Self-Timing Regenerative Repeaters

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In self-timing regenerative repeaters, a timing wave for control in pulse regeneration is derived from the binary pulse train at each repeater with the aid of a resonant circuit tuned to the pulse repetition frequency. The timing wave can be made to exercise complete control in retiming of pulses independent of the received pulse train, or it can be combined with the received pulse train to provide partial retiming. The timing principles are discussed here for a particular type of self-timed regenerative repeater invented by Wrathall, in which a timing wave derived from either the received or the regenerated pulse train is combined in a particular way with the received pulse train. The regeneration characteristics of such repeaters as determined by various design parameters are investigated, together with the cumulation of timing deviations in repeater chains and the circuit requirements that must be met to insure satisfactory performance.

INTRODUCTION

Pulse transmission systems employing binary codes, such as PCM, have two inherent properties that are desirable from the standpoint of avoiding excessive transmission impairments by noise and other imperfections in the transmission medium. For one thing binary pulse codes permit substantial transmission distortion of pulses within certain tolerable limits with negligible degradation of received signals. For another, regenerative repeaters can be used at intervals along a route to prevent accumulation of transmission distortion of pulses from various sources, so that virtually the entire allowable distortion can be permitted in each link or repeater section.

The above desirable properties are secured in exchange for increased channel bandwidth, and can be used to full advantage in applications of binary pulse systems to such transmission media as radio and wave guides, where transmission is at such high frequencies that increased channel bandwidth does not entail increased attenuation. In wire circuits, however, where baseband transmission is the more economical method, attenuation increases nearly in proportion to the square root of the channel bandwidth. For this reason, rather short repeater spacings may be required for binary pulse systems, so that for economical applications to wire circuits it is imperative to have reliable regenerative repeaters of simple design.

In their principle of operation regenerative repeaters are by nature more complicated than ordinary repeaters. In addition to providing gain to off-set attenuation in the transmission medium, as in ordinary repeaters, they must also perform gating operations for sampling and regenerating the received pulse train. This, however, does not preclude the possibility that these operational principles can be implemented in repeater design by instrumentation that is simpler than required for ordinary repeaters.

The possibility of simple instrumentation resides partly in the circumstance that equalization circuitry for regenerative repeaters can be substantially simpler than for ordinary repeaters, owing to less exacting requirements on equalization. Furthermore, satisfactory performance in pulse regeneration can be achieved without very precise timing in sampling and regeneration of pulse trains. It is thus possible to secure nearly the same performance as for ideal regenerative repeaters by partial rather than complete exact retiming of pulse trains at each repeater. This facilitates simple gating arrangements for regeneration of pulses. Moreover, it permits a timing wave for control of gating operations to be derived from either the received or regenerating pulse trains with the aid of a simple resonant of circuit.

The simplicity of instrumentation permitted by these considerations is exemplified in a self-timed regenerative repeater for baseband pulses invented by L. R. Wrathall of Bell Telephone Laboratories. The circuitry of the repeater together with the results of tests on laboratory models are dealt with elsewhere[1] and not considered here. The purpose of this paper is an analysis of the timing principles underlying this type of repeater together with its regeneration characteristics as determined by various basic design parameters, on the assumption of ideal implementation of the timing principles by appropriate instrumentation. In the Wrathall repeater "quantized feed-back" is employed as a means of reducing the effect of low-frequency cut-off in transformers. Since this is not an essential feature of self-timing repeaters and has no direct bearing on the timing principles, it is disregarded herein.

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[1] L. R. Wrathall, Transistorized Binary Pulse Regenerator, B.S.T.J., **35**, pp. 1059–1084, Sept., 1956.

I REGENERATION AND RETIMING

1.0 General

In an ideal regenerative repeater the received pulse train is sampled at proper fixed intervals, to determine whether a pulse is present. The regenerated pulses transmitted into the next repeater section are all of the same shape and amplitude, independent of the shape of the input pulses. Thus pulse distortion from noise and other system imperfections is removed, provided the maximum distortion is held within proper limits. Errors in the form of pulses in place of spaces, or conversely, are encountered when these limits are exceeded. In a repeater chain there will be cumulation of errors in proportion to the number of repeater sections in tandem. However, the rate of errors in each section and thus in the whole chain can be limited by a relatively small increase in the signal-to-noise ratio of each section as the number of repeaters in tandem is increased. This increase in signal-to-noise ratio with increasing length of the repeater chain is much less than with ordinary nonregenerative repeaters. For this reason regenerative rather than ordinary repeaters are desirable, though not essential for systems employing binary codes.

An ideal regenerative repeater with the above features would entail rather complicated instrumentation for precise timing, sampling and pulse regeneration. With partial rather than complete exact retiming the repeaters can be simplified, in exchange for some sacrifice in performance, as shown later.

1.1 Regeneration Without Retiming

It would be possible to have a repeater in which pulses would be regenerated in amplitude and shape, but without retiming. Pulses would in this case be regenerated when the amplitude of the pulse exceeded a certain triggering level L. If the pulse shape is given by $P(t)$, this would occur at a time t_0 such that

$$P(t_0) = L. \tag{1.1}$$

This would permit simple instrumentation, since regenerated pulses would be triggered without separate sampling of the received pulse train. With this method, however, timing deviations in the regenerated pulses would result from transmission distortion of the received pulses by noise and other system imperfections. These timing deviations would cumulate in a repeater chain and cause a reduction in the tolerance of the repeaters to noise, such that the signal-to-noise ratio would have to be increased with the number of repeaters in tandem in the same way as for ordinary repeaters.

1.2 Regeneration with Complete Retiming

With complete retiming, the instants of pulse regeneration would be controlled by a periodic retiming wave, $R(t)$, with a fundamental period equal to the interval between pulses. The received pulse train would be sampled at instants when the retiming wave had a certain level L_s. The sampling instants t_0 would thus be given by

$$R(t_0) = L_s. \tag{1.2}$$

$R(t_0)$ would satisfy this equation for $t_0 = nT \pm \Delta T$, where T is the nominal interval between pulses, n is an integer and ΔT is a certain tolerable deviation from the desired sampling instants. Pulses would be regenerated provided $P(t_0) > L$ and would be omitted if $P(t_0) < L$.

With this method the timing deviations in regenerated pulses would be limited to $\pm\Delta T$, regardless of the timing deviations in received pulses. There would be no cumulation of timing deviations in a repeater chain. However, the tolerance of the repeaters to noise would be somewhat reduced by the timing deviations $\pm\Delta T$.

1.3 Regeneration with Partial Retiming

Partial retiming is obtained by a combination of the above two methods, by triggering regenerated pulses without sampling at instants t_0 determined by

$$P(t_0) + R(t_0) = L. \tag{1.3}$$

To permit regeneration without sampling and without a marked reduction in the tolerance of the repeaters to noise, the timing wave $R(t)$ must meet certain conditions illustrated in Fig. 1. One is that it must be a nearly periodic function as for complete retiming. The second condition is that $R(t)$ must be zero near the sampling points to obtain substantially the same tolerance to noise in the presence of a pulse as in the absence of a pulse. A third condition is that $R(t)$ must have substantial negative values between sampling points in order that the repeater be rather insensitive to noise between sampling points, as with complete retiming. It will be recognized that, in general, the maximum value of $R(t)$ need not necessarily be zero, as in the above illustration. It can be greater or smaller than zero, provided the triggering level is modified accordingly. A maximum value of zero is, however, convenient from the standpoint of instrumentation.

A limiting shape of retiming wave that would result in complete retiming, but without the need for special sampling is also illustrated in Fig. 1.

1.4 Derivation of Timing Wave from Pulse Train

As shown above, the retiming wave must be essentially periodic, with a fundamental frequency equal to the pulse repetition frequency $f = 1/T$, where T is the interval between pulses. The simplest form is a sinusoidal wave, which can be derived from the pulse train at repeaters with the aid of a narrow band-pass filter, such as a simple resonant circuit cen-

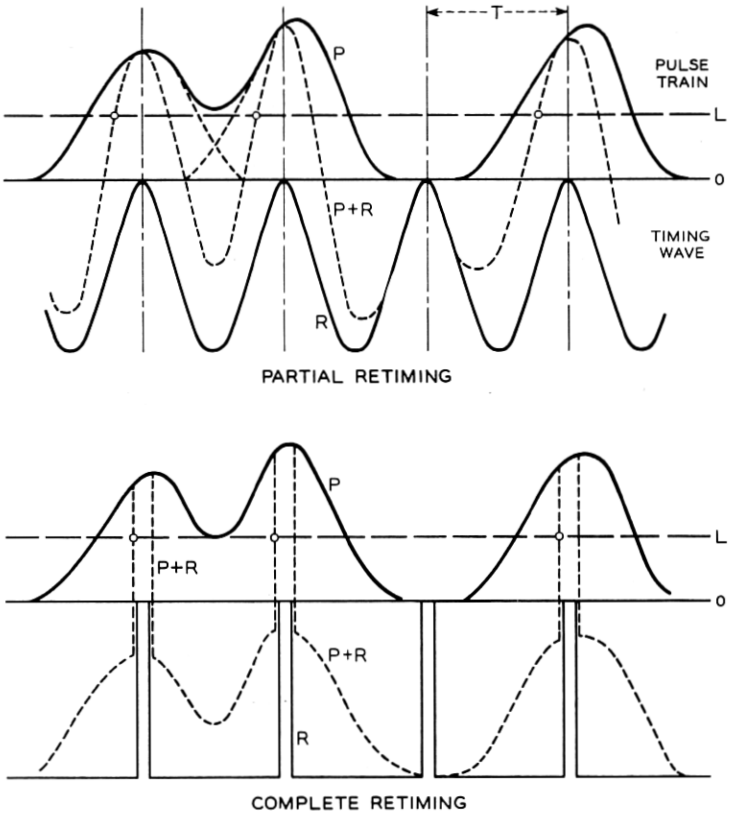

Fig. 1 — Principle of partial retiming method.

tered on the pulse repetition frequency. This possibility resides in the circumstance that a random "on-off" pulse train can be resolved into two components. One is an infinite sequence of pulses of the same polarity and equal amplitude, the other a sequence of randomly positive and negative polarity. The response of a resonant circuit to the first component is a steady state sinusoidal wave of the pulse repetition frequency. The second component gives rise to random variations in amplitude and phase, which in principle can be limited to any desired extent by limiting the band of the resonant circuit and the deviation in the resonant frequency from the pulse repetition frequency.

A principal feature of this method of "self-timing", aside from its simplicity, is that the timing wave becomes a slave of the pulse train. Thus, if there is a fixed delay in pulse regeneration at a repeater, the same delay is imparted to the timing wave derived from the pulse train at the next repeater. This prevents a cumulation of such fixed delays with respect to the timing wave, but not with respect to an absolute time scale; i.e., with respect to an ideal timing wave transmitted along the repeater chain and independent of the pulse train.

1.5 Self-Timed Repeaters with Partial Retiming

A timing wave derived from the pulse train with the aid of a resonant circuit can be used in conjunction with complete or partial retiming. With complete retiming, pulses could be regenerated at the zero points in the timing wave, and the effects of amplitude variations in the timing wave can thus be avoided. Timing deviations in the regenerated pulses would in this case depend only on phase deviations in the timing wave, caused partly by the component of randomly positive and negative polarity in the pulse train and partly by timing deviations in the pulse train from which the timing wave is derived.

With partial retiming the situation is more complex. Timing deviations in regenerated pulses in this case depend not only on amplitude and phase variations in the timing wave, but also on the regeneration characteristics of the repeaters.

1.6 Types of Timing Deviations

In a regenerated pulse train there will be fixed and random timing deviations. Of the latter there are three types. One is the timing deviation taken in relation to an exact timing wave with a period T equal to the nominal pulse interval. The second is the timing deviation taken in relation to the timing wave derived from the pulse train, which in itself will contain random deviations. The third type is random deviations in the interval of adjacent pulses. If the first type is held within tolerable limits, this will also be the case for the second and third types. For this reason only the first type is considered herein.

II REGENERATION CHARACTERISTICS WITH PARTIAL RETIMING

2.0 General

With partial retiming, there will be timing deviations in the regenerated pulses as a result of timing deviations, amplitude variations and distortion by noise of both the received pulses and the timing wave.

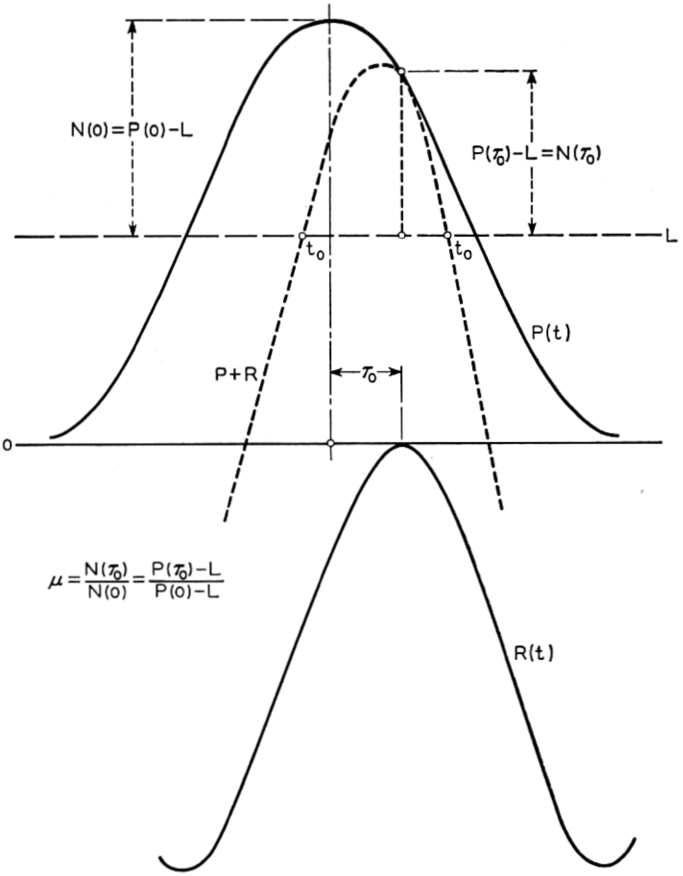

Fig. 2 — Reduction in tolerance to noise by displacement in timing wave.

