Fluctuations of Telephone Traffic

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The number of calls in progress in a simple telephone exchange model characterized by unlimited call capacity, a general probability density of holding-time, and randomly arriving calls is defined as $N(t)$. A formula, due to Riordan, for the generating function of the transition probabilities of $N(t)$ is proved. From this generating function, expressions for the covariance function of $N(t)$ and for the spectral density of $N(t)$ are determined. It is noted that the distributions of $N(t)$ are completely specified by the covariance function.

I INTRODUCTION

The aim of this paper is to study the average fluctuations of telephone traffic in an exchange, by means of a simple mathematical model to which we apply concepts used in the theory of stochastic processes and in the analysis of noise.

The mathematical model we use is based on the following assumptions: (1) requests for telephone service arise individually and collectively at random at an average rate of a per second; (2) the holding-times of calls are mutually independent random variables having the common probability density function $h(u)$; and (3) the capacity of the exchange is effectively unlimited, and no call is blocked or delayed by lack of equipment. This telephone exchange model has been described by J. Riordan.[5]

As a measure of traffic, it is natural to use the number of calls in progress in the exchange. We are thus led to consider a random step-function of time $N(t)$, defined as the number of calls in progress at time t. $N(t)$ fluctuates about an average in a manner depending on the calling-rate, a, and the holding-time density, $h(u)$.

II PROOF OF RIORDAN'S FORMULA FOR TRANSITION PROBABILITIES

Let $P_{m,n}(t)$ be the probability that n calls are in progress at t if m calls were in progress at 0. Define the generating function of these probabilities as

$$P_m(t, x) = \sum_{n \geqq 0} P_{m,n}(t)x^n,$$

and let

$$f(u) = \int_u^\infty h(x)\, dx,$$

so that the average holding-time, h, is given by

$$h = \int_0^\infty f(u)\, du.$$

Riordan[5] has given the following formula for $P_m(t, x)$:

$$P_m(t, x) = [1 + (x - 1)g(t)]^m \exp \{(x - 1)ah[1 - g(t)]\}, \qquad (1)$$

with

$$g(t) = \frac{1}{h} \int_t^\infty f(u)\, du.$$

For exponential holding-time density, this formula had already been derived (as the solution of a differential equation) by Palm.[2]

In private communication, J. Riordan has suggested that his proof of (1) is incomplete. We therefore give a new proof of (1).

We seek the generating function of $N(t)$, conditional on the event $N(0) = m$. We obtain it by first computing the joint generating function of $N(0)$ and $N(t)$; that is,

$$E\{y^{N(0)}x^{N(t)}\}. \qquad (2)$$

The desired conditional generating function is then the coefficient of y^m in (2), divided by the probability that $N(0) = m$.

To obtain a formula for (2), we exhaust the interval $(-\infty, 0)$ by division into a countable set of disjoint intervals, I_n, the n^{th} having length $T_n > 0$. Let S_n be the sum of the first n lengths, T_j. Let $\xi_n(t)$, for $t > -S_{n-1}$, be the number of those calls which arrive in I_n and are still in progress at t. And let $\eta(t)$ be the number of calls arriving during $(0, t)$, $t > 0$, and still in existence at t. Then

$$N(0) = \sum_{n \geqq 1} \xi_n(0), \qquad (3)$$

$$N(t) = \eta(t) + \sum_{n \geqq 1} \xi_n(t), \qquad t > 0. \qquad (4)$$

Since calls arriving during disjoint intervals are independent, we know that $\eta(t)$ is independent of all the ξ's, and that $\xi_n(t)$ is independent of $\xi_j(\tau)$ if $n \neq j$. Of course, $\xi_n(t)$ and $\xi_n(\tau)$ are not independent. It follows that if the infinite product converges, then for $t > 0$

$$E\{y^{N(0)}x^{N(t)}\} = E\{x^{\eta(t)}\} \prod_{n=1}^{\infty} E\{y^{\xi_n(0)}x^{\xi_n(t)}\}. \tag{5}$$

We now compute the terms of the product. If a call originates in interval I_n, it still exists at 0 with probability

$$Q_n = \frac{1}{T_n}\int_0^{T_n} f(u + S_{n-1})\, du = \frac{1}{T_n}\int_{S_{n-1}}^{S_n} f(u)\, du.$$

