A Sufficient Set of Statistics for a Simple Telephone Exchange Model

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This paper considers a simple telephone exchange model which has an infinite number of trunks and in which the traffic depends on two parameters, the calling-rate and the mean holding-time. It is desired to estimate these parameters by observing the model continuously during a finite interval, and noting the calling-time and hang-up time of each call, insofar as these times fall within the interval. It is shown that the resulting information may, for the purpose of this estimate, be reduced without loss to four statistics. These statistics are the number of calls found at the start of observation, the number of calls arriving during observation, the number of calls terminated during observation, and the average number of calls existing during the interval of observation. The joint distribution of these sufficient statistics is determined, in principle, by deriving a generating function for it. From this generating function the means, variances, covariances, and correlation coefficients are obtained. Various estimators for the parameters of the model are compared, and some of their distributions, means, and variances presented.

I THEORETICAL PROBLEMS AND METHODS OF TRAFFIC MEASUREMENT

Four important kinds of theoretical problems arise in the measurement of telephone traffic. These are: (1) the choice of a mathematical model, containing parameters characteristic of the traffic, to serve as a description; (2) the devising of efficient methods of estimating the parameters; (3) the determination of the anticipated accuracy of measurements; and (4) the assessment of actual accuracy, after measurements have been made.

The present paper deals with aspects of the second and third kinds of problem, for the simplest and least realistic mathematical model of telephone traffic. Specifically, for this model, we treat the problems of (i) complete extraction of the information from a given observation period, without regard to costs of observation, and (ii) determination of the anticipated accuracy of certain methods of estimation which arise naturally from the discussion of complete extraction.

The method by which we attack problems (i) and (ii) in this paper has three stages. First we choose a small number of significant properties of, or factors in, the physical system we are studying. Then we abstract these properties into a mathematical model of the physical system. Finally, from the properties of the model, we derive results which may be interpreted as answers to the two problems treated. The advantage of this method is that we can use the precise, powerful apparatus of mathematics in studying the model; its limitation is that it yields results which are only as accurate as the model in describing reality.

A method similar to the above forms the theoretical underpinning of telephone traffic engineering itself. To design equipment effectively, the traffic engineer needs a description of the traffic that is handled by central offices. He decides what properties of the entire system of telephone equipment and customers will be most useful to him in describing the traffic. He then designates certain parameters to serve as mathematically precise idealizations of these properties, and in terms of these parameters constructs a model of the traffic, upon which he bases much of his engineering.

In choosing a mathematical model for a physical system, one is confronted with two generally opposed desiderata: fidelity to the system described, and mathematical simplicity. The model may involve important departures from physical reality; a model that is sufficiently amenable to mathematical analysis often results only after one has introduced admittedly false assumptions, ignored certain effects and correlations, and generally oversimplified the system to be studied. However, the abstract model will be an exact and simple tool for analysis.

We can construct a simple mathematical model for the operation of a telephone central office by leaving out of consideration many important facts about such systems, and by concentrating on factors most relevant to operation. Since we are interested in telephone traffic and in the availability of plant, it seems natural to require that a realistic model take account of at least the following five significant factors: (1) the demand for telephone service; (2) the rate at which requests for service can be processed and connections established; (3) the lengths of conversations; (4) the supply of central office equipment; and (5) the manner in which the first four factors are interrelated. Unfortunately, the mathematical complexity of such a realistic model precludes easy investigation. Therefore, the model used in this paper is based only on factors (1) and (3).

The demand for telephone traffic is usually made precise by describing a stochastic process which represents the way in which requests for telephone service occur in time. A realistic description will take account of the facts that, the demand is not constant, but has daily extremes, and that in small systems, the demand may be materially lessened when many conversations are in progress. Since taking account of the first fact leads to a more complicated model in which our investigations are more difficult, we ignore it, with the proviso that the results we derive are only applicable to systems and observations for which the demand is nearly constant. The second kind of variation in demand becomes insignificant as the number of subscribers increases and the traffic remains constant. Hence, we further confine the applicability of our results to systems with large numbers of subscribers, and we assume that the demand does not depend on the number of conversations in existence.

With these assumptions, a mathematically convenient description of the demand is specified by the condition that the time-intervals between requests for service have lengths which are mutually independent positive random variables, with a negative exponential distribution.

A telephone central office contains two kinds of equipment: control circuits which establish a desired connection, and talking paths over which a conversation takes place. The time that a request for service occupies a unit of equipment, be the unit a control circuit or a talking path, is called the holding-time of the unit. A request for service affects the availability of both kinds of equipment but, except for special cases, the holding-times of talking paths are usually much longer than the holding-times of control units such as markers, connectors, or registers. In view of this disparity, we assume that the only holding-times of consequence are the lengths of conversations; i.e., the holding-times of talking paths. We assume also that these lengths are mutually independent positive random variables, with a negative exponential distribution.

For the simplest mathematical model of telephone traffic, we may consider the arrangement of switches and transmission lines which constitutes a talking path in the physical office to be replaced by an abstract unit called a "trunk". A trunk is then an abstraction of the equipment made unavailable by one conversation, and we may measure the supply of talking paths in the office by the number of trunks in a model. The word "trunk" is also used to mean a transmission line linking two central offices, but as long as we have explained our use of the word there need be no confusion. Often the number of transmission lines leading out of an office is a major limitation on its capacity to carry conversations, and in this case the two uses of the word "trunk" are very similar. Unfortunately, we do not take advantage of this similarity, since we make the mathematically convenient but wholly unrealistic assumption that the number of trunks in the model is infinite.

The model we investigate thus depends on only two of the factors previously listed as essential to a realistic model: namely, (1) the demand for service, and (3) the lengths of conversations. In view of the simplicity and inaccuracy of this model, the question arises whether much is gained from a detailed analysis. Such scrutiny may indeed reveal little that is of great practical value to traffic engineers. It is important methodologically, however, to have a detailed treatment of at least one approximate case. We undertake this detailed treatment largely for the insight that it may give into methods which could be useful in dealing with more complex and more accurate models.

Once a designer has chosen a model and has specified the parameters he would like to have measured, it is up to the statistician to invent efficient means of measurement, by choosing, for each parameter, some function of possible observations to serve as an estimate of that parameter. One measure of efficiency that is of mostly theoretical interest is the observation time required to achieve a given degree of anticipated accuracy; the most realistic measure of efficiency is in terms of dollars and man-hours. It may often be more efficient, in the sense of the latter measure, to spread observation over enough more time to compensate for the inability of an intrinsically cheaper method of measurement to extract all of the information present in a fixed time of observation. For example, periodic scanning of switches in a telephone exchange is usually less costly than continuous observation. As a result, telephone traffic measurement is usually carried out by averaging sequences of instantaneous periodic observations of the number of calls present, rather than by continuous time averaging, although it can be shown that continuous observation is more efficient at extracting information. Thus statistical efficiency, which may be expensive in terms of measuring equipment, can be exchanged for observation time, which may be less costly. This exchange brings about a reduction in cost without impairing accuracy.