The conversion of these variations into timing deviations in the regenerated pulses depends on certain relationships between the pulse train and the timing wave, discussed in the following sections.

2.1 Tolerance to Noise

From Fig. 2 it can be seen that if the timing wave is displaced by τ_0, the value of $P(t) + R(t - \tau_0)$ in the presence of a pulse exceeds the triggering level by a maximum amount

$$[P(t) + R(t - \tau_0) - L]_{\max} \cong [P(\tau_0) - L]. \tag{2.1}$$

It will be recognized that the right-hand side of this equation represents the tolerance to noise of negative amplitudes with instantaneous sampling at $t = \tau_0$, as in an ideal repeater with complete retiming.

With partial retiming, the tolerance to noise will be less than the above maximum value. However, it will be greater than the average of $P(t) + R(t - \tau_0) - L$ in the range where the latter difference is positive. Let it be assumed that it is smaller than the maximum by a factor k somewhat smaller than unity. The tolerance to noise with a displacement τ_0 in the timing wave is then smaller than without a displacement (i.e., $\tau_0 = 0$) by the factor

$$\mu = \frac{k[P(\tau_0) - L]}{k[P(0) - L]} = \frac{P(\tau_0) - L}{P(0) - L}. \tag{2.2}$$

The tolerance to noise will thus be reduced in a way similar to that for an ideal repeater with complete retiming. The absolute tolerance to noise will be less than for a repeater with complete retiming by a factor k somewhat smaller than unity, say in the order 0.8, corresponding to about 2 db.

2.2 Conversion of Timing Deviations

With partial retiming, timing deviations in received pulses and in the timing wave are converted into smaller deviations in regenerated pulses.

Let τ_p be a time displacement in a received pulse and τ_r in the timing wave, both in the positive direction. Pulses will then be regenerated at a time t_0' given by

$$P(t_0' - \tau_p) + R(t_0' - \tau_r) = L \tag{2.3}$$

where the minus signs are used since this corresponds to a displacement of P and R in the positive direction. Subtracting (1.3) from (2.3),

$$P(t_0' - \tau_p) - P(t_0) + R(t_0' - \tau_r) - R(t_0) = 0. \tag{2.4}$$

By adding and subtracting $P(t_0') + R(t_0')$ and rearranging terms, (2.4) can also be written

$$\begin{aligned}[P(t_0') - P(t_0)] &+ [R(t_0') - R(t_0)] \\ &= [P(t_0') - P(t_0' - \tau_p)] + [R(t_0') - R(t_0' - \tau_r)].\end{aligned} \tag{2.5}$$

For small values of τ_p and τ_r, such that $\delta_\tau = t_0' - t_0$ is sufficiently small, both sides of (2.8) can be represented in differential form as

$$\delta_\tau[P'(t_0) + R'(t_0)] = \tau_p P'(t_0) + \tau_r R'(t_0) \tag{2.6}$$

where $P'(t_0) = dP_0(t)/dt$ at $t = t_0$, and R' is correspondingly defined.

Equation (2.9) can be written in the form

$$\delta_\tau = p_\tau \tau_p + r_\tau \tau_r \tag{2.7}$$

where

$$p_t = \frac{P'(t_0)}{P'(t_0) + R'(t_0)}, \qquad r_\tau = \frac{R'(t_0)}{P'(t_0) + R'(t_0)}, \tag{2.8}$$

and

$$p_\tau + r_\tau = 1. \tag{2.9}$$

With random uncorrelated displacements of rms values $\bar{\tau}_p$ and $\bar{\tau}_r$, the rms value of δ_τ is

$$\underline{\delta}_\tau = (p_\tau^2 \bar{\tau}_p^2 + r_\tau^2 \bar{\tau}_r^2)^{1/2} \tag{2.10}$$

Equation (2.9) and (2.10) give the timing deviations in regenerated pulses in terms of the deviations τ_p and τ_r in the received pulses and in the timing wave. To limit timing deviations in the regenerated pulses, it is necessary to make p_τ and the product $r_\tau \tau_r$ small. This will entail the use of a timing wave comparable in amplitude to that of the pulses, or greater, in conjunction with a small timing deviation τ_r in the timing wave.

2.3 Conversion of Amplitude Variations Into Timing Deviations

With partial retiming there is a conversion of amplitude variations in the received pulses and in the timing wave into timing deviations in the regenerated pulses.

Let the pulses have an amplitude variation a_p and the timing wave a_r expressed as fractions of the normal values. Pulses will then be regenerated at a time t_0' given by

$$(1 + a_p)P(t_0') + (1 + a_r)R(t_0') = L. \tag{2.11}$$

Subtracting (1.3) from (2.11),

$$[P(t_0') - P(t_0)] + [R(t_0') - R(t_0)] = -a_pP(t_0') - a_rP(t_0').$$

For small values of a_p and a_r, such that $\delta_a = t_0' - t_0$ is sufficiently small, the same procedure as in Section 2.2 gives

$$\delta_a = (p_a a_p + r_a a_r), \tag{2.12}$$

and

$$p_a = \frac{-P(t_0)}{P'(t_0) + R'(t_0)}, \qquad r_a = \frac{-R(t_0)}{P'(t_0) + R'(t_0)}. \tag{2.13}$$

For uncorrelated variations of rms amplitude $\underline{a}_p$ and $\underline{a}_r$ the corresponding rms timing deviation is

$$\underline{\delta}_a = (p_a^2 \underline{a}_p^2 + r_a^2 \underline{a}_r^2)^{1/2}. \tag{2.14}$$

Equations (2.12) and (2.14) give the timing deviations in regenerated pulses resulting from amplitude variations in the pulses and in the timing wave.

2.4 Resultant Timing Deviations in Regenerated Pulses

For small variations in the pulses and in the timing wave as considered previously, the resultant timing deviation in a particular regenerated pulse is

$$\Delta = \delta_\tau + \delta_a. \tag{2.15}$$

Considering a large number of pulses, the resultant rms timing deviation in terms of the rms deviation in the received pulses and in timing wave is

$$\underline{\Delta} = (\underline{\delta}_\tau^2 + \underline{\delta}_a^2)^{1/2}. \tag{2.16}$$

These expressions can also be written

$$\Delta = \Delta_p + \Delta_r, \tag{2.17}$$

$$\underline{\Delta} = (\underline{\Delta}_p^2 + \underline{\Delta}_r^2)^{1/2}, \tag{2.18}$$

$$\Delta_p = p_\tau \tau_p + p_a a_p,$$

$$\underline{\Delta}_p^2 = p_\tau^2 \bar{\tau}_p^2 + p_a^2 \underline{a}_p^2, \tag{2.19}$$

$$\Delta_r = r_\tau \tau_r + r_a a_r,$$

$$\underline{\Delta}_r^2 = r_\tau^2 \bar{\tau}_r^2 + r_a^2 \underline{a}_r^2. \tag{2.20}$$

III ILLUSTRATIVE REGENERATION CHARACTERISTICS

3.0 General

In this section the general equations given in the preceding sections are applied to a particular case, in order to obtain specific expressions for the regeneration characteristics and illustrative curves, as an aid to further analysis. The particular case selected for illustration approximates the conditions in experimental Wrathall repeaters, and may be regarded as an idealized model of such a repeater, in which certain effects to be discussed later are ignored.

3.1 Pulse Shape

It will be assumed that the pulses are transmitted at intervals T and that the shape of the received pulses after equalization is given by:

$$P(t) = \frac{1}{2}\left[1 + \cos\frac{\pi}{\eta}\frac{t}{T}\right]. \tag{3.1}$$

This is the familiar "raised cosine" type of pulse. With $\eta = 1$ the pulse width is the maximum that can be tolerated without intersymbol interference. With $\eta = \frac{3}{4}$, the amplitude of a pulse train at a point midway between two success pulses is equal to half the peak amplitude of a pulse. The latter assumption will be made here, for reasons discussed later.

3.2 Retiming Wave

The retiming wave is assumed to be given by

$$R(t) = -\frac{1}{2}\cos\psi\left[1 - \cos\left(2\pi\frac{t}{T} - \psi\right)\right]. \tag{3.2}$$

This type of retiming wave can be obtained if a sinusoidal wave of the pulse repetition frequency $f = 1/T$ is applied to a resonant circuit to reduce distortion of the timing wave by noise. The resonant circuit would have a nominal resonant frequency $f = 1/T$, but because of mistuning it would actually be f_0. The output of the resonant circuit after appropriate adjustment of amplitude would be of the form [Appendix I, equation (2)]:

$$R_0(t) = \frac{1}{2}\cos\psi\cos\left(2\pi\frac{t}{T} - \psi\right), \tag{3.3}$$

where ψ is the phase shift of the resonant circuit at the frequency f, given by:

$$\tan \psi = Q\left(\frac{f}{f_0} - \frac{f_0}{f}\right), \tag{3.4}$$

and Q is the loss constant of the resonant circuit. If the peaks of the wave given by (3.3) are held at zero potential, a retiming wave as given by (3.2) is obtained. This type of retiming wave can also be obtained by applying an infinite sequence of rectangular pulses of equal amplitudes with spacing T to a resonant circuit.

3.3 Triggering Instants

With a pulse shape and retiming wave as assumed above, the resultant wave is given by

$$P(t) + R(t) = \frac{1}{2}\left[1 + \cos\frac{\pi}{\eta}\frac{t}{T}\right] - \frac{\cos\psi}{2}\left[1 - \cos\left(2\pi\frac{t}{T} - \psi\right)\right]. \tag{3.5}$$

This wave is shown in Fig. 3 for $\psi = 0$ and $\pm 60°$. For $\psi = \pm 90°$ the retiming wave disappears, so that the combined wave is $P(t)$.

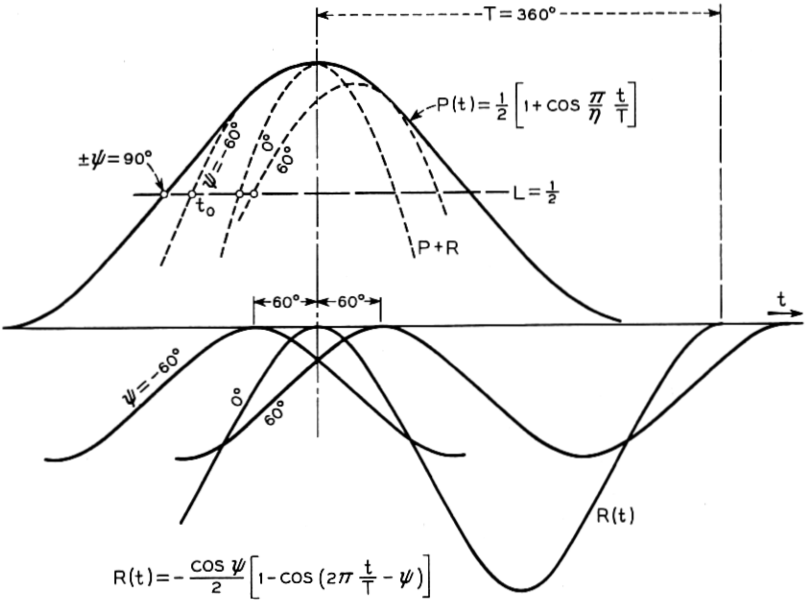

Fig. 3 — Illustrative example of pulse shape and retiming wave.

The triggering instants t_0 are obtained from the relation

$$P(t_0) + R(t_0) = L. \tag{3.6}$$

With complete retiming the optimum performance, with positive and negative noise amplitudes of equal probabilities, is obtained with a triggering level $\frac{1}{2}$. With partial retiming, optimum performance is obtained with a somewhat lower triggering level, but this is of secondary importance in connection with the present analysis. For this reason $L = \frac{1}{2}$ is assumed, in which case the following equation is obtained for determination of t_0:

$$\cos\frac{\pi}{\eta}\frac{t_0}{T} - \cos\psi\left[1 - \cos\left(2\pi\frac{t_0}{T} - \psi\right)\right] = 0\,. \tag{3.7}$$

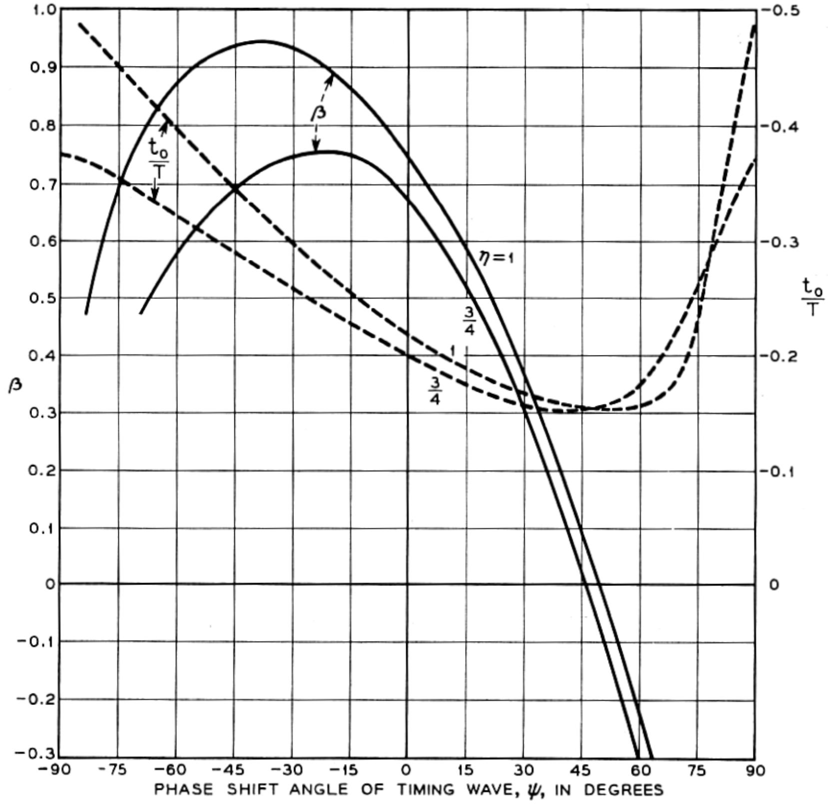

Fig. 4 — Triggering times versus phase shifts in timing wave.

