Coincidences in Poisson Patterns

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A number of practical problems, including questions about reliability of Geiger counters and short-circuits in electric cables, reduce to the mathematical problem of coincidences in Poisson patterns. This paper presents the probability of no coincidences as well as asymptotic formulas and simple bounds for that probability under a variety of circumstances. The probability of exactly N coincidences is also found in some cases.

INTRODUCTION

A number of practical problems are questions about what we call "coincidences" in Poisson patterns. In d-dimensional space, a Poisson pattern of density λ is a random array of points such that each infinitesimal volume element, dV, has probability λdV of containing a point, and such that the numbers of points in disjoint regions are independent random variables. Then a volume, V, has probability

$$\frac{(\lambda V)^k}{k!} e^{-\lambda V}$$

of containing exactly k points. A coincidence, in our usage of the word, is defined as follows: We imagine a certain fixed distance δ to be given in advance; two points are then said to be *coincident* if they lie within distance δ of one another.

Examples

The best-known case of a coincidence problem concerns Geiger counters. In the simplest mathematical model, there is a short dead-time δ after each count during which other particles can pass through the counter without registering a count. In our present terminology, a count is missed whenever two particles traverse the counter with coincident times of arrival. The same problem is encountered with telephone call registers.

Another example arises in the manufacture of electric cable. Each wire in a cable is covered with an insulation which contains occasional flaws. When the cable is assembled it will fail a short circuit test if it contains a pair of wires such that a flaw on one wire lies within some distance δ of a flaw on the other wire. In a similar way, coincident flaws in the insulation of the wire from which a coil is wound can lead to failure of the coil.

There are also some problems in the development of certain military systems which lead to the consideration of coincidences in Poisson patterns.

Outline of Work

Our primary aim is to study the probability of no coincidences under various circumstances. In Part I, we examine coincidences of two different Poisson patterns, of densities λ and μ respectively, on a line of length L. Here we do not count two points of the *same* pattern within a distance δ as giving a coincidence. A set of integral equations yields the probability of no coincidences as well as an asymptotic formula and upper and lower bounds.

In Part II, we study the probability, $F_0(L)$, of no coincidences for a single one-dimensional Poisson pattern of density λ. These results may also be interpreted as the distribution function for the minimum distance between pairs of points of a Poisson pattern. Sample formulas are the asymptotic formula (for large L)

$$F_0(L) \approx \frac{\lambda}{(\lambda - a)[1 + \delta(\lambda - a)]} e^{-aL},$$

and the bounds (valid for all L)

$$\left(1 - \frac{a}{\lambda}\right) e^{-aL} e^{-(a-\lambda)\delta} \leqq F_0(L) \leqq e^{-aL} e^{-(a-\lambda)\delta},$$

where $s = -a$ is the largest real root of

$$s + \lambda = \lambda e^{-(s+\lambda)\delta}.$$

The problem of n Poisson patterns, all of the same density λ, is examined in Part III. Coincidences are now counted between points of any two distinct patterns.

The one-dimensional problems of Parts I–III succumb readily to analytic techniques. We can find exact expressions for the probabilities of no coincidences in Parts I–III. Two entirely different methods of deriving exact results are available and are illustrated in Parts II and III. Unfortunately, the exact formulas, although they are finite sums, contain a number of terms which grows with L. Much of our effort has been directed toward finding good, easily computed bounds and asymptotic formulas.

The probabilities of having exactly N coincidences are also obtainable but they have more complicated formulas. A detailed derivation is given only in Part II.

In Part IV, we consider the probability of no coincidence in higher dimensional problems. The methods of Parts I–III fail in higher dimensions, but we are still able to derive some bounds. An exact formula is derived for the probability of no coincidences within a single two-dimensional Poisson pattern in a rectangle with sides $\leqq 2\delta$. We also give particular attention to coincidences in a three-dimensional cylinder.

Part V contains numerical results.

Reduction of the Examples to the Theory

We now wish to see how answers bearing on the practical problems previously listed may be found from this study.

The literature on Geiger counters (see bibliography in Feller[3]) is concerned with statistics of the number of counts registered in a given long time, t. The basic problem is to test the hypothesis that the particles arrive in a Poisson sequence. To this problem, then, are relevant the formulas for the probability of N coincidences in one pattern given in Part II, and the bounds and asymptotic results there derived.

The problem of coincident flaws in an electric cable is three-dimensional, and we have various approaches leading to the probability of no coincidences which are valid under different circumstances. If the cable contains only two wires (with possibly different flaw densities), then the problem reduces to the one-dimensional case of coincidences between two Poisson patterns treated in Part I. If the diameter of the cable is small with respect to δ, and if the density of flaws is the same on each of the n wires in the cable, we have the situation of n identical patterns treated in Part III. If, in addition, n is very large, we may ignore the fact that coincident flaws on a single wire do not cause short circuits, and think of coincidences within a single pattern (Part II). Without the assumption that the diameter of the cable is small with respect to δ, the problem is no longer reducible to a one-dimensional form. Section 4.4 is especially devoted to thick cable, and to producing a lower bound for the probability of no coincidences in this three-dimensional situation.

The literature on Poisson patterns in a line segment contains the following related papers. C. Domb[1] finds the distribution function for the total length of the set of points lying within distance δ of a pattern point. P. Eggleton and W. O. Kermack[2] and also L. Silberstein[5] consider *aggregates*, which are sets of k pattern points all contained in an interval of length δ. In the special case $k = 2$, aggregates are our coincidences. These authors find the expected number of aggregates but not the probability of N aggregates.

I CONCIDENCES BETWEEN TWO PATTERNS

1.1 Integral Equation

Consider two Poisson patterns of points on the real line, the first with density λ (points per unit length) and the second with density μ. We want the probability $F(L)$ that in the segment from 0 to L there is no coincidence between a point of pattern No. 1 and a point of Pattern No. 2. $F(L)$ will be formulated in terms of the conditional probabilities

$P_1(L)$ = Prob (no coincidence, given Pattern No. 1 has point at L),

$P_2(L)$ = Prob (no coincidence, given Pattern No. 2 has point at L).

If $L \leqq \delta$, $P_1(L)$ and $P_2(L)$ are the probabilities that patterns No. 2 and No. 1 are empty:

$$P_1(L) = e^{-\mu L}, \qquad P_2(L) = e^{-\lambda L}, \quad \text{if} \quad L \leqq \delta. \tag{1-1}$$

If $L > \delta$ and Pattern No. 1 contains a point at L, there are two ways that no coincidences can occur. First, Pattern No. 2 may fail to have any points anywhere in the interval $[0, L]$. The probability of this event is $\exp - \mu L$. The second possibility is illustrated in Fig. 1 (using circles for points of Pattern No. 1 and crosses for points in Pattern No. 2). Pattern No. 2 has points in $(0, L)$; the one closest to L is at $y < L - \delta$. Since the interval (y, L) contains no points of Pattern No. 2, the probability of finding this closest point, y, in an interval, dy, is

$$\exp [-\mu(L - y)]\mu \, dy.$$

The interval $(y, y + \delta)$ must be free from points of Pattern No. 1 (prob-

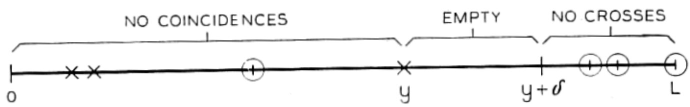

Fig. 1 — Patterns without coincidence.

ability $\exp(-\lambda\delta)$) and the interval $(0, y)$ must contain no coincidences (probability $P_2(y)$). One obtains finally

$$P_1(L) = e^{-\mu L}\left[1 + \mu e^{-\lambda\delta}\int_0^{L-\delta} e^{\mu y}P_2(y)\,dy\right], \tag{1-2}$$

and similarly

$$P_2(L) = e^{-\lambda L}\left[1 + \lambda e^{-\mu\delta}\int_0^{L-\delta} e^{\lambda y}P_1(y)\,dy\right]. \tag{1-3}$$

The solutions $P_1(L)$ and $P_2(L)$ are determined uniquely by (1-1), (1-2) and (1-3). For (1-1) determines them for $0 \leqq L \leqq \delta$ and the integrations indicated in (1-2) and (1-3) will provide the solutions in $0 \leqq L \leqq (n + 1)\delta$ when they are known in $0 \leqq L \leqq n\delta$. $P_1(L)$ and $P_2(L)$ are piecewise analytic; the analytic form of the solution changes each time L passes an integer multiple of δ. These analytic expressions soon become complicated and are less useful than the bounds and approximations given later on.