Our concern in this paper is with the less practical problems of complete extraction, and of the anticipated accuracy of estimation methods based on complete extraction. Let us consider how our mathematical model can shed light on these problems. A mathematical model may or may not be a faithful description of the behavior of real telephone systems. Nevertheless random numbers, with or without modern computing machines, enable one to make experiments and observations on physical situations which approximate, arbitrarily closely, any mathematical model. Thus we can speak meaningfully of events in the model, and of making measurements and observations on the model. The mathematical model elucidates our problems in the following ways: (1) it enables us to state precisely what information is provided by observation; (2) it enables us to explain what we mean by complete extraction of information; and (3) it enables us to derive results about the anticipated accuracy of measurements in the model. These results will have approximately true analogues in physical situations to which the model is applicable.

The calls existing during the observation interval (O, T) fall into four categories: (i) those which exist at O, and terminate before T; (ii) those which fall entirely within (O, T); (iii) those which exist at O and last beyond T; and (iv) those which begin within (O, T) and last beyond T. For calls of category (i), we assume that we observe the hang-up time of each call; for category (ii), we observe the matching calling-time and hang-up time of each conversation; for category (iii), we observe simply the number of such calls; and for category (iv), we observe the calling-times. Table I summarizes the kinds of calls and the information observed about each.

What we mean by the complete extraction of information is made precise by the statistical concept of *sufficiency*. By a statistic we mean any function of the observations, and by an estimator we mean a statistic which has been chosen to serve as an estimate of a particular parameter. Roughly and generally, a set S of statistics is sufficient for a set P of parameters when S contains all the information in the original data that was relevant to parameters in P. If S is sufficient for P, there is a set E of estimators for parameters in P, such that the estimators in E depend only on statistics from S, and such that an estimator from E does at least as well as any other estimator we might choose for the same parameter. Thus we incur no loss in reducing the original data (of specified form) to the set S of statistics. It remains to state what it means for S to contain all the relevant information. We do this in terms of our model.

The mathematical model we are adopting contains two distribution

Table I — Information Observed

Types of Calls	Start in (O, T)	Start before O
End in (O, T)	(ii), matching calling-times and hang-up times known, number of calls known	(i), hang-up times known, number of calls known
End after T	(iv), calling-times known, number of calls known	(iii), number of calls known

functions, that of the intervals between demands for service, and that of the lengths of conversations. We have supposed that these distributions are both of negative exponential type, each depending on a single parameter. Thus we know the functional form of each distribution, and each such form has one unknown constant in it. Since the mathematical structure of the model is fully specified except for the values of the two unknown constants, we can assign a likelihood or a probability density to any sequence Σ of events in the model during the interval $(O,\ T)$. This likelihood will depend on the parameters, on Σ, and on the number of calls in existence at the start O of the interval. If the likelihood $L(\Sigma)$ can be factored into the form $L = F \cdot H$, where F depends on the parameters and on statistics from the set S only, and H is independent of the parameters, then the set S of statistics may be said to summarize all the information (in a sequence Σ) relevant to the parameters. If L can be so factored, then S is sufficient for the estimation of the parameters.

The mathematical model to be used in this paper is described and discussed in Sections II and III, respectively. Section IV contains a summary of notations and abbreviations which have been used to simplify formulas.

In Appendix A we show that the original data we have allowed ourselves can be replaced by four statistics, which are sufficient for estimation. In Appendix B and Sections V–VIII we discuss various estimators (for parameters of the model) based on these four statistics. To determine the anticipated accuracy of these methods of measurement, we consider the statistics themselves as random variables whose distributions are to be deduced from the structure of the model.

A primary task is the determination of the joint distribution of the sufficient statistics. In view of the sufficiency, this joint distribution tells us, in principle, just what it is possible to learn from a sample of length T in this simple model. By analyzing this distribution we can derive results about the anticipated accuracy of measurements in the model.

The joint distribution of the sufficient statistics is obtainable in principle from a generating function computed in Appendix C, using methods exemplified in Section X. This generating function is the basic result of this paper. The implications of this result are summarized in Section IX, which quotes the generating function itself, and presents some statistical properties of the sufficient statistics in the form of four tables: (i) a table of generating functions obtainable from the basic one; (ii) a table of mean values; (iii) a table of variances and covariances; and (iv), a table of squared correlation coefficients. (The coefficients are all non-negative.)

II DESCRIPTION OF THE MATHEMATICAL MODEL

Throughout the rest of the paper we follow a simplified form of the notational conventions of J. Riordan's paper[11] wherever possible. A summary of notations is given in Section IV. The model we study has the following properties:

(i) Demands for service arise individually and collectively at random at the rate of a calls per second. Thus the chance of one or more demands in a small time-interval Δt is

$$a\Delta t + o(\Delta t),$$

where $o(\Delta t)$ denotes a quantity of order smaller than Δt. The chance of more than one demand in Δt is of order smaller than Δt. It can be shown (Feller,[2] p. 364 et seq.) that this description of the demand is equivalent to saying that the intervals between successive demands for service are all independent, with the negative exponential distribution

$$1 - e^{-at}.$$

This again is equivalent to saying that the call arrivals form a Poisson process;[2] i.e., that for any time interval, t, the probability that exactly n demands are registered in t is

$$\frac{e^{-at}(at)^n}{n!}.$$

Thus the number of demands in t has a Poisson distribution with mean at.

(ii) The holding-times of distinct conversations are independent variates having the negative exponential distribution

$$1 - e^{-\gamma t},$$

where γ is the reciprocal of the mean holding-time h. This description of the holding-time distribution is the same as saying that the probability that a conversation, which is in progress, ends during a small time-interval Δt is

$$\gamma\Delta t + o(\Delta t),$$

without regard to the length of time that the conversation has lasted Feller, p. 375).

(iii) The model contains an infinite number of trunks. Thus, at no time will there be insufficient central office equipment to handle a demand for service, and no provision need be made for dealing with demands that cannot be satisfied.

The original work on this particular model for telephone traffic is in Palm,[9] and Palm's results have been reported by Feller[3] and Jensen.[4] The results have been extended heuristically to arbitrary absolutely continuous holding-time distributions by Riordan,[11] following some ideas of Newland[8] suggested by S. O. Rice.

