Distortion Produced in a Noise Modulated FM Signal by Nonlinear Attenuation and Phase Shift

By S. O. Rice

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An expression is given for the FM distortion introduced by a transducer whose attenuation and phase shift depend upon the frequency in an arbitrary way. This expression appears to be difficult to evaluate, but it yields useful approximations for the second and third order modulation terms. In all of the work, it is assumed that the distortion is small compared to the signal, and that the signal can be represented by a random noise having the same power spectrum.

INTRODUCTION

A number of workers have been concerned with the problem of computing the distortion introduced by a transducer when an FM wave passes through it. Some of the earliest results were published by Carson and Fry[1] and by van der Pol.[2] Several contributions to the subject have been made recently in connection with studies of microwave radio systems.

An excellent paper on this subject has been published recently by R. G. Medhurst and G. F. Small.[3] Although their results differ considerably in form from those given here, they are nevertheless closely related to ours — their "sinusoidal variations of transmission characteristics" being special cases of our "nonlinear attenuation and phase shift."

Here we treat the problem by applying a method used in a recent paper[4] to study the distortion produced by an echo. Two assumptions are made, (1) that the distortion is small compared to the signal, and (2) that the signal can be represented by a random noise which has the same power spectrum as the signal. In Section I, we review some known results and put them in a form suited to our needs. Sections II and III are devoted to the derivation of our main formulas. The principal result is given by the triple integral (3.2) for the power spectrum of the distortion. Unfortunately, the integrals are difficult to evaluate. However, it is possible to obtain approximations for the second and third order modulation terms. These are given in Section IV. Some miscellaneous comments are made in Section V.

I APPROXIMATE EXPRESSION FOR THE DISTORTION $\theta(t)$

Let the FM signal be $\varphi'(t) = d\varphi/dt$ (for phase modulation the signal would be $\varphi(t)$). Then the FM wave is the real part of

$$v_i(t) = e^{ipt+i\varphi(t)} \tag{1.1}$$

where $p = 2\pi f_p$ is the carrier frequency. Let this wave pass through a transducer having attenuation α and phase shift β, where α and β are even and odd functions, respectively, of the frequency f. When a unit impulse of voltage $\delta(t)$ is applied to the transducer input, the output is

$$g(t) = \int_{-\infty}^{\infty} e^{-\alpha-i\beta+2\pi ift}\, df. \tag{1.2}$$

For physical systems, $g(t)$ is zero for negative t.

When $v_i(t)$ is applied to the transducer input, the output is

$$v_0(t) = \int_{-\infty}^{\infty} v_i(t')g(t-t')\, dt'. \tag{1.3}$$

When $v_0(t)$ is applied to an FM receiver, the detector output consists of the original signal $\varphi'(t)$ plus the distortion $\theta'(t)$ introduced by the transducer. Comparison with (1.1) shows that $\theta(t)$ may be obtained by solving

$$V(t)e^{ipt+i\varphi(t)+i\theta(t)} = v_0(t) \tag{1.4}$$

when p, $\varphi(t)$, $v_0(t)$ are assumed to be known, and $V(t)$, $\theta(t)$ unknown. When $V(t)$ is taken to be positive, (1.4) determines $\theta(t)$ except for an additive term of $2\pi n$ where n is an integer.

We now assume that the transducer acts like a good transmission medium in that the output differs but little from the input. More precisely, we assume

$$|v_0(t) - v_i(t)| \ll 1. \tag{1.5}$$

Since $|v_i(t)| = 1$, it follows that $|v_0(t)| \approx 1$. Transducers having appreciable attenuation and delay may be regarded as two transducers in tandem, one with constant (independent of f) values of α and β/f which are roughly equal to those of the original transducer, and the second with variable α and β/f. The first transducer produces no distortion of the signal, and if condition (1.5) is satisfied by the second, the considerations of this paper will apply.

