The Character of Waveguide Modes in Gyromagnetic Media

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A magnetized gyromagnetic medium is birefringent. The effect of birefringence is studied in rectangular and circular waveguides with special attention paid to propagation characteristics in guides of arbitrarily small cross-section. Propagating, small-size structures are found in certain ranges of magnetization for both types of guide.

I. INTRODUCTION

A gyromagnetic medium, isotropic in the absence of a magnetizing field, becomes axially symmetric with respect to that field when magnetized. A tensor susceptibility[1] is thus produced which reflects the resulting anisotropy. Two essentially different types of rays appear in the medium in much the same manner in which the ordinary and extraordinary optical rays form in a calcite crystal. These rays may combine to produce results in a ferrite loaded waveguide quite alien in character to those of a conventional isotropic guide. Since the ferrite is, to first order, characteristic of general gyromagnetic media we shall discuss all gyromagnetic phenomena in terms of ferrites alone.

One very startling phenomenon observed in ferrite loaded waveguides is the occurrence of propagation in a waveguide of arbitrarily small transverse dimensions.[2] We shall show that this type of wave guide behavior is a consequence of the particular form of the birefringent character of the medium.

In order to understand the nature of the ferrite loaded case let us first consider the conventional isotropic small wave guide. Fig. 1 shows, schematically, the field distribution encountered in a small rectangular waveguide operating in a (1,1) mode. The x axis is shown along the wide transverse dimension and z is along the narrow height dimension. The y axis is chosen to coincide with the guide axis.

The field solutions of such a waveguide may be obtained as a super-

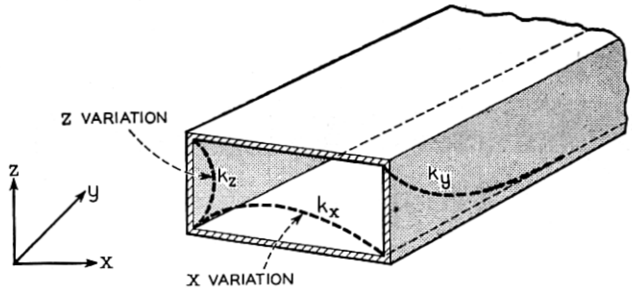

Fig. 1 — Rectangular waveguide mode in isotropic medium for cutoff guide.

position of plane waves of dependence $\varepsilon^{-i\boldsymbol{k}\cdot\boldsymbol{R}}$. If we represent $\boldsymbol{k}$ in a cartesian frame, the wave equation is satisfied for the condition

$$k^2 = k_x^2 + k_y^2 + k_z^2 = \omega^2\mu\varepsilon$$

where μ and ε are the permeability and permittivity respectively of the medium. Satisfaction of wall boundary condition requires that k_x and k_z be real and that each be of the order of the reciprocal of the transverse guide dimensions. Small transverse dimensions thus cause k_y^2 to be negative, driving the waveguide into a cutoff condition.

We shall now find that birefringence permits another class of modes in the small size ferrite loaded waveguide. Letting the magnetic axis be in the z direction, it will be shown in the text that corresponding to any mode of the guide k_y and k_z are unique. Birefringence generally requires that two different magnitudes of $\boldsymbol{k}$ occur simultaneously, causing two different values of k_x to appear. In particular, let us postulate that both these values of k_x are imaginary. Given two exponentials, it is possible now to satisfy the requirements of electric field nulls at either side wall, as shown in Fig. 2. At the other side wall we shall show that the exponentials decay so fast as to effectively cause the field to vanish there. Since $k_{x1,2}^2$ are now negative quantities, there is no contradiction in presuming that k_y^2 may now be positive, thus permitting propagation in an arbitrarily small size waveguide.

The effect of birefringence may then be that of transforming a class of longitudinally cutoff modes into another class that propagates longitudinally but cuts off transversely. The condition of this occurrence will be shown to be that for which the diagonal term of the Polder tensor, μ, is positive and is less in magnitude than the magnitude of the off diagonal term κ. In the case of a small rectangular guide, propagation occurs anomalously for negative values of μ, as well but in a manner not as substantially dependent on the birefringent character of the medium for large width to height aspect ratios of the waveguide. We shall find, further, that propagation occurs with entirely real values of k_x and k_z.