Let $P_{ij}(t)$ be the probability that there are j trunks busy at t if there were i busy at 0. And let $P_i(t, x)$ be the generating function of these probabilities, defined by

$$P_i(t, x) = \sum_{j=0}^{\infty} x^j P_{ij}(t).$$

Then Palm[9] has shown (pp. 56 et seq.) that

$$P_i(t, x) = [1 + (x - 1)\, e^{-\gamma t}]^i \exp\{(x - 1)ah\,(1 - e^{-\gamma t})\}.$$

This is formula (12) of Riordan[11] with his g replaced by $e^{-\gamma t}$. It can be verified that the random variable $N(t)$ is Markovian; the limit of $P_i(t, x)$ as $t \to \infty$ is

$$\exp\{(x - 1)\, ah\},$$

so that the equilibrium distribution of the number of trunks in use is a Poisson distribution with mean $b = ah$. The shifted random variable $[N(t) - b]$ then has mean zero, and covariance function $be^{-\gamma t}$.

For additional work on this model the reader is referred to F. W. Rabe,[10] and to H. Stormer.[12]

III DISCUSSION OF THE MODEL

Let us envisage the operation of the model we have described by considering the random variable $N(t)$ equal to the number of trunks busy at time t. As a random function of time, $N(t)$ jumps up one unit step each time a demand for service occurs, and it jumps down one unit step each time a conversation ends. If $N(t)$ reaches zero, it stays there until there is another demand for service. If $N(t) = n$, the probability that a conversation ends in the next small time-interval Δt is

$$n\gamma\Delta t + o(\Delta t),$$

because the n conversations are mutually independent. A graph of a sample of $N(t)$ is shown in Fig. 1.

The model we described departs from reality in several important ways, which it is well to discuss. First, the assumption that the number of trunks is infinite is not realistic, and is justified only by the mathematical complication which results when we assume the number of trunks

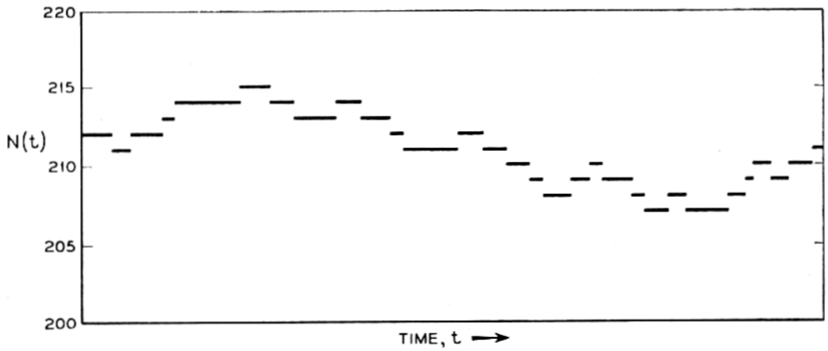

Fig. 1 — A graph of $N(t)$.

to be finite. It can also be argued that unlimited office capacity is approached by offices with adequate facilities and low calling rates, and therefore, in some practical cases at least, the model is not flagrantly inaccurate.

Second, the choice of a constant calling rate for the model ignores the fact that in most offices the calling rate is periodic. Thus, the applicability of our results to offices whose calling rates undergo drastic changes in time is restricted to intervals during which the normally variable calling rate is nearly constant. Finally, although the assumption of a negative exponential distribution of holding-time affords the model great mathematical convenience, it is doubtful whether in a realistic model the most likely holding-time would have length zero, as it does in the present one.

IV SUMMARY OF NOTATIONS

a = Poisson calling rate

h = mean holding-time

$\gamma = h^{-1}$ = hang-up rate per talking subscriber

$b = ah$ = average number of busy trunks

$N(t)$ = number of trunks in use at t

(O, T) = interval of observation

$n = N(O)$ = number of trunks in use at the start of observation

A = number of calls arriving in (O, T)

H = number of hang-ups in (O, T)

$K = A + H$

$$Z = \int_0^T N(t)\,dt$$

$$M = Z/T = \text{average of } N(t) \text{ over } (O, T)$$

$\{p_n\}$ = the (discrete) probability distribution of n, the number of trunks found busy at the start of observation

An estimator for a parameter is denoted by adding a cap (ˆ) and a subscript. The subscripts differentiate among various estimators for the same parameter. We use $\hat{a}_c = A/T$, $\hat{\gamma}_c = H/Z$, $\hat{a}_1 = K/2T$, $\hat{\gamma}_1 = K/2Z$, and $\hat{\gamma}_2 = A/Z$.

Also, it is convenient to use the following abbreviations: r for γT, and C for $(1 - e^{-r})/r$, where r is the dimensionless ratio of observation-time to mean holding-time. The symbol E is used throughout to mean mathematical expectation.

V THE AVERAGE TRAFFIC

We have adopted a model which depends on two parameters, the calling rate a, and the mean holding-time h, or its reciprocal γ. Before searching for a set of statistics that is sufficient for the estimation of these parameters, let us consider the product $ah = b$. This product is important because, as we saw in Section II, the equilibrium distribution of the number of trunks in use depends only on b, and not on a and h individually. Indeed, the equilibrium probability that n trunks are busy is

$$\frac{e^{-b}b^n}{n!},$$

and the average number of busy trunks in equilibrium is just b.

The average number of trunks busy during a time interval T is

$$M = \frac{1}{T}\int_0^T N(t)\,dt;$$

i.e., the integral of the random function $N(t)$ over the interval T, divided by T. This suggests that for large time intervals T, M will come close to the value of b, and can be used as an estimator of b. Since M is a random variable, the question arises, what are the statistical properties of M? This question has been considered in the literature, the principal references being to F. W. Rabe[10] and to J. Riordan.[11] Riordan's paper is a determination of the first four semi-invariants of the distribution of M during a period of statistical equilibrium, but without restriction on the assumed frequency distribution of holding-time. It follows from Riordan's results that M converges to b in the mean, which is to say that

$$\lim_{T\to\infty} E\{|M - b|^2\} = 0.$$

It also follows that M is an unbiased estimator of b; i.e., that $E\{M\} = b$, and that M is a consistent estimator of b, which means that

$$\lim_{T\to\infty} pr\{|M - b| > \varepsilon\} = 0$$

for each $\varepsilon > 0$.

VI MAXIMUM CONDITIONAL LIKELIHOOD ESTIMATORS

As shown in Appendix A, the likelihood L_c of an observed sequence, conditional on $N(O)$, is defined by

$$\ln L_c = A \ln a + H \ln \gamma - \gamma Z - aT.$$

According to the method of maximum likelihood, we should select, as estimators of a and γ respectively, quantities $\hat{a}_c$ and $\hat{\gamma}_c$ which maximize the likelihood L_c. Now a maximum of L_c is also one of $\ln L_c$, and vice versa. Therefore a_c and γ_c are determined as roots of the following two equations, called the likelihood equations:

$$\frac{\partial}{\partial a} \ln L_c = 0; \qquad \frac{\partial}{\partial \gamma} \ln L_c = 0.$$

The solutions to the likelihood equations are

$$\hat{a}_c = \frac{A}{T}, \qquad \hat{\gamma}_c = \frac{H}{Z}.$$

These are the maximum conditional likelihood estimators of a and γ. The estimator $\hat{a}_c$ is the number of requests for service in T divided by T; this is intuitively satisfactory, since $\hat{a}_c$ estimates a calling rate.