To compute $F(L)$, consider the last place before L at which either Pattern No. 1 or No. 2 has a point. The probability that this last point lies between x and $x + dx$ and belongs to Pattern No. 1 is $\exp[-(\lambda + \mu)(L - x)]\lambda\,dx$ (Fig. 2). This term multiplied by $P_1(x)$ and integrated from 0 to L gives the probability of no coincidences if the last point is a circle. A similar integral gives the probability if the last point is a cross. Finally there is probability $\exp[-(\lambda + \mu)L]$ that neither pattern has a last point [i.e., $(0, L)$ empty]. Then

$$F(L) = e^{-(\lambda+\mu)L}\left[1 + \int_0^{L} e^{(\lambda+\mu)x}(\lambda P_1(x) + \mu P_2(x))\,dx\right]. \tag{1-4}$$

1.2 Solution by Laplace Transforms

For $i = 1$ or 2, let

$$p_i(s) = \int_0^{\infty} P_i(L)e^{-sL}\,dL. \tag{1-5}$$

Replacing $P_1(L)$ in (1-5) by (1-1) for $0 \leqq L \leqq \delta$, by (1-2) for $\delta \leqq L$,

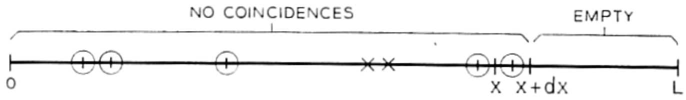

Fig. 2 — Patterns without coincidence.

and interchanging the order of integration of a double integral,

$$(s + \mu)p_1(s) = 1 + \mu e^{-(\lambda+\mu+s)\delta} p_2(s).$$

Similarly,

$$(s + \lambda)p_2(s) = 1 + \lambda e^{-(\lambda+\mu+s)\delta} p_1(s),$$

so that

$$p_1(s) = \frac{s + \lambda + \mu e^{-(\lambda+\mu+s)\delta}}{(s + \lambda)(s + \mu) - \lambda\mu e^{-2(\lambda+\mu+s)\delta}}, \tag{1-6}$$

and

$$p_2(s) = \frac{s + \mu + \lambda e^{-(\lambda+\mu+s)\delta}}{(s + \lambda)(s + \mu) - \lambda\mu e^{-2(\lambda+\mu+s)\delta}}. \tag{1-7}$$

Likewise, using (1-4), the Laplace transform $f(s)$ of $F(L)$ is

$$f(s) = \frac{1 + \lambda p_1(s) + \mu p_2(s)}{\lambda + \mu + s}.$$

As one might expect from the piecewise analytic character of $P_1(L)$ and $P_2(L)$ there is no convenient way of transforming $f(s)$ back to $F(L)$. By evaluating residues of $f(s)$ exp (sL) at the poles of $f(s)$ one might express $F(L)$ as an infinite series of exponential terms. The most slowly damped term in this series can be expected to approximate $F(L)$ when L is large. The poles of $f(s)$ are at the zeros of the denominator $D(s)$ of $p_1(s)$ and $p_2(s)$:

$$D(s) = (s + \lambda)(s + \mu) - \lambda\mu e^{-2(\lambda+\mu+s)\delta}. \tag{1-9}$$

Since $D(x) > 0$ for $x \geqq 0$ and both $D(-\lambda)$ and $D(-\mu)$ are negative, it follows that $D(s)$ has a real zero $s = -a$ with $a < \text{Min}(\lambda, \mu)$.

The zero $s = -a$ of $D(s)$ is the one with the largest real part. For, letting $s = x + iy$, we have in the half plane $x \geqq -a$

$$\begin{aligned} |(s + \lambda)(s + \mu)| - \lambda\mu\, |e^{-2(\lambda+\mu+s)\delta}| &= |s + \lambda| \cdot |s + \mu| - \lambda\mu e^{-2(\lambda+\mu+x)\delta} \\ &\geqq (x + \lambda)\cdot(x + \mu) - \lambda\mu e^{-2(\lambda+\mu+x)\delta} \geqq 0. \end{aligned}$$

Also, if $y \neq 0$ the $\geqq$ sign in the above proof can be replaced by $>$ and one concludes that all other zeros of $D(s) = 0$ satisfy

$$\text{Re}\, s < -b$$

for some $b > a$ (note that the left hand side of the preceding inequality does not approach 0 as y approaches $\pm\infty$).

The pole of $f(s)$ at $s = -a$ contributes to $F(L)$ a dominant term

$$F(L) \approx \frac{\lambda^2 + \mu^2 - (\lambda + \mu)a + 2\lambda\mu e^{-(\lambda+\mu-a)\delta}}{(\lambda + \mu - a)[\lambda + \mu - 2a + 2\delta(\lambda - a)(\mu - a)]} e^{-aL}. \quad (1\text{-}10)$$

In (1-10) the error is $0(\exp - bL)$ for large L.

When δ is small, we find $a = 2\lambda\mu\delta + 0(\delta^2)$ and (1-10) becomes

$$F(L) \approx [1 + 0(\delta^2)] \exp - [2\lambda\mu\delta + 0(\delta^2)]L. \quad (1\text{-}11)$$

It is interesting to note that a simple heuristic argument also leads to a formula like (1-11). When δ is small and L is large, one expects that the intervals of length 2δ which contain points of Pattern No. 1 at their centers will comprise a total length near $(\lambda L)(2\delta)$ of the line segment $(0, L)$. The probability that a set of length $2\lambda L\delta$ shall be free of points of Pattern No. 2 is $\exp - 2\lambda\mu\delta L$.

1.3 Bounds

In this section we derive some relatively simple expressions which are good upper and lower bounds on $F(L)$. Both bounds have the same functional form:

$$K(A, B; L) = \frac{\lambda A + \mu B}{\lambda + \mu - a} e^{-aL} + \left(1 - \frac{\lambda A + \mu B}{\lambda + \mu - a}\right) e^{-(\lambda+\mu)L}. \quad (1\text{-}12)$$

In (1-12), a is again the smallest real solution of $D(-a) = 0$. A and B are positive constants which are related by

$$\frac{A}{B} = \frac{\mu}{\mu - a} e^{-(\lambda+\mu-a)\delta} = \frac{\lambda - a}{\lambda} e^{(\lambda+\mu-a)\delta}. \quad (1\text{-}13)$$

$K(A, B; L)$ becomes an upper bound or a lower bound depending on additional restrictions which will be placed on A and B.

To get the lower bound, we restrict A and B by the inequalities

$$A < e^{(a-\mu)\delta}, \qquad B < e^{(a-\lambda)\delta}, \quad (1\text{-}14)$$

and

$$A < \left(1 - \frac{a}{\lambda}\right) e^{\mu\delta}, \qquad B < \left(1 - \frac{a}{\mu}\right) e^{\lambda\delta}. \quad (1\text{-}15)$$

We first prove that (1-13), (1-14), and (1-15) imply

$$P_1(L) > Ae^{-aL}, \qquad P_2(L) > Be^{-aL}. \quad (1\text{-}16)$$

When $0 \leqq L \leqq \delta$, (1-16) holds because of (1-1), (1-14), and the inequalities $a < \lambda$, $a < \mu$. If (1-16) were not true for all L there would be a smallest value, say $L = X > \delta$, at which at least one of the inequalities (1-16) would become an equality. Suppose the inequality (1-16) on $P_1(X)$ fails. Using (1-16) for $L < X$, and (1-2),

$$P_1(X) > e^{-\mu X}\left(1 + B\mu e^{-\lambda\delta}\,\frac{e^{(\mu-a)(X-\delta)} - 1}{\mu - a}\right)$$

$$> Ae^{-aX} + \left(1 - B\,\frac{\mu e^{-\lambda\delta}}{\mu - a}\right)e^{-\mu X} \quad \text{by} \quad (1\text{-}13),$$

$$> Ae^{-aX} \quad \text{by} \quad (1\text{-}15).$$

This contradicts our assumption that (1-16) fails for $P_1(X)$. A similar proof shows (1-16) cannot fail for $P_2(X)$.