Equation (1.4) may be written as

$$V(t)e^{i\theta(t)} = v_0(t)/v_i(t)$$

so that

$$\theta(t) = \operatorname{Im} \log \frac{v_0(t)}{v_i(t)}. \tag{1.6}$$

When we write

$$v_0(t)/v_i(t) = 1 + [v_0(t) - v_i(t)]/v_i(t),$$

expand the logarithm in (1.6), and use (1.5), we obtain our approximate expression for $\theta(t)$:

$$\begin{aligned}\theta(t) &= \operatorname{Im} [v_0(t) - v_i(t)]/v_i(t) = \operatorname{Im} v_0(t)/v_i(t) \\ &= \operatorname{Im} [v_i(t)]^{-1} \int_{-\infty}^{\infty} v_i(t')\, g\,(t - t')\, dt' \\ &= \operatorname{Im} \int_{-\infty}^{\infty} \exp\,[ip(t' - t) + i\varphi(t') - i\varphi(t)] g(t - t')\, dt'.\end{aligned} \tag{1.7}$$

So far there is nothing essentially new in our work.[5]

II AUTOCORRELATION FUNCTION OF $\theta(t)$

In Section I, $\varphi'(t)$ could be any reasonable sort of signal. In the following work we assume that it is a Gaussian noise whose power spectrum, $w_{\varphi'}(f)$, is given to us. The power spectrum of $\varphi(t)$ is

$$w_\varphi(f) = w_{\varphi'}(f)/(2\pi f)^2, \tag{2.1}$$

and its autocorrelation function is

$$\psi_\tau = \int_0^{\infty} w_\varphi(f) \cos 2\pi f\tau\, df. \tag{2.2}$$

We have written ψ_τ instead of $\psi(\tau)$ or $R_\varphi(\tau)$ to simplify the appearance of the formulas which occur in our work.

Our problem is to find the power spectrum, $w_\theta(f)$, of the distortion $\theta(t)$, given $w_\varphi(f)$. The method of solution is much the same as that used in Reference 4. We first find the autocorrelation function $R_\theta(\tau)$ of $\theta(t)$ and then obtain $w_\theta(f)$ by taking the Fourier cosine transform of $R_\theta(\tau)$.

Let the last integral in (1.7) be $F(t)$ so that $\theta(t) = \operatorname{Im} F(t)$. Then

$$\theta(t)\theta(t + \tau) = \tfrac{1}{2} \operatorname{Re} \{F(t)F^*(t + \tau) - F(t)F(t + \tau)\} \tag{2.3}$$

where $F^*(t + \tau)$ is the complex conjugate of $F(t + \tau)$. The autocorrelation function of $\theta(t)$ is obtained by averaging over the ensemble of the noise functions $\varphi(t)$:

$$\begin{aligned} R_\theta(\tau) &= \operatorname{av} \theta(t)\theta(t + \tau) \\ &= \operatorname{av} \frac{1}{2} \operatorname{Re} \left\{ \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt'' \exp [ip(t' - t) + i\varphi(t') - i\varphi(t)] \right. \\ &\quad \cdot g(t - t')g(t + \tau - t'')[\exp [-ip(t'' - t - \tau) \\ &\quad - i\varphi(t'') + i\varphi(t + \tau)] - \exp [ip(t'' - t - \tau) + i\varphi(t'') \\ &\quad \left. - i\varphi(t + \tau)] \right\}. \end{aligned} \tag{2.4}$$

Since $g(t)$ is real, $g^*(t) = g(t)$. The averaging process may be carried out by a method analogous to that used in Reference 4. The formula to be used is

$$\begin{aligned} &\operatorname{av} \exp [i\varphi(t') - i\varphi(t) + ia\varphi(t'') - ia\varphi(t + \tau)] \\ &\quad = \exp [-\psi_0(1 + a^2) + \psi_{t'-t} - a\psi_{t'-t''} + a\psi_{t'-t-\tau} \\ &\qquad + a\psi_{t-t''} - a\psi_\tau + a^2\psi_{t''-t-\tau}] \end{aligned} \tag{2.5}$$

where a is either -1 or $+1$, and ψ_τ is an even function of τ. When (2.5) is used in (2.4) a double integral for $R_\theta(\tau)$ is obtained. The substitutions

$$\begin{aligned} x &= t - t', \\ y &= t + \tau - t'', \\ R_v &= \psi_{\tau+x-y} - \psi_{\tau+x} - \psi_{\tau+x} - \psi_{\tau-y} + \psi_\tau \end{aligned} \tag{2.6}$$

convert the double integral into

$$\begin{aligned} R_\theta(\tau) = \frac{1}{2} \operatorname{Re} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, g(x)e^{-ipx-2\psi_0+\psi_x+\psi_y} \\ \cdot g(y)[e^{ipy+R_v} - e^{-ipy-R_v}]. \end{aligned} \tag{2.7}$$

The symbol R_v is chosen to agree as closely as possible with the notation of Reference 4. There R_v was the autocorrelation of the random function, $v(t)$, where $v(t + T) = \varphi(t) - \varphi(t + T)$, T being the echo delay. Here, R_v is the average value of the product,

$$[\varphi(t) - \varphi(t + y)]\,[\varphi(t + \tau) - \varphi(t + \tau + x)]$$

which becomes the autocorrelation function of $v(t)$ when $y = x = T$.