Since maximum likelihood estimators of functions of parameters are generally the same functions of maximum likelihood estimators of the parameters, we see that AZ/HT is a maximum likelihood estimator of b.

VII PRACTICAL ESTIMATORS SUGGESTED BY MAXIMIZING THE LIKELIHOOD L, DEFINED IN APPENDIX A

We obtain as likelihood equations

$$\frac{\partial}{\partial a} \ln L = 0, \qquad \frac{\partial}{\partial \gamma} \ln L = 0.$$

These may be written as

$$a = \frac{n + A}{h + T},$$

and

$$\gamma = \frac{H + \dfrac{a}{\gamma}}{Z + \dfrac{n}{\gamma}}.$$

The first of these shows the estimated calling rate as a pooled combination of the conditional estimate A/T, considered in the last section, and an estimate n/h based on the initial state. This latter estimate has the form

$$\frac{\text{calls in progress}}{\text{mean holding time}},$$

and so is intuitively reasonable, since $b/h = a$. The second equation exhibits our estimate of γ as a pooled combination of the conditional estimate H/Z and the ratio a/n. This ratio is acceptable as an estimate of γ, since $a/b = \gamma$ and $b = E\{n\}$ is the average value of n.

If we substitute, in the right-hand sides of these equations, the conditional estimators A/T, H/Z, and Z/H for a, γ, and h, respectively, we obtain simple, intuitive estimators which include the influence of the initial state n, and show how it decreases with increasing T. Thus

$$\frac{n + A}{\dfrac{Z}{H} + T} \quad \text{estimates } a,$$

$$\frac{H + \dfrac{AZ}{TH}}{Z + \dfrac{nZ}{H}} \quad \text{estimates } \gamma.$$

VIII OTHER ESTIMATORS

Additional estimators may be arrived at by intuitive considerations, or by modifying certain maximum likelihood estimators. Some estimators so obtained are important because they use more of the information available in an observation than do the conditional estimators $\hat{a}_c$ and $\hat{\gamma}_c$, without being so complicated functionally that we cannot easily study their statistical properties.

It seems reasonable, and can be shown rigorously (Appendix C), that for an interval (O, T) of statistical equilibrium, the distribution of A and that of H are the same. Thus we can argue that, for long time intervals, A and H will not be very different. This suggests using

$$\hat{a}_1 = \frac{A + H}{2T} = \frac{K}{2T}$$

as an estimator of a. This estimator does not involve γ, and it uses not only information given by A, but also information supplied by arrivals occurring possibly before the start of observation.

Similarly, since $b = a/\gamma$, and M is a consistent and unbiased estimator of b, we may use

$$\hat{\gamma}_1 = \frac{K}{2Z} = \frac{1}{\hat{h}_1}$$

to estimate γ, and its reciprocal to estimate h. Finally, since for long (O,T) we have $A \sim H$, we may try

$$\hat{\gamma}_2 = \frac{A}{Z} = \frac{1}{\hat{h}_2}$$

as an estimator of γ, and its reciprocal as an estimator of h.

IX THE JOINT DISTRIBUTION OF THE SUFFICIENT STATISTICS

The basic result of this paper is a formula for the generating function

$$E\{z^n x^{N(T)} w^A u^H e^{-\zeta Z}\} \tag{9.1}$$

for the joint distribution of the random variables n, $N(T)$, A, H, and Z. This formula is derived in Appendix C, by methods illustrated in Section X. For an initial n distribution $\{p_n\}$, the generating function is

$$\begin{aligned} \sum_{n \geqq 0} p_n z^n & \left[\frac{(\zeta x + \gamma x - \gamma u)e^{-(\zeta+\gamma)T} + \gamma u}{\zeta + \gamma}\right]^n \\ & \cdot \exp\left\{\frac{aw(\zeta x + \gamma x - \gamma u)[1 - e^{-(\zeta+\gamma)T}]}{(\zeta + \gamma)^2} + \frac{a\gamma wuT}{\zeta + \gamma} - aT\right\}. \end{aligned} \tag{9.2}$$

It is proved in Appendix A that the set of statistics $\{n, A, H, Z\}$ is sufficient for estimation on the basis of the information assumed, which was described in Section I. Thus the generating function (9.2) specifies, at least in principle, what can be discovered about the process from an observation interval (O, T), for which $N(O)$ has the distribution $\{p_n\}$. All the results summarized in this section are consequences of (9.2).

TABLE II

X	$\ln E\{X\}$
1. $e^{-\zeta Z}$	$b\left[-\zeta T + \dfrac{\zeta^2 T}{\zeta + \gamma} - \dfrac{\zeta^2(1 - e^{-(\zeta+\gamma)T})}{(\zeta + \gamma)^2}\right]$
2. $e^{-\zeta M}$	$b\left[-\zeta + \dfrac{\zeta^2}{\zeta + r} - \dfrac{\zeta^2(1 - e^{-(\zeta+r)})}{(\zeta + r)^2}\right]$
3. y^K	$2aTC(y - 1) + aT(1 - C)(y^2 - 1)$
4. $e^{-\zeta \hat{a}_1}$	$2aTC(e^{-\zeta/2T} - 1) + aT(1 - C)(e^{-\zeta/T} - 1)$
5. $y^K e^{-\zeta M}$	$b\left[\left(1 - \dfrac{ry}{\zeta + r}\right)^2 [e^{-\zeta(+r)} - 1] - r\left(1 - \dfrac{ry^2}{\zeta + r}\right)\right]$

By substitution, and by either letting the appropriate power series variables $\to 1$, or letting $\zeta \to 0$, or both, we can obtain from (9.2) the generating function of any combination of linear functions of the basic random variables n, $N(T)$, A, H, and Z. Some of the generating functions thereby obtained are listed in Table II, in which the entries all refer to an interval (O, T) of equilibrium.

Since, for equilibrium (O, T), the generating functions are all exponentials, it has been convenient to make Table II a table of logarithms of expectations, with random variables X on the left, and functions $\ln E\{X\}$ on the right. C as a function of r is plotted in Fig. 2.