TABLE I — VALUES OF t_0/T FOR $\eta = \frac{3}{4}$ AND $\eta = 1$

ψ	$-90°$	$-60°$	$-30°$	0	30°	60°	90°
$\eta = \frac{3}{4}$	−0.375	−0.322	−0.258	−0.198	−0.156	−0.17	−0.375
$\eta = 1$	−0.50	−0.391	−0.293	−0.215	−0.170	−0.15	−0.50

TABLE II — VALUES OF p_τ AND r_τ FOR $\eta = \frac{3}{4}$

ψ	$-90°$	$-60°$	$-30°$	0	30°	60°	90°
p_τ	1	0.61	0.43	0.32	0.32	0.50	1
r_τ	0	0.39	0.57	0.68	0.68	0.50	0

This equation is satisfied for the values of t_0/T given in Table I. The values of t_0/T are also shown in Fig. 4 as a function of ψ.

3.4 Conversion Factors for Time Deviations

The conversion factors defined by (2.8) become:

$$p_\tau = \frac{1}{D} \sin \frac{\pi}{\eta} \frac{t_0}{T} = 1 - r_\tau , \tag{3.8}$$

$$r_\tau = \frac{1}{D} 2\eta \cos \psi \sin \left(2\pi \frac{t_0}{T} - \psi \right), \tag{3.9}$$

and

$$D = \sin \frac{\pi}{\eta} \frac{t_0}{T} + 2\eta \cos \psi \sin \left(2\pi \frac{t_0}{T} - \psi \right), \tag{3.10}$$

where t_0/T has the values given previously as a function of ψ.

For various values of ψ, the factors for $\eta = \frac{3}{4}$ are given in Table II and in Fig. 5.

3.5 Conversion Factors for Amplitude Into Time Deviations

The conversion factors defined by (2.13) become

$$p_a = -\frac{T\eta}{\pi} \frac{1}{D} \left[1 + \cos \frac{\pi}{\eta} \frac{t_0}{T} \right], \tag{3.11}$$

and

$$r_a = \frac{T\eta}{\pi} \frac{1}{D} \cos \psi \left[1 - \cos \left(2\pi \frac{t_0}{T} - \psi \right) \right], \tag{3.12}$$

where D and t_0/T are defined as before.

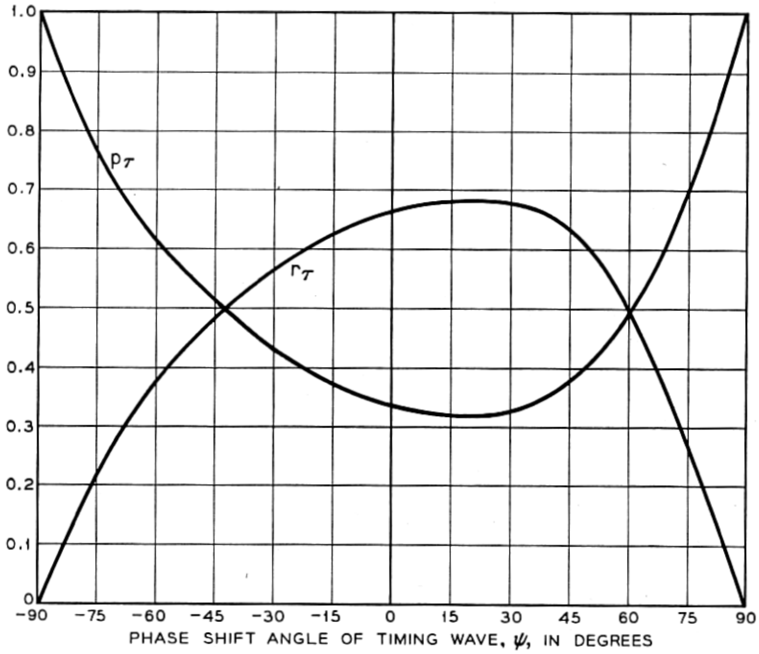

Fig. 5 — Conversion of timing deviations in received pulses and in timing wave into timing deviations in regenerated pulses, for pulse shapes and timing waves shown in Fig. 3. Timing deviations in regenerated pulses in relation to timing deviation t_p in pulses and t_r in retiming wave is $p_t t_p + r_t t_r$.

For various values of ψ the factors for $\eta = \frac{3}{4}$ are given in Table III and in Fig. 6.

3.6 Correlated Amplitude and Time Deviations

The amplitude and time deviations in the pulses are generally uncorrelated, but this does not always apply to the timing wave. In particular, if a deviation τ_r in the timing wave is the result of a change in the phase ψ, it will be accompanied by a given amplitude variation. A change in phase by $\Delta\psi$ is related to the corresponding time deviation τ_r by

$$\Delta\psi = \frac{2\pi}{T} \tau_r . \tag{3.13}$$

TABLE III — VALUES OF p_a/T AND r_a/T FOR $\eta = \frac{3}{4}$

ψ	−90°	−60°	−30°	0	30°	60°	90°
p_a/T	−0.24	−0.185	−0.175	−0.19	−0.22	−0.325	−0.24
r_a/T	0	0.035	0.055	0.072	0.106	0.14	0

With this change in phase, the factor $\cos \psi$ of (3.2) is modified to

$$\begin{aligned} \cos(\psi + \Delta\psi) &= \cos\psi \cos\Delta\psi - \sin\psi \sin\Delta\psi, \\ &\cong \cos\psi - \frac{2\pi}{T}\tau_r \sin\psi \end{aligned} \tag{3.14}$$

where the approximation applies for small values of ψ. The amplitude variation resulting from the above change in phase is accordingly

$$a_r = -\tau_r \frac{2\pi}{T} \sin\psi. \tag{3.15}$$

Considering both the time deviation τ_r and the corresponding amplitude variation a_r, the resultant time deviation in regenerated pulses is in accordance with (2.20)

$$\Delta_r = r_\tau \tau_r + r_a a_r. \tag{3.16}$$

The resultant equation can be written

$$\Delta_r = \beta \tau_r \tag{3.17}$$

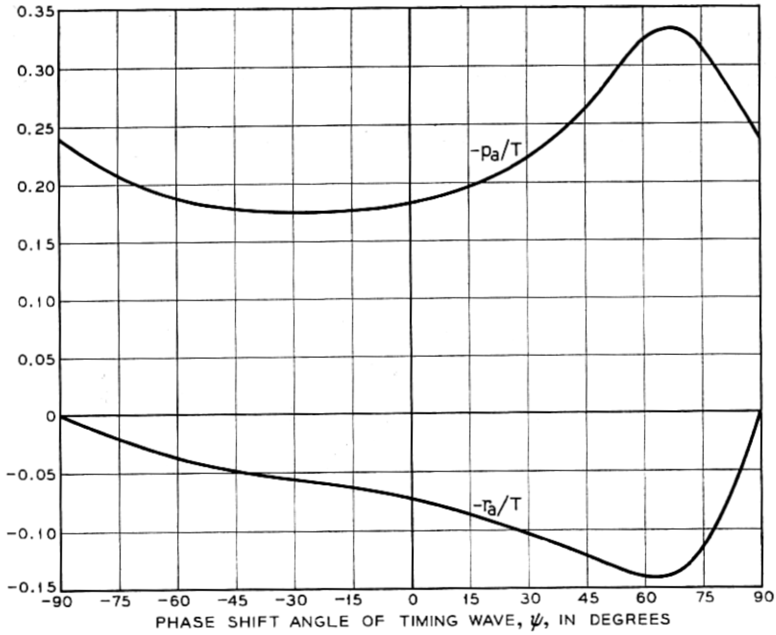

Fig. 6 — Conversion of amplitude variations in received pulses and in timing wave into timing deviations in regenerated pulses, for pulse shapes and timing waves shown in Fig. 3. Timing deviations in regenerated pulses for amplitude variations a_p and a_r in received pulses and in timing wave is $p_a a_p + r_a a_r$.

where

$$\beta = \frac{2\eta \cos \psi}{D} \left\{ \sin \left(2\pi \frac{t_0}{T} - \psi \right) + \sin \psi \left[1 - \cos \left(2\pi \frac{t_0}{T} - \psi \right) \right] \right\} \tag{3.18}$$

and D and t_0/T are defined as before.

The factor β indicates the time deviation in regenerated pulses in relation to the time deviation τ_r in the timing wave which results from a phase shift $\Delta\psi$ as given by (3.13). It may be regarded as a timing feedback factor that is of interest in connection with timing from regenerated pulses as discussed later. The factor β is shown in Fig. 4 for $\eta = \frac{3}{4}$ and $\eta = 1$.

3.7 Reduction in Tolerance to Noise by Timing Deviations

When the pulse shape is given by (3.1) and the timing wave is displaced by τ_0, the tolerance to noise is in accordance with (2.2) reduced by the factor

$$\mu = \frac{\frac{1}{2}\left(1 + \cos \frac{\pi}{\eta} \frac{\tau_0}{T}\right) - \frac{1}{2}}{\frac{1}{2}\left(1 + \cos \frac{\pi}{\eta} \frac{0}{T}\right) - \frac{1}{2}} = \cos \frac{\pi}{\eta} \frac{\tau_0}{T}. \tag{3.19}$$

For a phase displacement ψ,

$$\tau_0 = T\psi/2\pi, \tag{3.20}$$

and

$$\mu = \cos \frac{\psi}{2\eta}. \tag{3.21}$$

For $\eta = \frac{3}{4}$, the factor μ and the corresponding reduction in the tolerance to noise in db are as follows:

$\psi =$	0	±30°	±45°	±60°	±90°
$\mu =$	1	0.94	0.866	0.766	0.5
$\mu_{db} =$	0	0.5	1.2	2.3	6

IV DERIVATION OF TIMING WAVE FROM PULSE TRAIN

4.0 General

The retiming wave $R(t)$ must have a fixed relation to the received pulses, with certain tolerable fixed and random deviations to be considered later. Such a timing wave can be derived from the pulse train with the aid of a sufficiently narrow band-pass filter, the simplest form of which is a resonant circuit consisting of a coil and capacitor in series or in parallel.

A train of rectangular "on-off" pulses is shown in Fig. 7 as it would appear at the output of a regenerative repeater and at the input of the next repeater, (dotted) with uniform intervals T between sampling points.

As indicated in Fig. 7, the pulse train can be regarded as being made up of two components. One of these is an infinite sequence of pulses of one polarity, the other an infinite sequence of randomly positive and negative polarity.

It will be recognized that the first of the above components at the output has a fundamental frequency equal to the pulse repetition frequency, $f = 1/T$, and the forced response of a resonant circuit to this component will be the pulse repetition frequency, regardless of any imperfections in tuning. In order that this frequency be present in the received pulse train, it is necessary that the spectrum of the received pulses extend beyond the pulse repetition frequency, so that there will be a ripple in a long sequence of received pulses of one polarity, as indicated in the illustration.

The second random component of the pulse train will have a frequency spectrum that is nearly uniform over the band of the tuned circuit, and which will vary in amplitude depending on the composition of the pulse train. The response of the tuned circuit to this component is thus rather complex, and must be treated on an approximate statistical basis. It will consist of an almost periodic wave with random amplitude and phase modulation, and with mean frequency equal to the resonant frequency.

Owing to the presence of the second component, there will be a variation with time in the amplitude and phase of the response of a mistuned resonant circuit, and resultant deviations in timing. The regenerated pulses will thus not be uniformly spaced, but will in general have random deviations from the desired exact positions. Such deviations can be created by superposing on a train with uniform spacing a random dipulse train, as indicated in Fig. 7. The resonant circuit response to this dipulse train would be expected to be smaller than to the random amplitude component of the pulse train. It may be regarded as a third component representing a second order effect resulting from the second component.

In the Appendix, this method of superposition has been used as a basis of an analysis of a resonant circuit response to a random binary pulse train. This problem has also been dealt with by somewhat different methods in prior unpublished work by W. R. Bennett and J. R. Pierce, both of Bell Telephone Laboratories.

