The Determination of Pressure Coefficients of Capacitance for Certain Geometries

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(Manuscript received February 15, 1955)

Expressions are derived for the pressure coefficients of capacitance of parallel plate capacitors subjected to one-dimensional and hydrostatic pressures and of cylindrical capacitors subjected to radial compression. The derivations apply to systems in which the dielectrics are isotropic, elastic solids.

I. INTRODUCTION

The electrical capacitance between two conductors separated by a dielectric is a quantity which can be calculated with ease only in certain geometrical arrangements of high symmetry. Even the classic example of parallel plates presents major difficulties as one may only perform the calculation exactly for the case of plates of infinite area or vanishing separation. The approximation becomes poor when $(\text{area})^{\frac{1}{2}}/(\text{separation})$ becomes small and the theoretical treatment of edge effects is sufficiently difficult that it has not been solved though the solution would greatly facilitate dielectric constant measurement.

When pressure enters into the situation as a variable the difficulties are enhanced as one must be able to describe the geometry effects as well as the change in dielectric constant.

The engineers responsible for designing submarine cables are confronted with the necessity of knowing the manner in which capacitance depends upon pressure as may be illustrated in the following way. A submarine telephone cable is composed of a central copper conductor surrounded by a sheath of dielectric material. Due to the extreme length repeaters must be placed at intervals, the separation being determined by the attenuation of the cable. The attenuation, α, of a coaxial telephone cable may be written

$$\alpha = (G/2)(L/C)^{\frac{1}{2}} + (R/2)(C/L)^{\frac{1}{2}}$$

where G is the conductance of the dielectric per unit length, C the capacitance, L the inductance, and R the conductor resistance per unit length. The second term contributes about 99% of the attenuation. Considering only this term we deduce

$$(1/\alpha)(\partial\alpha/\partial P) \cong (1/R)(\partial R/\partial P) + (1/2C)(\partial C/\partial P) - (1/2L)(\partial L/\partial P)$$

Accurate knowledge of the coefficient $(1/C)(\partial C/\partial P)$ is thus essential in designing very long cables which are to be exposed to high pressures.

In evaluating dielectric materials for use in cables it is often desirable to make measurements on sheet specimens rather than cable. It thus becomes necessary to be able to translate sheet data into cable data. It is the purpose of this paper to analyze the problem of calculating pressure coefficients of capacitance for certain simple geometries and to consider the methods of measurement which have been used. It will be shown that results of theory and experiment are in as good agreement as can be expected but more accurate measurements of electric and elastic properties are needed.

The equations which will be derived are also necessary if one wishes to determine the dependence of dielectric constant on pressure using any of the geometries described herein.

The problems treated in this paper are particularly simple and amenable to mathematical treatment but many problems encountered in submarine cable design are at present subject to solution only by empirical means. Fundamental investigations of the effects of pressure on dielectric materials are needed.

II. THEORETICAL TREATMENT

In the following treatment we consider that the dielectric substance is an elastic solid which obeys Hooke's law. We denote the relative permittivity or dielectric constant by ε, the permittivity of free space by ε_0,* the principal stresses and strains by τ_{ii} and e_{ii}, the density by ρ, the compressibility by k, and Poisson's ratio by σ.

A. *Calculation of* $\frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial P}$

One of the quantities which will be needed in the evaluation of

$$\frac{1}{C}\frac{\partial C}{\partial P} \quad \text{is} \quad \frac{1}{\varepsilon}\frac{\partial\varepsilon}{\partial P}$$

As ε is not dependent on the geometric configuration it can be calculated

* $\epsilon_0 = 8.86 \times 10^{-12}$ farads/meter.

in general and the result applied to each of the special cases to follow. A relation between dielectric constant and density is required and usually, when dealing with non-polar dielectrics, one assumes that the Clausius-Mosotti relation gives the proper dependence. That is

$$\frac{\varepsilon - 1}{\varepsilon + 2} = (\text{constant})\ \rho \tag{1}$$

This formula may be differentiated to give

$$\frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial P} = \frac{(\varepsilon - 1)(\varepsilon + 2)}{3\varepsilon}\frac{1}{\rho}\frac{\partial \rho}{\partial P} \tag{2}$$

In the theory presented herein, (1) will be used though it is at best an approximation. Corrections to the Clausius-Mosotti formula[1] which have been given do not seem applicable to polymer dielectrics and introduce parameters which must be fitted.