It will be shown that the proper wave equation for one of the two birefringent rays is satisfied in the small waveguide limit by the relationship

$$k_x^2 + k_y^2 + k_z^2/\mu = 0.$$

In the region of $\mu > 0$, and k_z real, we confirm somewhat more rigorously the requirement stated earlier that either k_x or k_y be imaginary. However, k_x and k_y may both be real over a range of negative values of μ, permitting boundary conditions to be satisfied, approximately, in waveguides having aspect ratios of the type discussed earlier, by just one class of rays in the small size waveguide.

Propagation in small size circular guide employing the essential character of birefringence, occurs over the entire range of $|\mu| < |\kappa|$. This range is divided into that of $\mu > 0$ and that of $\mu < 0$. Transmission occurs in one sense of circular polarization in each of these regions and for both senses for $\mu < 0$. Thompson[3] has suggested that propagation in a small circular waveguide might be attributed to the negative permeability of one preferred polarization; it appears, however, that propagation is possible over a considerably wider range of conditions and for somewhat different reasons.

In the case shown in Fig. 2, higher propagating modes occur in a rectangular waveguide when one half or more sinusoids of field variation occurs in the z direction. These simply produce the result of stronger

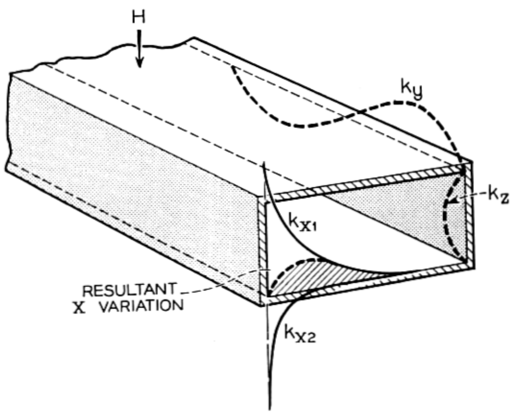

Fig. 2 — Mode in ferrite filled rectangular guide.

transverse cutoff. Therefore, by demonstrating the existence of the lowest order mode we show that an infinite number of these anomalous modes may propagate simultaneously. These modes are, however, bound very tightly as surface waves to the side walls of the guide because of their strong transverse cutoff. The medium is therefore used in a very inefficient manner and high loss results, the loss increasing with mode number.

The higher propagating rectangular waveguide modes have an analogue in the higher propagating modes in a ferrite filled circular waveguide. This analogue occurs in terms of the integral number of peripheral variations. We find, similarly, an infinite number of such propagating modes each one corresponding to a given polarization sense and having a given number of peripheral variations. The reservations on practical transmission still hold in the same manner as in the rectangular case.

In the course of preparing this publication it was brought to the author's attention that Mikaelyan[4] employed an analysis similar, in part, to that developed here. It is felt, in the present analysis, that the physical results are made more readily evident by a consideration of the limiting case of small guides, with large ratios of width to height in the case of rectangular waveguides. The choice of such large ratios is made to simplify analyses involving imaginary values of k_{x1} and k_{x2}, wherein the wave is considered to be bound to one wall of the guide and reflections from the opposite wall are of negligible amplitudes.

II. ANALYSIS OF TRANSVERSELY MAGNETIZED FERRITE IN RECTANGULAR GUIDE

The character of the ferrite medium is introduced through the Polder permeability tensor:

$$T = \begin{pmatrix} \mu & i\kappa & 0 \\ -i\kappa & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1}$$

The quantities μ and κ relate to the self and inductive permeabilities transverse to the z axis. The relative permeability along the z axis is given as unity. These permeabilities may be expressed as follows in gaussian units.[1]

$$\mu = 1 + \frac{4\pi M_s \gamma \omega_0}{\omega_0^2 - \omega^2} \tag{2a}$$

$$\kappa = \frac{4\pi M_s \gamma \omega}{\omega_0^2 - \omega^2} \tag{2b}$$

$$\gamma = 2.80 \text{ Mc/sec/oersted}$$
$$\omega_0 = \gamma H_0$$
$$H_0 = \text{Internal dc magnetic field}$$
$$4\pi M_s = \text{Saturation magnetization}$$