Entry 1 of Table II is actually the cumulant generating function of Z for equilibrium (O, T); similarly, Entry 2 is that of M, and depends only on the average traffic b and the ratio r. The form of the general cumulant of M is

$$k_n = b\,\frac{n(n-1)}{T^n}\int_0^T (T - x)x^{n-2}e^{-\gamma x}\,dx.$$

This result coincides with a special case (exponential holding-time) of a conjecture of Riordan.[11] This conjecture was first established (for a general holding-time distribution) in unpublished work of S. P. Lloyd. The cumulant generating function permits investigation of asymptotic properties. We prove in Section X that the standardized variable

$$\begin{aligned} v &= (\gamma T/2b)^{1/2}(M - b) \\ &= (r/2b)^{1/2}(M - b) \end{aligned}$$

is asymptotically normally distributed with mean 0 and variance 1.

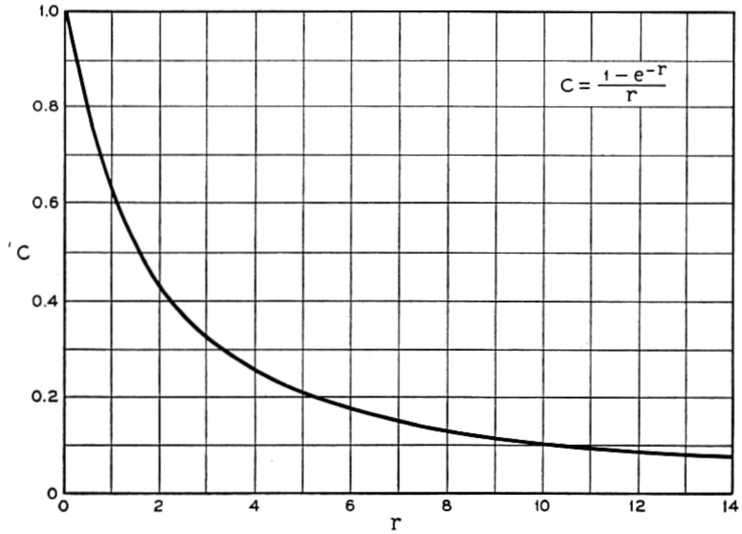

Fig. 2 — C as a function of r.

From Entry 3 of Table II it can be seen that K is distributed as $2u + v$, where u and v follow independent Poisson distributions with the respective parameters $aT(1 - C)$ and $2aTC$. The probability that $K = n$ for an interval of equilibrium is

$$r_n = \exp \{aT(C - 1)\} \sum \frac{(2aTC)^{n-2j}}{(n - 2j)!} \frac{(aT - aTC)^j}{j!},$$

where the sum is over j's for which $0 \leqq 2j \leqq n$.

The estimator $\hat{a}_1$ for a is equal to $K/2T$, and has mean and variance given by

$$E\{\hat{a}_1\} = a,$$

$$\text{var}\{\hat{a}_1\} = \frac{a}{2T}(2 - C).$$

The distribution of $\hat{a}_1$ is given by

$$\text{pr}\{\hat{a}_1 \leqq x\} = \sum r_n,$$

the summation being over $n \leqq 2Tx$.

From (9.2) one can obtain, by substitution of the stationary n distribution for $\{p_n\}$, and subsequent differentiation, the means, variances, covariances, and correlation coefficients of the sufficient statistics, for

TABLE III — $E\{X, Y\}$

	1	n	A	H	K	Z
1	1	b	aT	aT	$2aT$	bT
n		$b(1 + b)$	baT	$aT(C + b)$	$aT(C + 2b)$	$bT(C + b)$
A			$aT(1 + aT)$	$aT(1 - C + aT)$	$aT(2 - C + aT)$	$bT(1 - C + aT)$
H				$aT(1 + aT)$	$aT(2 - C + aT)$	$bT(1 - C + aT)$
K					$2aT(2 - C + 2aT)$	$2bT(1 - C + aT)$
Z						$bTh(2 - 2C + aT)$

TABLE IV — $\text{cov}\{X, Y\}$

	n	A	H	K	Z
n	b	0	aTC	aTC	bTC
A		aT	$aT(1 - C)$	$aT(2 - C)$	$bT(1 - C)$
H			aT	$aT(2 - C)$	$bT(1 - C)$
K				$2aT(2 - C)$	$2bT(1 - C)$
Z					$2bTh(1 - C)$

equilibrium intervals (O, T). It has been convenient to display these in three triangular arrays, the first consisting of expectations of products, the second comprising the variances and covariances, and the third exhibiting, for simplicity, the squared correlation coefficients, since the correlation coefficients are never negative for these random variables.

In Table III, the entry with coordinates (X, Y) is $E\{XY\}$ for equilibrium (O, T). All three tables are expressed in terms of a, b, T, h, r, and C, the last of which is plotted in Fig. 2.

The variances and covariances of the sufficient statistics are listed in Table IV; the entries are of the form:

$$\text{cov}\,\{X, Y\} = E\{XY\} - E\{X\}E\{Y\}.$$

Table V, finally, lists the squared correlation coefficients; i.e., the quantities

$$\rho^2(X, Y) = \frac{\text{cov}^2\,\{X, Y\}}{\text{var}\,\{X\}\,\text{var}\,\{Y\}}.$$

For any time interval (O, T), A has a Poisson distribution with parameter aT, so that $T\hat{a}_c$ does also. Therefore the distribution of $\hat{a}_c$ is given by

$$\text{pr}\,\{\hat{a}_c \leqq x\} = \sum \frac{e^{-aT}(aT)^n}{n!},$$

where the summation is over $n \leqq xT$. Evidently

$$E\{\hat{a}_c\} = a,$$

and

$$\text{var}\ \{\hat{a}_c\} = \frac{a}{T},$$

so that $\hat{a}_c$ is an unbiased and consistent estimator of a. We now compare the variances of estimators $\hat{a}_c$ and $\hat{a}_1$. From Table IV we have

$$\text{var}\ \{\hat{a}_1\} = \frac{a}{T}\left(1 - \frac{C}{2}\right) < \frac{a}{T} = \text{var}\ \{\hat{a}_c\},$$

so that $\hat{a}_1$ is a better estimator of a for any $T > 0$, in the sense that its variance is less.

X THE DISTRIBUTIONS OF Z AND M

Since we have defined

$$Z = \int_0^T N(t)\, dt,$$

we can regard Z as the result of growth whose rate is given by the random step-function $N(t)$; when $N(t) = n$, Z is growing at rate n. An idea similar to this is used by Kosten, Manning, and Garwood[6], and by Kosten alone.[5] Now the $Z(T)$ process by itself is not Markovian, but it can be seen that the two-dimensional variable $\{N(t), Z(t)\}$ itself is Markovian. Let $F_n(z, t)$ be the probability that $N(t) = n$ and $Z(t) \leqq z$. Since the two-dimensional process is Markovian, we can derive infinitesimal relations for $F_n(z, t)$ by considering the possible changes in the system during a small interval of time Δt.