B. The effect of a One-Dimensional Pressure Acting on a Disc

Consider a one-dimensional pressure, $-P$, acting along the axis of a circular disc of dielectric material with electrodes affixed to opposite faces. Assume the disc is constrained such that no lateral displacement can occur. Let t be the thickness and A the area of the disc.

The capacitance of such a capacitor is given by the equation

$$C = \varepsilon\varepsilon_0 \frac{A}{t}$$

so the desired pressure coefficient is

$$\frac{1}{C}\frac{\partial C}{\partial P} = \frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial P} - \frac{1}{t}\frac{\partial t}{\partial P} \tag{3}$$

where use has been made of the condition that the area is constant (i.e., no lateral displacement). Hooke's law states

$$e_{xx} = \frac{k}{3(1 - 2\sigma)}\left[\tau_{xx} - \sigma(\tau_{yy} + \tau_{zz})\right] \tag{4}$$

$$e_{yy} = \frac{k}{3(1 - 2\sigma)}\left[\tau_{yy} - \sigma(\tau_{xx} + \tau_{zz})\right] \tag{5}$$

$$e_{zz} = \frac{k}{3(1 - 2\sigma)}\left[\tau_{zz} - \sigma(\tau_{xx} + \tau_{yy})\right] \tag{6}$$

[1] C. J. F. Böttcher, Theory of Electric Polarisation, Elsevier Publishing Co., Amsterdam, 1952, p. 199 et. seq.

We assume the z-axis lies along the disc axis so $\tau_{zz} = -P$ and by symmetry $\tau_{xx} = \tau_{yy}$. The condition of no lateral strain states $e_{xx} = e_{yy} = 0$ which when combined with (4) gives

$$\tau_{xx} = \tau_{yy} = -\frac{\sigma}{1-\sigma} P$$

Using the last result with (6) we obtain

$$e_{zz} = -\frac{kP}{3}\left(\frac{1+\sigma}{1-\sigma}\right)$$

and

$$\frac{e_{zz}}{P} = \frac{1}{t}\frac{\partial t}{\partial P} = -\frac{1}{\rho}\frac{\partial \rho}{\partial P} = -\frac{k}{3}\left(\frac{1+\sigma}{1-\sigma}\right) \tag{7}$$

Combination of (2), (3), and (7) results in

$$\frac{1}{C}\frac{\partial C}{\partial P} = \left[\frac{(\varepsilon-1)(\varepsilon+2)}{3\varepsilon} + 1\right]\frac{k}{3}\left(\frac{1+\sigma}{1-\sigma}\right) \tag{8}$$

C. The Effect of a Hydrostatic Pressure Acting on a Disc

Assume that conditions are similar to those considered in section B except that $e_{xx} = e_{yy} = 0$ is now replaced by

$$\tau_{xx} = \tau_{yy} = \tau_{zz} = -P.$$

The area is no longer independent of pressure so

$$\frac{1}{C}\frac{\partial C}{\partial P} = \frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial P} - \frac{1}{t}\frac{\partial t}{\partial P} + \frac{1}{A}\frac{\partial A}{\partial P} \tag{9}$$

Hooke's law, (4), (5), (6), now becomes

$$e_{ii} = \frac{-kP}{3} \ (i = x, y, z)$$

Thus

$$\frac{1}{\rho}\frac{\partial \rho}{\partial P} = -\frac{(e_{xx} + e_{yy} + e_{zz})}{P} = k \tag{10}$$

$$\frac{1}{A}\frac{\partial A}{\partial P} = \frac{(e_{xx} + e_{yy})}{P} = -\frac{2k}{3} \tag{11}$$

$$\frac{1}{t}\frac{\partial t}{\partial P} = \frac{e_{zz}}{P} = -\frac{k}{3} \tag{12}$$

Combination of (2), (9), (10), (11), and (12) results in

$$\frac{1}{C}\frac{\partial C}{\partial P} = \left[\frac{(\varepsilon - 1)(\varepsilon + 2)}{3\varepsilon} - \frac{1}{3}\right] k \tag{13}$$

D. The Effect of a Radial Pressure Acting on a Cylindrical Annulus

Consider a radial pressure acting normally to the axis of a cylindrical annulus to which electrodes are affixed to the inner and outer surfaces. Let the inner and outer radii be a and b respectively and assume the cylinder is filled with an incompressible substance so that the inner radius is not pressure dependent. The capacitance per unit length is given by

$$C_l = \frac{2\pi\varepsilon\varepsilon_0}{\ln \frac{b}{a}}$$

so

$$\frac{1}{C}\frac{\partial C}{\partial P} = \frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial P} - \frac{1}{\ln \frac{b}{a}} \frac{1}{b}\frac{\partial b}{\partial P} \tag{14}$$