Maxwell's equations are given as:

$$\text{Curl } \boldsymbol{H} = i\omega\varepsilon\boldsymbol{E} \tag{3a}$$

$$\text{Curl } \boldsymbol{E} = -i\omega\mu_0 T\cdot\boldsymbol{H} \tag{3b}$$

Assuming a plane wave of dependence $\varepsilon^{i(wt-\boldsymbol{k}\cdot\boldsymbol{R})}$, and appropriately combining (3a) and (3b), we have,

$$[\boldsymbol{kk} - k^2 I + \omega^2\varepsilon\mu_0 T]\cdot\boldsymbol{H} = 0 \tag{4}$$

The operator in square brackets is a dyadic which may be represented in matrix form. The quantity I is the idemfactor, having a unit diagonal representation. If we are to require that a non-trivial field $\boldsymbol{H}$ exist, the determinant of the operator in (4) must vanish. Since all rays traveling perpendicularly to the magnetizing axis are equivalent the medium is degenerate in the transverse plane, and some simplification is achieved in causing $\boldsymbol{k}$ to lie in the yz plane and letting $k_x = 0$. Some further simplification is achieved in normalizing the Polder tensor such that

$$T = \frac{1}{\omega^2\varepsilon\mu_0}\begin{pmatrix} f & ig & 0 \\ -ig & f & 0 \\ 0 & 0 & h \end{pmatrix} \tag{5}$$

The following secular equation is then formed.

$$\begin{vmatrix} -k^2 + f & ig & 0 \\ -ig & -k_z^2 + f & k_y k_z \\ 0 & k_y k_z & -k_y^2 + h \end{vmatrix} = 0 \tag{6}$$

Introducing the substitution $p \equiv k_z^2/k^2$, and recognizing that

$$k_y^2 = k_z^2\left(\frac{1-p}{p}\right)$$

we have upon expanding (6),

$$p^2[(f^2 - g^2)h + k_z^2(f^2 - fh - g^2)] \\ + p[(h-f)k_z^4 - k_z^2(f^2 + fh - g^2)] + k_z^4 f = 0. \tag{7}$$

We note, in general, two solutions in p corresponding to each value of k_z, indicating birefringence of the medium. In particular k_z must be non-vanishing for birefringence to occur or, stated alternatively, rf field gradients must exist parallel to the applied magnetic field to obtain birefringence.

The characteristic vector solutions of (4) may be expressed for each solution of (7); they are the magnetic fields,

$$\boldsymbol{H} = H_x \begin{Bmatrix} 1 \\ \frac{i}{g}\left(f - \frac{k_z^2}{p}\right) \\ \mp \frac{i}{g} k_z^2 \left(\frac{1-p}{p}\right)^{\frac{1}{2}} \dfrac{\left(f - \frac{k_z^2}{p}\right)}{\left(h - k_z^2\left(\frac{1-p}{p}\right)\right)} \end{Bmatrix} \varepsilon^{-i(k_y y + k_z z)} \tag{8}$$

and the corresponding E fields,

$$\boldsymbol{E} = \frac{k_z}{\omega\varepsilon} H_x \begin{Bmatrix} i\frac{h}{g} \dfrac{\left(f - \frac{k_z^2}{p}\right)}{h - k_z^2\left(\frac{1-p}{p}\right)} \\ -1 \\ \pm\left(\frac{1-p}{p}\right)^{\frac{1}{2}} \end{Bmatrix} \varepsilon^{-i(k_y y + k_z z)} \tag{9}$$

The sign indeterminacy above is defined with respect to the ratio k_y/k_z, the upper sign being given by the positive value of this ratio.

We shall analyze the rectangular waveguide by first seeking parallel plane solutions and then utilizing these solutions to form those of the rectangular guide. We choose as parallel planes those perpendicular to the applied magnetic field, or z direction and having a separation b. Because of the absolute uniformity of this type of structure, the field configurations as a function of the coordinates transverse to the magnetic field, x and y, may change only by a uniform phase factor. Again, the choice of transverse axes is made such that these phase variations occur only along y.

From (7) we would find that a specification of k_y leads to a quadratic equation in p, with an appropriate consequent multiplicity in k_z^2. Let us define as a partial wave any standing wave in the z direction corresponding to some linear combination of the positive and negative values of k_z for one of the values of k_z^2. Examination of (9) reveals that the ratio of E_y to E_x, the field components tangent to the bounding walls, to be independent of the sign of k_z. Hence, each partial wave has an individual value of this ratio irrespective of its standing wave distribution in the z direction. It is thus impossible, in general, to provide a mutual cancellation of two or more partial waves at the electric walls by combinations of such partial waves, with the consequence that each partial wave must individually satisfy the boundary requirement. We find, then, that each partial wave takes on the familiar condition $k_z = m\pi/b$.

