = \frac{\left(\frac{1}{T} \sum_{n=-\infty}^{+\infty} Z_{12}(p + jn\omega_s) \frac{I_0}{p + jn\omega_s - j\omega_0}\right) S_0(p) Z_{12}(p)}{2[Z(p)S_0(p)]^*}. \tag{25}$$

Since neither $S_0(p)$ nor $Z_{12}(p)$ have poles on the imaginary axis, the steady state includes only the terms corresponding to the imaginary axis poles of the summation terms. Thus the steady-state response is of the form

$$\sum_{-\infty}^{+\infty} A_n \, e^{j(\omega_0 - n\omega_s)t},$$

where, from (25),

$$A_n = \frac{I_0 Z_{12}(j\omega_0) S_0(j\omega_0 - jn\omega_s) Z_{12}(j\omega_0 - jn\omega_s)}{2 \sum_{k=-\infty}^{+\infty} Z[j\omega_0 + j(k - n)\omega_s] S_0[j\omega_0 + j(k - n)\omega_s]}. \tag{26}$$

V. TRANSMISSION LOSS

A practically important question is to find out *a priori* whether a switched filter necessarily introduces some transmission loss.

The following considerations apply exclusively to the zeroth order approximation. It will be shown that assuming ideal elements, the transmission at dc may have as small a loss as desired.

By transmission at dc we mean the ratio of the dc component of the steady state output voltage to the intensity of the applied direct current. Thus we refer to (26) and set $\omega_0 = 0$ and $n = 0$. Suppose the lossless networks N_1 and N_2 are designed so that their transfer impedance Z_{12} is of the Butterworth type, that is

$$| Z_{12}(j\omega) |^2 = \frac{1}{1 + \omega^{2M}},$$

where for our purposes M is a large integer.

In the following sum, which is the denominator of (26) when $\omega_0 = n = 0$

$$2 \sum_{k=-\infty}^{+\infty} Z(jk\omega_s) S_0(jk\omega_s),$$

(where $\omega_s > 2$ since the cutoff of the networks N occurs at $\omega = 1$), the terms corresponding to values of $k \neq 0$ will make a contribution that vanishes as $M \to \infty$. This is a consequence of the following facts:

(a) $\mathrm{Re}[Z(jk\omega_s)] = | Z_{12}(jk\omega_s) |^2$, since the networks N_1 and N_2 are dissipationless. Hence for $k \neq 0$ and as $M \to \infty$ $\mathrm{Re}[Z(jk\omega_s)] \to 0$,

(b) $\mathrm{Im}[Z(jk\omega_s)] = -\mathrm{Im}[Z(-jk\omega_s)]$,

(c) $S_0(j\omega)$ is real.

Thus the imaginary part of the products $Z(jk\omega_s)S_0(jk\omega_s)$ cancel out and the real part (for $k \neq 0$) decreases exponentially to zero as $M \to \infty$. Hence for sufficiently large M the denominator of (26) may be made as close to two as desired.

It is easy to check that the numerator of (26) reduces to I_0, the intensity of the applied direct current. Therefore the ratio of A_0, the dc component of the output voltage to I_0 may be made as close to one-half as desired.

VI. A SIMPLE EXAMPLE

Since the approximate formulae derived in Section IV are somewhat unfamiliar it seems proper to consider in a rather detailed manner a simple example.*

Consider the system of Fig. 5. Assume that the current source applies a constant current to the system and assume that the steady state is reached. For simplicity let $I_0R = E$.

The steady-state behavior of the voltages $e_2(t)$ and $e_3(t) = e_4(t)$ is

* In addition, the limiting case of the sampling rate $\to \infty$, i.e., $T \to 0$, is treated in Appendix II.

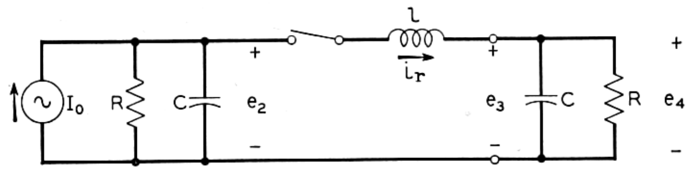


Fig. 5 — A simple example.