TABLE V — $\rho^2(X, Y)$

	n	A	H	K	Z
n	1	0	$1 - e^{-r}$	$\frac{rC^2}{2 - C}$	$\frac{rC^2}{2(1 - C)}$
A		1	$1 - C$	$\frac{2 - C}{2}$	$\frac{1 - C}{2}$
H			1	$\frac{2 - C}{2}$	$\frac{1 - C}{2}$
K				1	$\frac{1 - C}{2 - C}$
Z					1

If $N(t) = n$, then the probability is $[1 - \gamma n\Delta t - a\Delta t - o(\Delta t)]$ that there is neither a request for service nor a hang-up during Δt following t, and that $Z(t + \Delta t) = Z(t) + n\Delta t$. Therefore the conditional probability that $N(t + \Delta t) = n$ and $Z(t + \Delta t) \leqq z$, given that no changes occurred in Δt, is

$$F_n(z - n\Delta t, t).$$

For $N(t) = (n + 1)$, the probability is $\gamma(n + 1)\Delta t + o(\Delta t)$ that one conversation will end during Δt following t. The increment to $Z(t)$ during Δt will depend on the length x of the interval from t to the point within Δt at which the conversation ended. The increment has magnitude $(n + 1)x + n(\Delta t - x) = x + n\Delta t$, as can be verified from Fig. 3, in which the shaded area is the increment. Since x is distributed uniformly between 0 and Δt, the increment $x + n\Delta t$ is distributed uniformly between $n\Delta t$ and $(n + 1)\Delta t$. Therefore the conditional probability that $N(t + \Delta t) = n$ and $Z(t + \Delta t) \leqq z$, given that one conversation ended in Δt, is

$$\frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} F_{n+1}(z - u, t)\, du.$$

By a similar argument it can be shown that the probability that one request for service arrives in Δt is $a\Delta t + o(\Delta t)$, and that the conditional probability that $N(t + \Delta t) = n$ and $Z(t + \Delta t) \leqq z$, given that one request arrived during Δt, is

$$\frac{1}{\Delta t} \int_{(n-1)\Delta t}^{n\Delta t} F_{n-1}(z - u, t)\, du.$$

Define $F_n(z, t)$ to be identically 0 for negative n. Adding up the probabil-

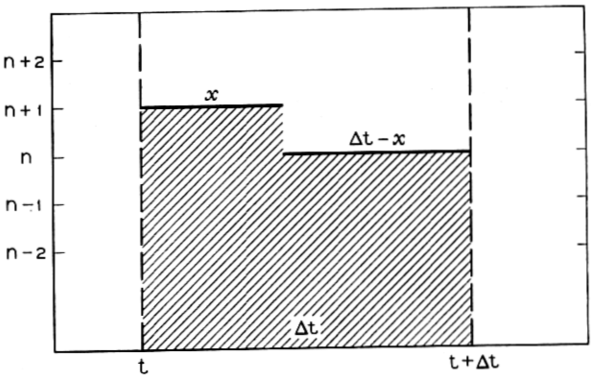

Fig. 3 — Increment to Z in Δt.

ities of mutually exclusive events, we obtain the following infinitesimal relations for $F_n(z, t)$:

$$F_n(z, t + \Delta t) = \gamma(n + 1) \int_{n\Delta t}^{(n+1)\Delta t} F_{n+1}(z - u, t)\, du$$
$$+ a \int_{(n-1)\Delta t}^{n\Delta t} F_{n-1}(z - u, t)\, du + F_n(z - n\Delta t, t)$$
$$\cdot[1 - \Delta t(\gamma n + a)] + o(\Delta t), \quad \text{for any } n.$$

Expanding the penultimate term of the right side in powers of $n\Delta t$, and the left side in powers of Δt, we divide by Δt, and take the limit as Δt approaches 0. Now

$$\lim_{\Delta t \to 0} \frac{1}{\Delta t} \int_{n\Delta t}^{(n+1)\Delta t} F_{n+1}(z - u, t)\, du = F_{n+1}(z, t).$$

Thus, omitting functional dependence on z and t for convenience, we reach the following partial differential equations for $F_n(z, t)$:

$$\frac{\partial}{\partial t} F_n = \gamma(n + 1)F_{n+1} + aF_{n-1} - n \frac{\partial}{\partial z} F_n - [\gamma n + a]F_n, \quad \text{for any } n. \tag{10.1}$$

Since $Z(0) = 0$, we impose the following boundary conditions:

$$\begin{aligned} F_n(0, t) &= 0 && \text{for } n > 0 \text{ and } t > 0, \\ F_n(z, 0) &= p_n && \text{for } z \geqq 0, \\ F_n(z, 0) &= 0 && \text{for } z < 0, \end{aligned} \tag{10.2}$$

where the sequence $\{p_n\}$ forms an arbitrary $N(0)$ distribution that is zero for negative n.

To transform the equations, we introduce the Laplace-Stieltjes integrals

$$\varphi_n(\zeta, t) = \int_{0-}^{\infty} e^{-\zeta z}\, dF_n(z, t), \qquad t \geqq 0, \qquad \text{Re}\,(\zeta) > 0,$$

in which the Stieltjes integration is understood always to be on the variable z. We note that

$$\int_{0-}^{\infty} e^{-\zeta z} F_n(z, t)\, dz = \frac{1}{\zeta} \varphi_n(\zeta, t),$$

and that

$$\varphi_n(\zeta, t) = F_n(0, t) + \int_0^{\infty} e^{-\zeta z} \frac{\partial}{\partial z} F_n(z, t)\, dz.$$

Applying now the Laplace-Stieltjes transformation to (10.1), we obtain

$$\frac{\partial \varphi_n}{\partial t} = \gamma(n+1)\varphi_{n+1} + a\varphi_{n-1} - n\zeta\varphi_n + n\zeta F_n(0, t) - [\gamma n + a]\varphi_n, \tag{10.3}$$

in which we have left out functional dependence on ζ and t where it is unnecessary. By the boundary conditions (10.2), $n\zeta F_n(0, t) = 0$ for $n \geqq 0$ and $t > 0$; in (10.3) we may therefore omit this term in the region $t > 0$. Let φ be defined by

$$\varphi(x, \zeta, t) = \sum_{n=0}^{\infty} x^n \varphi_n(\zeta, t).$$

The series is absolutely convergent for $|x| < 1$, since

$$|\varphi_n(\zeta, t)| \leqq 1, \text{ for all } n.$$

The following partial differential equation for φ is obtained from (10.3):

$$\frac{\partial \varphi}{\partial t} + [\zeta x + \gamma(x-1)]\frac{\partial \varphi}{\partial x} = a(x-1)\varphi. \tag{10.4}$$