The parallel plane waves now will be appropriately oriented and superposed to satisfy the side wall boundary conditions in the rectangular guide. Since, as shown in Fig. 2, mutual cancellation is required on the side walls of the rectangular guide, the rate of vertical variation must be identical for all the component parallel plane waves; thus m is a constant of the waveguide mode and k_z is uniquely specified.

Two essential characteristics thus define a rectangular waveguide mode in a transversely magnetized, ferrite filled, medium.

1. The modes are ordered by integral values of m in the relationship $k_z = m\pi/b$.
2. The propagation constant k_y is uniquely specified.

Standing waves may now be formed in the z direction satisfying electric boundary conditions at the parallel planes. Each partial wave of the electric field may then be expressed as follows corresponding to its appropriate value of p:

$$E = \frac{m\pi}{b\omega\varepsilon} H_x \begin{Bmatrix} \dfrac{h}{g}\,\dfrac{\left(f - \dfrac{1}{p}\left(\dfrac{m\pi}{b}\right)^2\right)\sin\dfrac{m\pi}{b}z}{h - \left(\dfrac{m\pi}{b}\right)^2\left(\dfrac{1-p}{p}\right)} \\ i\sin\dfrac{m\pi z}{b} \\ \left(\dfrac{1-p}{p}\right)^{\frac{1}{2}}\cos\dfrac{m\pi}{b}z \end{Bmatrix} \varepsilon^{-i(m\pi/b)(1-p/p)^{\frac{1}{2}}y} \tag{10}$$

Let us now specialize our analysis to the small guide case. The requirement of birefringence to produce small guide propagation demands that k_z be non-vanishing and that m take on an integral value of unity or greater. We have, from (7), the two limiting values of p corresponding to a small value of b,

$$p_1 = \frac{f}{f - h} = \frac{\mu}{\mu - 1} \tag{11a}$$

$$p_2 = k_z^2 \frac{f - h}{f^2 - fh - g^2} = \frac{k_z^2}{\omega^2 \mu_0 \varepsilon} \frac{\mu - 1}{\mu^2 - \mu - \kappa^2} \tag{11b}$$

Discarding the z dependence in equation (10) and dropping a constant multiplier, the two characteristic electric field solutions become:

$$\boldsymbol{E}^{(1)} = \begin{pmatrix} \dfrac{1 - \mu}{\kappa} \\ i \\ i\mu^{-\frac{1}{2}} \end{pmatrix} \varepsilon^{(m\pi/b)\mu^{-\frac{1}{2}} y} \tag{12}$$

$$\boldsymbol{E}^{(2)} = \begin{pmatrix} 0 \\ i \\ i \end{pmatrix} \varepsilon^{(m\pi/b) y} \tag{13}$$

Equations (12) and (13) are parallel plane solutions obtained for some arbitrary direction, y, transverse to the magnetic field. This direction need not be intrinsically real; mathematically, it simply satisfies Maxwell's equations. We may transform to a desired waveguide frame of reference by rotations φ_1 and φ_2, corresponding to p_1 and p_2, about the z axis, where these rotations may possibly be made through complex angles. We then have for the electric fields in the new space:

$$\boldsymbol{E}^{(1)} \to \begin{pmatrix} \dfrac{1 - \mu}{\kappa} \cos \varphi_1 + i \sin \varphi_1 \\ -\left(\dfrac{1 - \mu}{\kappa}\right) \sin \varphi_1 + i \cos \varphi_1 \\ i\mu^{-\frac{1}{2}} \end{pmatrix} \varepsilon^{(m\pi/b)\mu^{-\frac{1}{2}}(y\cos\varphi_1 + x\sin\varphi_1)} \tag{14}$$

$$\boldsymbol{E}^{(2)} \to \begin{pmatrix} i \sin \varphi_2 \\ i \cos \varphi_2 \\ i \end{pmatrix} \varepsilon^{(m\pi/b)(y\cos\varphi_2 + x\sin\varphi_2)} \tag{15}$$

The new y axis of the transformed coordinates is now considered the longitudinal axis of the waveguide.