Hence if k calls arrived in I_n, the probability that m of them are still in progress at 0 is

$$\begin{aligned}\text{pr}\{\xi_n(0) &= m \mid k \text{ calls arrive in } I_n\}\\ &= \binom{k}{m} Q_n^m (1 - Q_n)^{k-m}, \qquad m \leqq k.\end{aligned}$$

Similarly, if a call originates in I_n and exists at 0, it also exists at $t > 0$ with probability

$$K_n = (Q_n T_n)^{-1} \int_0^{T_n} f(u + t + S_{n-1})\, du.$$

Therefore

$$\begin{aligned}E\{x^{\xi_n(t)} \mid \xi_n(0) &= m \text{ and } k \text{ calls arrive in } I_n\}\\ &= [1 + (x - 1)K_n]^m,\end{aligned}$$

and so

$$\begin{aligned}&E\{y^{\xi_n(0)}x^{\xi_n(t)} \mid k \text{ calls arrive in } I_n\}\\ &\qquad = \{1 + \langle y[1 + (x - 1)K_n] - 1\rangle Q_n\}^k\\ &\qquad = \alpha^k.\end{aligned}$$

The number of calls arriving during I_n has a Poisson distribution with mean aT_n; hence

$$\begin{aligned}E\{y^{\xi_n(0)}x^{\xi_n(t)}\} &= \exp\{aT_n(\alpha - 1)\}\\ &= \exp\{aT_nQ_n\langle y[1 + (x - 1)K_n] - 1\rangle\}.\end{aligned} \tag{6}$$

By reasoning like that leading to (6), it can be shown that

$$\begin{aligned} E\{x^{\eta(t)}\} &= \exp\left\{at(x-1)\frac{1}{t}\int_0^t f(u)\,du\right\} \\ &= \exp\{ah(x-1)[1-g(t)]\}. \end{aligned} \tag{7}$$

Now

$$\sum_{n\geqq 1} aT_nQ_n = a\int_0^\infty f(u)\,du = ah,$$

$$\begin{aligned} \sum_{n\geqq 1} aT_nQ_nK_n &= a\sum_{n\geqq 1}\int_0^{T_n} f(u+t+S_{n-1})\,du, \\ &= a\int_t^\infty f(u)\,du = ahg(t). \end{aligned}$$

Therefore the infinite product is convergent, and

$$\begin{aligned} E\{y^{N(0)}x^{N(t)}\} &= \exp\{ah(x-1)[1-g(t)] \\ &\quad + \sum_{n\geqq 1} aT_nQ_n\langle y[1+(x-1)K_n]-1\rangle\} \\ &= \exp\{ah\langle(x-1)[1-g(t)]+(y-1)+y(x-1)g(t)\rangle\}. \end{aligned} \tag{8}$$

Thus the generating function of the joint distribution of $N(0)$ and $N(t)$ is independent of the division of $(-\infty, 0)$ into intervals I_n. By letting x approach 1 in (8) and finding the coefficient of y^m in the resulting limit, we find that

$$\text{pr}\{N(0) = m\} = \frac{e^{-ah}(ah)^m}{m!}. \tag{9}$$

The coefficient of y^m in (8) itself is

$$\frac{e^{-ah}(ah)^m}{m!}[1+(x-1)g(t)]^m \exp\{(x-1)ah[1-g(t)]\},$$

and so using (9) we find that the required conditional generating function of $N(t)$, given $N(0) = m$, is given by Riordan's formula (1).

III THE AUTOCORRELATION

In terms of $N(t)$ one can define various stochastic integrals which will be characteristic of the process. A simple one which has been extensively treated in connection with estimating the average traffic is

$$M = \frac{1}{T}\int_0^T N(t)\,dt,$$

the average of $N(t)$ over an interval $(0, T)$. The chief references in the literature on M are References 3 and 5. If we consider $N(t)$ during an interval $(0, T + \tau)$, a measure of the coherence of $N(t)$ during this interval, i.e., of the extent to which $N(t)$ hangs together, is given by the integral

$$U(T, \tau) = \frac{1}{T} \int_0^T N(t)\, N(t + \tau)\, dt,$$

depending on values of $N(t)$ taken τ apart. When the limit $\psi(\tau)$ of u as T approaches ∞ exists, it is usually called the autocorrelation function; most statisticians, however, reserve the term "correlation" for suitably normalized, dimensionless quantities. It can be shown that this limit exists and is the same for almost all $N(t)$ in the ensemble. It then coincides with the ensemble average, i.e.,

$$\begin{aligned} \psi(\tau) &= \lim_{T \to \infty} U(T, \tau), \text{ almost all } N(t), \\ &= E\{N(t)N(t + \tau)\}. \end{aligned}$$