In this analysis it is assumed that the regenerated pulses are of sufficiently short duration to be regarded as impulses. The response of the resonant circuit to the second and third components above, when taken in relation to that for the first component will, however, remain very

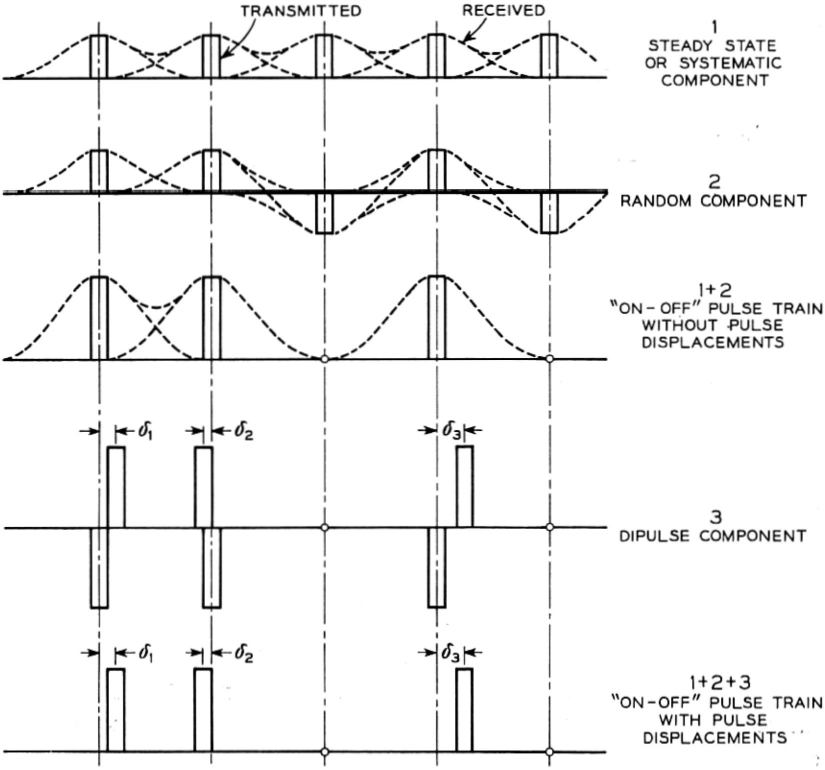

Fig. 7 — Resolution of "on-off" pulse train with timing deviations into systematic component (1), random component (2), and time displacement component (3).

nearly the same for other pulse shapes, provided the frequency spectrum of the pulses can be regarded as approximately constant over the important portion of the band of the resonant circuit. This approximation is legitimate for resonant circuits with a loss constant Q and pulse shapes at the input of repeaters as considered here.

4.1 Resonant Circuit Response to Steady State Component

The first component consists of an infinite sequence of impulses of amplitude $\frac{1}{2}$ and all of the same polarity, at intervals T. This sequence has a fundamental frequency $f = 1/T$. When it impinges on a resonant circuit with resonant frequency $f_0 = f - \Delta f$ and loss constant Q, the response is of the form

$$A_s(t) = \cos \psi \cos (\omega t - \psi), \tag{4.1}$$

and

$$\tan \psi = Q(f/f_0 - f_0/f) \cong 2Q \frac{\Delta f}{f}. \tag{4.2}$$

The response is thus a steady state sinusoidal wave of frequency f displaced from the fundamental component of the input wave by the phase shift ψ and reduced in amplitude by $\cos \psi$. This is the phase shift and amplitude reduction of the resonant circuit at the frequency f when the resonant frequency is f_0.

4.2 Resonant Circuit Response to Random Signal Component

The second component consists of an infinite random sequence of impulses of amplitude $\pm\frac{1}{2}$, at intervals T. The response of the resonant circuit to this component will be a randomly fluctuating wave $A_r(t)$ of mean value 0. The maximum positive amplitude is obtained when all impulses of the second component are positive and is $A_r(t) = A_s$. The maximum negative amplitude is $A_r(t) = -A_s$. Owing to the presence of this component the total output of the resonant circuit $A_s + A_r(t)$ can thus fluctuate between the limits 0 and $2A_s$, but the actual fluctuations of significant probability will be smaller.

The above fluctuations can be resolved into a component in phase with the steady state response given by (4.1) and another component at quadrature with the steady state timing wave. The rms values of these components taken in relation to the amplitude of the steady state wave are

$$\underline{a}_r' = \underline{A}_r'/A_s = \left(\frac{\pi}{2Q}\right)^{1/2} [1 - \psi^2/2]^{1/2} \frac{1}{\cos \psi}, \tag{4.3}$$

and

$$\underline{a}_r'' = \underline{A}_r''/A_s = \left(\frac{\pi}{4Q}\right)^{1/2} \frac{|\psi|}{\cos\psi}. \tag{4.4}$$

These relations apply for small values of ψ and for $\pi/Q \ll 1$.

The resultant rms amplitude variation in the timing wave is $\underline{a}_r = \underline{a}_r'$ as given by (4.3).

The rms phase error $\bar{\varphi}_r$ resulting from the quadrature component $\underline{a}_r''$ is given by

$$\tan \bar{\varphi}_r \cong \bar{\varphi}_r = a_r''. \tag{4.5}$$

The corresponding rms time deviation is $(T/2\pi)\bar{\varphi}_r$ or

$$\underline{\delta}_r = \frac{T}{2\pi}\left(\frac{\pi}{4Q}\right)^{1/2} \frac{|\psi|}{\cos\psi}. \tag{4.6}$$

With regard to the probability of exceeding the above rms values by various factors the normal law can probably be invoked with reasonable accuracy. As mentioned before, the maximum possible amplitudes are $\hat{A}_r(t) = \pm A_s$ which would correspond to a peak factor $(2Q/\pi)^{1/2}$. With $Q = 100$, the factor is about 8, while with $Q = 1000$ it is about 25. Based on the normal law the probability of exceeding the rms value by a factor of 4 is about 5×10^{-5}, and by a factor of 5, about 10^{-7}. The normal law would be expected to apply, since the limiting peak values are substantially greater than the peak values expected with significant probabilities.

4.3 Resonant Circuit Response to Pulse Displacements

Because of the random components given by (4.3) and (4.4), the timing wave will contain small random amplitude and phase deviations from a sinusoidal wave represented by (4.1). This will result in small random deviations in the positions of regenerated pulses triggered from the timing wave, which is represented by the third component shown in Fig. 7. When the rms deviation in the pulse positions is $\underline{\delta}$, there will be an additional random quadrature component in the timing wave which, when taken in relation to the steady state component, is given by

$$\underline{a}_\delta'' = \underline{A}_\delta''/A_s = \omega\underline{\delta}\left(\frac{\pi}{Q}\right)^{1/2}. \tag{4.7}$$

The corresponding rms phase deviation is given by

$$\bar{\varphi}_\delta \cong \underline{a}_\delta''. \tag{4.8}$$

The resultant rms time deviation is $(T/2\pi)\bar{\varphi}_\delta$ or

$$\delta_\delta = \delta\underline{\alpha}, \tag{4.9}$$

and

$$\underline{\alpha} = (\pi/Q)^{1/2}. \tag{4.10}$$

The above factor $\underline{\alpha}$ applies to a single resonant circuit. When the rms timing deviations represented by (4.9) are present in the regenerated pulse train, the rms deviation at the output of the second resonant circuit is

$$\delta_{\delta,2} = \delta\underline{\alpha}_1\underline{\alpha}_2\,,$$

where

$$\underline{\alpha}_1 = \underline{\alpha}.$$

With n resonant circuits in tandem,

$$\delta_{\delta,n} = \delta\underline{\alpha}_1\underline{\alpha}_2\underline{\alpha}_3 \cdots \underline{\alpha}_n\,. \tag{4.11}$$

The factors α_n are given by

$$\underline{\alpha}_1 = \underline{\alpha} = (\pi/Q)^{1/2}, \tag{4.12}$$

$$\underline{\alpha}_j = \left(1 - \frac{1}{2(j-1)}\right)^{1/2} \qquad j \geqq 2, \tag{4.13}$$

$$\underline{\alpha}_2 = (1 - \tfrac{1}{2})^{1/2},$$

$$\underline{\alpha}_3 = (1 - \tfrac{1}{4})^{1/2},$$

$$\underline{\alpha}_4 = (1 - \tfrac{1}{6})^{1/2}, \text{ etc.}$$

$$[\underline{\alpha}_1\cdot\underline{\alpha}_2\cdot\underline{\alpha}_3 \cdots \underline{\alpha}_n]^2 = \underline{\alpha}^2\,\frac{1\cdot3\cdot5\cdots[2(n-1)-1]}{2\cdot4\cdot6\cdots2(n-1)} \tag{4.14}$$

$$= \underline{\alpha}^2\,\frac{(2n)!}{2^{2n}(n!)^2} \tag{4.15}$$

$$= \underline{\alpha}^2\left(\frac{1}{\pi n}\right)^{1/2} \qquad \text{when} \qquad n \gg 1. \tag{4.16}$$

The factors $\underline{\alpha}_j$ for $j \geqq 2$ represent the reduction in timing deviations resulting from the reduction in bandwidth as resonant circuits are added in tandem. If resonant circuits with a narrow flat pass-band were used, the bandwidth of any number of resonant circuits in tandem would be the same as for a single resonant circuit. In this case $\underline{\alpha}_2 = \underline{\alpha}_3 = \underline{\alpha}_n = 1$.

4.4 Deviations in Timing Wave

The timing wave derived from an "on-off" pulse train with the aid of a resonant circuit will in accordance with the expressions given in the previous sections contain three types of amplitude and timing deviations.

The first type is a fixed amplitude reduction by a factor a_0 and a fixed time deviation τ_0 given by

$$a_0 = \cos \psi, \tag{4.17}$$

and

$$\tau_0 = \frac{T}{2\pi} \psi, \tag{4.18}$$

where ψ is given by (4.2).

The second type is a random amplitude and time deviation resulting from the random amplitude component of the pulse train, which have rms values

$$a_r \cong \left(\frac{\pi}{2Q}\right)^{1/2} [1 - \psi^2/2]^{1/2} \frac{1}{\cos \psi}, \tag{4.19}$$

and

$$\underline{\delta}_r \cong \frac{T}{2\pi} \left(\frac{\pi}{4Q}\right)^{1/2} \frac{|\psi|}{\cos \psi}. \tag{4.20}$$

The third type is a random amplitude and time deviation resulting from random timing deviations $\bar{\tau}_p = \underline{\delta}$ in the pulse train. The amplitude variation can be disregarded and the rms time deviation is

$$\underline{\delta}_\delta = \underline{\alpha} \bar{\tau}_p, \qquad \underline{\alpha} = \left(\frac{\pi}{Q}\right)^{1/2}. \tag{4.21}$$

The total rms amplitude variation is accordingly given by (4.19). The total rms timing deviation obtained by combining (4.20) and (4.21) is

$$\bar{\tau}_r = \underline{\delta}_r^{\,2} + \underline{\alpha}^2 \bar{\tau}_p^{\,2})^{1/2}. \tag{4.22}$$

The expressions for $\underline{\delta}_r$ and $\bar{\tau}_r$ are the quantities appearing in (2.20) for $\underline{\Delta}_r$, the total rms timing deviation in regenerated pulses resulting from random amplitude and timing deviations in the timing wave.

V SELF-TIMED REPEATERS WITH PARTIAL RETIMING

5.0 General

As shown in the preceding section, timing for pulse regeneration can be derived from the pulse trains, with certain random phase and amplitude variations in the timing wave that can be reduced by increasing the loss constant Q of the resonant circuit. This method of "self-timing" can be combined with partial retiming, and the regeneration characteristics of this type of repeater will be discussed in the following sections.

For purposes of numerical illustration, the same type of pulse shape and timing wave will be assumed as in the previous numerical illustration in Section III. This pulse shape and timing wave closely approximates those in experimental Wrathall repeaters, in which timing is derived from the regenerated pulse train. In the following discussion timing from the received pulse train will also be considered.

5.1 Timing from Received Pulse Train

It will be assumed that the timing wave is derived from the received pulse train with the aid of a resonant circuit and that random timing deviations are absent. The response of the resonant circuit is then a sinusoidal wave as given by (4.1). From this wave it is possible to obtain a retiming wave of the form

$$R(t) = -\cos\psi\left[1 - \cos\left(2\pi\frac{t}{T} - \psi\right)\right]. \tag{5.1}$$

This can be accomplished by holding the peaks of the timing wave from the resonant circuit at zero potential with a diode. This is the form of retiming wave previously considered in Section III, in conjunction with a pulse shape given by (3.1).

As shown in Section 3.7, the tolerance to noise will vary with the phase shift ψ of the resonant circuit, in accordance with (3.21). If a reduction in the tolerance to noise of about 2 db is allowed, the maximum permissible phase shift would be about $\psi = 1$ radian (57.6°). On this basis the maximum permissible deviation $\Delta f_{\max}$ in the resonant frequency from the pulse repetition frequency f as obtained from (4.2) with $\psi = 1$ radian becomes

$$\frac{\Delta f_{\max}}{f} = \frac{\tan\psi}{2Q} = \frac{1.58}{2Q}. \tag{5.2}$$

For various values of Q in the range that can be realized by simple resonant circuits, the permissible deviations are as follows:

Q	10	25	50	100	200
$\Delta f_{max}/f$	0.08	0.030	0.016	0.008	0.004

This assumes that there are no random timing deviations and that the tolerance to noise is reduced by not more than 2 db.

5.2 Timing from Regenerated Pulse Train

It will again be assumed that there are no random timing deviations. Without a phase shift in the resonant circuit, let the regenerated pulses be triggered at a time t_0. When there is a phase shift ψ', the pulses will be triggered at a time t_0'. The timing wave derived from the regenerated pulses will then have a time shift

$$\Delta = t_0' - t_0 + \frac{T}{2\pi}\psi'.$$

This time shift will cause pulses to be regenerated with a time shift $\beta'\Delta$, which must equal $t_0' - t_0$. Accordingly,

$$t_0' - t_0 = \beta'\left(t_0' - t_0 + \frac{T}{2\pi}\psi'\right),$$

and

$$t_0' - t_0 = \frac{T}{2\pi}\frac{\beta'\psi'}{1 - \beta'}. \tag{5.3}$$

With timing from the received pulse train with a phase shift ψ in the resonant circuit, the following relation applies:

$$t_0' - t_0 = \frac{T}{2\pi}\beta\psi. \tag{5.4}$$

If $t_0' - t_0$ is to be the same in both cases, so that the timing wave and tolerance to noise is the same, the following relation must exist between the phase shifts in the resonant circuit:

$$\psi' = \psi(1 - \beta')\frac{\beta}{\beta'}. \tag{5.5}$$

In this expression, β and β' are the factors shown in Fig. 4. It will be recognized from (5.5) that the smallest permissible phase shifts are obtained for large values of β'. From Fig. 4, it is seen that the largest values of β are for phase shifts between 0 and $-60°$. For $\eta = \frac{3}{4}$, $\beta \cong 0.7$ and for $\eta = 1$, $\beta \cong 0.9$.