Having proved (1-16) we now substitute these bounds into (1-4) and integrate to get $F(L) > K(A, B; L)$.

To make (1-12) into an upper bound it is only necessary to replace (1-14) and (1-15) by

$$A > 1, \qquad B > 1, \tag{1-17}$$

and

$$A > \left(1 - \frac{a}{\lambda}\right)e^{\mu\delta}, \qquad B > \left(1 - \frac{a}{\mu}\right)e^{\lambda\delta}. \tag{1-18}$$

The proof that now $F(L) < K(A, B; L)$ proceeds exactly as before but with all the inequality signs reversed.

Both bounds are dominated by an exponential term $\exp - aL$, as is the asymptotically correct formula (1-10). In typical numerical cases the coefficients multiplying this term in the three formulas agree closely. A numerical case is given in Part V.

1.4 *Probability of N Coincidences*

The methods of Sections 1.1 and 1.2 can also be used to find the probability $F_N(L)$ that there be exactly N coincidences in the interval $(0, L)$. It might appear most natural to define N to be the number of pairs of points (x, z), x from Pattern No. 1, z from Pattern No. 2, such that

$$| x - z | < \delta. \tag{i}$$

However, we add the additional requirement that x and z be "adjacent" points; i.e.

$$\text{the interval } (x, z) \text{ is empty.} \tag{ii}$$

For example, in Fig. 3, we would count $N = 6$ coincidences even though there are 18 pairs which satisfy (i). In cable problems it appears reasonable to count coincidences as above. If we assume that all flaws are equally bad, then a short circuit is likely to develop only across an adjacent coincidence; our N is the number of places on the cable at which a short circuit can form. Another interpretation is that the cable can be cut into exactly $N + 1$ pieces each of which contain no coincidences.

Let $P_{1,N}(L)$ be the conditional probability of having N coincidences in $(0, L)$ knowing that there is a point of Pattern No. 1 at L. The Laplace transform of $P_{1,N}(L)$ turns out to be the coefficient of t^N in a generating function of the form

$$p_1(t, s) = \frac{\lambda + s + \mu\Omega}{(\lambda + s)(\mu + s) - \lambda\mu\Omega^2},$$

where $\Omega = e^{-(\lambda+\mu+s)\delta}(1 - t) + t$. Interchanging λ and μ one gets the generating function $p_2(t, s)$ for the Laplace transform of the probability $P_{2,N}(L)$ of N coincidences, given a point of Pattern No. 2 at L. Finally the Laplace transform of $F_N(L)$ is the coefficient of t^N in the generating function

$$f(t, s) = \frac{1 - e^{-(\lambda+\mu+s)\delta} + \lambda p_1(t, s) + \mu p_2(t, s)}{\lambda + \mu + s}.$$

Since $f(t, s)$ is a rational function of t, it is easy to find the coefficient of t^N. The poles of this function are again just zeros of $D(s)$. Now, however, the poles are higher order poles. For large L an asymptotic formula for $F_N(L)$ has the form exp $- aL$ times a polynomial in L with degree depending on N.

For more details about this method we refer the reader to Part II where a similar, but less involved, calculation is carefully done.

II SELF-COINCIDENCES IN ONE POISSON PATTERN

2.1 Integral Equation

In this part we shall consider a single one-dimensional Poisson pattern with density λ and ask for the probability $F_N(L)$ that in the interval $(0, L)$ the pattern have exactly N coincidences. We count coincidences

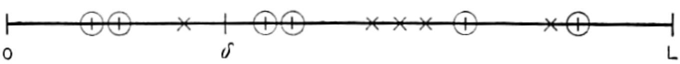

Fig. 3 — Patterns with six coincidences.

as in Section 1.4; a pair (x, z) of pattern points contributes one coincidence to the total number N only if both $|x - z| < \delta$ and the interval between x and z is empty.

Note that $F_0(L)$ is related to the distribution function for the minimum distance between the points of the pattern in $(0, L)$:

$$\text{Prob (min. dist.} \leqq \delta) = 1 - F_0(L),$$

where it must be remembered that $F_0(L)$ is a function of δ.

As in Part I, we first define the conditional probabilities $P_N(L) =$ Prob (exactly N coincidences in $(0, L)$, given a point at L). We then have the following equations:

$$\text{If } L \leqq \delta, \qquad P_N(L) = \frac{(\lambda L)^N}{N!} e^{-\lambda L}, \qquad \text{all } N. \tag{2-1}$$

$$\text{If } L \leqq \delta, \qquad F_N(L) = \frac{(\lambda L)^{N+1}}{(N+1)!} e^{-\lambda L}, \qquad N \geqq 1. \tag{2-2}$$

If $L > \delta$, and $N \geqq 1$, the probability of exactly N coincidences in $(0, L)$ equals the probability of N coincidences up to the last point of the pattern in the interval $(0, L)$ — and if there are to be any coincidences, there must be points of the pattern in $(0, L)$. Hence, if $L > \delta$, $N \geqq 1$,

$$F_N(L) = \int_0^L P_N(L - y)e^{-\lambda y}\lambda \, dy. \tag{2-3}$$

If $N = 0$, the same argument applies, but there is also the possibility that there are no points at all of the pattern in $(0, L)$. Hence, if $L > \delta$,

$$F_0(L) = e^{-\lambda L} + \int_0^L P_0(L - y)e^{-\lambda y}\lambda \, dy. \tag{2-4}$$

Now let us consider the case where there is a point of the pattern at L. Then if the last point preceding L is between $L - \delta$ and L, this point and the point at L will create a coincidence; if there is no point within $(L - \delta, L)$, then all coincidences are within $(0, L - \delta)$. Hence, if $L > \delta$, and $N \geqq 1$,

$$P_N(L) = \int_0^\delta P_{N-1}(L - y)\lambda e^{-\lambda y} \, dy + e^{-\lambda\delta}F_N(L - \delta). \tag{2-5}$$

For the case $N = 0$, we cannot allow a point in the interval $(L - \delta, L)$, and hence, if $L > \delta$,

$$P_0(L) = e^{-\lambda\delta}F_0(L - \delta). \tag{2-6}$$

2.2 *Laplace Transform of $F_N(L)$*

To analyze the system of equations which is given by relations (2-1) through (2-6), we introduce the generating functions

$$f(L, t) = \sum_{N=0}^{\infty} F_N(L)t^N,$$

and

$$p(L, t) = \sum_{N=0}^{\infty} P_N(L)t^N.$$

If $L > \delta$, we obtain from (2-3) and (2-4) the relation

$$e^{\lambda L}f(L, t) = 1 + \int_0^L p(w, t)e^{\lambda w}\lambda\, dw, \tag{2-7}$$

and from (2-5) and (2-6) the relation (again if $L > \delta$)

$$e^{\lambda L}p(L, t) = \lambda t \int_{L-\delta}^{L} p(w, t)e^{\lambda w}\, dw + e^{\lambda(L-\delta)}f(L - \delta, t). \tag{2-8}$$

If we differentiate (2-7) and (2-8) with respect to L, and then apply (2-7) differentiated to simplify the last terms of (2-8) differentiated, we obtain, still only for $L > \delta$,

$$f'(L, t) + \lambda f(L, t) = \lambda p(L, t), \tag{2-9}$$

$$p'(L, t) + \lambda(1 - t)p(L, t) = \lambda e^{-\lambda\delta}(1 - t)p(L - \delta, t). \tag{2-10}$$

It is easy to check from (2-1) and (2-2) that if $L \leqq \delta$, then

$$p(L, t) = e^{-\lambda L(1-t)},$$

and

$$f(L, t) = e^{-\lambda L}\left(\frac{(e^{\lambda Lt} - 1)}{t} + 1\right),$$

and hence (2-9) is valid for *all* L, but the left side of (2-10) *vanishes* if $L \leqq \delta$. Hence we may take Laplace transforms of (2-9) and (2-10). If we define

$$A(s, t) = \int_0^\infty f(L, t)e^{-Ls}\, dL,$$

and

$$B(s, t) = \int_0^\infty p(L, t)e^{-Ls}\, dL,$$

we obtain from (2-9), which we now know to be valid for all L,

$$(\lambda + s)A(s, t) - 1 = \lambda B(s, t), \tag{2-11}$$

and from (2-10), by recalling that the left side vanishes for $L \leqq \delta$,

$$sB(s, t) - 1 + \lambda(1 - t)B(s, t) = \lambda(1 - t)e^{-(s+\lambda)\delta}B(s, t). \tag{2-12}$$