The function, ψ, for the system we are discussing is derived by Riordan,[5] and we reproduce his argument for ease of understanding. For equilibrium, and $b = ah$, we have

$$E\{N(t)N(t + \tau)\} = \sum_{m=0}^{\infty} \frac{e^{-b}(b)^m}{m!}\, m\, \frac{\partial}{\partial x} P_m(\tau, x)\Big]_{x=1}.$$

Now

$$\frac{\partial}{\partial x} P_m(\tau, x)\Big]_{x=1} = mg(\tau) + b[1 - g(\tau)],$$

so that

$$\begin{aligned} \psi(\tau) &= \sum_{m=0}^{\infty} \frac{e^{-b} b^m}{m!}\, m\{mg(\tau) + b[1 - g(\tau)]\}, \\ &= b^2 + bg(\tau). \end{aligned} \tag{10}$$

(Cf.,[5] p. 1136)

The limiting value of $\psi(\tau)$ for τ approaching ∞ is the square of the mean occupancy, b, and the limiting value of $\psi(\tau)$ for τ approaching 0 is the mean square occupancy, $b^2 + b$, the second moment of the Poisson distribution with mean b.

IV THE COVARIANCE AND SPECTRAL DENSITY

The average value of $N(t)$ is $b = ah$. One way to study the fluctuations of $N(t)$ about its average is by means of the power spectrum used in the analysis of noise. (Cf. Rice.[4]) We resolve the difference $[N(t) - b]$ into sinusoidal components of non-negative frequency, and postulate a noise current proportional to this difference dissipating power through a unit resistance. The spectrum $w(f)$ is then the average power due to frequencies in the interval $(f, f + df)$.

More formally, we consider the Fourier integral

$$S(f, T) = \int_0^T [N(t) - b]e^{-2\pi i f t}\, dt,$$

and we recall, for completeness, the relationship between $S(f, T)$ and the covariance function, $R(\tau)$, of $[N(t) - b]$. If

$$w(f) = \lim_{T\to\infty} \frac{2 \mid S(f, T)|^2}{T},$$

then

$$\begin{aligned} w(f) &= 4\int_0^\infty R(\tau) \cos 2\pi f\tau\, d\tau, \\ R(\tau) &= \int_0^\infty w(f) \cos 2\pi f\tau\, df. \end{aligned} \tag{11}$$

(Cf. Rice,[4] p. 312 ff.)

At the same time, we have

$$\begin{aligned} R(\tau) &= E\{[N(t) - b][N(t + \tau) - b]\} \\ &= \psi(\tau) - b^2 \\ &= bg(\tau). \end{aligned}$$

Let $X(t)$ be any stochastic process which is known to be the occupancy of a telephone exchange of unlimited capacity, having a probability density of holding-time, and subject to Poisson traffic. From the preceding result it can be seen that the covariance function of $X(t)$ determines the distributions of the $X(t)$ process completely, since

$$a = -\left.\frac{dR}{d\tau}\right]_{t=0},$$

$$f(\tau) = \int_\tau^\infty h(u)\, du = -a^{-1}\frac{dR}{d\tau}.$$

If the holding-times are bounded by a constant, k, then readings of $N(t)$ taken further apart than k are uncorrelated. In fact, such values are independent, because no call which contributes to $N(t)$ can survive until $(t + k)$, with probability 1.

Using (11), we see that

$$\begin{aligned} w(f) &= 4 \int_0^\infty \cos 2\pi f\tau R(\tau)\, d\tau \\ &= 4b \int_0^\infty \cos 2\pi f\tau g(\tau)\, d\tau \\ &= 4a \int_0^\infty \cos 2\pi f\tau \int_\tau^\infty \int_y^\infty h(u)\, du\, dy\, d\tau \qquad (12) \\ &= \frac{2a}{\pi f} \int_0^\infty \sin 2\pi f\tau \int_\tau^\infty h(u)\, du\, d\tau \\ &= \frac{a}{\pi^2 f^2}\left[1 - \int_0^\infty \cos 2\pi f\tau h(\tau)\, d\tau\right]. \end{aligned}$$

Equation (12) expresses the mean square of the frequency spectrum of the fluctuations of the traffic away from the average in terms of the calling-rate and the cosine transform of the holding-time density, $h(u)$. The calling-rate appears only as a factor, and so does not affect the shape of $w(f)$. The function $w(f)$ is what Doob[1] (p. 522) calls the "spectral density function (real form)."