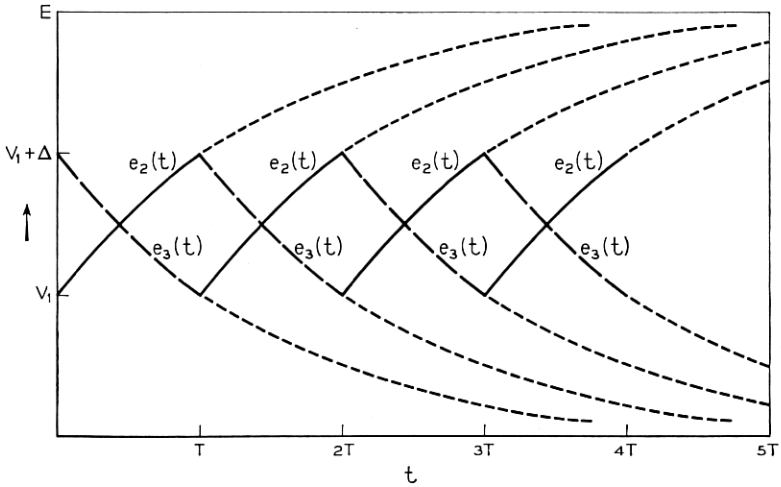


Fig. 6 — Waveforms of the network of Fig. 5.

shown on Fig. 6. It is further assumed that the duration τ during which the switch is closed is negligible compared to T, the interval between two successive closures.

Let $\bar{e}_4$ be the average value of the steady-state voltage $e_4(t)$. Thus $\bar{e}_4$ is equal to A_0 as given by (26) with $\omega_0 = n = 0$, namely,

$$e_4 = I_0 \frac{Z_{12}^2(0)\ S_0(0)}{2 \sum_{-\infty}^{+\infty} Z(jk\omega_s)\ S_0(jk\omega_s)} .$$

In this particular case

$$Z(p) = Z_{12}(p) = \frac{R}{1 + pRC} = \frac{C^{-1}}{p + \dfrac{1}{RC}} .$$

Since we assume τ to be infinitesimal $S_0(p)$ and $S_1(p)$ may be considered equal to unity over the band of interest. Using the expansion

$$\coth z = \frac{1}{z} + \sum_{1}^{\infty} \frac{2z}{z^2 + n^2\pi^2},$$

we obtain

$$Z^*(p) = \frac{1}{T} \sum_{-\infty}^{+\infty} \frac{C^{-1}}{(p + jn\omega_S) + \dfrac{1}{RC}} = \frac{1}{2C} \coth \left[\left(p + \frac{1}{RC} \right) \frac{T}{2} \right].$$

Hence

$$\frac{2}{T} \sum_{-\infty}^{+\infty} Z(jk\omega_s) = 2Z^*(p) \bigg|_{p=0} = \frac{1}{C} \coth \left(\frac{T}{2RC} \right).$$

Thus finally

$$\bar{e}_4 = \frac{I_0 R^2}{T \dfrac{1}{C} \coth \left(\dfrac{T}{2RC} \right)} = E \frac{RC}{T} \frac{1}{\coth \left(\dfrac{T}{2RC} \right)}. \tag{27}$$

This last result obtained from the theory developed above is now going to be checked directly. Referring to Fig. 6, where the notation is defined, and noting the periodicity of the boundary conditions, we get

$$(V_1 + \Delta)e^{-(T/RC)} = V_1,$$
$$(E - V_1)e^{-(T/RC)} = E - (V_1 + \Delta).$$

Noting that $e_4(t) = (V_1 + \Delta)e^{-t/RC}$, and solving for V_1 and Δ we finally get

$$e_4(t) = \frac{e^{-t/RC}}{1 + e^{-T/RC}}.$$

By definition

$$\overline{e_4} = \frac{1}{T} \int_0^T e_4(t)\, dt = \frac{E\, RC}{T} \frac{1 - \bar{e}^{-T/RC}}{1 + \bar{e}^{-T/RC}},$$

or

$$\overline{e_4} = E \frac{RC}{T} \frac{1}{\coth \left(\dfrac{T}{2RC} \right)}.$$

This last equation checks with (27).