For $\eta = \frac{3}{4}$ and $\eta = 1$ the tolerable maximum phase shifts ψ' in the resonant circuit with timing from the regenerated pulse train, in relation to the maximum tolerable ψ with timing from the input, are

$$\psi' \cong 0.3\psi \quad \text{for} \quad \eta = \tfrac{3}{4}, \tag{5.6}$$

and

$$\psi' \cong 0.1\psi \quad \text{for} \quad \eta = 1.$$

Although greater phase shifts can be tolerated when ψ is positive, and β' is smaller than above, the requirements on the resonant circuit must be based on the worst condition that can be encountered, as above.

From (5.6) it follows that for $\eta = 1$ the requirements on the permissible phase shift in the resonant circuit are much more severe than for $\eta = \frac{3}{4}$. For this reason the latter value of η is decidedly preferable for the particular case in which the peak amplitudes of the pulse train and the timing waves are equal, as assumed here. A value $\eta = \frac{3}{4}$ is also desirable from the standpoints of avoiding intersymbol interference between adjacent pulses at the triggering instants, to permit the timing wave to be derived from the pulse train and to permit self-starting of the repeaters, as discussed later.

In accordance with (5.6) the maximum tolerable frequency deviation for $\eta = \frac{3}{4}$ will be less than with timing from the received pulse train by a factor of about 0.3. The maximum permissible frequency deviation for a phase shift of about one radian in the timing wave and 0.3 radian in the resonant circuit, will accordingly be about as follows:

Q	10	25	50	100	200
$\Delta f_{max}/f$	0.025	0.009	0.005	0.0025	0.0012

For a repeater with complete rather than partial retiming, the factor β would be unity, and timing from the regenerated pulse train would not be possible.

5.3 Random Timing Deviations

In combining random timing deviations from various sources at a particular repeater, it will be assumed that there is no correlation between the various deviations, so that they will combine on a root-sum-square basis.

In accordance with (2.21) the rms timing deviation at the output is then:

$$\underline{\Delta}^2 = (p_\tau^2 \bar{\tau}_p^2 + p_a^2 \underline{a}_p^2) + (r_\tau^2 \bar{\tau}_r^2 + r_a^2 \underline{a}_r^2), \tag{5.7}$$

where in accordance with (4.13) and (4.16)

$$\underline{a}_r = \left[\frac{\pi}{2Q}(1 - \psi^2/2)\right]^{1/2} \frac{1}{\cos \psi}, \tag{5.8}$$

$$\bar{\tau}_r = (\underline{\delta}_r^2 + \underline{\alpha}^2 \bar{\tau}_p^2)^{1/2}, \tag{5.9}$$

$$\underline{\alpha} = \left(\frac{\pi}{Q}\right)^{1/2}, \tag{5.10}$$

$$\underline{\delta}_r = \frac{T}{2\pi}\left(\frac{\pi}{4Q}\right)^{1/2} \frac{\psi}{\cos \psi}. \tag{5.11}$$

When (5.9) is inserted in (5.7)

$$\underline{\Delta}^2 = (p_\tau^2 + \underline{\alpha}^2 r_\tau^2)\bar{\tau}_p^2 + p_a^2 \underline{a}_p^2 + r_\tau^2 \underline{\delta}_r^2 + r_a^2 \underline{a}_r^2. \tag{5.12}$$

This expression gives the rms timing deviation at the output in terms of the rms deviation $\bar{\tau}_p$ at the input and the various repeater parameters.

With timing from the output, rather than the input as assumed above, $\bar{\tau}_p$ is replaced by $\underline{\Delta}$ in (5.9), and the following relation is obtained:

$$\underline{\Delta}^2(1 - \underline{\alpha}^2 r_\tau^2) = p_\tau^2 \bar{\tau}_p^2 + p_a^2 \underline{a}_p^2 + r_\tau^2 \underline{\delta}_r^2 + r_a^2 \underline{a}_r^2. \tag{5.13}$$

In the above expressions $p_\tau^2 \cong 0.15$, $r_\tau^2 \cong 0.4$ and $\underline{\alpha}^2 \cong 0.03$ ($Q = 100$). The term $\underline{\alpha}^2 r_\tau^2$ can thus be neglected in comparison with p_τ^2 in (5.12) and in comparison with 1 in (5.13).

The following expression is thus obtained with timing from either the input or the output:

$$\begin{aligned}\underline{\Delta}^2 &= (p_\tau^2 \bar{\tau}_p^2 + p_a^2 \underline{a}_p^2) + (r_\tau^2 \underline{\delta}_r^2 + r_a^2 \underline{a}_r^2) \\ &= \underline{\Delta}_p^2 + \underline{\Delta}_r^2.\end{aligned} \tag{5.14}$$

5.4 Magnitude of Random Timing Deviations

The first two terms of (5.14) represents the rms timing deviations in the regenerated pulses resulting from timing deviations and amplitude variations in the received pulses. The last two terms represent the timing deviations resulting from timing deviations and amplitude variations in the timing wave. The conversion factors p_τ, p_a, r_τ and r_a are discussed in Section II and representative values given in Figs. 5 and 6. The values of $\underline{a}_r$ and $\underline{\delta}_r$ are obtained from (5.8).

TABLE IV — RMS DEVIATIONS FROM TIMING WAVE DISTORTION FOR $Q = 100$

ψ	$-60°$	$-30°$	$0°$	$30°$	$60°$
$r_\tau \underline{\delta}_r/T$	0.011	0.005	0	0.006	0.015
$r_a \underline{a}_r/T$	0.006	0.007	0.009	0.014	0.024
$\underline{\Delta}_r/T$	0.0126	0.009	0.009	0.015	0.028
$\bar{\varphi}_r$	4.5°	3.2°	3.2°	5.4°	10°

In Table IV are given the values of the two last terms in (5.14), which represents the rms deviations $\underline{\Delta}_r$ owing to random deviations in the timing wave. The results are given for the particular case in which $Q = 100$, and for other values of Q are inversely proportional to $Q^{1/2}$. The table shows the deviations as a fraction of the interval T between pulses, and also as the corresponding rms phase deviation $\bar{\varphi}_r$.

In Table V are given the values of the first two terms in (5.14), which represents the rms deviation $\underline{\Delta}_p$ in the regenerated pulses resulting from random amplitude and timing deviation in the received pulses. In binary systems it is customary to limit the rms pulse distortion to $\underline{a}_p = \frac{1}{10}$, corresponding to $\frac{1}{10}$ the peak amplitude of the received pulses, or $\frac{1}{5}$ the triggering level (17 db signal-to-noise ratio). The corresponding rms phase deviation would be about $\frac{1}{10}$ radian, corresponding to an rms deviation $\bar{\tau}_p$ in the pulses of 0.016 the pulse spacing, or $\bar{\tau}_p/T \cong 0.016$. The total rms timing deviation obtained from (5.14) and the corresponding rms phase deviation are given in Table VI.

TABLE V — RMS DEVIATIONS RESULTING FROM PULSE DISTORTION

ψ	$-60°$	$-30°$	$0°$	$30°$	$60°$
$p_a \underline{a}_p/T$	0.019	0.018	0.019	0.022	0.032
$p_\tau \bar{\tau}_p/T$	0.010	0.007	0.005	0.005	0.008
$\underline{\Delta}_p/T$	0.021	0.020	0.020	0.023	0.033
$\bar{\varphi}_p$	7.5°	7.2°	7.2°	8.2°	12°

TABLE VI — TOTAL RMS DEVIATIONS FROM TIMING WAVE AND PULSE DISTORTION

ψ	$-60°$	$-30°$	$0°$	$30°$	$60°$
$\underline{\Delta}/T$	0.025	0.022	0.022	0.028	0.043
$\bar{\varphi}$	9°	8°	8°	10°	16°

The probability that random phase deviations will exceed the above rms values by a factor of more than 4 is small enough to be ignored. On this basis the sum of the fixed and random deviations would be limited to about 70°, if the fixed phase shift ψ is less than $\pm 30°$. With this requirement on the fixed phase shift for satisfactory performance, the values of Δf_{max} would be about half as great as previously given in Sections 5.1 and 5.2, for a single repeater as considered here.

VI REPEATER CHAINS

6.0 General

In the previous section, a single self-timed repeater was considered, from the standpoint of fixed and random timing deviations, as determined by various repeater design parameters. In a repeater chain there will be some cumulation of random timing deviations as the number of repeaters in tandem is increased, and a resultant reduction in the tolerance to noise of repeaters toward the end of the chain. Exact evaluation of such cumulation is rendered difficult by the circumstance that timing deviations from various sources may not follow the same law of combination along the repeater chain. In the following, expressions are given based both on root-sum-square and direct addition of random timing deviations, which can be regarded as lower and upper limits.

6.1 Combination of Random Timing Deviations

To determine the rms value of random timing deviations at the end of a repeater chain, it is necessary to combine random deviations from various repeaters. Random deviations from various sources at a repeater do not necessarily follow the same law of cumulation along a repeater chain. Since there is no correlation between timing deviations caused by noise in various repeater sections, these can be combined on a root-sum-square basis. This, however, may not be appropriate as regards the combination of timing deviations resulting from imperfections in the timing wave. Thus, with perfect tuning of all resonant circuits, the timing waves at various repeaters would have virtually identical amplitude variations, but no phase deviations. While in this case there would be complete correlation between the timing wave variations at the repeaters, it does not follow that the resultant timing deviations should be combined directly rather than on a root-sum-square basis along the repeater chain. The timing deviations at the end of a chain of N repeaters resulting from amplitude variations in the timing wave of the first repeater will be modified by N intermediate resonant circuits. Those resulting from amplitude variations at subsequent repeaters will be modified by N-1, N-2 etc. intermediate resonant circuits. The situation is similar to that of applying identical noise waves at the input of each of N resonant circuits in tandem. At the output the N noise waves will have different shapes owing to restriction of the band and increasing phase distortion as the number of resonant circuits in tandem increases. For this reason combination on a root-sum-square basis appears justified also in this case, particularly with various degrees of mistuning of the resonant circuits, so that the amplitude variations in the timing waves will differ in phase among repeaters.

6.2 Propagation of Timing Deviations

To determine the cumulation of timing deviations along a repeater chain, it is convenient to first consider a single repeater as a source of timing deviations, and to determine the propagation of these timing deviations along a repeater chain. In the following, γ_n will designate the rms propagation factor for n repeaters in tandem; i.e., the factor by which the rms timing deviations at the end of a chain of n repeaters is smaller than at the first repeater, with timing deviations originating at the first repeater only.

Let the rms timing deviation at the output of the first repeater as given by (5.14) for convenience be taken as unity. At the output of the second repeater the squared rms timing deviation is then reduced by the factor

$$\gamma_2^2 = p_\tau^2 + \underline{\alpha}_1^2 r_\tau^2, \qquad \underline{\alpha}_1 = \underline{\alpha}. \tag{6.1}$$

As indicated symbolically in Fig. 8, the first term represents the reduction owing to partial retiming. The second term is the additional devia-

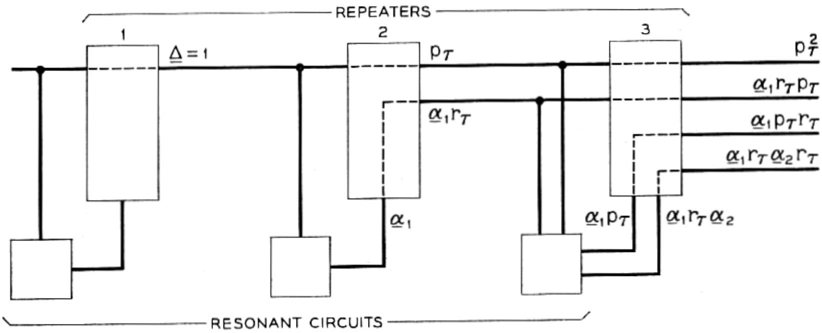

Fig. 8 — Propagation of random timing deviations along repeater chain.

tion resulting from the effect on the timing wave of unit rms deviation in the received pulse train at the second repeater.

At the output of the third repeater, the squared rms deviation is smaller than at the output of the first repeater by the factor

$$\gamma_3^2 = (p_\tau^2 + \underline{\alpha}_1^2 r_\tau^2)p_\tau^2 + p_\tau^2 \underline{\alpha}_1^2 r_\tau^2 + \underline{\alpha}_1^2 r_\tau^2 \underline{\alpha}_2^2 r_\tau^2 \tag{6.2}$$

$$= p_\tau^4 + 2\underline{\alpha}_1^2 p_\tau^2 r_\tau^2 + \underline{\alpha}_1^2 \underline{\alpha}_2^2 r_\tau^4. \tag{6.3}$$

As indicated in Fig. 8, the first term in (6.2) represents the reduction owing to partial retiming of the received pulse train at the third repeater. The second term, $(p_\tau \alpha_1 r_\tau)^2$, is the additional rms deviation resulting from the effect on the timing wave at the third repeater of an rms deviation p_τ in the received pulse train. The third term $(\underline{\alpha}_1 r_\tau \cdot \underline{\alpha}_2 r_\tau)^2$ is the additional deviation caused by the effect on the timing wave of an rms deviation $\underline{\alpha}_1 r_\tau$ in the received pulse train. The factor $\underline{\alpha}_2 r_\tau$ represents the modification in the rms deviation $\underline{\alpha}_1 r_\tau$ by a second resonant circuit, with $\underline{\alpha}_2$ defined as in Section 4.3.