The partial wave fields of (14) and (15) may be joined to form a single mode by equating the propagation constant. Therefore,

$$\cos \varphi_2 = \mu^{-\frac{1}{2}} \cos \varphi_1 \tag{16}$$

where $\cos \varphi_2$ is imaginary for propagation. Propagation may therefore occur for $\mu > 0$ and $\cos \varphi_1$ imaginary and/or, $\mu < 0$ and $\cos \varphi_1$ real.

Boundary conditions require E_y and E_z to vanish at both guide side walls. Four equations result which may be satisfied, in turn, by a superposition of four transverse waves involving k_{x_1}, $-k_{x_1}$, k_{x_2}, and $-k_{x_2}$ corresponding to values $\pm\varphi_{1,2}$. For $\mu > 0$ both of the birefringent rays have transverse decay. Since the magnitudes of $k_{x_{1,2}}$ are large in small size guide (see Introduction) boundary conditions need be satisfied for practical purposes at only a single wall. We are then left with the simplification of only two equations in two unknowns.

Setting $x = 0$ in (14) and (15) and taking equation (16) into account, we have the boundary conditions

$$A\left[-\frac{(1-\mu)}{\kappa}\sin \varphi_1 + i\cos \varphi_1\right] + B[i\mu^{-\frac{1}{2}}\cos \varphi_1] = 0 \tag{17}$$

$$A[\mu^{-\frac{1}{2}}] + B = 0 \tag{18}$$

With the result that

$$\cot \varphi_1 = -i\frac{\mu}{\kappa} = \left(\frac{k_y}{k_{x_1}}\right) \tag{19}$$

Choosing k_y positive real, k_{x_1} is positive imaginary for κ positive and negative imaginary for κ negative. The rf field therefore hugs the right wall for $\kappa > 0$ and the left for $\kappa < 0$, or, alternatively, switches sides in the change from a forward to backward direction of propagation.

Equation (19) may be written equivalently as

$$\cos^2 \varphi_1 = \frac{\mu^2}{\mu^2 - \kappa^2} \tag{20}$$

Propagation, occurring for imaginary values of $\cos \varphi$ and $\mu > 0$, is obtained for $|\mu| < |\kappa|$.

Let us now analyze, the possibility of small guide propagation for $\mu < 0$. We find, from (16), that $\cos \varphi_1$ is real for this case. Two cases arise; the first for which $|\cos \varphi_1| < 1$ and the second for the reverse situation.

Let us first consider the case of $|\cos \varphi_1| < 1$. From (14), k_{x_1} is real whereas from (15) k_{x_2} is imaginary. Let us associate wave amplitudes with x dependences as follows:

$$A\varepsilon^{-ik_{x1}x}$$

$$B\varepsilon^{ik_{x1}x}$$

$$C\varepsilon^{-ik_{x2}x}$$

$$D\varepsilon^{ik_{x2}(x-a)}$$

where a is the guide width. Let us assume that k_{x_2} is a sufficiently large imaginary quantity of such sign that

$$\varepsilon^{-ik_{x2}a} \ll 1$$

This assumption will be seen to be consistent with the solution. [(26b) for small size guide.]

Setting up the boundary conditions for E_y and E_z at $x = 0$, we have from (14), (15), and (16),

$$(A - B)\left(\frac{\mu - 1}{\kappa}\right) \sin \varphi_1 + i(A + B) \cos \varphi_1 + iC\mu^{-\frac{1}{2}} \cos \varphi_1 = 0 \quad (21a)$$

$$(A + B)\mu^{-\frac{1}{2}} + C = 0 \quad (21b)$$

let $r = B/A$. Combining these last two equations we have

$$\frac{1 - r}{1 + r} = \frac{\kappa}{\mu} \cot \varphi_1 \quad (22)$$

Satisfying the boundary conditions at $x = a$ produces an equation similar to (22) with the substitution

$$r \rightarrow r\varepsilon^{-i2k_{x1}a} = r\varepsilon^{-i2\lambda}$$

Thus

$$\frac{1 - r}{1 + r} = \frac{1 - r\varepsilon^{-i2\lambda}}{1 + r\varepsilon^{i2\lambda}} \quad (23)$$