VII. NUMERICAL EXAMPLES

Consider the network of Fig. 7. The cutoff of both N_1 and N_2 occurs at $\omega = 1$. In view of the sampling theorem good transmission requires

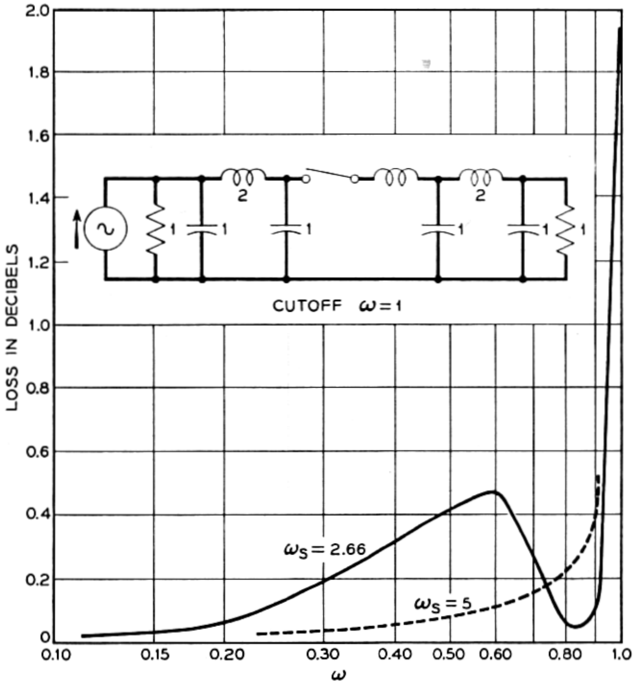


Fig. 7 — Computed transmission loss.

that the signal be sampled at a rate at least twice as large as its highest frequency component. Since the cutoff occurs at $\omega = 1$, the sampling angular frequency should at least be equal to 2. For illustration purposes we have taken $\omega_s = 2.67$ and $\omega_s = 5$ for the angular sampling frequency The value $\omega_s = 2.67$ corresponds to a cutoff at 3 kc and a sampling rate. of 8 kc. The ratio τ/T was taken to be 1/125. The transmission through the switched network as given by the zeroth approximation is shown for both cases on Fig. 7.

As expected the transmittance of the switched filter gets closer to that of an ordinary filter as the switching frequency increases.

VIII. THE SUCCESSIVE APPROXIMATION SCHEME

The ideas involved in the successive approximation scheme are simple and straightforward. One point remains to be settled, namely the convergence of the procedure.

We shall assign a subscript 1 to the correction to be applied to the zeroth approximation in order to obtain the first approximation. Thus adding $i_{r1}(t)$ to $i_{r0}(t)$ we get the first approximation $i_{r0}(t) + i_{r1}(t)$. More generally the kth approximation is $\sum_{n=0}^{k} i_{rn}(t)$. The procedure will converge if, in particular, the infinite series $\sum_{n=0}^{\infty} i_{rn}(t)$ converges.

8.1 *Preliminary Steps*

(a) Let us normalize the frequency (and consequently the time) so that the switching period T is unity. Since the networks N_1 and N_2 must have high insertion loss for $\omega > \frac{1}{2}(2\pi/T) = \pi$, the pass band of N_1 and N_2 must be the order of 1 radian/sec. As a result the element values of the capacitor C and the inductance L_n (see Fig. 3) are also 0(1).

(b) For the excitation $i_0 = e^{i\omega t}$, the zeroth approximation derived above may be written in terms of Fourier components:

$$i_{r0}(t) = e^{i\omega t} \sum_{k=-\infty}^{+\infty} I_{r0,k} e^{i2\pi kt},$$

$$i_{n0}(t) = e^{i\omega t} \sum_{k=-\infty}^{+\infty} I_{n0,k} e^{i2\pi kt}.$$

Let $\bar{i}_{r0}$ denote the complex conjugate of i_{r0}, then

$$i_{r0}(t)\bar{i}_{r0}(t) = \sum_{k,\ell=-\infty}^{+\infty} I_{r0,k}\bar{I}_{r0,\ell} e^{2\pi i(k-\ell)t}.$$

Since the functions $e^{2\pi ikt}[k = \cdots -1, 0, 1, \cdots]$ are orthonormal over the interval (0, 1) and form a complete set,[3] we have from Bessel's equality:[3]

$$\int_0^1 | i_{r0}(t) |^2 \, dt = \sum_{k=-\infty}^{+\infty} | I_{r0,k} |^2 = N(I_{r0}),$$

where $N(I_{r0})$ denotes the norm of the vector I_{r0} which is defined by its components $I_{r0,k}(k = \cdots -1, 0, +1 \cdots)$. Similarly,

$$\int_0^1 | i_{n0}(t) |^2 \, dt = \sum_{k=-\infty}^{+\infty} | I_{n0,k} |^2 = N(I_{n0}).$$