At the output of the fourth repeater, the rms timing deviation is reduced by the following factor, obtained in the same manner:

$$\gamma_4^2 = p_\tau^6 + 3\underline{\alpha}_1^2 p_\tau^4 r_\tau^2 + 3\underline{\alpha}_1^2 \underline{\alpha}_2^2 p_\tau^2 r_\tau^4 + \underline{\alpha}_1^2 \underline{\alpha}_2^2 \underline{\alpha}_3^2 r_\tau^6. \tag{6.4}$$

At the output of repeater n, the squared rms timing deviation is smaller than at the output of the first repeater by the propagation factor

$$\begin{aligned}\gamma_n^2 = {} & p_\tau^{2(n-1)} + \frac{(n-1)}{1!} p_\tau^{2(n-2)} r_\tau^2 \underline{\alpha}_1^2 \\ & + \frac{(n-1)(n-2)}{2!} p_\tau^{2(n-3)} r_\tau^4 \underline{\alpha}_1^2 \underline{\alpha}_2^2 \\ & + \frac{(n-1)(n-2)(n-3)}{3!} p_\tau^{2(n-4)} r_\tau^1 \underline{\alpha}_1^2 \underline{\alpha}_2^2 \underline{\alpha}_3^2 \\ & + \cdots + r_\tau^{2(n-1)} \underline{\alpha}_1^2 \underline{\alpha}_2^2 \underline{\alpha}_3^2 \cdots \underline{\alpha}_{n-1}^2,\end{aligned} \tag{6.5}$$

where p_τ and r_τ are defined as in Section 2.2, and α_1, $\alpha_2 \cdots \alpha_n$ as in Section 4.3.

In the above formulation the rms deviation at the output of the first repeater was assumed given by (5.14), which is an approximation of (5.12) in which the term $\underline{\alpha}^2 r_\tau^2 \bar{\tau}_p^2$ was neglected. This term will have a different propagation factor ρ_n, the expression for which differs from that for γ_n as given by (6.5) in that the subscripts of the factors $\underline{\alpha}_j$ are raised by one unit. Thus,

$$\rho_n^2 = p_\tau^{2(n-1)} + \frac{(n-1)}{1!} p_\tau^{2(n-2)} r_\tau^2 \underline{\alpha}_2^2 + \cdots + r_\tau^{2(n-1)} \underline{\alpha}_2^2 \underline{\alpha}_3^2 \cdots \underline{\alpha}_n^2. \tag{6.6}$$

The rms deviation at the output of repeater n thus becomes

$$\underline{\Delta}_n^2 = (\underline{\Delta}_p^2 + \underline{\Delta}_r^2)\gamma_n^2 + \underline{\alpha}^2 r_\tau^2 \bar{\tau}_p^2 \rho_n^2. \tag{6.7}$$

In the case of repeaters with partial retiming the last term in (6.7) can be neglected, in which case the cumulation of timing deviation will be virtually the same when the timing wave is derived from the regenerated as when it is derived from the received pulse train.

The above expressions apply for resonant circuits consisting of a coil and capacitor which have a gradual cut-off. If resonant circuits with a flat pass-band and sharp cut-offs were used, $\alpha_2 = \alpha_3 = \alpha_n$ and (6.5) can be simplified to

$$\gamma_n^2 = (1 - \underline{\alpha}_1^2)p_\tau^{2(n-1)} + \underline{\alpha}_1^2(p_\tau^2 + r_\tau^2)^{(n-1)}. \tag{6.8}$$

6.3 Cumulation of Timing Deviations

The cumulation of random timing deviations from various repeaters in a chain can be determined from the propagation constant given above for any prescribed law of combination of timing deviations from various repeaters. When equal rms deviations are contributed by each of N repeaters, and they are combined on a root-sum-square basis, the rms deviation at the end of a repeater chain is greater than for a single repeater by the cumulation factor

$$C = \left(\sum_{n=1}^{N} \gamma_n^2\right)^{1/2}. \tag{6.9}$$

An upper limit to C is obtained by taking $\underline{\alpha}_2 = \underline{\alpha}_3 = \underline{\alpha}_n = 1$ in (6.5) in which case γ_n^2 is given by (6.8); (6.9) then becomes for $N = \infty$

$$C = \left[(1 - \underline{\alpha}_1^2)\frac{1}{1 - p_\tau^2} + \underline{\alpha}_1^2 \frac{1}{1 - p_\tau^2 - r_\tau^2}\right]^{1/2} \tag{6.10}$$

$$\cong \left(\frac{1}{1 - p_\tau^2}\right)^{1/2}, \tag{6.11}$$

where the terms in $\underline{\alpha}_1^2$ have been neglected in (6.11), since $\underline{\alpha}_1^2 = \underline{\alpha}^2 \ll 1$, about 0.03 for $Q = 100$.

From Fig. 5 it will be seen that when $\psi < \pm 60°$, $p_\tau < 0.6$. Hence $C < 1.25$. Cumulation of random timing deviations can thus for practical purposes be disregarded, with root-sum-square combination as assumed above. The value of C obtained from (6.11) will differ from that obtained from (6.9) when γ_n is given by (6.5), by a small fraction of one per cent.

Although root-sum-square combination appears justified for reasons given before, it is of interest to determine an upper limit to the cumulation based on direct addition of random timing deviations. The maximum cumulation factor thus obtained is

$$C_{\max} = \sum_{n=1}^{N} \gamma_n . \tag{6.12}$$

Employing (6.8) for γ_n and neglecting the terms in α_1^2, the upper limit to the cumulation factor for $N = \infty$ becomes

$$C_{\max} = \frac{1}{1 - p_\tau}. \tag{6.13}$$

With $p_\tau < 0.6$ for $\psi < \pm 60°$, $C_{\max} < 2.5$.

If the above maximum cumulation factor is applied to random timing deviations resulting from amplitude variations in the timing wave, as given in Table IV of Section 5.4, the resultant rms phase deviation at the end of a long repeater chain could be as great as 25°, rather than 10° for a single repeater, when $\psi = 60°$ and $Q = 100$. To attain satisfactory performance it would in this case be necessary to limit the maximum fixed phase shift to substantially less than ±60°, which would entail greater frequency precision than indicated in Sections 5.1 and 5.2.

If $\psi < \pm 15°$, $p_\tau < 0.40$ and $C_{\max} < 1.7$. In this case the rms phase deviation as given in Table I for a single repeater is $\bar{\varphi}_r \cong 4°$, and the rms phase deviation in a long repeater chain would be less than 7°. In a long repeater chain the rms phase deviation resulting from pulse distortion would be greater than given in Table II by an rms cumulation factor $C = 1.08$ for $p_\tau = 0.4$, and would thus be about 8° when $\psi < \pm 15°$. The total rms phase deviation would thus be about $(7^2 + 8^2)^{1/2} \cong 11°$. Random phase deviations exceeding 4 times the latter value, or about 45°, would be rather unlikely. The sum of the fixed and random phase deviations would thus be limited to about 60°, so that satisfactory performance would be expected when the fixed phase deviation is limited to about ±15°.

With the approximations for γ_n employed above, the rms cumulation factor for a chain of N repeaters as obtained from (6.9) is less than for $N = \infty$ by the factor $(1 - p_\tau^{2N})^{1/2} \cong 0.99$ for $p_\tau = 0.5$ and $N = 3$. The maximum cumulation factor obtained from (6.12) is less than for $N = \infty$ by the factor $1 - p_\tau^N \cong 0.99$ for $N = 6$. Thus, cumulation of random timing deviations is virtually completed in a chain of 3 to 6 repeaters, so that for experimental determinations of the degree of cumulation it suffices to operate a few repeaters in tandem.

6.4 Repeaters with Complete Retiming

In the particular case of complete retiming, $p_\tau = 0$ and $r_\tau = 1$ in (6.5) and (6.6) so that

$$\gamma_n = \underline{\alpha}_1 \underline{\alpha}_2 \underline{\alpha}_3 \cdots \underline{\alpha}_{n-1}, \tag{6.14}$$

$$\rho_n = \underline{\alpha}_2 \underline{\alpha}_3 \cdots \underline{\alpha}_n . \tag{6.15}$$

For $n \gg 1$, approximation (4.16) can be employed, in which case

$$\gamma_n = \underline{\alpha} \left(\frac{1}{\pi n}\right)^{1/4}, \qquad \rho_n = \left(\frac{1}{\pi n}\right)^{1/4}. \tag{6.16}$$

In this case (5.14) simplifies to

$$\underline{\Delta}_p^2 + \underline{\Delta}_r^2 = \underline{\delta}_r^2, \tag{6.17}$$

since $p_a = 0$, $r_a = 0$, $p_\tau = 0$ and $r_\tau = 1$.

Hence (6.7) becomes

$$\underline{\Delta}_n^2 = \underline{\delta}_r^2 \gamma_n^2 + \bar{\tau}_p^2 \underline{\alpha}^2 \rho_n^2. \tag{6.18}$$

With approximations (6.16),

$$\underline{\Delta}_n^2 = (\underline{\delta}_r^2 + \bar{\tau}_p^2) \underline{\alpha}^2 \left(\frac{1}{\pi n}\right)^{1/2}. \tag{6.19}$$

At the output of the first repeater,

$$\underline{\Delta}_1^2 = \underline{\delta}_r^2 + \underline{\alpha}^2 \bar{\tau}_p^2. \tag{6.20}$$

For $n \gg 1$ the squared propagation factor is accordingly

$$\underline{\Delta}_n^2 / \underline{\Delta}_1^2 = \underline{\alpha}^2 \frac{\underline{\delta}_r^2 + \bar{\tau}_p^2}{\underline{\delta}_r^2 + \underline{\alpha}^2 \bar{\tau}_p^2} \left(\frac{1}{\pi n}\right)^{1/2}. \tag{6.21}$$

The squared rms cumulation factor for $N \gg 2$ repeaters becomes

$$\begin{aligned} C^2 &\cong 1 + \underline{\alpha}^2 \frac{\underline{\delta}_r^2 + \bar{\tau}_p^2}{\underline{\delta}_r^2 + \underline{\alpha}^2 \bar{\tau}_p^2} \int_2^N \left(\frac{1}{\pi n}\right)^{1/2} dn \\ &\cong 1 + \underline{\alpha}^2 \frac{\underline{\delta}_r^2 + \bar{\tau}_p^2}{\underline{\delta}_r^2 + \underline{\alpha}^2 \bar{\tau}_p^2} \left[\left(\frac{4N}{\pi}\right)^{1/2} - \left(\frac{8}{\pi}\right)^{1/2}\right]. \end{aligned} \tag{6.22}$$

In the particular case of perfect tuning of all resonant circuits $\underline{\delta}_r = 0$ and

$$(\Delta_n/\Delta_1)^2 \cong \left(\frac{1}{\pi n}\right)^{1/2}, \tag{6.23}$$

$$C^2 \cong 1 + \left(\frac{4N}{\pi}\right)^{1/2} - \left(\frac{8}{\pi}\right)^{1/2},$$

$$C \cong \left(\frac{4N}{\pi}\right)^{1/4}. \tag{6.24}$$

The last expression gives the factor by which the rms timing deviation at the output of repeater N is greater than at the output of the first repeater. The rms deviation at the output of the first repeater is greater than at the input by the factor $\underline{\alpha}$. The rms deviation at the output of repeater N is thus greater than at the input of the first repeater by the factor,

$$C_1 = \underline{\alpha}\left(\frac{4N}{\pi}\right)^{1/4}. \tag{6.25}$$

For this particular case ($\underline{\delta}_r = 0$) expressions equivalent to those above have been derived in unpublished work by H. E. Rowe of Bell Telephone Laboratories.

In accordance with (6.22) and (6.24) the cumulation of random timing deviations increases indefinitely with N when retiming is complete. The cumulation factor as given by (6.24) is in fact the same as would be obtained if a timing wave were transmitted on a separate pair, with a resonant circuit at each repeater to limit noise and with amplification of the timing wave at each repeater to obtain the same amplitude of the timing wave as when it is derived from the pulse train. With partial retiming cumulation is limited, for the reason that there is partial regeneration of both the pulse train and the timing wave.

Although with complete retiming the cumulation factor increases indefinitely with N, this is of but little practical significance, because of the slow rate of cumulation. At the output of a chain of N repeaters an rms deviation approximately equal to that at the input of the first repeater could be tolerated, in which case $C_1 \cong 1$. On this basis the permissible number of repeaters would be

$$N \cong \frac{\pi}{4}\frac{1}{\underline{\alpha}^4} = \frac{\pi}{4}\left(\frac{Q}{\pi}\right)^2, \tag{6.26}$$

$$\cong 800 \quad \text{when} \quad Q = 100.$$

This assumes exact tuning of all resonant circuits. With mistuning of the resonant circuits, the permissible number of repeaters in tandem for a specified rms deviation at the output of the final repeater can be determined with the aid of the cumulation factor given by (6.22). For example, if the rms deviation at the output of repeater N is assumed the same as at the input of the first repeater, the permissible number of repeaters in tandem is less than given by (6.26) by the factor $[(1 - m^2)/(1 + m^2)]^2$, $m = \delta_r/\bar{\tau}_p$. When the fixed phase shift is 30°, $m \cong 0.5$ and $N \cong 300$.