Equation (23) is satisfied by the condition $\lambda = n\pi$. Since $k_{x_1} = i(m\pi/b)\mu^{-\frac{1}{2}} \sin \varphi_1$, we have

$$\sin \varphi_1 = i\mu^{\frac{1}{2}} \frac{n}{m} \frac{b}{a} \quad (24)$$

The assumption that $\cos \varphi_1$ is real and less, in magnitude, than unity is realized by the condition

$$(-\mu)^{\frac{1}{2}} \frac{n}{m} \frac{b}{a} < 1 \quad (25)$$

Only in the limiting condition of a infinitely greater than b do all modes (m, n) propagate in the negative region of μ. In this particular case, $\sin \varphi_1 = 0$ and we find from (21a) and (21b) that $C = D = 0$. Thus we find a situation in which the guide boundary conditions are satisfied by but a single class rays of the two classes available.

This result is entirely comprehensible if we observe the wave number relationship obeyed by $\boldsymbol{k}_1$ and $\boldsymbol{k}_2$. Employing the definition of p which states that $k^2 = k_z^2/p$, and using (11a) and (11b), we have

$$k_{x_1}^2 + k_{y_1}^2 + k_z^2/\mu = 0 \tag{26a}$$

$$k_{x_2}^2 + k_{y_2}^2 + k_z^2 = 0 \tag{26b}$$

As stated in the Introduction, it is an entirely consistent procedure to satisfy boundary requirements with real wave numbers over the negative range of μ using the class of rays indicated for (26a) above.

More generally, (25) shows a complex relationship of the ordering of propagation modes by n and m, for finite a, for a given negative value of μ. In contrast to the $\mu > 0$ case, propagation may possibly not occur for a range of lower order integral values of m. As μ becomes increasingly large in magnitude, m must likewise take on increasingly higher values for transmission to occur.

The case of $\cos \varphi_1$ real and greater, in magnitude, than unity, leads to trivial result. Both partial waves have imaginary values of k_x, for this case, and the far wall receives essentially no coupling. Analysis simply repeats the result of (20) and we find that $| \mu | > | \kappa |$ and $\mu < 0$. If the Polder tensor components given in (2a) and (2b) are plotted (see Fig. 5). We find that this last set of inequalities form an impossible combination.

Summarizing we find that a rectangular waveguide of any dimension (and, in particular a guide of arbitrarily small dimensions), filled with a lossless transversely magnetized ferrite medium, will support an infinite number of freely propagating modes at any frequency for which $| \mu | < | \kappa |$. The character of these modes differs considerably in the two regions of $\mu < 0$ and $\mu > 0$ and somewhat different viewpoints of propagation must be taken. We shall find similar results relating to the longitudinally magnetized ferrite filled circular waveguide in the following section.

III. ANALYSIS OF LONGITUDINALLY MAGNETIZED FERRITE IN CIRCULAR GUIDE

We now proceed to a second structural geometry in which an anomalous behavior occurs attributable to the birefringence of the medium.

This is the circular guide which has been the subject of considerable analysis by Suhl and Walker. It is instructive, however, to repeat the analysis of this case, in the small guide limit, showing more pointedly its behavior from the viewpoint of combinations of the two types of waves in the medium.

The character of transmission in undersized circular waveguide is very similar to that of the undersized rectangular case. We may demonstrate the physical significance of this statement by the following argument. The excitation in a rectangular waveguide, for $|\mu| < |\kappa|$ and $\mu > 0$, is essentially that of a surface wave bound very tightly to a single wall. Considering this wall alone, which may now be extended to arbitrary dimensions but with k_z kept large, it may be wrapped upon itself either about the magnetic field as an axis or containing the magnetic field peripherally. In either event, the wrapped guide must start and terminate at the same phase, requiring a multiplicity of 2π around the circumference, and the wave must thus continue to have a large k_z value. Considering the large value of k_z and the state of excitation of the ferrite, the small circular guide may propagate.

Analysis will demonstrate that propagation also takes place in the region $\mu < 0$. The quantity k_{x_1} is real and k_{x_2} imaginary, see (26), leading to a case essentially similar to that of the rectangular waveguide. The analogy is appropriate to the case of b/a of finite value for which the rectangular guide requires the appearance of both refractions. We now proceed to obtain the field solutions for the circular guide.