V EXAMPLE 1. $N(t)$ MARKOVIAN

Let the frequency $h(u)$ be negative exponential, so that

$$h(u) = \frac{1}{h} e^{-u/h}, \qquad (13)$$

where h is the mean holding-time. It is shown in Riordan[5] p. 1134, that $N(t)$ is Markovian if and only if $h(u)$ has the form (13). From page 523 of Doob[1] we know that the covariance function of a real, stationary Markov process (wide sense) has the form

$$R(\tau) = R(0)e^{-\alpha\tau}, \qquad \alpha \text{ constant.} \qquad (14)$$

Under the assumption (13), the covariance of $N(t)$ is

$$\begin{aligned} R(\tau) = bg(\tau) &= \frac{b}{h} \int_\tau^\infty \int_y^\infty h(u)\, du\, dy \\ &= b \int_\tau^\infty \int_y^\infty \frac{1}{h^2} e^{-u/h}\, du\, dy \\ &= be^{-\tau/h}, \end{aligned}$$

in agreement with (14). The spectral density can now be obtained from (11) or (12); it is

$$w(f) = \frac{4bh}{1 + 4\pi^2 f^2 h^2}.$$

This is the same as would be obtained for a Markov process that alternately assumed the values $+\sqrt{ah}$, $-\sqrt{ah}$ at the Poisson rate of $(2h)^{-1}$ changes of sign per sec. (Cf. Rice[4] p. 325.)

VI EXAMPLE 2. HOLDING-TIME DISTRIBUTED UNIFORMLY IN (α, β)

Let $h(u)$ be constantly equal to $(\beta - \alpha)^{-1}$ in the interval (α, β), and constantly 0 elsewhere. Then by (12),

$$\begin{aligned} w(f) &= \frac{a}{\pi^2 f^2}\left[1 - \frac{1}{\beta - \alpha}\int_\alpha^\beta \cos 2\pi f t \, dt\right] \\ &= \frac{a}{\pi^2 f^2}\left[1 - \frac{\sin 2\pi f\beta - \sin 2\pi f\alpha}{2\pi f(\beta - \alpha)}\right]. \end{aligned}$$

Now we see that

$$f(y) = \int_y^\infty h(u)\, du = \begin{cases} 1 & \text{for} \quad y \leqq \alpha \\ \dfrac{\beta - y}{\beta - \alpha} & \text{for} \quad \alpha \leqq y \leqq \beta \\ 0 & \text{for} \quad y \geqq \beta \end{cases}$$

so that

$$R(\tau) = \begin{cases} a\left[\alpha - \tau + \dfrac{\beta - \alpha}{2}\right] & 0 \leqq \tau \leqq \alpha \\ \dfrac{a}{2}\dfrac{(\beta - \tau)^2}{\beta - \alpha} & \alpha \leqq \tau \leqq \beta \\ 0 & \tau \geqq \beta \end{cases} \tag{15}$$

is the covariance function of the process $N(t)$ when holding-time is distributed uniformly in (α, β).

If, formally, we let $(\beta - \alpha)$ approach 0 while keeping $\frac{1}{2}(\alpha + \beta)$ fixed, then the holding-times become concentrated in the neighborhood of the mean, h; in the limit, as $h(u)$ tends to a singular normal distribution with mean, h, and variance zero, we obtain

$$w(f) = \frac{a}{\pi^2 f^2}[1 - \cos 2\pi f h] \tag{16}$$

as the spectral density function for the $N(t)$ process with constant holding-time, $h = \frac{1}{2}(\alpha + \beta)$. Similarly, from (15), we note that as the holding-times become singularly normal with mean, h, and variance zero, the covariance function becomes

$$R(\tau) = \begin{cases} = a(h - \tau) & 0 \leqq \tau \leqq h \\ = 0 & \tau \geqq h. \end{cases}$$

We can express (16) as

$$w(f) = 2ah^2 \left(\frac{\sin \pi f h}{\pi f h} \right)^2,$$

and note that this is exactly like the power spectrum of a random telegraph wave constructed by choosing values $+\sqrt{ah}$, $-\sqrt{ah}$ with equal probability and independently for each interval of length, h. (Cf. Rice,[4] page 327.)

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