6.5 Self-Starting of Self-Timed Repeaters

With self-timing it is necessary that repeaters be self-starting if the timing wave should be absent for any reason. If each repeater is self-starting, this will also be the case for a repeater chain, since starting will be progressive along the chain. Initially, before the timing wave has reached the appropriate amplitude at all repeaters, there will be a high rate of digital errors.

With timing from the received pulse train, the resonant circuit will be excited by every pulse and the timing wave will reach its normal amplitude in about $n \cong Q$ pulses. With timing from the regenerated pulse

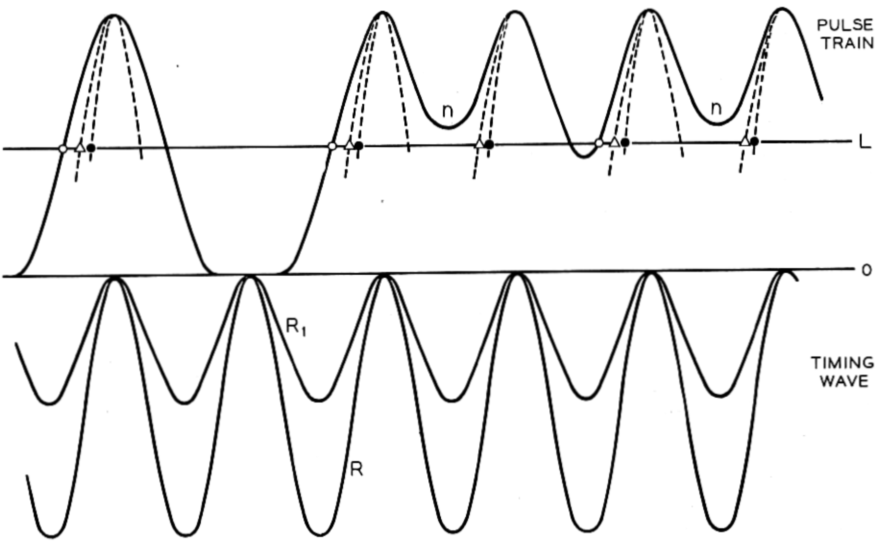

Fig. 9 — Progression of repeater starting in absence of timing wave when timing is derived from regenerated pulse train.
○ Triggering points with timing wave absent. Noise prevents triggering at certain points, n. Timing wave reaches fraction of normal value, R_1 .
△ Triggering points with timing wave R_1. Timing wave increases to normal amplitude R.
● Triggering points with normal timing wave.

train the resonant circuit will not be excited by every pulse, unless the shape of the received pulses is such that there are virtually no overlaps between pulses so that the triggering level will be penetrated by each pulse.

With a pulse shape as assumed in the previous analysis, the amplitude of a pulse train midway between pulses is half the peak amplitude of the pulses, as indicated in Fig. 9. In the presence of noise, triggering will in this case occur on the average for every second pulse, as indicated in the above figure. If it is assumed that the resonant circuit has the maximum permissible phase shift of about 20° allowed with timing from the output, the amplitude of the timing wave with excitation from every pulse will be virtually equal to the peak pulse amplitude. With excitation from half the pulses, the amplitude of the timing wave will rapidly reach half the peak amplitude of the pulses. When this initial timing wave is combined with the pulse train, triggering will occur for virtually all pulses, as indicated in Fig. 9. It will thus reach its normal value. If the phase shift is greater than 20° as assumed above, say 60°, the initial amplitude of the timing wave will be $\frac{1}{4}$ the peak pulse amplitude. Combination of this initial timing wave with the pulse train will increase the number of pulses exciting the resonant circuit, which in turn increases the amplitude of the timing waves, etc.

Self-starting with a pulse shape as assumed in this analysis is thus insured.

VII SUMMARY

In self-timing regenerative repeaters as considered here, a timing wave is derived from either the received or regenerated pulse train with the aid of a simple resonant circuit tuned to the pulse repetition frequency. This timing wave is combined linearly with received pulse trains as indicated in Fig. 1, and pulses are regenerated when the combined wave penetrates a certain triggering level.

It is concluded that if these timing principles are implemented by appropriate repeater instrumentation, a performance can be realized that approaches that of ideal regenerative repeaters. To this end it is necessary to meet certain requirements with regard to the loss constant Q of the resonant circuit, its frequency precision, the shape of received pulses and the amplitude of the timing wave in relation to that of received pulses.

Equalization of each repeater section should preferably be such that the received pulses have a shape and duration in relation to the pulse interval as indicated in Fig. 3, and the peak amplitude of the timing wave should be about equal to that of the received pulses. Under these conditions the pulse repetition frequency will be present in the received pulse train in sufficient amplitude to permit derivation of the timing wave from the received pulse train, and to permit rapid self-starting in the absence of a timing wave if it is derived from the regenerated pulse train.

A loss constant of the resonant circuit $Q \cong 100$ appears desirable. This value is sufficiently low to be readily realized with simple resonant circuits consisting of a coil and capacitor in series or parallel, without unduly severe requirements on its frequency precision. It is also adequately high from the standpoint of avoiding excessive random timing deviations in regenerated pulses from amplitude and phase deviations in the timing wave.

The tolerable deviation in the resonant frequency from the pulse repetition frequency with $Q = 100$ is about 0.2 per cent when the timing wave is derived from the received pulse train, and about 0.06 per cent when it is derived from the regenerated pulse train. These frequency precisions correspond to a maximum fixed phase shift of 15° in the timing wave, and allow for the possibility that random timing deviations resulting from amplitude variations in the timing wave may cumulate directly along a repeater chain, rather than on a root-sum-square basis. With root-sum-square cumulation of timing deviations from all sources, the frequency deviations could be about twice as great.

When the above requirements are met the reduction in the tolerance to noise owing to timing deviations in a repeater chain is limited to about 2 db. If the requirements on frequency precision of the resonant circuit are met, substantial degradation or improvement in performance would not be expected as a result of moderate changes in the other design parameters.

VIII ACKNOWLEDGMENTS

In this presentation the writer had the benefit of unpublished work, referred to previously, by W. R. Bennett and J. R. Pierce on the derivation of a timing wave from a pulse train with the aid of a resonant circuit, and by H. E. Rowe on the cumulation of timing deviations in a chain of repeaters with complete retiming. Bennett, Pierce and Rowe are at Bell Telephone Laboratories. He is also indebted to H. E. Rowe for a critical review that resulted in several improvements in the analysis.

Appendix

IX RESONANT CIRCUIT RESPONSE TO RANDOM BINARY PULSE TRAINS

1 General

In the following analysis of the response of a resonant circuit to a binary "on-off" pulse train, the pulses are assumed of sufficiently short duration to be regarded as impulses. This is a legitimate approximation when the duration does not exceed about half the interval between pulses.

The pulse train is regarded as made up of three components, as indicated in Fig. 10. The first is a systematic component consisting of pulses of amplitude $\frac{1}{2}$. This component gives rise to a steady state response at the fundamental frequency of the pulse sequence. The second component consists of pulses of amplitude $\pm\frac{1}{2}$, with random $\pm$ polarity. This component gives rise to a random component in the resonant cir-

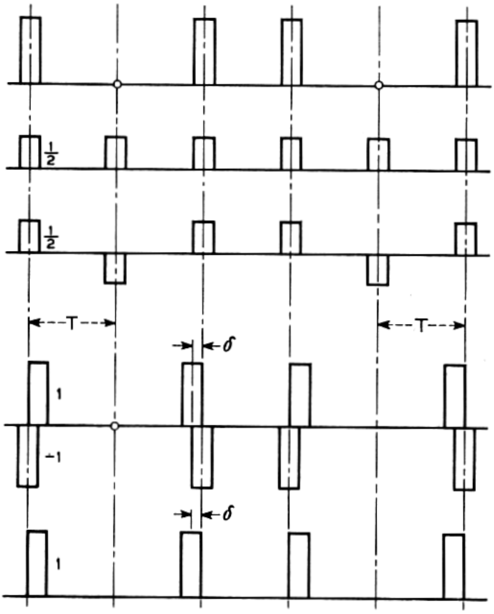

Fig. 10 — Components of random binary on-off pulse train. A. — Transmitted "on-off" pulses. B. — Steady state pulse train of fundamental frequency $f = 1/T$. C. — Random pulse train with zero mean value. D. — Random pulse train with displacements $\pm\delta$. F. — "On-off" pulse train with displacements $\pm\delta$ from average pulse interval T.

cuit response; i.e., a fluctuation about the steady state value derived from the first component.

The third component consists of a train of dipulses. Each dipulse consists of a pair of pulses of amplitude 1 and -1, displaced by an interval $\pm\delta$. The response of the resonant circuit to this component gives the effect of random displacements $\pm\delta$ in the original "on-off" pulse train.

2 Impedance of Resonant Circuit

The impedance of a resonant circuit consisting of R, L and C in parallel is

$$Z(i\omega) = \bar{Z}(i\omega)e^{-i\psi}, \tag{1}$$

$$\bar{Z}(i\omega) = \frac{R}{[1 + Q^2(\omega/\omega_0 - \omega_0/\omega)^2]^{1/2}} = R \cos \psi, \tag{2}$$

$$\tan \psi = Q(\omega/\omega_0 - \omega_0/\omega), \tag{3}$$

where

$$Q = \omega_0 RC = \text{Loss constant}, \tag{4}$$

and

$$\omega_0 = (1/LC)^{1/2} = \text{Resonant frequency}. \tag{5}$$

The above expressions also apply for the admittance of a resonant circuit consisting of R, L and C in series, except that in this case $Q = \omega_0 L/R$.

3 Impulse Response of Resonant Circuit

When a rectangular pulse of unit amplitude and sufficiently short duration δ_0 is applied to a resonant circuit, the impulse response for $Q \gg 1$ is of the form

$$P(t) = P(0) \cos \omega_0 t e^{-\omega_0 t/2Q}, \tag{6}$$

where

$$P(0) = \omega_0 \delta_0 R/Q. \tag{7}$$

$P(t)$ designates voltage in response to an impulse current in the case of a parallel resonant circuit, or the current in response to an impulse voltage in the case of a series resonant circuit.

4 Response to Steady State Impulse Train

Let a long sequence of impulses of amplitude $\frac{1}{2}$ and the same polarity impinge on a resonant circuit at uniform intervals T. The response after N impulses is then

$$A_s(t) = \frac{1}{2} \sum_{n=0}^{N} P(t - nT) \tag{8}$$

$$= \frac{P(0)}{2} \sum_{n=0}^{N} \cos \omega_0(t - nT) e^{-\omega_0(t-nT)/2Q}. \tag{9}$$

The subscript s indicates a systematic component.

The above series is conveniently summed by taking the real part of the series

$$A_s'(t) = \frac{P(0)}{2} \sum_{n=0}^{N} e^{i\omega_0(t-nT)} e^{-\omega_0(t-nT)/2Q}. \tag{10}$$

With $t = NT + t_0$, $0 < t_0 < T$:

$$A_s'(t) = \frac{P(0)}{2} e^{i\omega_0 t_0} e^{-\omega_0 t_0/2Q} \sum_{n=0}^{N} e^{i\omega_0 t(N-n)} e^{-\omega_0 t(N-n)/2Q}. \tag{11}$$

When $N \to \infty$, the steady state responses becomes

$$A_s'(t) = \frac{P(0)}{2} e^{i\omega_0 t_0} e^{-\omega_0 t_0/2Q} \frac{1}{1 - e^{i\omega_0 t} e^{-\omega_0 t/2Q}}. \tag{12}$$

The interval between pulses can be written

$$T = 2\pi/\omega, \tag{13}$$

where ω is the fundamental frequency of the impulse train, or the pulse repetition frequency.

Let

$$\omega_0 = \omega - \Delta\omega,$$

so that

$$\omega_0 = \frac{2\pi}{T} (1 - \Delta\omega/\omega). \tag{14}$$

The following approximations then apply:

$$\begin{aligned} e^{+i\omega_0 T} &= e^{2\pi i} e^{-2\pi i \Delta\omega/\omega} = e^{-2\pi i \Delta\omega/\omega}, \\ &\cong 1 - 2\pi i \Delta\omega/\omega; \end{aligned} \tag{15}$$

$$\begin{aligned} e^{-\omega_0 T/2Q} &= e^{-\pi/Q} e^{+(\pi/Q)\Delta\omega/\omega}, \\ &\cong 1 - \pi/Q \text{ when } \pi/Q \ll 1. \end{aligned} \tag{16}$$

With these approximations

$$1 - e^{i\omega_0 T} e^{-\omega_0 T/2Q} \cong \frac{\pi}{Q}[1 + i\psi], \tag{17}$$

where

$$\psi = \frac{2\Delta\omega}{\omega} Q, \tag{18}$$

will be recognized as the phase shift of the resonant current at the frequency ω, as obtained from (3) with $\omega = \omega_0 + \Delta\omega$.

A further approximation that can be introduced in (12) is

$$e^{i\omega_0 t_0} e^{-\omega_0 t_0/2Q} = e^{i\omega t_0} e^{-i\Delta\omega t_0} e^{-\omega_0 t_0/2Q},$$
$$\cong e^{i\omega t_0}, \tag{19}$$

since $t_0 < T$ and $\Delta\omega t_0$ and $\omega_0 t_0/2Q \ll 1$.

With the above approximations (12) becomes

$$A_s' = \frac{P(0)}{2} \frac{Q}{\pi} e^{i[\omega t_0 - \psi]} \cos \psi. \tag{20}$$

The real part of this expression is

$$A_s = \frac{P(0)}{2} \frac{Q}{\pi} \cos (\omega t_0 - \psi) \cos \psi, \tag{21}$$

which is the response to the steady state component of the pulse train.