Hence

$$B(s, t) = \frac{1}{s + \lambda(1 - t)[1 - e^{-(s+\lambda)\delta}]}, \tag{2-13}$$

and

$$A(s, t) = \frac{1}{\lambda + s}(1 + \lambda B(s, t)).$$

If we denote the Laplace transforms of $P_N(L)$ and $F_N(L)$ by $p_N(s)$ and $f_N(s)$ respectively, then

$$p_N(s) = \frac{\lambda^N[1 - e^{-(s+\lambda)\delta}]^N}{[s + \lambda - \lambda e^{-(s+\lambda)\delta}]^{N+1}}, \tag{2-14}$$

and

$$\begin{aligned} f_0(s) &= \frac{1}{\lambda + s}(\lambda p_0(s) + 1), \\ f_N(s) &= \frac{\lambda}{\lambda + s}p_N(s) \quad \text{for} \quad N = 1, 2, \cdots . \end{aligned} \tag{2-15}$$

2.3 Exact Formula for $F_0(L)$

It is possible to solve (2-1) through (2-6) in piecewise analytic form by computing recursively from each interval of length δ to the next one. We shall obtain the piecewise analytic form for $F_0(L)$ by a direct derivation essentially due to E. C. Molina.[4]

Suppose k is the number of pattern points which fall into $(0, L)$. Let x_i denote the distance between the $i - 1^{\text{st}}$ point and the i^{th} point (x_1 is the distance from 0 to the first point) as shown in Fig. 4. The configura-

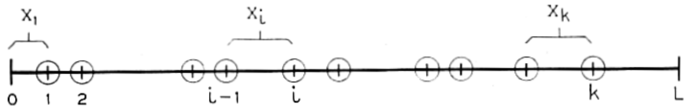

Fig. 4 — Definition of x_i .

tion of points $1, \cdots, k$ on the line is represented by a single point $(x_1, \cdots, x_k)$ in the polyhedron T in k-dimensional space defined by the inequalities

$$T: \quad 0 \leqq x_1, \cdots, 0 \leqq x_k, \qquad x_1 + x_2 + \cdots + x_k \leqq L,$$

and the probability distribution of the point $(x_1, \cdots, x_k)$ in T is uniform. The configurations with no coincidences lie in a smaller polyhedron T' consisting of all points of T for which $\delta \leqq x_2, \cdots, \delta \leqq x_k$. Given k, the conditional probability that there be no coincidences is the ratio of two k-dimensional volumes Vol (T')/Vol (T).

$$\text{Vol } (T') = 0 \quad \text{if} \quad L \leqq (k-1)\,\delta.$$

For larger values of L let $y_1 = x_1$, $y_2 = x_2 - \delta$, $y_3 = x_3 - \delta, \cdots, y_k = x_k - \delta$. Then T' becomes a polyhedron of the form

$$T'': \quad 0 \leqq y_1, 0 \leqq y_2, \cdots, 0 \leqq y_k, \qquad y_1 + y_2 \cdots + y_k \leqq L - (k-1)\delta.$$

Since the transformation from x's to y's has determinant equal to one, T'' has the same volume as T'. However, T'' is now seen to be similar to T but with sides of length $L - (k-1)\delta$ instead of L. The volume ratio sought must be

$$\left(\frac{L - (k-1)\delta}{L}\right)^k.$$

Since k has the Poisson distribution with mean λL we obtain finally

$$F_0(L) = e^{-\lambda L} \sum_{k=0}^{1+[L/\delta]} \frac{(\lambda L)^k}{k!} \left(1 - \frac{(k-1)\delta}{L}\right)^k.$$

The piecewise-analytic character of $F_0(L)$ is evident; increasing L by an amount δ increases the upper limit on the sum by one and thereby adds a new term to the analytic expression for $F(L)$.

2.4 Asymptotic Formula for $F_N(L)$

Similar exact formulas could be found for all the $F_N(L)$, but they are both complicated and inconvenient for computing if L/δ becomes large. It is thus natural to aim for asymptotic results and for bounds connected with them.

The Laplace transform of $F_N(L)$ is given through (2–14) and (2–15) above. The pole of $f_N(s)$ with largest real part is a pole of order $N + 1$

at a real negative point

$$s = -a > -\lambda.$$

For large L, the asymptotic behavior is given by

$$F_N(L) \approx \frac{\lambda e^{-aL}}{(\lambda - a)[1 + \delta(\lambda - a)]N!}\left[\frac{aL}{1 + \delta(\lambda - a)}\right]^N,$$

where the error term is $0(L^{N-1} e^{-aL})$ if $N \geqq 1$. Such a formula, then, is a good approximation for fixed N as L increases; for fixed L, however, it will fail to be good for sufficiently large N.

If $N = 0$, the asymptotic form is

$$F_0(L) \approx \frac{\lambda}{(\lambda - a)[1 + \delta(\lambda - a)]} e^{-aL},$$

but the error term now decreases at a more rapid rate, as may be seen by including the contributions of some of the complex poles of $f_0(s)$. To find these poles, set

$$s + \lambda = \lambda e^{-(s+\lambda)\delta}.$$

If

$$s = -\lambda + r \exp (i\theta),$$

one obtains the simultaneous real system

$$2\pi m - \theta = \delta r \sin \theta \qquad (m \text{ integer}),$$

$$\log (r/\lambda) = -\delta r \cos \theta.$$

The first equation defines an infinite family of curves in the s-plane (see Fig. 5). The second equation defines a single curve which intersects the family at poles of $p(s)$.

2.5 Bounds on $F_0(L)$

As in Part I, we may derive bounds on $F_0(L)$ from the integral equation, and obtain

$$\left(1 - \frac{a}{\lambda}\right) e^{-aL} e^{-(a-\lambda)\delta} \leqq F_0(L) \leqq e^{-aL} e^{-(a-\lambda)\delta}.$$

Since $a = \lambda^2\delta + 0(\delta^2)$ for small δ, the bounds are very close if $\lambda\delta$ is not too large.

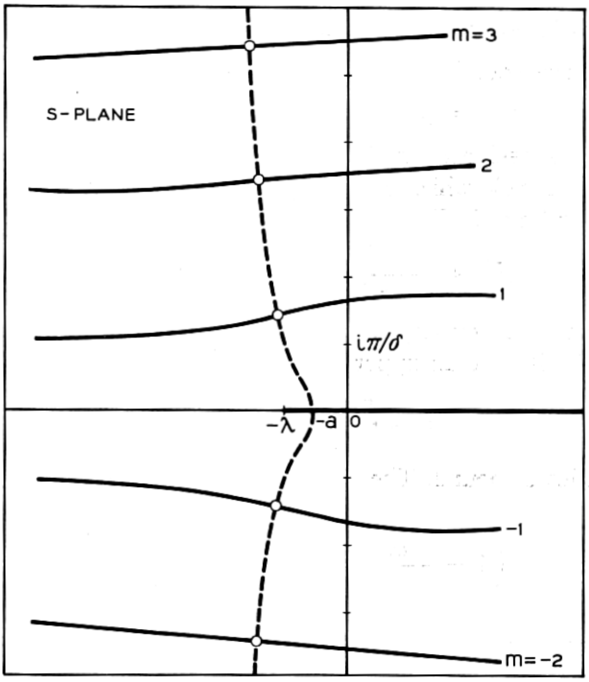

FIG. 5 — Solution of $s + \lambda = \lambda e^{-(s+\lambda)\delta}$

III COINCIDENCES BETWEEN n POISSON PATTERNS

3.1 Integral Equation

In this part we consider n one-dimensional Poisson patterns and ask for the probability, $F(L)$, that in the interval $(0, L)$ no pair of points from different patterns are coincident. Unlike Part I, we now consider only the case in which all n patterns have the same density λ. Let $P(L)$ be the conditional probability, given that Pattern No. 1 has a point at L, that there are no coincidences in $(0, L)$.