It may be verified that the expression (2.7) for $R_\theta(\tau)$ is an even function of τ, as it should be. Expression (2.7) is the autocorrelation function we set out to find.

The distortion $\theta(t)$ has an average value, $\bar{\theta}$, whose square is $R_\theta(\infty)$. Since $\varphi(t)$ is a noise function, its autocorrelation function ψ_τ goes to zero as τ approaches ∞. Hence, $R_\theta(\infty)$ is given by the expression obtained from (2.7) by setting $R_v = 0$. The autocorrelation function of $\theta(t) - \bar{\theta}$ is

$$
\begin{aligned}
R_{\theta-\bar{\theta}}(\tau) &= R_\theta(\tau) - R_\theta(\infty) \\
&= \frac{1}{2}\operatorname{Re}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, g(x) e^{-ipx-2\psi_0+\psi_x+\psi_y} \\
&\qquad \cdot g(y)[e^{ipy}(e^{R_v} - 1) - e^{-ipy}(e^{-R_v} - 1)].
\end{aligned}
\tag{2.8}
$$

III POWER SPECTRUM OF THE DISTORTION

Since $\theta(t)$ has an average value which is generally not zero, its power spectrum, $w_\theta(f)$, has a spike of infinite height at $f = 0$ corresponding to the power in the dc component $\bar{\theta}$. When this spike is subtracted from $w_\theta(f)$ the remainder is the power spectrum of $\theta(t) - \bar{\theta}$ given by

$$w_{\theta-\bar{\theta}}(f) = 4\int_0^\infty R_{\theta-\bar{\theta}}(\tau)\cos 2\pi f\tau\, d\tau. \tag{3.1}$$

When we use (2.8) and note that $R_{\theta-\bar{\theta}}(\tau)$ is an even function of τ, we obtain

$$
\begin{aligned}
w_{\theta-\bar{\theta}}(f) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy g(x) g(y) e^{-2\psi_0+\psi_x+\psi_y} \int_{-\infty}^{\infty} [\cos(px - py) \\
\cdot (e^{R_v} - 1) - \cos(px + py)(e^{-R_v} - 1)] \cos 2\pi f\tau\, d\tau.
\end{aligned}
\tag{3.2}
$$

Reasoning similar to that given in Reference 4 shows that the interchannel interference spectrum, $w_c(f)$, (i.e., $w_c(f)\Delta f$ is the average amount of distortion power received in an idle channel of width Δf centered on the frequency f, all other channels being busy) may be obtained from (3.2) by replacing $(e^{\pm R_v} - 1)$ by $(e^{\pm R_v} \mp R_v - 1)$.

The power spectrum of $\theta(t) - \bar{\theta}$ may be regarded as made up of modulation products of all orders. It turns out that the contribution of n^{th} order products is given by the integral of the R_v^n terms obtained from the power series expansions of $\exp[\pm R_v]$.

IV FIRST AND SECOND ORDER MODULATION TERMS

Here we shall study the first and second order modulation terms. These arise from the first and second powers of R_v in the expansion of the quantity within the square brackets in (3.2):

$$2R_v \cos px \cos py + R_v^2 \sin px \sin py. \tag{4.1}$$

The integrations with respect to τ may be performed with the help of

$$\int_{-\infty}^{\infty} \psi_{\tau+b} \cos 2\pi f\tau \, d\tau = \frac{w_\varphi(f)}{2} \operatorname{Re} e^{-i2\pi fb}, \tag{4.2}$$

$$\begin{aligned} &\int_{-\infty}^{\infty} \psi_{\tau+b}\psi_{\tau+c} \cos 2\pi f\tau \, d\tau \\ &\qquad = \operatorname{Re} \frac{1}{4} \int_{-\infty}^{\infty} du \, w_\varphi(u) w_\varphi(f-u) \exp \{-i2\pi[bu + c(f-u)]\} \end{aligned} \tag{4.3}$$

which follow from (2.2) and the fact that we have defined $w(-f)$ to be equal to $w(f)$. In our notation the total power in a random noise function is the integral of $w(f)$ from $f = 0$ to $f = \infty$.