We employ cylindrical coordinates (r, θ, z) where the z-axis is taken along the cylinder axis. Hooke's law becomes

$$e_{rr} = \frac{k}{3(1 - 2\sigma)} [\tau_{rr} - \sigma(\tau_{\theta\theta} + \tau_{zz})] \tag{15}$$

$$e_{\theta\theta} = \frac{k}{3(1 - 2\sigma)} [\tau_{\theta\theta} - \sigma(\tau_{rr} + \tau_{zz})] \tag{16}$$

$$e_{zz} = \frac{k}{3(1 - 2\sigma)} [\tau_{zz} - \sigma(\tau_{rr} + \tau_{\theta\theta})] \tag{17}$$

Equilibrium of an arbitrary volume element demands[2]

$$\tau_{rr} = A + \frac{B}{r^2} \tag{18}$$

and

$$\tau_{\theta\theta} = A - \frac{B}{r^2} \tag{19}$$

where A and B are constants with respect to spatial coordinates. (A

[2] J. Prescott, Applied Elasticity, Dover Publications, New York, 1946, p. 330.

should not be confused with the electrode area used in previous sections.) We assume

$$e_{zz} = 0 \text{ for all } (r, \theta, z) \tag{20}$$

$$e_{\theta\theta} = 0 \text{ for } r = a \tag{21}$$

and

$$\tau_{rr} \; -P \text{ for } r = b \tag{22}$$

Manipulation of (15) through (22) allows the evaluation of the constants A and B as

$$A = -\frac{b^2}{a^2} \frac{P}{\dfrac{b^2}{a^2} + (1 - 2\sigma)}$$

and

$$B = \frac{b^2(1 - 2\sigma)P}{\dfrac{b^2}{a^2} + (1 - 2\sigma)}$$

Thus

$$-\frac{1}{\rho}\frac{\partial \rho}{\partial P} = \frac{e_{rr} + e_{\theta\theta} + e_{zz}}{P} \text{ yields}$$

$$\frac{1}{\rho}\frac{\partial \rho}{\partial P} = \frac{2(1 + \sigma)k}{3} \frac{b^2}{a^2} \frac{1}{\dfrac{b^2}{a^2} + (1 - 2\sigma)} \tag{23}$$

Also

$$\frac{1}{b}\frac{\partial b}{\partial P} = \frac{e_{rr}\big|_{r=b}}{P} = \frac{k(1 + \sigma)}{3}\left[1 - \frac{b^2}{a^2}\right] \frac{1}{\dfrac{b^2}{a^2} + (1 - 2\sigma)} \tag{24}$$

Combining (2), (14), (23), and (24) we obtain

$$\frac{1}{C}\frac{\partial C}{\partial P} = \frac{2}{3}\left[\frac{(1 + \sigma)\dfrac{b^2}{a^2}}{\dfrac{b^2}{a^2} + (1 - 2\sigma)}\right]\left[\frac{(\varepsilon - 1)(\varepsilon + 2)}{3\varepsilon} + \frac{1 - \dfrac{a^2}{b^2}}{2 \ln \dfrac{b}{a}}\right]k \tag{25}$$

E. The Case $\sigma = \frac{1}{2}$

The equations derived above reduce to the expressions one would obtain if the dielectric were considered to be a compressible fluid when σ is set equal to $\frac{1}{2}$.

Equation (8) for the parallel plate arrangement becomes

$$\frac{1}{C}\frac{\partial C}{\partial P} = \left[\frac{(\varepsilon - 1)(\varepsilon + 2)}{3\varepsilon} + 1\right]k \tag{26}$$

while (13) is unaltered. The difference between the two cases arises from the fact that in (13) the area was allowed to vary while in the former case it was not. The deviation of

$$\frac{1}{C}\frac{\partial C}{\partial P}$$

from (26) when $\sigma \neq 0.5$ is given by the factor

$$\frac{1}{3}\left(\frac{1+\sigma}{1-\sigma}\right)$$

The capacitance-pressure coefficient for the cylindrical configuration, (25), becomes

$$\frac{1}{C}\frac{\partial C}{\partial P} = \left[\frac{(\varepsilon - 1)(\varepsilon + 2)}{3\varepsilon} + \frac{1 - \dfrac{a^2}{b^2}}{2\ln\dfrac{b}{a}}\right]k \tag{27}$$