If $0 \leqq L \leqq \delta$, $\qquad P(L) = \exp - (n - 1)\lambda L.$

If $\delta < L$,

$$P(L) = e^{-(n-1)\lambda L}\left(1 + (n - 1)\lambda e^{-\lambda\delta} \int_0^{L-\delta} e^{(n-1)\lambda y} P(y)\, dy\right)$$

by the same sort of argument used in Part I. Then $F(L)$ will be given by

$$F(L) = e^{-n\lambda L}\left(1 + n\lambda \int_0^L e^{n\lambda x} P(x)\, dx\right).$$

3.2 Bounds and Asymptotic Formula

The Laplace transform of $P(L)$ is

$$p(s) = \{s + (n - 1)\lambda(1 - e^{-(n\lambda+s)\delta})\}^{-1} \tag{3-1}$$

which has one real pole at a negative point $s = -a$, $a < (n - 1)\lambda$. Again it is this pole which contributes the dominant term to both $P(L)$ and $F(L)$ for large L. We find

$$F(L) \approx \frac{n\lambda e^{-aL}}{(1 + [(n - 1)\lambda - a]\delta)(n\lambda - a)}.$$

To bound $P(L)$ by expressions of the form $A \exp(-aL)$ one finds that $A > 1$ will give an upper bound and

$$A < \left(1 - \frac{a}{(n - 1)\lambda}\right) e^{\lambda\delta}$$

will give a lower bound. The corresponding bounds on $F(L)$ are of the form

$$\left(1 - \frac{n\lambda A}{n\lambda - a}\right) e^{-n\lambda L} + \frac{n\lambda A}{n\lambda - a} e^{-aL}.$$

3.3 Exact Solution

As in Part II an exact formula for $F(L)$ may be given as a finite sum. We now derive it from the Laplace transform,

$$f(s) = (s + n\lambda)^{-1} (1 + n\lambda p(s)),$$

of $F(L)$. We may use (3-1) to expand $f(s)$ into the series

$$f(s) = \frac{1}{s + n\lambda} \left\{1 + n\lambda \sum_{k=0}^{\infty} \frac{((n - 1)\lambda e^{-(n\lambda+s)\delta})^k}{(s + (n - 1)\lambda)^{k+1}}\right\}. \tag{3-2}$$

The identity

$$\begin{aligned}(s + n\lambda)^{-1}&(s + (n - 1)\lambda)^{-k-1} \\ &= \frac{1}{\lambda} \sum_{j=0}^{k} (-\lambda)^{-k+j}(s + (n - 1)\lambda)^{-j-1} + (-\lambda)^{-k-1}(s + n\lambda)^{-1}\end{aligned} \tag{3-3}$$

provides a partial fraction expansion for the k^{th} term of the series (3-2). Transforming (3-2) term by term with the help of (3-3) we find

$$\begin{aligned}F(L) &= e^{-n\lambda L}[-(n - 1)]^{[L/\delta]+1} \\ &\quad + ne^{-(n-1)\lambda L} \sum_{k=0}^{[L/\delta]} [-(n - 1)e^{-\lambda\delta}]^k \sum_{j=0}^{k} \frac{[-\lambda(L - k\delta)]^j}{j!}.\end{aligned}$$

This is the desired formula for $F(L)$.

IV MULTIDIMENSIONAL PROBLEMS

4.1 Two-Pattern Lower Bound

We now derive some results on the probabilities of no coincidences in some multi-dimensional situations. The simplest one is a lower bound for the case of two Poisson patterns.

Theorem: Consider a d-dimensional region of volume V containing two Poisson patterns with densities λ and μ. Let $S(\delta)$ be the volume of the d-dimensional sphere of radius δ. The probability of no coincidences between the two patterns has the lower bound

$$e^{-\lambda V(1-e^{-\mu S(\delta)})}.$$

Proof

Let the pattern with density λ be called the λ-pattern and the other the μ-pattern. Given any λ-pattern of k points there will be no coincidences provided only that a certain region T contains no points of the μ-pattern. T consists of all points of the volume V which lie in any of the spheres of radius δ centered on the k points of the λ-pattern. Since these spheres may overlap and may extend partly outside the volume V, we have

$$\text{volume of } T \leqq k\, S(\delta),$$

and

$$\begin{aligned}\text{Prob (no coinc., given } k \text{ points)} &= \exp\,(-\mu \text{ volume of } T)\\ &\geqq \exp\,(-\,k\mu\, S(\delta)).\end{aligned}$$

Since the number, k, of points of the λ-pattern has the Poisson distribution with mean λV the (unconditional) probability of no coincidences has the lower bound

$$\sum_{k=0}^{\infty} \frac{(\lambda V)^k}{k!}\, e^{-\lambda V} e^{-k\mu S(\delta)}.$$

Summing the series one proves the theorem. Interchanging λ and μ in the theorem gives another lower bound. The one stated above is the better of the two if $\lambda < \mu$.

The difference between the lower bound and the true probability comes from two sources: (a) The overlap between the k spheres; this will be a small effect if $\lambda^2 S(2\delta) V$ is small, and (b) the spheres which extend partly outside the volume V; there will be relatively few such spheres if only a small fraction of the volume V lies within distance δ

of its boundary. Hence in some cases the lower bound will be a good approximation to the correct value.

It may also be noted that no real use was made of the spherical shape of the volumes $S(\delta)$. If one wants to consider a point of the μ-pattern to be coincident with a point of the λ-pattern if it lies in some other neighborhood, not of spherical shape, the same lower bound applies but with $S(\delta)$ replaced by the volume of the neighborhood.

4.2 Single-Pattern Lower Bound

A similar derivation in the case of a single Poisson pattern leads to:

Theorem: Let a Poisson pattern of density λ *be distributed over a d-dimensional region of volume* V. *Let* $S(\delta)$ *be the volume of the d-dimensional sphere of radius* δ. *Then the probability of no coincidences is at least as large as*

$$e^{-\lambda V}\{1 + \lambda S(\delta)\}^{V/S(\delta)}.$$

The theorem will follow from another bound which is slightly more accurate but much more cumbersome.

Lemma

In the above theorem a lower bound is

$$e^{-\lambda V}\left(1 + \lambda V + \sum_{k=2}^{[V/S(\delta)]} \frac{(\lambda V)^k}{k!} \prod_{j=1}^{k-1} [1 - jS(\delta)/V]\right). \tag{4-1}$$

Proof of Lemma

The probability sought is of the form

$$\sum_k e^{-\lambda V} \frac{(\lambda V)^k}{k!} p_k , \tag{4-2}$$

where p_k is the probability that, when exactly k points are distributed at random over V, there are no coincidences. To estimate p_k, imagine the k points to be numbered 1, 2, $\cdots$ k and placed in the region one at a time. If no coincidences have been created among points 1, $\cdots$, j (which is an event of probability p_j) the probability that the addition of point $j + 1$ creates no coincidence is just the probability that this new point lies in none of the j spheres of radius δ centered on points 1, $\cdots$, j. The union of these j spheres intersected with the volume V

is always of volume $\leqq jS(\delta)$. Hence

$$p_{j+1} \geqq p_j[1 - jS(\delta)/V],$$

or

$$p_k \geqq \prod_{j=1}^{k-1} [1 - jS(\delta)/V]. \tag{4-3}$$

When $(k - 1)S(\delta) > V$ the above argument fails because the later terms of the product are negative; in this case we use the trivial bound $p_k \geqq 0$. Combining (4-2) with (4-3) the lemma follows.

Once more the bound may be expected to be almost correct if $\lambda^2 VS(2\delta)$ is small and if most of the region V lies farther than δ away from its boundary. The bound is also correct for non-spherical neighborhoods (see discussion of previous theorem).