5 *Response to Random Component of Impulse Train*

Let a sequence of impulses of amplitude $\frac{1}{2}$ and randomly positive and negative polarity impinge on the resonant circuit at intervals T. The response is then,

$$A_r(t) = \frac{P(0)}{2} \sum_{n=0}^{N} \pm \cos \omega_0(t - nT) e^{-\omega_0(t-nT)/2Q}. \tag{22}$$

This expression for the random component differs from (9) for the systematic component in that the impulses have random $\pm$ polarity. If all signs are chosen the same, the values of $A_r(t)$ will be either $-A_s(t)$ or $+A_s(t)$. The resultant response of the resonant circuit, i.e. $A_s(t) + A_r(t)$, can thus vary between the limit 0 and $2A_s(t)$. $A_r(t)$ represents a random fluctuation about $A_s(t)$ as a mean value. In the following the rms value of this fluctuation is evaluated.

In order to determine the components of $A_r(t)$ in phase and at quadrature with the steady state response as given by (21), it is convenient to write

$$\omega_0 = \omega - \Delta\omega,$$

$$\begin{aligned}\cos \omega_0(t - nT) &= \cos [\omega(t - nT) - \psi + \psi - \Delta\omega(t - nT)] \\ &= \cos [\omega(t - nT) - \psi] \cos [\psi - \Delta\omega(t - nT)] \\ &\quad - \sin [\omega(t - nT) - \psi] \sin [\psi - \Delta\omega(t - nT)].\end{aligned} \tag{23}$$

With $t = NT + t_0$, and $\omega T = \pi$, (22) can be written:

$$\begin{aligned}A_r(t) &= \cos (\omega t_0 - \psi) \sum_{n=0}^{N} \pm \cos [\psi_1 - \Delta\omega T(N - n)] e^{-\omega_0 (T)(N-n)/2Q} \\ &\quad - \sin (\omega t_0 - \psi) \sum_{n=0}^{N} \pm \sin [\psi_1 - \Delta\omega T(N - n)] e^{-\omega_0 (T)(N-n)/2Q}\end{aligned} \tag{24}$$

where $\psi_1 = \psi - \Delta\omega t_0 = \psi\left(1 - \dfrac{\omega t_0}{2Q}\right) \cong \psi$, since $\omega t_0/2Q \leq \pi/2Q \ll 1$.

With equal probabilities of a plus or a minus sign in the summations, the rms value of the in-phase component becomes

$$\begin{aligned}\underline{A_r}' &= \left[\sum_{n=0}^{N} \cos^2 [\psi - \Delta\omega T(N - n)] e^{-\omega_0 T(N-n)/Q}\right]^{1/2} \\ &= \left[\sum_{n=0}^{N} \frac{1}{2} (1 + \cos 2[\psi - \Delta\omega T(N - n)] e^{-\omega_0 T(N-n)/Q})\right]^{1/2}.\end{aligned} \tag{25}$$

The rms value of the quadrature component becomes

$$\begin{aligned}\underline{A_r}'' &= \left[\sum_{n=0}^{N} \sin^2 [\psi - \Delta\omega T(N - n)] e^{-\omega_0 T(N-n)/Q}\right]^{1/2} \\ &= \left[\sum_{n=0}^{N} \frac{1}{2} (1 - \cos 2[\psi - \Delta\omega T(N - n)] e^{-\omega_0 T(N-n)/Q})\right]^{1/2}.\end{aligned} \tag{26}$$

These expressions can be transformed into sums of geometric series by writing

$$\cos x = \tfrac{1}{2}(e^{ix} + e^{-ix}), \qquad x = 2[\psi - \Delta\omega T(N - n)].$$

Evaluation of (25) and (26) by this method gives

$$\underline{A_r}' = \frac{P(0)}{2} \frac{1}{2^{1/2}} \left[\frac{1}{1 - e^{-\omega_0 T/Q}} + \frac{N}{D}\right]^{1/2}, \tag{27}$$

$$\underline{A_r}'' = \frac{P(0)}{(2)} \frac{1}{2^{1/2}} \left[\frac{1}{1 - e^{-\omega_0 T/Q}} - \frac{N}{D}\right]^{1/2}, \tag{28}$$

where

$$N = \cos 2\psi(1 - \cos 2\Delta\omega T e^{-\omega_0 T/2Q}) + \sin 2\psi \sin 2\Delta\omega T e^{-\omega_0 T/2Q}, \quad (29)$$

$$D = 1 + e^{-2\omega_0 T/Q} - 2e^{-\omega_0 T/Q} \cos 2\Delta\omega T. \quad (30)$$

With the same approximations as used previously in connection with (12) and with

$$\cos 2\Delta\omega T \cong 1 - 2(\Delta\omega T)^2, \qquad \sin 2\Delta\omega T \cong 2\Delta\omega T,$$

$$N \cong \frac{2\pi}{Q}, \quad (31)$$

$$D \cong \left(\frac{2\pi}{Q}\right)^2 [1 + \psi^2], \quad (32)$$

$$1 - e^{-\omega_0 T/Q} = 2\pi/Q. \quad (33)$$

With these approximations in (27) and (28),

$$\underline{A}_r' \cong \frac{P(0)}{2} \left(\frac{Q}{2\pi}\right)^{1/2} [1 - \psi^2/2]^{1/2}, \quad (34)$$

$$\underline{A}_r'' \cong \frac{P(0)}{2} \left(\frac{Q}{2\pi}\right)^{1/2} \frac{|\psi|}{2^{1/2}}, \quad (35)$$

which apply when ψ is small and $(2\pi/Q) \ll 1$.

6. Response to Random Dipulse Train

Each dipulse is assumed to consist of two impulses of unit amplitude and opposite polarity, displaced by an interval δ, which in general will be a function of the pulse position; i.e., $\delta = \delta(n)$. The response of the resonant circuit to a train of such dipulses, obtained by taking the difference in response to the two impulses, is given by

$$A_\delta(t) = P(0) \left[\sum_{n=0}^{N} \cos \omega_0(t - nT) e^{-\omega_0(t-nT)/2Q} - \cos \omega_0[t - nT + \delta(n)] e^{-\omega_0[t-nT+\delta(n)]/2Q} \right]. \quad (36)$$

In determining the response, mistuning of the resonant current can be disregarded; i.e., $\omega_0 = \omega$. Furthermore, in the second term of (36) it is permissible to take

$$\exp[-\omega_0 \delta(n)/2Q] \cong 1.$$

With the following further approximations

$$\cos \omega_0(t - nT) - \cos \omega_0[t - nT + \delta(n)] = \sin \omega_0[t + \delta(n)/2]\, 2 \sin [\omega_0\delta(n)/2], \cong \omega_0\delta(n) \sin \omega_0 t, \tag{37}$$

expression (36) becomes:

$$A_\delta(t) = P(0)\omega_0 \sin \omega_0 t \sum_{n=0}^{N} \delta(n) e^{-\omega_0(t-nT)/2Q} = P(0)\omega_0 \sin \omega_0 t_0 \sum_{n=0}^{N} \delta(n) e^{-\omega_0 T(N-n)/2Q}, \tag{38}$$

where the substitution $t = NT + t_0$ has been made as in previous expressions.

The above expression shows that the resonant circuit response will be at quadrature with the steady state timing wave $\cos \omega t_0$.

In the above expressions, the dipulses are assumed to be present at intervals T, whereas in a random pulse train they will be present at average intervals $2T$. The rms value of the quadrature component with randomly positive and negative dipulses at intervals $2T$, with an rms displacement $\underline{\delta}$, is

$$\underline{A}_\delta'' = P(0)\omega_0\underline{\delta} \left[\sum_{n=0}^{N} e^{-2\omega_0 T(N-n)/Q} \right]^{1/2} = \frac{P(0)}{2} \omega_0\underline{\delta} \left(\frac{Q}{\pi} \right)^{1/2}. \tag{39}$$

In (38) the function $e^{-\omega_0 t/2Q}$ will be recognized as the impulse response function of a circuit with impedance

$$Z(i\omega) = \frac{1}{\beta + i\omega}, \qquad \beta = \omega_0/2Q, \tag{40}$$

$$= \overline{Z}(i\omega) e^{-i\psi}, \tag{41}$$

$$\overline{Z}(i\omega) = \frac{1}{\beta} \left[\frac{1}{1 + \omega^2/\beta^2} \right]^{1/2}, \tag{42}$$

$$\tan \psi = \omega/\beta. \tag{43}$$

It will also be recognized that (39) corresponds to the rms response of such a circuit, when impulses $\delta(n)$ of random amplitude with an rms value $\underline{\delta}$ are applied to average intervals $2T$. Thus (39) can alternately be obtained from

$$\begin{aligned} \underline{A}_\delta'' &= P(0)\omega_0\underline{\delta}\left[\frac{1}{2T}\frac{1}{2\pi}\int_{-\infty}^{\infty}[\overline{Z}(i\omega)]^2\,d\omega\right]^{1/2} \\ &= P(0)\omega_0\underline{\delta}\left[\frac{1}{4\pi T\beta^2}\,\beta(\tan^{-1}\omega/\beta)_{-\infty}^{\infty}\right]^{1/2} \qquad (44)\\ &= P(0)\omega_0\underline{\delta}\left(\frac{1}{4T\beta}\right)^{1/2} \\ &= \frac{P(0)}{2}\,\omega_0\underline{\delta}\left(\frac{Q}{\pi}\right)^{1/2}. \qquad (45) \end{aligned}$$

Let the output of the first resonant circuit be applied to a second resonant circuit, and in turn to n successive resonant circuits, with an amplitude amplification β between successive resonant circuits. At the output of the n^{th} resonant circuit, the rms amplitude of the response is then obtained from

$$\begin{aligned} \underline{A}_{\delta,n}'' &= P(0)\omega_0\underline{\delta}\left[\frac{\beta^{2(n-1)}}{4\pi T}\int_{-\infty}^{\infty}[\overline{Z}^2(i\omega)]^n\,d\omega\right]^{1/2} \\ &= P(0)\omega_0\underline{\delta}\left[\frac{1}{4\pi T\beta^2}\int_{-\infty}^{\infty}\frac{d\omega}{(1+\omega^2/\beta^2)^n}\right]^{1/2} \qquad (46)\\ &= \frac{P(0)}{2}\,\omega_0\underline{\delta}\left(\frac{Q}{\pi}\right)^{1/2}, \qquad I_n = \underline{A}_\delta'' I_n, \end{aligned}$$

where

$$I_n^2 = \frac{1}{\pi\beta}\int_{-\infty}^{\infty}\frac{d\omega}{(1+\omega^2/\beta^2)^n}, \qquad (47)$$

$$\begin{aligned} &= 1, && n = 1, \\ &= \frac{2n-3}{2(n-1)}\,I_{n-1}^2, && n \geq 2, \qquad (48)\\ &= I_{n-1}^2\left(1-\frac{1}{2(n-1)}\right), \end{aligned}$$

$$I_2^2 = (1-\tfrac{1}{2}), \qquad I_3^2 = (1-\tfrac{1}{4})I_2^2, \qquad I_4^2 = (1-\tfrac{1}{6})I_3^2.$$

Thus (46) can be written:

$$\underline{A}_{\delta,n}'' = \underline{A}_\delta''\underline{\alpha}_2\underline{\alpha}_3\cdots\underline{\alpha}_n, \qquad (49)$$

where

$$\underline{\alpha}_j^2 = 1 - \frac{1}{2(j-1)}, \qquad (50)$$

$$\underline{\alpha}_2^2\underline{\alpha}_3^2 \cdots \underline{\alpha}_n^2 = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{6}\right) \cdots \left(1 - \frac{1}{2(n-1)}\right) \quad (51)$$

$$= \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots [2(n-1) - 1]}{2 \cdot 4 \cdot 6 \cdot 8 \; 2(n-1)} \quad (52)$$

$$= \frac{(2n)!}{2^{2n}(n!)^2}. \quad (53)$$

When $n \gg 1$, (51) approaches the value

$$\underline{\alpha}_2^2\underline{\alpha}_3^2 \cdots \underline{\alpha}_n^2 \cong \left(\frac{1}{\pi n}\right)^{1/2}. \quad (54)$$

The latter approximation is based on the following expression, for $x = -\frac{1}{2}$, given in Whittaker and Watson's: "Modern Analysis" page 259:

$$\lim_{n\to\infty} (1 + x)(1 + x/2)(1 + x/3) \cdots (1 + x/n) = \frac{n^x}{\Gamma(1 + x)}, \quad (55)$$

where Γ is the gamma function, $\Gamma(-\frac{1}{2} + 1) = \pi^{1/2}$.

The above analysis assumes that the timing wave at each resonant circuit is applied directly to the next resonant circuit, except for the amplification between resonant circuits. This would be the case if the timing wave were transmitted on a separate pair, in which case $\underline{A}''_{\delta,n}$ would be the rms quadrature component owing to noise in the timing circuit.

In regenerative repeaters, deviations in the timing wave resulting from the quadrature component are imparted at intervals T into the next repeater section as deviations in the spacing of pulses. These timing deviations occurring at intervals T will have a certain random amplitude distribution, which can be regarded as having a certain frequency spectrum. When the deviations are discrete and occur at intervals T, the spectrum will extend to a maximum frequency $f_{max} = 1/2T$, or $\omega_{max} = \pi/T = \omega_0/2$. In this case the upper and lower limits of the integrals above would be replaced by $\pm\omega_0/2$, except for the first repeater section. The recurrence relation (48) is then no longer exact, but the resultant modification is insignificant and can be disregarded. This will be seen when the value $\omega_0/2$ is inserted for ω in the integrand of (47), which then becomes $1/(1 + Q^2)$, as compared with 1 for $\omega = 0$. Thus the contribution to the integrals for $\omega > \omega_0/2$ can for practical purposes be disregarded.