(c) Since the switch is periodically closed we shall be interested in the Fourier series expansion of $\Delta(t)$:

$$\Delta(t) = u(t) - u(t - \tau) = \sum_{-\infty}^{+\infty} \Delta_k e^{ik2\pi t},$$

where $\Delta_0 = \tau$ and

$$\sum_{-\infty}^{+\infty} | \Delta_k |^2 = \tau.$$

Since $\tau/T \ll 1$, and since the frequency has been normalized so that $T = 1$, we have $\tau \ll 1$.

Using the convolution in the frequency domain, we have

$$i_{n0}(t)\Delta(t) = \sum_{k=-\infty}^{+\infty} \left(\sum_{\alpha=-\infty}^{+\infty} \Delta_{k-\alpha} I_{n0,\alpha} \right) e^{i(\omega+2\pi k)t}.$$

If we introduce the infinite matrix G defined by

$$G_{ik} = \Delta_{i-k} \ (i, k = -\infty, \cdots, -1, 0, +1, \cdots \infty),$$

the convolution may be represented by the product, GI_{n0}, where I_{n0} is the vector whose components are $I_{n0,k}(k = \cdots -1, 0, 1, \cdots)$.

(d) Considering the network shown on Fig. 3, let $E(p)$ be the ratio of $I_n'(p)$ to $I_r(p)$. Taking into account the assumed identity between N_1 and N_2 it follows that

$$\left.\frac{+I_n(p)}{I_r(p)}\right|_{I_0=0} = \frac{I_n'(p)}{I_r(p)} = E(p).$$

Using the system of (1) and, for example, by Neumann series expansion of the inverse matrix, we get

$$E(p) = \frac{1}{L_n C p^2} - \frac{2}{L_n^2 C^2 p^4} + \cdots$$

(e) Considering now the effect of $i_n(t)$ and $i_n'(t)$ on $i_r(t)$, (42) of Appendix I gives $I_r(p)$ as a function of $I_n(p)$ and $I_n'(p)$. In the present discussion where we are interested in the steady state of $i_r(t)$ it is essential to keep in mind that since the switch opens at $t = \tau$, the memory of the resonant circuit extends only over an interval $0 < t \leqq \tau$. To take this into account we must modify the factor $(\omega_0^2/2)/(p^2 + \omega_0^2)$ of (40), because the impulse response (which represents this memory) must be identically zero for $t > \tau$. The resulting new expression is

$$F(p) = \frac{\frac{\omega_0^2}{2}}{p^2 + \omega_0^2} e^{-p\tau/2} [e^{p\tau/2} + e^{-p\tau/2}],$$

or

$$F(p) = \frac{\omega_0^2}{p^2 + \omega_0^2} e^{-p\tau/2} \cosh \frac{p\tau}{2}.$$

Since the time function whose transform is $F(p)$ is non-negative for all t's and since $F(0) = 1$, it follows that

$$|F(j\omega)| \leqq 1. \tag{28}$$

8.2 *Matrix description of the successive approximations*

From the developments of Section IV, we know $i_{r0}(t)$, $i_{n0}(t)$ and $i_{n0}'(t)$ or what is equivalent, the vectors I_{r0}, I_{n0} and I_{n0}'. The first approximation takes into account the effect of $i_{n0}(t)$ and $i_{n0}'(t)$ on $i_r(t)$. [See equation (1.c) and (1.d)]. The time functions $i_{n0}(t)$ and $i_{n0}'(t)$ affect the system R only during the interval $(0, \tau)$. Therefore we must consider the vector $G(I_{n0} + I_{n0}')$ which corresponds to the excitation of the resonant circuit. Since the opening of the switch after a closure time τ forcibly brings $i_r(t)$ to zero we have

$$I_{r1} = G\,F\,G(I_{n0} + I_{n0}'), \tag{29}$$

where the matrix G has been defined above and the matrix F is a diagonal matrix whose diagonal elements F_k $(k = \cdots -1, 0, +1 \cdots)$ are defined by $F_k = F(j\omega_s + j2\pi k)$. Note that (28) implies that $|F_k| \leqq 1$ for all k's. It should be kept in mind that $I_{r0} + I_{r1}$ is the first approximation to the exact $I_r(p)$.