When $V/S(\delta)$ is large, the sum (4-1) is unwieldy. If we let H equal $V/S(\delta)$, we may rewrite the typical term in the sum as

$$\frac{(\lambda V)^k}{k!}\prod_{j=1}^{k-1}(1 - j/H) = \frac{(\lambda V/H)^k}{k!} H(H - 1) \cdots (H - k + 1).$$

If H happens to be an integer, this equals

$$\binom{H}{k} (\lambda V/H)^k,$$

so that the complete sum (4-1) equals

$$e^{-\lambda V}\left(1 - \frac{\lambda V}{H}\right)^H. \tag{4-4}$$

We will now prove that if H is not an integer, the sum always *exceeds* (4-4), so that (4-4) is a lower bound in all cases. We wish to prove that

$$1 + \sum_{k=1}^{[H]+1} \frac{x^k}{k!} H(H - 1) \cdots (H - k + 1) \geqq (1 + x)^H \tag{4-5}$$

for any positive H, in which event the theorem follows with

$$x = \frac{\lambda V}{H} \quad \text{and} \quad H = V/S(\delta).$$

The inequality (4-5) will be proved by induction on $[H]$. If $[H] = 0$, then we are required to show that

$$1 + Hx \geqq (1 + x)^H$$

for $0 \leqq H < 1$. This follows immediately from the concavity of $(1 + x)^H$.

Suppose now that (4-5) holds for a value H. If we integrate both sides of (4-5) from 0 to x, we obtain

$$x + \sum_{k=1}^{[H]+1} \frac{x^{k+1}}{(k+1)!} H(H-1) \cdots (H-k+1) \geqq \frac{(1+x)^{H+1} - 1}{H+1},$$

which may be rewritten as

$$1 + \sum_{k=1}^{[H+1]+1} \frac{x^k}{k!} (H+1)(H) \cdots (H-k+2) \geqq (1+x)^{H+1}.$$

This completes the induction, and the proof of the theorem.

4.3 Another Lower Bound (Any Number of Patterns)

Another kind of lower bound can be derived which sometimes will be better than the above bounds when the region V has a large fraction of its volume within δ of the boundary. For example, V might be a three-dimensional circular cylinder (a cable) with a radius which is comparable to δ.

To derive this bound one first finds the expected number, E, of coincidences in V. An upper bound on E will also suffice. Then it is noted that $1 - E$ is a lower bound on the probability of no coincidences. For if Q_N is the probability of finding N coincidences,

$$E = \sum NQ_N \geqq \sum_{N=1}^{\infty} Q_N = 1 - Q_0 . \tag{4-6}$$

4.4 Thick Cable

For example, we now give a lower bound which is of interest in connection with the problem of a cable with many wires.

Theorem: Let a Poisson pattern of points with density λ be placed in a cylinder of length L and radius $R > \delta$. The probability of finding no coincidences in the cylinder is at least as great as

$$1 - \lambda^2\pi^2 L\left(\frac{2R^2\delta^3}{3} - \frac{R\delta^4}{4} + \frac{\delta^5}{15}\right).$$

Proof

Introduce cylindrical coordinates r, φ, Z so that the cylinder is described by

$$r \leqq R, \qquad 0 \leqq Z \leqq L.$$

Consider first any pattern point (r, φ, Z) with Z-coordinate satisfying $\delta \leqq Z \leqq L - \delta$. Let arrows be drawn from this point to all other pattern points (if any) within distance δ. The expected number of arrows drawn from this point will be $\lambda G(r)$ where $G(r)$ is the volume of the intersection of the cylinder with a sphere of radius δ centered at the point. For points near the ends of the cylinder ($Z \leqq \delta$ or $L - \delta \leqq Z$), the expected number of arrows will be less than $\lambda G(r)$. Since the probability of finding a pattern point in a little volume element dV is λdV, we conclude that the expected number of arrows drawn in the entire cylinder will be less than

$$\iiint_{\text{cylinder}} \lambda^2 G(r)\, dV.$$

If the cylinder has N coincidences, there will be $2N$ arrows (each point of a coincident pair appears once at the head of an arrow and once at the tail). Hence the expected number of coincidences is

$$E \leqq \lambda^2 \pi L \int_0^R G(r)\, r\, dr. \tag{4-7}$$

Since an exact formula for $G(r)$ is rather cumbersome, we are content with a simple but close upper bound. If $r \leqq R - \delta$ then clearly $G(r) = 4\pi\delta^3/3$. If $r > R - \delta$ we get an upper bound on $G(r)$ by computing the shaded volume in Fig. 6; the intersection of the sphere with a half-space.

$$G(r) \leqq [2\delta^3 + 3(R - r)\delta^2 - (R - r)^3]\, \pi/3.$$

Substituting these expressions for $G(r)$ in (4-7), integrating, and using (4-6) the theorem follows.

The approximation to $G(r)$ which was made above is bad when R is much less than δ, but in this case good estimates may be obtained from the one-dimensional results of Part II. Note also that if λ is large enough, the bound becomes negative and is therefore useless.

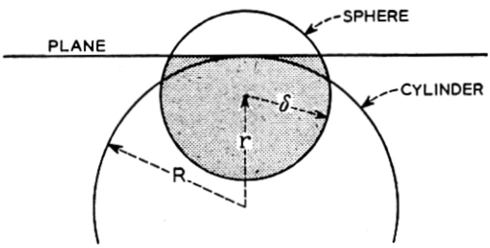

Fig. 6 — A region for estimating $G(r)$.

4.5 Upper Bounds

Good upper bounds appear even harder to get than lower bounds. One procedure is to divide the region V into a number of smaller cells. If each cell has probability, p, of no coincidences and if there are K cells, then p^K is the probability of no coincidence in any cell. If there is no coincidence in V there will be none in any cell; hence p^K is an upper bound on the probability of no coincidence in V.

Of course, p^K is too large because of the possibility of a coincidence between two points in different cells. It follows that p^K will be a close bound only if the cell size is made large; but then p becomes hard to compute.

For example, consider self-coincidences in a single Poisson pattern in a large region of area V in the plane. Cover this area with an array of hexagonal cells of side $\delta/2$ as shown in Fig. 7. The area of each hexagon is $3\sqrt{3}\,\delta^2/8$ so the number of cells used will be about $K = 8V/3\sqrt{3}\,\delta^2$. A cell has no coincidence if it contains at most one pattern point, hence

$$p = (1 + \lambda 3\sqrt{3}\,\delta^2/8)\exp - 3\sqrt{3}\,\lambda\delta^2/8.$$

The upper bound is

$$p^K = e^{-\lambda V}\left(1 + \frac{3\sqrt{3}}{8}\lambda\delta^2\right)^{(8V/3\sqrt{3}\delta^2)}$$

which has an interesting resemblance to the lower bound

$$e^{-\lambda V}(1 + \pi\lambda\delta^2)^{V/\pi\delta^2}.$$

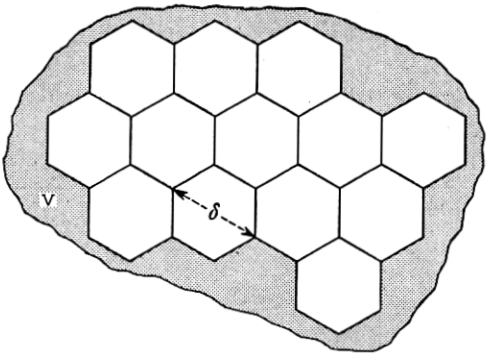

Fig. 7 — Pattern for studying coincidences in a plane region.

4.6 An Exact Calculation

The upper and lower bounds in Section 4.5 are not very close, largely because of the small size of the hexagonal cells. An improved upper bound may be obtained using square cells of side 2δ. We can calculate p for small rectangular cells but only if we redefine our notion of coincidence in terms of square neighborhoods instead of circular neighborhoods. That is, points (x_1, y_1) and (x_2, y_2) are now considered coincident if simultaneously

$$| x_1 - x_2 | \leqq \delta, \quad \text{and} \quad | y_1 - y_2 | \leqq \delta.$$

The result we get is the only exact calculation of a non-trivial multi-dimensional coincidence probability known to us.

Consider the rectangle $0 \leqq x \leqq L$, $0 \leqq y \leqq M$ with L and M both $\leqq 2\delta$. If L is less than δ, two points are coincident if and only if their y-coordinates differ by less than δ. The problem then reduces to a one-dimensional coincidence computation such as we gave in Part II. Therefore, suppose both L and M are greater than δ.