The first order modulation term is obtained from (3.2) by replacing the term within the square bracket by $2R_v \cos px \cos py$. When the expression (2.6) for R_v is used, the integration with respect to τ may be performed with the help of (4.2):

$$\int_{-\infty}^{\infty} R_v \cos 2\pi f\tau \, d\tau = \frac{w_\varphi(f)}{2} \operatorname{Re} [(e^{-2\pi ixf} - 1)(e^{i2\pi yf} - 1)]. \tag{4.4}$$

This leads to the following expression for the first order modulation term in (3.2)

$$w_\varphi(f) \left| \int_{-\infty}^{\infty} dx g(x) e^{-\psi_0+\psi_x} \cos px (e^{-2\pi ixf} - 1) \right|^2. \tag{4.5}$$

This is the quantity which is to be subtracted from $w_{\theta-\bar{\theta}}(f)$ to obtain the interchannel interference spectrum $w_c(f)$.

The second order modulation term is handled in much the same manner. With the help of (4.3) it may be shown that

$$\begin{aligned} \int_{-\infty}^{\infty} R_v^2 \cos 2\pi f\tau \, d\tau = \operatorname{Re} \frac{1}{4} \int_{-\infty}^{\infty} & du \, w_\varphi(u) w_\varphi(f-u) \\ & \cdot (e^{-2\pi ixu} - 1)\,(e^{-2\pi ix(f-u)} - 1) \\ & \cdot (e^{2\pi iyu} - 1)\,(e^{2\pi iy(f-u)} - 1). \end{aligned} \tag{4.6}$$

From this it follows that the second order modulation term in (3.2) is

$$\begin{aligned} \frac{1}{2!\,2} \int_{-\infty}^{\infty} du \, w_\varphi(u) w_\varphi(f-u) \bigg| \int_{-\infty}^{\infty} & dx \, g(x) e^{-\psi_0+\psi_x} \sin px \\ & \cdot (e^{-2\pi ixu} - 1)(e^{-2\pi ix(f-u)} - 1) \bigg|^2. \end{aligned} \tag{4.7}$$

When $\psi_0 - \psi_x$ is so small that $\exp(-\psi_0 + \psi_x)$ may be replaced by unity, as it is in some important practical cases, approximations may be obtained for (4.5) and (4.7). The integral in x may be expressed as the sum of integrals of the type

$$\begin{aligned}\int_{-\infty}^{\infty} g(x)e^{-ipx-2\pi iax}\,dx &= [e^{-\alpha-i\beta}]_{f=a+f_p} \\ &= G_a + iB_a\,, \qquad (4.8) \\ \int_{-\infty}^{\infty} g(x)e^{ipx-2\pi iax}\,dx &= G_{-a} - iB_{-a}\,.\end{aligned}$$

The values of the integrals follow from (1.2) and the Fourier integral theorem. G and B are, respectively, even and odd functions of frequency, and G_a, B_a are their values at the frequency $f = f_p + a$ where $f_p = p/2\pi$ is the carrier frequency:

$$G \text{ at frequency } f_p + a = G_a\,,$$

$$B \text{ at frequency } f_p + a = B_a\,.$$

In this way we get the approximation

$$4^{-1}w_\varphi(f)\,[(G_f - 2G_0 + G_{-f})^2 + (B_f - B_{-f})^2] \qquad (4.9)$$

for the first order modulation term, and

$$\begin{aligned}\frac{1}{2!8}\int_{-\infty}^{\infty} du\, w_\varphi(u)w_\varphi(f-u)[(G_u - G_{-u} + G_{f-u} - G_{-f+u} - G_f \\ + G_{-f})^2 + (B_u + B_{-u} + B_{f-u} + B_{-f+u} - B_f - B_{-f} - 2B_0)^2]\end{aligned} \qquad (4.10)$$

for the second order modulation term.

Expression (4.10) is an approximation to the second order modulation term (4.7). When most of the interchannel interference is due to second order modulation products, (4.10) is also an approximation to $w_c(f)$, the interchannel interference spectrum. The following remarks may be of some help in deciding whether (4.10) may be used.