If we integrate out the information about Z by letting ξ approach 0 in this equation, we obtain the equation derived by Palm (loc. cit.) for the generating function of $N(t)$. Therefore our equation has a solution of the same form as Palm's. For the boundary conditions (10.2), this solution is

$$\varphi = \exp\left\{\frac{a[1 - e^{-(\zeta+\gamma)t}]}{(\zeta+\gamma)^2}[\zeta x + \gamma(x-1)] - \frac{a\zeta t}{\zeta+\gamma}\right\} \cdot \sum_{n=0}^{\infty} p_n \left[\frac{[\zeta x + \gamma(x-1)]e^{-(\zeta+\gamma)t} + \gamma}{\zeta+\gamma}\right]^n. \tag{10.5}$$

Actually φ contains more information than we want since it yields the joint distribution of N and Z. We may integrate out the former variable by letting x approach 1 in 10.5. Then,

$$E\{\exp(-\zeta Z)\} = \exp\left\{\frac{a\zeta(1 - e^{-(\zeta+\gamma)T})}{(\zeta+\gamma)^2} - \frac{a\zeta T}{\zeta+\gamma}\right\} \cdot \sum_{n=0}^{\infty} p_n \left[\frac{\zeta e^{-(\zeta+\gamma)T} + \gamma}{\zeta+\gamma}\right]^n$$

is the Laplace transform of the distribution of Z for an arbitrary $N(O)$ distribution $\{p_n\}$. This result is not restricted to an interval (O, T)

of statistical equilibrium; however, if the sequence $\{p_n\}$ does form the stationary distribution discussed in Section II, then

$$\sum x^n p_n = \exp\{b(x - 1)\}, \tag{10.6}$$

and

$$\psi = \exp\left\{\frac{b\zeta^2(e^{-(\zeta+\gamma)T} - 1)}{(\zeta + \gamma)^2} - \frac{a\zeta T}{\zeta - \gamma}\right\} \tag{10.7}$$

is the Laplace transform of the distribution of Z for an interval (O, T) of statistical equilibrium.

The Laplace transform is a moment generating function expressible as

$$\psi = \sum_{n=0}^{\infty} \frac{(-\zeta)^n m_n}{n!},$$

where m_n is the n^{th} ordinary moment of Z. Differentiation of 10.7 then gives a recurrence relation for the moments upon equating powers of $(-\zeta)$. Thus,

$$(\zeta + \gamma)^3\left(-\frac{\partial\psi}{\partial\zeta}\right)$$
$$= \psi\cdot\{2a\zeta(1 - e^{-(\zeta+\gamma)T}) + (\zeta + \gamma)[\gamma aT + bT\zeta^2 e^{-(\zeta+\gamma)T}]\},$$

and

$$\gamma^3 m_{n+1} - 3\gamma^2 n m_n + 3\gamma n(n - 1)m_{n-1} - n(n - 1)(n - 2)m_{n-2}$$
$$= a\gamma^2 T m_n - (2a + a\gamma T)nm_{n-1} + 2ane^{-\gamma T}(m + T)^{n-1} + n \tag{10.8}$$
$$\cdot(n - 1)aTe^{-\gamma T}(m + T)^{n-2} - n(n - 1)(n - 2)bTe^{-\gamma T}(m + T)^{n-3},$$

where $(m + T)^n$ is the usual symbolic abbreviation of

$$\sum_{j=0}^{n}\binom{n}{j} T^j m_{n-j}.$$

From the recurrence (10.8) it is easily verified that

$$m_1 = bT,$$

$$m_2 = (bT)^2 + \frac{2bT}{\gamma}[1 - C],$$

from which it follows that the variance of Z is

$$\text{var}\{Z\} = \frac{2bT}{\gamma}[1 - C].$$

Since ψ is the Laplace-Stieltjes transform of the distribution of Z over an interval of equilibrium, $\ln \psi$ is the cumulant generating function, and has the following simple form:

$$\begin{aligned}\ln \psi &= b\left[\frac{\zeta^2(e^{-(\zeta+\gamma)T} - 1)}{(\zeta + \gamma)^2} - \frac{\gamma\zeta T}{\zeta + \gamma}\right] \\ &= b\left[-\zeta T + \frac{\zeta^2 T}{\zeta + \gamma} - \frac{\zeta^2(1 - e^{-(\zeta+\gamma)T})}{(\zeta + \gamma)^2}\right].\end{aligned} \tag{10.9}$$

M is a linear function of Z, so we may obtain the cumulant generating function of M in accordance with Cramér[1] (p. 187). This function is

$$b\left[-\zeta + \frac{\zeta^2}{r + \zeta} - \frac{\zeta^2(1 - e^{-r}e^{-\zeta})}{(r + \zeta)^2}\right], \tag{10.10}$$

and depends only on b and r.

The mean and variance of M for an interval of equilibrium are respectively given by

$$E\{M\} = b,$$

$$\text{var}\,\{M\} = \frac{2b}{r}[1 - C], \qquad \text{with} \quad C = \frac{1 - e^{-r}}{r},$$

results which were first proved in Riordan.[11] A normal distribution having the mean and variance of M has the cumulant generating function

$$b\left[-\zeta + \frac{\zeta^2}{r} + \frac{\zeta^2(e^{-r} - 1)}{r^2}\right], \tag{10.11}$$

which is to be compared to (10.10). Since var $\{M\}$ goes to 0 as T approaches ∞, we may expect that a suitably normalized version of Z will be asymptotically normally distributed as T approaches ∞. The cumulant generating function of the normalized variable $(2bhT)^{-1/2}(Z - bT)$ is

$$\frac{\zeta^2}{\zeta\left(\frac{2}{aT}\right)^{1/2} + 2}\left[1 + \frac{\exp\left\{-\zeta\left(\frac{2b}{r}\right)^{-1/2} - r\right\} - 1}{\zeta\left(\frac{2b}{r}\right)^{-1/2} + r}\right],$$

which approaches $\zeta^2/2$ as $T \to \infty$. It follows that the normalized variable is asymptotically normal with mean 0 and variance 1, and that $(r/2b)^{\frac{1}{2}}(M - b)$ is also asymptotically normal (0, 1).

APPENDIX A

PROOF THAT $\{n, A, H, Z\}$ IS SUFFICIENT.