The deviation of

$$\frac{1}{C}\frac{\partial C}{\partial P}$$

from the value given in (27) when $\sigma \neq 0.5$ is thus given by the factor

$$\frac{2}{3}\left[\frac{(1+\sigma)\dfrac{b^2}{a^2}}{\dfrac{b^2}{a^2} + (1 - 2\sigma)}\right]$$

III. APPARATUS

The experimental arrangement employed to investigate the validity of (8) is shown in Fig. 1.* Pressure was applied by means of a Baldwin tensile testing machine. The cell makes use of a "sandwich" arrangement wherein two disc samples (2″ in diameter, 0.050″ thick) of dielectric are pressed between three brass electrodes, the outer electrodes being grounded. The capacitance thus formed is well shielded and stray capaci-

* This cell was designed by C. A. Bieling.

tances are minimized. Lateral displacements are kept small by an annular ring of steatite ceramic which is in turn surrounded by a ring of Ketos steel. Pressures of 23,000 lb on the two-inch sample discs have been applied without damaging the cell.

Although measurements could be made with ease in this cell it is not without disadvantages. The steatite ring has a rather high dielectric constant which tends to increase fringing effects. These effects are furthermore pressure dependent since the electrode separation varies as pressure is applied. Also loss measurements could not be obtained as leakage along the steatite surface was larger than the leakage through the samples of the polyethylene-butyl rubber compound investigated.

An attempt was made to eliminate fringing effects by making measurements on samples of varying thickness and extrapolating to zero thickness but results were too uncertain to be of quantitative value. The uncertainty resulted from the inability to cast the sample discs with uniform thickness an effect which becomes pronounced with very thin samples. It was possible, however, to estimate the total stray capacitance in this manner and it was found to be about 10 per cent of the sample capacitance and only slightly dependent upon pressure.

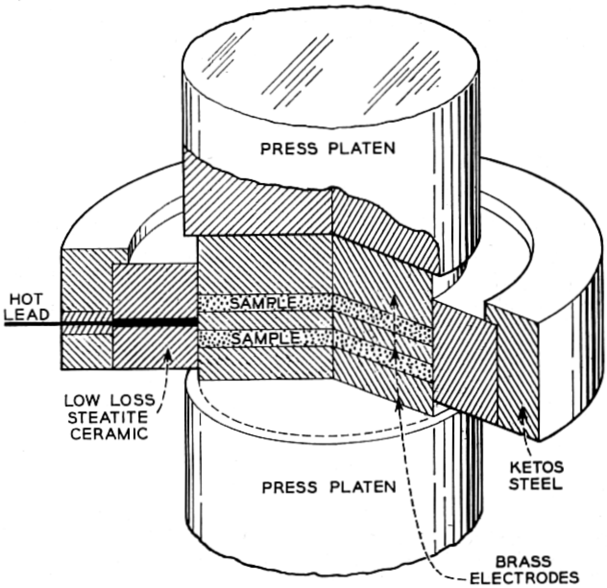

Fig. 1 — Cell employed to measure $(1/C)(\partial C/\partial P)$ with sheet specimens under one dimensional pressure.

Also, due to non-uniformity of the thickness of the specimens, it was found that the capacitance-pressure coefficient was larger at low pressures than at high pressures. This was apparently due to the initial squeezing out of voids between the electrodes and samples and the difficulty was removed by applying silver paint electrodes to the sample discs.

A Western Electric capacitance bridge was used to make capacitance measurements with this cell. The temperature was kept at 25°C and the humidity of the room was maintained at 50 per cent. The frequency used was 10 kc.

It is believed that the fact that the steatite is much more rigid than the specimens makes this experimental arrangement closely approximate the assumptions made in deriving (8) (i.e. lateral strains are negligible). It is important, however, that the specimens be cut to fit the steatite ring very closely.

Experiments which correspond to the cylindrical capacitor under biaxial stress have been performed as follows.* The specimens in this case were lengths of cable which consisted of a copper wire central conductor surrounded by the polyethylene-5 per cent butyl rubber compound. b/a was 3.87 in most of the measurements but some data were obtained for $b/a = 4.68$ (actual dimensions 0.620″/0.160″ and 0.750″/0.160″). Twenty foot lengths of cable were placed in a long tank provided with a seal at one end. The end of the cable inside the tank was closed such that the center conductor was isolated. The tank was then filled with water which served as the outer conductor, tap water having sufficiently high conductivity. Pressure was applied by the water.

A Leeds and Northrup capacitance bridge was used and the measurements reported were made at 10 kc.