1. For the case of phase modulation and a "flat" signal band, the first of equations (5.3) shows that ψ_0 and ψ_τ may be made as small as we please by choosing the signal power (as measured by P_0) small enough. Since R_v is proportional to P_0, P_0 may be chosen small enough to make R_v^3 and higher order terms negligible in the expansion of the integrand of (3.2) (unless there is some sort of symmetry which causes the second order terms to vanish). In this case the interference is mostly second order modulation and (4.7) is a good approximation to $w_c(f)$. Furthermore, as P_0 approaches zero, $\exp(-\psi_0 + \psi_x)$ approaches unity

and (4.10) becomes a good approximation to (4.7). Just how small P_0 has to be depends upon the signal bandwidth, f_b, and the characteristics of the transducer.

2. For the case of FM and a flat signal band, the second of equations (5.3) shows that even if P_0 is small, the difference $\psi_0 - \psi_\tau$ approaches ∞ as $|\tau|$ approaches ∞. To justify the use of (4.10) in this case it is necessary to take into account the behavior of $g(t)$, the response of the transducer to the unit impulse $\delta(t)$. For example, if the duration of $g(x)$ in (4.7) is so brief that $g(x)$ becomes negligibly small before $-\psi_0 + \psi_x$ becomes appreciably different from zero (which may be achieved by making P_0 small enough) then (4.10) is a good approximation to (4.7).

3. When the attenuation, α, and phase shift, β, are given for any particular transducer, the corresponding $g(t)$ may be obtained from (1.2). Once $g(t)$ and $\psi_0 - \psi_\tau$ are known, the conditions under which $\exp(-\psi_0 + \psi_x)$ may be replaced by unity in (4.7) and $O(R_v^3)$ terms neglected in (3.2) may be determined by direct examination of the integrals.

As might be expected, the third order modulation results are quite complicated. The third order modulation term in (3.2) is

$$\frac{1}{3!4}\int_{-\infty}^{\infty} df' \int_{-\infty}^{\infty} df'' w_\varphi(f') w_\varphi(f'') w_\varphi(f''') \left| \int_{-\infty}^{\infty} dx\, g(x) \cos px\, e^{-\psi_0+\psi_x}(z^{f'} - 1)(z^{f''} - 1)(z^{f'''} - 1) \right|^2 \qquad (4.11)$$

where $f''' = f - f' - f''$ and $z = \exp(-i2\pi x)$. When ψ_0 is small this is approximately

$$\frac{1}{3!16}\int_{-\infty}^{\infty} df' \int_{-\infty}^{\infty} df'' w_\varphi(f') w_\varphi(f'') w_\varphi(f''')[H^2 + K^2] \qquad (4.12)$$

where

$$\begin{aligned} H = m(f') + m(f'') + m(f) + m(f - f' - f'') \\ - m(f - f') - m(f - f'') - m(0) - m(f' + f''), \end{aligned} \qquad (4.13)$$

$$m(f) = G_f + G_{-f}, \qquad n(f) = B_f - B_{-f},$$

and K is an expression obtained from H by replacing n by m.

V MISCELLANEOUS COMMENTS

Here we make some miscellaneous comments related to the foregoing results.

If the transducer is perfect except for an echo, its response to a unit impulse $\delta(t)$ is

$$g(t) = \delta(t) + r\delta(t - T) \tag{5.1}$$

where r and T are the amplitude and the delay of the echo. The results obtained using (5.1) agree, as they should, with the results obtained in Reference 4. Of course, r must be assumed small compared to unity in order that condition (1.5) may hold.

When the power spectrum of the signal is equal to a constant P_0 over the band (f_a, f_b) and zero elsewhere we have for phase and frequency modulation, respectively,

$$\begin{aligned} \text{PM:} \quad & w_\varphi(f) = P_0, & f_a < f < f_b, \\ \text{FM:} \quad & w_\varphi(f) = P_0/(2\pi f)^2, & f_a < f < f_b. \end{aligned} \tag{5.2}$$

When $f_a = 0$ the autocorrelation functions are

$$\begin{aligned} &\text{PM:} \quad \psi_\tau = P_0 f_b (\sin v)/v, \\ &\text{FM:} \quad \psi_0 - \psi_\tau = A[-1 + \cos v + vSi(v)], \\ &v = 2\pi f_b \tau, \qquad A = P_0 f_b (2\pi f_b)^{-2} = (\sigma/f_b)^2. \end{aligned} \tag{5.3}$$

The mean square values of the signals are $P_0 f_b$ (radians)^2 for PM and $P_0 f_b$ (radians/sec)^2 for FM. If, for FM, σ is the rms frequency deviation in cps (so that the "peak" deviation is, say, 4σ cps) then $(2\pi\sigma)^2 = P_0 f_b$. The difference $\psi_0 - \psi_\tau$ is used in the FM case to avoid difficulty at $f = 0$. It will be noticed that our formulas are such that the ψ's may be replaced by $(\psi - \psi_0)$'s without altering the values of the various exponents, etc. In microwave systems the quantity A is often small in comparison with unity.