The next iteration is obtained by first taking into account the effect of I_{r1} on the rest of the network:

$$\begin{aligned} I_{n1} &= E\,I_{r1}, \\ I_{n1}' &= E\,I_{r1}, \end{aligned} \tag{30}$$

where E is a diagonal matrix whose elements E_k $(k = \cdots, -1, 0, +1, \cdots)$ are defined by $E_k = E(j\omega_s + 2\pi kj)$, and then the effects of I_{n2} and I_{n2}' on I_r, that is,

$$I_{r2} = G\,F\,G\,(I_{n1} + I_{n1}'), \tag{31}$$

combining (30) and (31), $I_{r2} = 2\,G\,F\,G\,E\,I_{r1}$. A repetition of the same procedure would lead to $I_{r3} = 2\,G\,F\,G\,E\,I_{r2}$, and in general $I_{rn+1} = 2\,G\,F\,G\,E\,I_{rn}$.

Since the nth approximation to $I_r(p)$ is given by the sum $\sum_{k=0}^{n} I_{rk}$, the successive approximation scheme will be convergent only if the series

$$\sum_{k=0}^{\infty} I_{rk}$$

converges. This will be the case if and only if the series

$$[1 + 2\,G\,F\,G\,E + \cdots + (2\,G\,F\,G\,E)^n + \cdots]I_{r1} \tag{32}$$

converges.

8.3 *Convergence Proof*

Consider a vector X of bounded norm corresponding to a time function $x(t)$ having the property that $x(t) = 0$ for $\tau \leqq t \leqq T$ and $x(t) \neq 0$ for $0 < t < \tau$. In the above scheme, the vector X would be I_{rn}. Let us define the vectors Y, Z, U and V by the relations

$$Y = EX, \tag{33}$$

$$Z = GY, \tag{34}$$

$$U = FZ, \tag{35}$$

$$V = 2GU, \tag{36}$$

hence

$$V = 2\,G\,F\,G\,E\,X. \tag{37}$$

We wish to show that $N(V) \leqq \alpha N(X)$ with $\alpha < 1$, since these inequalities imply that the infinite series (32) converges.

Since (a) N_1 and N_2 are low-pass filters with cutoff $\leqq \pi$ radians/sec, (b) $E(p) = 1$ for $p = 0$, (c) $E(p) \propto 1/L_nCp^2$ for $p \gg 1$, only a few of the E_k's will be of the order of unity In most cases E_{-1}, E_0, E_1 will be smaller than unity, thus,

$$N(Y) \leqq N(X). \tag{38}$$

In view of the pulsating character of $x(t)$ the power spectrum of $x(t)$ is almost constant up to frequencies of the order of π/τ radians/sec. Because of the low-pass characteristic of $E(p)$, the function $y(t)$ associated with the vector Y is smooth in comparison to $x(t)$, thus from (34),

$$N(Z) = \int_0^1 |\, z(t) \,|^2 \, dt = \int_0^\tau |\, y(t) \,|^2 \, dt = a\tau N(Y),$$

where $a = 0(1)$.

Since $|\, F_k \,| \leqq 1$ for all k's, from (33), $N(U) \leqq N(Z)$, hence $N(U) = b\tau N(Y)$ with $b = 0(1)$.

$$N(U) = b\tau N(Y) \quad \text{with} \quad b = 0(1).$$

From (36) we have

$$N(V) = 2 \int_0^\tau |\, u(t) \,|^2 \, dt \leqq 2 \int_0^1 |\, u(t) \,|^2 \, dt = 2N(U).$$

Thus we finally get

$$N(V) = 2b\tau N(Y) \quad \text{where} \quad b = 0(1), \tag{39}$$

and since $\tau \ll 1$ we get from (38) and (39) $N(V) = \alpha N(X)$ with $\alpha < 1$. Hence the convergence is established.

IX. A MODIFICATION OF THE BLOCK DIAGRAM TO IMPROVE THE ZEROTH APPROXIMATION

In principle it is possible to obtain a block diagram whose transmission characteristic is equal to the first approximation. In many cases it is not necessary to go that far. The first approximation takes into account the effect of the currents $i_{n0}(t)$ and $i_{n0}'(t)$ on the resonant circuit of Fig. 2. Since during the switch closure time the currents i_{n0} and i_{n0}' cannot vary much, let us assume that they remain constant for the duration of the switch closure.