IV. RESULTS

It is experimentally observed in all the cases considered that plots of C versus P are nearly linear for polyethylene-5% butyl rubber. This may be shown to be in agreement with the foregoing theories as follows. Equation (8) may be written

$$\int_{C(0)}^{C(P)} d \ln C = \int_0^P \left[\frac{(\varepsilon - 1)(\varepsilon + 2)}{3\varepsilon} + 1\right] \frac{k}{3}\left[\frac{1+\sigma}{1-\sigma}\right] dP \qquad (28)$$

* The investigation of radial compression on cylindrical (cable) specimens was carried out by A. W. Lebert and O. D. Grismore of Bell Telephone Laboratories.

The integrand is approximately constant so

$$\ln \frac{C(P)}{C(0)} \cong \left[\frac{(\varepsilon - 1)(\varepsilon + 2)}{3\varepsilon} + 1\right] \frac{k}{3} \left[\frac{1 + \sigma}{1 - \sigma}\right] P$$

and

$$C(P) = C(0) \exp \left\{ \left[\frac{(\varepsilon - 1)(\varepsilon + 2)}{3\varepsilon} + 1\right] \frac{k}{3} \left[\frac{1 + \sigma}{1 - \sigma}\right] P \right\}$$

$kP \cong 10^{-2}$ for the highest pressures used in the present experiments and

$$\left[\frac{(\varepsilon - 1)(\varepsilon + 2)}{3\varepsilon} + 1\right] \frac{1}{3} \left[\frac{1 + \sigma}{1 - \sigma}\right]$$

is of the order of unity so the exponential may be expanded as

$$C(P) \cong C(0) \left\{ 1 + \left[\frac{(\varepsilon - 1)(\varepsilon + 2)}{3\varepsilon} + 1\right] \frac{k}{3} \left[\frac{1 + \sigma}{1 - \sigma}\right] P \right\}$$

This treatment applies only to dielectrics for which ε, k, and σ are insensitive to pressure. Equations (13) and (25) may be treated similarly.

Values obtained experimentally and theoretically for polyethylene-5% butyl rubber are compared in Table I. The experimental values represent averages of many measurements. Agreement is considered adequate but more careful experiments are needed. The necessary parameters assumed in making these comparisons are:

$$\varepsilon = 2.28$$

$$k = 2.14 \times 10^{-6}/\text{psi}$$

$$\sigma = 0.50$$

TABLE I — EXPERIMENTAL AND THEORETICAL VALUES FOR POLYETHYLENE-5 PER CENT BUTYL RUBBER

Sample	Pressure	$\frac{1}{C}\frac{\partial C}{\partial P}$ (/10^6 psi)	
		Experimental	Theoretical
Sheet	one-dimensional	3.3*	3.74
Cable $b/a = 3.87$	radial	2.4	2.27
Cable $b/a = 4.68$	radial	2.2	2.22

* This value has not been corrected for stray capacitance. Such a correction would tend to make the agreement between experimental and theoretical results better.

V. SUMMARY

Equations relating electrical capacitance and pressure have been derived for plane capacitors under one dimensional and hydrostatic pressures and cylindrical capacitors under radial pressure. The dielectric material has been assumed to be an elastic solid but the relationships also apply to fluid dielectrics when Poisson's ratio is set equal to $\frac{1}{2}$. Experiments corresponding to the assumptions have been described briefly and experimental results are found to be in agreement with the theoretical predictions.

The results are of practical value in making estimates of the dependence of attenuation of submarine cables on pressure. The equations may also be put in forms useful for determining the dependence of the dielectric constant on pressure from capacitance measurements.

ACKNOWLEDGEMENT

The author would like to acknowledge several valuable discussions with G. T. Kohman and A. C. Lynch. Contributions to this work were also made by A. W. Lebert and O. D. Grismore. C. A. Weatherington assisted in some of the measurements. J. A. Lewis made several important comments on the manuscript.

ERRATA

The Effects of Surface Treatments on Point Contact Transistor Characteristics by J. H. Forster and L. E. Miller, B.S.T.J., **35,** pp. 767–811, July, 1956. Figs. 3, page 776, and 10, page 787, were inadvertently interchanged.

Cable Design and Manufacture for the Transatlantic Submarine Cable System by A. W. Lebert, H. B. Fisher and M. C. Biskeborn, B.S.T.J., **36,** pp. 189–216. Table I, page 3, the material for type B armor wire should be medium steel instead of mild steel. Page 207, the equation for Z_0 should read

$$Z_0 = \frac{b}{\sqrt{\epsilon}} \log \frac{D}{b} \text{ ohms}$$