There is probability

$$g_k = \frac{(\lambda LM)^k}{k!} e^{-\lambda LM}$$

that the rectangle contains k points. We therefore subdivide the problem into cases of the form "given k, find the probability that the k points have no coincidences". Only five of these cases have a non-zero answer. To show this, divide the rectangle into four rectangles of sides $L/2$, $M/2$; if $k \geqq 5$ one of these rectangles must contain more than one point, and so a coincidence. The remaining cases $k = 0, 1, 2, 3, 4$ may be further subdivided according to which pairs of x-coordinates are less than δ apart. Let us number the k points $(x_1, y_1), \cdots, (x_k, y_k)$ in such a way that the x-coordinates are in order $x_1 \leqq x_2 \leqq \cdots \leqq x_k$. If, for some i, $x_{i+2} \leqq x_i + \delta$, then the subcase in question contributes zero to the probability of no coincidences because all of $| x_i - x_{i+1} |$, $| x_{i+1} - x_{i+2} |$, $| x_{i+2} - x_i |$ are $\leqq \delta$ and at least one of $| y_i - y_{i+1} |$, $| y_{i+1} - y_{i+2} |$, $| y_{i+2} - y_i |$ is $\leqq \delta$. The only subcases which remain to give a non-zero contribution are the nine listed in Table I. The number in the "subcase" column is k. The next column contains the x-inequalities which define the subcase. The probability that the k ordered x-coordinates satisfy the stated inequalities is listed as prob_x. If the x-inequalities are satisfied there will be no coincidences if and only if $| y_b - y_a | > \delta$ for every inequality $| x_b - x_a | \leqq \delta$ given in the x-inequality column. These y-in-

TABLE I

Subcase	x inequ.	prob_x	y inequ.	prob_y
0	—	1	—	1
1	—	1	—	1
2(a)	$x_2 - x_1 > \delta$	$(1 - \delta/L)^2$	—	1
2(b)	$x_2 - x_1 \leqq \delta$	$\frac{2\delta L - \delta^2}{L^2}$	$\vert y_2 - y_1 \vert > \delta$	$(1 - \delta/M)^2$
3(a)	$x_2 - x_1 \leqq \delta$ $x_3 - x_2 > \delta$	$\frac{2}{3}(1 - \delta/L)^3$	$\vert y_2 - y_1 \vert > \delta$	$(1 - \delta/M)^2$
3(b)	$x_2 - x_1 > \delta$	$\frac{2}{3}(1 - \delta/L)^3$	$\vert y_2 - y_3 \vert > \delta$	$(1 - \delta/M)^2$
3(c)	$x_2 - x_1 \leqq \delta$ $x_3 - x_2 \leqq \delta$ $x_3 - x_1 > \delta$	$\frac{2}{3}(1 - \delta/L)^2 \left(\frac{4\delta}{L} - 1\right)$	$\vert y_2 - y_3 \vert > \delta$ $\vert y_1 - y_2 \vert > \delta$	$\frac{2}{3}(1 - \delta/M)^3$
4(a)	$x_3 - x_2 \leqq \delta$ $x_3 - x_1 > \delta$ $x_4 - x_2 > \delta$	$\frac{1}{3}(1 - \delta/L)^4$	$\vert y_2 - y_1 \vert > \delta$ $\vert y_3 - y_2 \vert > \delta$ $\vert y_4 - y_3 \vert > \delta$	$\frac{5}{12}(1 - \delta/M)^4$
4(b)	$x_3 - x_2 > \delta$	$\frac{1}{3}(1 - \delta/L)^4$	$\vert y_2 - y_1 \vert > \delta$ $\vert y_4 - y_3 \vert > \delta$	$(1 - \delta/M)^4$

equalities are listed in the third column and the probabilities that they are satisfied are listed as prob_y. The probability of no coincidences is

$$\sum g_k \, \text{prob}_x \, \text{prob}_y$$

where the sum is over all nine subcases. The sum is

$$\exp(-\lambda LM) \left\{ 1 + \lambda LM + \frac{\lambda^2}{2} [L^2M^2 - \delta^2(2L - \delta)(2M - \delta)] \right.$$
$$+ \frac{2\lambda^3}{27} (L - \delta)^2(M - \delta)^2(2LM + L\delta + M\delta - 4\delta^2)$$
$$\left. + \frac{17\lambda^4}{864} (L - \delta)^4(M - \delta)^4 \right\}.$$

If $L = M = 2\delta$, this reduces to

$$\exp(-4\delta^2\lambda) \left[1 + 4\delta^2\lambda + \frac{7}{2}\delta^4\lambda^2 + \frac{16}{27}\delta^6\lambda^3 + \frac{17}{864}\delta^8\lambda^4 \right].$$

A sample of one of the above computations may be instructive. Consider, for example, Case 4(a). We have $0 \leqq x_1 \leqq x_2 \leqq x_3 \leqq x_4 \leqq L$, and require:

$$x_3 - x_2 \leqq \delta,$$

$$x_3 - x_1 > \delta,$$

$$x_4 - x_2 > \delta.$$

The probability of this is

$$(L^4/8)^{-1} \int_{x_3=\delta}^{L} \int_{x_2=x_3-\delta}^{L-\delta} \int_{x_1=0}^{x_3-\delta} \int_{x_4=x_2+\delta}^{L} dx_4\, dx_1\, dx_2\, dx_3$$

$$= \frac{8}{L^4} \int_{x_3=\delta}^{L} \int_{x_2=x_3-\delta}^{L-\delta} (L - x_2 - \delta)(x_3 - \delta)\, dx_3\, dx_2$$

$$= \frac{8}{L^4} \frac{(L-\delta)^4}{24} = \frac{1}{3}\left(1 - \frac{\delta}{L}\right)^4.$$

In the y-direction we require $| y_2 - y_1 | > \delta$, $| y_3 - y_2 | > \delta$, $| y_4 - y_3 | > \delta$, and there are no order restrictions. Assume first that $y_2 < y_3$. Then the probability that y_1 and y_4 satisfy their restrictions is

$$\left(\frac{y_3 - \delta}{M}\right)\left(\frac{M - y_2 - \delta}{M}\right).$$

Hence, the probability for satisfying all the conditions is

$$\int_{\delta}^{M} \int_{0}^{y_3-\delta} \left(\frac{y_3 - \delta}{M}\right)\left(\frac{M - y_2 - \delta}{M}\right) \frac{dy_2}{M} \frac{dy_3}{M} = \frac{5}{24}\left(1 - \frac{\delta}{M}\right)^4.$$

Interchanging y_2 with y_3 and y_1 with y_4 shows that the assumption $y_2 > y_3$ yields the same answer, so that the required probability is

$$\frac{5}{12}\left(1 - \frac{\delta}{M}\right)^4.$$

V NUMERICAL WORK

5.1 *Coincidences between Two Patterns*

5.1.1 *Machine Computation of $F(L)$*

To compute the probability of no coincidences in a line of length L directly, it is convenient to transform equations (1–2) through (1–4) into the following differential difference equations:

$$P_1'(x) + \mu P_1(x) = \begin{cases} 0 \text{ if } x \leqq \delta \\ P_2(x - \delta)\mu e^{-(\lambda+\mu)\delta} & \text{if } x > \delta, \end{cases}$$

$$P_2'(x) + \lambda P_2(x) = \begin{cases} 0 \text{ if } x \leqq \delta \\ P_1(x - \delta)\lambda e^{-(\lambda+\mu)\delta} & \text{if } x > \delta, \end{cases}$$

$$F'(x) + (\lambda + \mu)F(x) = \lambda P_1(x) + \mu P_2(x),$$

$$P_1(0) = P_2(0) = F(0) = 1.$$

These have been solved on a general purpose analog computer with the aid of a lumped-element approximate delay line for a number of cases. We have chosen for illustrative purposes the parameters $\lambda = 5$, $\mu = 10$, $\delta = 0.02$, and $L \leqq 1$. The exact solution, together with various approximations to be described in the sequel, is plotted in Fig. 8, where the exact solution is labelled y_1.