Referring to the analysis of Appendix I and to (42) in particular, we see that the current i_r is increased by

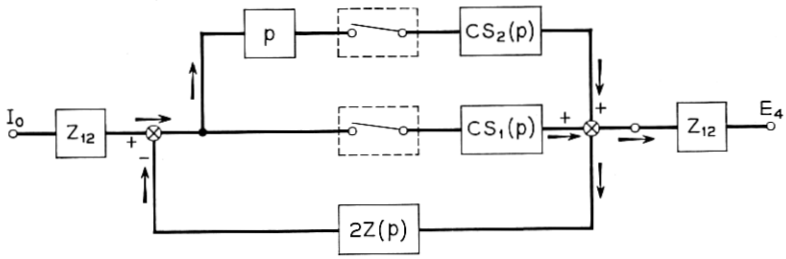
$$\delta i_n(p) = \frac{\omega_0^2}{p^2 + \omega_0^2} \frac{i_n(0-) + i_n'(0-)}{2p},$$

or

$$\delta i(t) = C \frac{\dot{e}_2(0-) - \dot{e}_3(0-)}{2} (1 - \cos \omega_0 t) \qquad 0 \leqq t < \tau.$$

Defining $S_2(p) = \mathfrak{L}^{-1}\{\tfrac{1}{2}(1 - \cos \omega_0 t)[u(t) - u(t - \tau)]\}$, or

$$S_2(p) = \frac{\omega_0^2}{2} \frac{1}{p(p^2 + \omega_0^2)} - \frac{(2p^2 + \omega_0^2)}{2p(p^2 + \omega_0^2)} e^{-p\tau},$$



$$E_4(p) = \frac{C[Z_{12}(p)I_0(p)]^* S_1(p) + C[pZ_{12}(p)I_0(p)]^* S_2(p)}{1 + 2C\{[Z(p)S_1(p)]^* + [pZ(p)S_2(p)]^*\}}$$

Fig. 8 — Modified block diagram.

and recalling that the input of the impulse modulator of Fig. 4 is $e_2(0) - e_3(0)$, it becomes obvious that the modified block diagram should be that given by Fig. 8. The output of the modified block diagram is given by[2]

$$E_4(p) = \frac{C[Z_{12}(p)I_0(p)]^*S_1(p) + C[pZ_{12}(p)I_0(p)]^*S_2(p)}{1 + 2C\{[Z(p)S_1(p)]^* + [pZ(p)S_2(p)]^*\}}.$$

X. CONCLUSION

Let us compare the method of solution presented above with the more formal approach proposed by Bennett. The latter method leads to the exact steady-state transmission through a network containing periodically operated switches. This method is perfectly general in that it does not require any assumption relative to the properties of the network nor to the ratio of τ/T. As expected this generality implies a lot of detailed computations. In particular it requires, for each reactance of the network, the computation of the voltage across it due to any initial condition. The method presented in this paper is not so general because it assumes first that the ratio τ/T is small; second the value of the inductance ℓ is very much smaller than that of L_n (see Fig. 3). The result of these assumptions is that the system of time varying equations may be solved by successive approximations with the further advantage that the convergence proof guarantees that, for very small τ/T, the zeroth approximation will be a close estimate of the exact solution.

The zeroth approximation may conveniently be obtained by considering a block-diagram analogous to those used in the analysis of sampled servomechanisms. Further the proposed method leads directly to some interesting results, for example, as far as the zeroth approximation is concerned, the dc transmission may be achieved with as small a loss as desired provided the lossless networks N_1 and N_2 are suitably designed. Another advantage of the proposed method is that the simplicity of the analysis permits the designer to investigate at a small cost a large number of possible designs.

Finally it should be pointed out that this approach to the solution of a system of time-varying linear differential equations may find applications in many other physical problems.

APPENDIX I

ANALYSIS OF THE RESONANT CIRCUIT

Consider the resonant circuit of Fig. 2. Suppose that at $t = 0$, the left-hand capacitor has a potential $e_2(0)$ and the right-hand capacitor has a potential $e_3(0)$ and that at $t = 0$ the current i_r through the inductance ℓ is zero.