We observe the system during the interval (O, T), and gather the information specified in Section I, and summarized in Table I. From this information we can extract four sets of numbers, described as follows:

- S_a the set of complete observed inter-arrival times, not counting the interval from the last arrival until T
- S_h the set of complete observed holding times
- S_1 the set of hang-up times for calls of category (i)
- S_4 the set of calling-times for calls of category (iv)

In addition, our data enable us to determine the following numbers:

- n the number $N(O)$ of calls found at the start of observation
- k the number of calls of category (iii); i.e., of calls which last throughout the interval (O, T)
- x the length of the time-interval between the last observed arrival and T

In view of the negative exponential distributions which have been assumed for the inter-arrival times and the holding-times, and in view of the assumptions of independence, we can write the likelihood of an observed sequence of events as

$$L = e^{-k\gamma T - ax} p_n \cdot \prod_{u \in S_a} a e^{-au} \cdot \prod_{z \in S_h} \gamma e^{-\gamma z} \cdot \prod_{w \in S_1} e^{-\gamma w} \cdot \prod_{y \in S_4} e^{-\gamma(T-y)},$$

so that

$$\ln L = -\gamma kT - ax + \ln p_n + A \ln a - \sum_{u \in S_a} au + H \ln \gamma - \sum_{z \in S_h} \gamma z - \sum_{w \in S_1} \gamma w - \sum_{y \in S_4} \gamma(T - y)$$

It is easily seen that the summations and the two initial terms can be combined into a single term, so that we obtain

$$\ln L = \ln p_n + A \ln a + H \ln \gamma - \gamma Z - aT.$$

This shows that L depends only on the statistics n, A, H, and Z; it follows that the information we have assumed can be replaced by the set of statistics $\{n, A, H, Z\}$, and that these are sufficient for estimation based on that information.

The likelihood is sometimes defined without reference to the initial state, by leaving the factor p_n out of the expression for L. Strictly speaking, this omission defines the conditional likelihood for the observed sequence, conditional on starting at n. We use the notation:

$$L_c = \frac{L}{p_n}.$$

A definition of likelihood as L_c has been used by Moran.[7] Clearly

$$\ln L_c = A \ln a + H \ln \gamma - \gamma Z - aT.$$

Appendix B

UNCONDITIONAL MAXIMUM LIKELIHOOD ESTIMATES

The definition of likelihood as L leads to complicated results which are of theoretical rather than practical interest. For this reason these results have been relegated to an appendix.

The results of setting $\partial/\partial\gamma \ln L$ and $\partial/\partial a \ln L$ equal to zero lead, respectively, to the likelihood equations

$$a - \gamma(n - H) - \gamma^2 Z = 0,$$

$$\gamma n - a + \gamma A - a\gamma T = 0.$$

Considered as a system of equations for γ and a, this pair has the non-negative roots

$$\hat{\gamma} = \frac{H - n - M + \{(H - n - M)^2 + 4MK\}^{1/2}}{2Z},$$

$$\hat{a} = \frac{K}{T} - \hat{\gamma} M.$$

These are the unconditional maximum likelihood estimators for γ and a. Although $\hat{a}_c$ depended only on A and T, and $\hat{\gamma}_c$ only on H and Z, the unconditional estimators depend on all of n, A, H, Z, and T. We may obtain a maximum unconditional likelihood estimator for b as well, either by considering L to be a function of b and γ, or from general properties of maximum likelihood estimators. Since $b = a/\gamma$, we expect that $\hat{b} = \hat{a}/\hat{\gamma}$, as can be verified by an argument similar to that used above for $\hat{a}$ and $\hat{\gamma}$.

The estimators $\hat{a}$, b, and $\hat{\gamma}$ obtained in this Appendix may turn out to be useful in practice, but their complicated dependence on the sufficient statistics n, A, H, and Z makes a study of their statistical properties difficult. As a first step along such a study, we have derived the generating function of the joint distribution of the sufficient statistics in Appendix C. The greater simplicity of the conditional estimators of Section VI makes it possible to study their statistical properties. This fact gives them a practical ascendancy over the unconditional estimators, even though the latter may be more efficient statistically by dint of using all the information available in an observation.

Appendix C

THE JOINT DISTRIBUTION OF $N(t)$, n, A, H, AND Z

By methods already used in Section X one can obtain a generating function for the joint distribution of all the random variables n, $N(t)$, A, H, and Z. Let

$$\Phi = E\{x^{N(t)} w^A u^H e^{-\zeta Z}\}.$$

Then Φ satisfies the differential equation

$$\frac{\partial \Phi}{\partial t} + [\zeta x + \gamma x - \gamma u] \frac{\partial \Phi}{\partial x} = a(wx - 1)\Phi,$$

whose solution has the form

$$\Phi = R\{[\zeta x + \gamma x - \gamma u] e^{-(\zeta+\gamma)t}\} \cdot \exp\left(\frac{aw[\zeta x + \gamma x - \gamma u][1 - e^{-(\zeta+\gamma)t}]}{(\zeta + \gamma)^2} + \frac{a\gamma wut}{\zeta + \gamma} - at \right),$$

where the function R is determined by the initial distribution $\{p_n\}$ through the relation

$$R\{\xi\} = \sum_{n \geqq 0} p_n \left[\frac{\xi + \gamma u}{\zeta + \gamma} \right]^n.$$

From these results it follows that the generating function

$$E\{z^n x^{N(T)} w^A u^H e^{-\zeta Z}\}$$

is given by

$$\sum_{n \geqq 0} p_n z^n \left(\frac{(\zeta x + \gamma x - \gamma u) e^{-(\zeta+\gamma)T} + \gamma u}{\zeta + \gamma} \right)^n \cdot \exp\left(\frac{aw(\zeta x + \gamma x - \gamma u)[1 - e^{-(\zeta+\gamma)T}]}{(\zeta + \gamma)^2} + \frac{a\gamma wuT}{\zeta + \gamma} - aT \right).$$

If $\{p_n\}$ forms the stationary distribution, this reduces to

$$\exp\left[b\left(\frac{z(\zeta x + \gamma x - \gamma u) e^{-(\zeta+\gamma)T} + \gamma uz}{\zeta + \gamma} - 1 \right) + \frac{aw(\zeta x + \gamma x - \gamma u)[1 - e^{-(\zeta+\gamma)T}]}{(\zeta + \gamma)^2} + \frac{a\gamma wuT}{\zeta + \gamma} - aT \right].$$

If, in this last expression, we let x approach 1, z approach 1, and u approach 1, we obtain

$$\exp\left[\left(1 - \frac{\gamma w}{\zeta + \gamma}\right)\left(\frac{b\zeta(e^{-(\zeta+\gamma)T} - 1)}{\zeta + \gamma} - aT\right)\right] \tag{C}$$

as the generating function $E\{w^A e^{-\zeta Z}\}$ for an interval of equilibrium. Alternately, if instead we let x approach 1, z approach 1, and w approach 1, we obtain (C) with u substituted for w; this implies the not-surprising result that for an interval of equilibrium, the two-dimensional variables $\{A, Z\}$ and $\{H, Z\}$ have the same distribution. From this and (C) it follows that for equilibrium (O, T), (i) A and H both have a Poisson distribution with mean aT, and (ii) the estimators $\hat{h}_c$ and $\hat{h}_2$ have the same distribution.

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