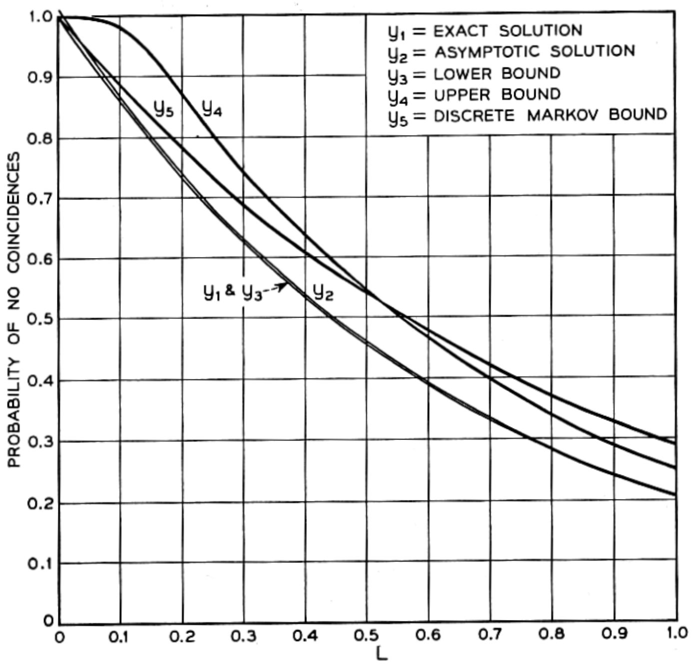

FIG. 8 — Probability of no coincidences between two one-dimensional Poisson patterns with $\lambda = 5$, $\mu = 10$, if $\delta = 0.02$.

5.1.2 The Asymptotic Formula

An approximation to the probability $F(x)$ of no coincidences is given by the asymptotic formula (1-10) which, of course, becomes a better approximation the larger L becomes. If $\lambda = 5$, $\mu = 10$, and $\delta = 0.02$, the smallest value, a, such that

$$(\lambda - a)(\mu - a) = \lambda\mu e^{-2(\lambda+\mu-a)\delta}$$

is $a = 1.548$. The asymptotic formula for $F(L)$ now becomes

$$F(L) \approx 1.013e^{-1.548L},$$

which is found in Fig. 8 as y_2.

5.1.3 Bounds Using the Asymptotic Exponent

Formulas (1-12) through (1-18) give a scheme for computing both upper and lower bounds for $F(L)$ which have the right behavior for large L, and also agree with the solution at $L = 0$. They become

$$F(L) \geqq 1.007e^{-1.548L} - 0.007e^{-15L},$$

and

$$F(L) \leqq 1.195e^{-1.548L} - 0.195e^{-15L},$$

respectively, and are represented by y_3 and y_4 in Figure 8.

5.1.4 An Upper Bound by a Discrete Markov Process

If we mark on the positive x-axis the points $n\delta/2$, $n = 0, 1, 2, \cdots$, we can assign to each interval of length $\delta/2$ thus created a state (ij), i, $j = 0$ or 1, as follows: $i = 0$ if no point of the λ-process is present in the interval, $i = 1$ if one or more points of the λ-process are present, and similarly for j and μ. An interval of length δ, made up of two adjacent intervals of length $\delta/2$, may then be represented by a number between 0 and 15 in binary notation, where 3, 6, 7, 9, and 11–15 represent a coincidence within the interval of length δ. We now define a Markov process as follows: in the interval $0 \leqq t < \delta$, let $p_i^{(0)}$, $i = 0, 1, 2, 4, 5, 8, 10$, be the probabilities of occurrence of the i^{th} state, so that, for example, $p_0^{(0)} = e^{-2\lambda\delta}e^{-2\mu\delta}$, and $p_1^{(0)} = e^{-2\lambda\delta}e^{-\mu\delta}(1 - e^{-\mu\delta})$. These are the states in which there is no coincidence in $(0, \delta)$. In addition, let $q^{(0)}$ represent the probability of all the other states put together; i.e., of a coincidence in $(0, \delta)$. We now define $p_i^{(n)}$, $i = 0, 1, 2, 4, 5, 8, 10$ as the probability of the i^{th} state in the interval $(n\delta/2, (n + 2)\delta/2)$, where we require in addition that all states in the intervals $(k\delta/2, (k + 2)\delta/2)$, $k < n$, are from the same "no coincidence" index set. We define $q^{(n)}$ as the probability of a state 3, 6, 7, 9, or 11–15, in *some* interval $(k\delta/2, (k + 2)\delta/2)$, $k \leqq n$. There are then transition probabilities from states in the $n - 1^{st}$ to states in the n^{th} interval. For example,

$$p_0^{(n)} = e^{-\lambda\delta}e^{-\mu\delta}(p_0^{(n-1)} + p_4^{(n-1)} + p_8^{(n-1)}),$$

and

$$q^{(n)} = q^{(n-1)} + (1 - e^{-\lambda\delta})(1 - e^{-\mu\delta})(p_0^{(n-1)} + p_4^{(n-1)} + p_8^{(n-1)}) + (1 - e^{-\lambda\delta})(p_1^{(n-1)} + p_5^{(n-1)}) + (1 - e^{-\mu\delta})(p_2^{(n-1)} + p_{10}^{(n-1)}).$$

The quantity $1 - q^{(n)}$ is then an upper bound for the probability of no coincidences (upper because it is possible for a coincidence to occur in the process which is not counted in this subdivision of it). The curve y_5 in Fig. 8 is drawn through points at $L = n\delta/2$ computed in this manner.

To summarize the results, we see that the asymptotic formula and the lower bound are both indistinguishable from the right answer; the upper bounds are fairly far off. The upper bound derived by the Markov process is better than that derived from the integral equation until

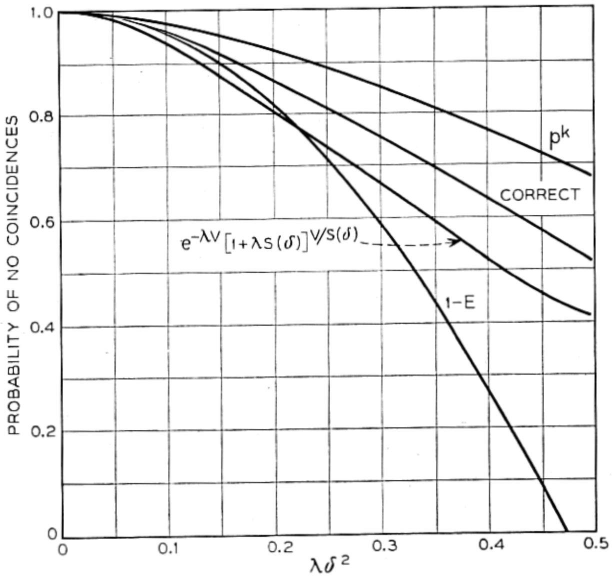

FIG. 9 — Probability of no coincidences in a $2\delta \times 2\delta$ square; neighborhoods are square.

about $L = 0.5$ (25 iterations), when the integral equation upper bound becomes better.

5.2 A Single Pattern in a Square

To test our higher-dimensional bounds, we consider again coincidences in a single Poisson pattern in a square of side 2δ. The exact probability of no coincidences was given in Part IV assuming square neighborhoods. The lower bound (Sec. 4.2)

$$e^{-\lambda V}(1 + \lambda S(\delta))^{V/S(\delta)}$$

applies using $V = (2\delta)^2$ and $S(\delta) = (2\delta)^2$ for square neighborhoods. To use the lower bound $1 - E$ we note that the exact expected number of coincidences is

$$E = \frac{1}{2}\lambda^2 \int_0^{2\delta}\int_0^{2\delta} A(x, y)\, dx\, dy$$

where $A(x, y)$ is the area of the intersection of the given square with the square neighborhood centered at (x, y). The lower bound is $1 - E = 1 - 9\lambda^2\delta^4/2$. The upper bound p^K can be used if the square is cut into $K = 4$ squares of side δ, each with a probability $p = (1 + \lambda\delta^2) \exp - \lambda\delta^2$ of no coincidence.

These bounds, together with the exact probability, are plotted as functions of $\lambda\delta^2$ in Fig. 9. When $\lambda\delta^2$ is small, the $1 - E$ bound is correct to terms of order $0(\lambda^3\delta^6)$. This might have been predicted from (4-6) since it seems reasonable that $Q_2, Q_3, \cdots$ should be of higher order in λ than Q_1 when λ is small. Ultimately the first lower bound becomes a better estimate. It must be recognized that this other lower bound is being tested under very severe conditions. Since every point of the square has a neighborhood which intersects the boundary, the errors from source (b) of Part V are considerable.

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