The network equation is

$$\ell \frac{d}{dt} i_r + \frac{2}{C} \int i_r \, dt = 0.$$

Now $i_r(0) = 0$ and $di_r(0)/dt = [e_2(0) - e_3(0)]/\ell$. Let $2/\ell C = \omega_0^2$, then $di_r(0)/dt = \omega_0^2 C[e_2(0) - e_3(0)]/2$.
Using Laplace transforms,

$$\begin{aligned} (p^2 + \omega_0^2) I_r(p) &= p i_r(0) + \frac{di_r(0)}{dt}, \\ I_r(p) &= \frac{\omega_0^2 C}{2} [e_2(0) - e_3(0)] \frac{1}{p^2 + \omega_0^2}, \end{aligned} \tag{40}$$

hence

$$i_r(t) = \omega_0 C \frac{e_2(0) - e_3(0)}{2} \sin \omega_0 t \tag{41}$$

and

$$q(t) = \int_0^t i_r(t) \, dt = C \frac{e_2(0) - e_3(0)}{2} [1 - \cos \omega_0 t].$$

If $2\pi/\omega_0 = 2\tau$, i.e., $\tau = \pi\sqrt{\ell C/2}$, which means that the duration of the switch closure is a half-period of the resonance of the tuned circuit, then

$$\begin{aligned} i_r(t) &= \frac{\pi}{\tau} \frac{C[e_2(0) - e_3(0)]}{2} \sin \frac{\pi t}{\tau}, \\ q(t) &= \frac{C[e_2(0) - e_3(0)]}{2} \left[1 - \cos \frac{\pi t}{\tau}\right]. \end{aligned}$$

It is clear then that, during the period τ, the charge transferred onto the

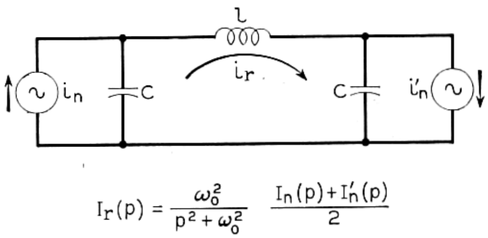


Fig. 9 — Resonant circuit excited by current sources I_n and I_n'.

right-hand capacitor is $q(\tau) = C[e_2(0) - e_3(0)]$ and as a result at time $t = \tau$ the right-hand capacitor has a voltage $e_2(0)$ and the left-hand capacitor has a voltage $e_3(0)$. Considering now the network of Fig. 9, the equation is

$$\ell \frac{d^2 i_r}{dt^2} + \frac{2}{C} i_r = \frac{1}{C} [i_n(t) + i_n'(t)].$$

Assuming all initial conditions* to be zero we get,

$$I_r(p) = \frac{\omega_0^2}{p^2 + \omega_0^2} \frac{I_n(p) + I_n'(p)}{2}. \tag{42}$$

Appendix II

STUDY OF THE LIMITING CASE $T \to 0$

We expect that if the sampling period $T \to 0$, which is equivalent to stating that the sampling frequency $\omega_s \to \infty$, then the inductance $\ell \to 0$ and as a result the voltage $e_3(t)$ will be infinitely close, at all times, to the voltage $e_2(t)$. Thus, in the limit, everything happens as if the terminal pairs (2) of N_1 and N_2 were directly connected. In that case the gain of the system is

$$\frac{Z_{12}(p)}{2Z(p)} Z_{12}(p),$$

as is easily seen by referring to the Thevenin equivalent circuit of N_1.

Let us show that as $T \to 0$, (21) leads to the same result. First note that both $Z_{12}I_0$ and ZS_1 go to zero at least as fast as $1/p^2$ for $p \to \infty$. Hence the summations in (21) reduce to the term corresponding to $n = 0$. Therefore,

$$E_4(p) = \frac{C[Z_{12}I_0]^*S_1}{1 + 2C[ZS_1]^*} Z_{12} \to \frac{CZ_{12}I_0Z_{12}S_1}{T + 2CZS_1} \to \frac{Z_{12}^2}{2Z} I_0 .$$

Appendix III

ZEROTH APPROXIMATION IN THE CASE WHERE N_1 IS NOT IDENTICAL TO N_2

Let, for $k = 1, 2$; C_k be the shunt capacitor at the terminal pair 2 of N_k, $Z_k(p)$ be the driving point impedance of N_k, and $Z_{12}^{(k)}(p)$ be the transfer impedance of N_k. In the present case the capacitors C_1 and C_2 are in series in the resonant circuit of Fig. 2. It can be shown that the