As an example of the use of the second order modulation approximation (4.10) consider the case where the attenuation, α, is zero and the phase shift $\beta = a_2(f - f_p)^2/2$ radians, a_2 being small. Then, since $G \approx 1 - \alpha$, $\beta \approx -\beta$, we have $G_u \approx 1$ and

$$\begin{aligned} B_u &\approx -[\beta \text{ for } f = f_p + u] \\ &= -a_2 u^2/2. \end{aligned} \tag{5.4}$$

When we take the FM case of (5.2) and substitute in the approximation (4.10), the interchannel interference power spectrum is found to be

$$\begin{aligned} &\frac{1}{2!\,8} \int_{f-f_b}^{f_b} \frac{P_0}{(2\pi u)^2} \frac{P_0}{(2\pi)^2(f-u)^2} [0 + (2a_2 u(f-u))^2]\, du \\ &\qquad = (2\pi)^{-4}(a_2 P_0/2)^2(2f_b - f). \end{aligned} \tag{5.5}$$

Dividing by $w_\varphi(f) = P_0/(2\pi f)^2$ gives the ratio of the interference power to the signal power

$$(a_2\sigma f/2)^2(2 - f/f_b) \tag{5.6}$$

where the relation $P_0 = (2\pi\sigma)^2/f_b$ has been used to eliminate P_0. Here σ is the rms frequency deviation of the FM signal in cps. The expression (5.6) agrees with results of some earlier work done at Bell Telephone Laboratories. In that work the second order modulation products were summed directly.

It is interesting to apply the formulas given here to some of the cases considered by Medhurst and Small.[3] They have shown that when (in our notation) $\alpha = -r \cos 2\pi fT$ and $\beta = 0$ the power spectrum of the distortion is

$$w_{\theta-\bar{\theta}}(f) = \sin^2 \pi fT[w_{\theta-\bar{\theta}}(f)]_{\text{echo}}, \tag{5.7}$$

and when $\alpha = 0$ and $\beta = r \sin 2\pi fT$,

$$w_{\theta-\dot{\theta}}(f) = \cos^2 \pi fT[w_{\theta-\bar{\theta}}(f)]_{\text{echo}}. \tag{5.8}$$

Here $[w_{\theta-\bar{\theta}}(f)]_{\text{echo}}$ is the power spectrum of the distortion due to a simple echo of amplitude r and delay T (corresponding to $\alpha = -r \cos 2\pi fT$ and $\beta = r \sin 2\pi fT$). Expressions (5.7) and (5.8) may also be obtained by setting the impulse response $g(t)$ equal to

$$\delta(t) + \frac{r}{2}\,\delta(t - T) \pm \frac{r}{2}\,\delta(t + T)$$

in (3.2).

The second order modulation approximation for the $\alpha = -r \cos 2\pi fT$, $\beta = 0$ case may be obtained from (4.10) and turns out to be

$$\int_{-\infty}^{\infty} w_\varphi(u)w_\varphi(f - u)[2r \sin pT \sin \pi fT \sin \pi uT \sin \pi(f - u)T]^2 \, du. \tag{5.9}$$

It is seen that this contains the factor $\sin^2\pi fT$ predicted by (5.7). When (5.9) is applied to the FM case of (5.2) an integral something like (5.5) (but more complicated) is obtained. The ratio of the second order modulation interference power to the signal power is found to be

$$2[r \sin pT \sin \pi fT]^2(\sigma/f_b)^2UK \tag{5.10}$$

where K is the quantity

$$K = 2\alpha^2 \int_{\alpha-U}^{U} \left[\frac{\sin (y/2) \sin (\alpha - y)/2}{y(\alpha - y)}\right]^2 dy \tag{5.11}$$

tabulated in Table 4.2 of Reference 4 and

$$\alpha = 2\pi f T, \qquad U = 2\pi f_b T. \tag{5.12}$$

The parameters a and k that appear in the table are defined by

$$a = f/f_b \quad \text{and} \quad k = 8 f_b T.$$

These formulas serve to supplement the formulas and curves given by Medhurst and Small.

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