* Their contribution has been found in (40).

charge exchanged during one-half period of the resonance is

$$\frac{2C_1C_2}{C_1 + C_2}[e_2(0) - e_3(0)].$$

For the present case, (16) and (17) become

$$e_2(t) = v(t) - \int_{-\infty}^{t} i_{r0}(\tau)z_1(t - \tau)\,d\tau,$$

$$e_3(t) = \int_{-\infty}^{t} i_{r0}(\tau)z_2(t - \tau)\,d\tau.$$

Following the same procedure as before we are finally led to the block diagram of Fig. 10 whose output is given by

$$E_4(p) = \frac{[Z_{12}^{(1)}(p)I_0(p)]^* \dfrac{2C_1C_2}{C_1 + C_2} S_1(p)Z_{12}^{(2)}(p)}{1 + \dfrac{2C_1C_2}{C_1 + C_2}\{[Z_1(p) + Z_2(p)]S_1(p)\}^*}.$$

Appendix IV

THE DERIVATION OF EQUATION (24)

Considering the method used in Section IV to derive the zeroth approximation, it is clear that during the switch closure the voltages $e_2(t)$ and $e_3(t)$ vary sinusoidally, that is,

$$e_2(t) = e_2(0) - \frac{e_2(0) - e_3(0)}{2}\left[1 - \cos\frac{\pi t}{\tau}\right],$$

$$e_3(t) = e_3(0) + \frac{e_2(0) - e_3(0)}{2}\left[1 - \cos\frac{\pi t}{\tau}\right].$$

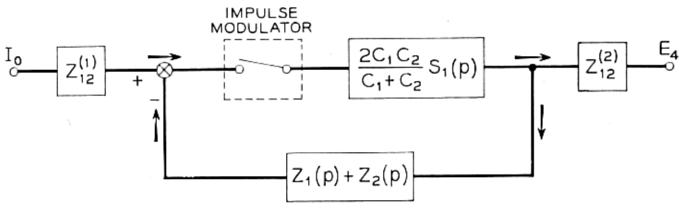


Fig. 10 — Zeroth approximation: modified block diagram for the case where N_1 and N_2 are not identical.

Thus, it always happens that for $t = \tau/2$, i.e., at the middle of switch closure time, $e_2(t) - e_3(t) = 0$.

Therefore if we consider the time function $e_2(t) - e_3(t)$ we have for all k's $(-\infty, \cdots 0, \cdots + \infty)$,

$$e_2\left(kT + \frac{\tau}{2}\right) - e_3\left(kT + \frac{\tau}{2}\right) = 0.$$

If, for simplicity of analysis, we assume that the switch is closed during the intervals $-(\tau/2) + kT \leqq t \leqq +\tau/2 + kT$, then for all k's,

$$e_2(kT) - e_3(kT) = 0.$$

Using (17), this condition implies that $[V(p)]^* - 2[I_{r0}(p)Z(p)]^* = 0$.

Now, remembering that $i_{r0}(t)$ consists of a sequence of half sine waves whose shape is defined by $s_0(t)$ (which is by definition identical to $s_1(t)$ except for an advance in time of $\tau/2$) it follows that $I_{r0}(p) = B(p)S_0(p)$, where $B(p)$ is the $\mathfrak{L}$-transform of the sequence of impulses whose measure is equal to the charge interchanged between N_1 and N_2 at each switch closure. Since $[B(p)S_0(p)Z(p)]^* = B(p)[S_0(p)Z(p)]^*$, then

$$B(p) = \frac{[Z_{12}(p)I_0(p)]^*}{2[S_0(p)Z(p)]^*}.$$

From which it immediately follows that

$$I_{r0}(p) = \frac{[Z_{12}(p)I_0(p)]^*S_0(p)}{2[S_0(p)Z(p)]^*}$$

and

$$E_{40}(p) = \frac{[Z_{12}(p)I_0(p)]^*S_0(p)Z_{12}(p)}{2[S_0(p)Z(p)]^*},$$

where

$$[S_0(p)\ Z(p)]^* = \frac{1}{T}\sum_{n=-\infty}^{+\infty} S_0(p + jn\omega_s)\ Z(p + jn\omega_s).$$

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