Referring to (9) for the plane wave solution of the electric field, let us define to within a constant multiplier.

$$\boldsymbol{E} = \begin{pmatrix} iE_x \\ E_y \\ E_z \end{pmatrix} \varepsilon^{-i(k_y y + k_z z)} \tag{27}$$

where, for the case of large k_z (9, 12, 13)

$$E_x^{(1)} = \frac{1-\mu}{\kappa} \qquad E^{(2)} = 0$$

$$E_y^{(1)} = E_2^{(2)} = -1$$

$$E_z^{(1)} = i\mu^{-\frac{1}{2}} \qquad E_z^{(2)} = i$$

$$\frac{k_{y_1}}{k_z} = i\mu^{-\frac{1}{2}} \qquad \frac{k_{y_2}}{k_z} = i$$

We shall consider here, of the two possible wrapped-wall structures, that case in which the magnetic field is applied axially as shown in Fig. 3. Referring to Fig. 4, the cylindrical drical electric wave satisfying Max-

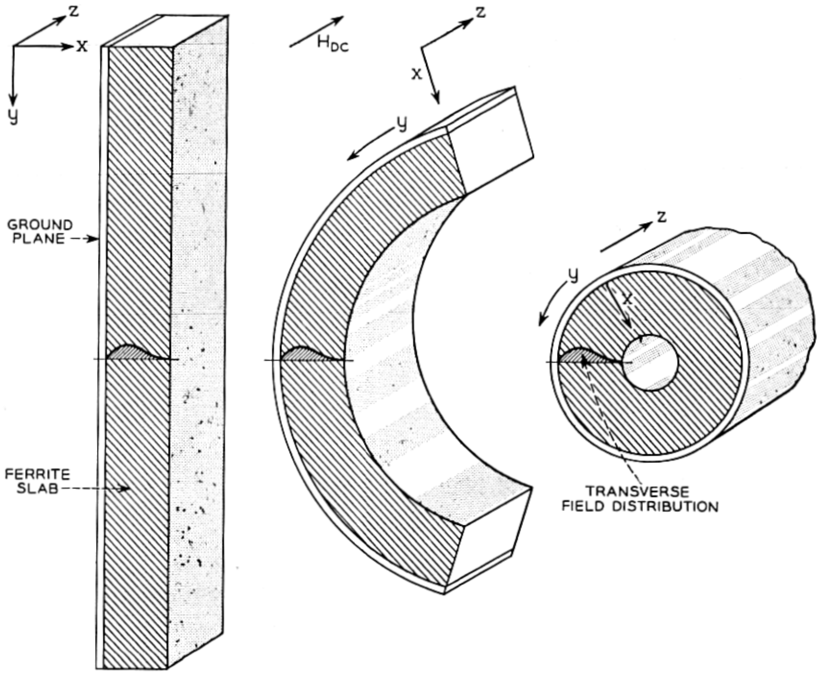

Fig. 3 — Axially magnetized filled circular guide formed by wrapping wall.

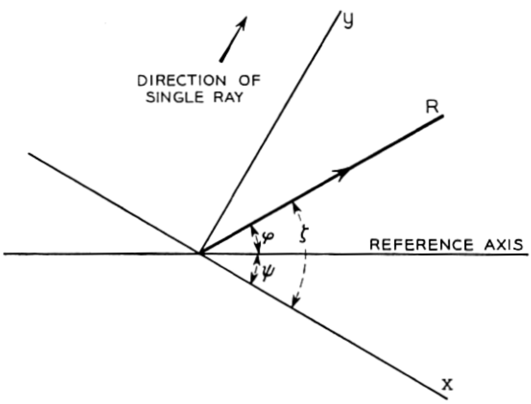

Fig. 4 — Transformation to polar coordinate frame.

well's equations and the boundary conditions for this structure is obtained by integrating plane waves of the form of (27) traveling at all possible angles ψ, the integration being subject to a weighting factor $G(\psi)$ to obtain the most general field. The coordinates (r, φ) refer to the physical system and the coordinate ψ identifies a plane wave traveling along a particular y axis. We have thus in an (r, φ, z) coordinate frame:

$$\underline{E} = \frac{1}{2\pi} \int_0^{2\pi} G(\psi) \begin{pmatrix} \cos \zeta & \sin \zeta & 0 \\ -\sin \zeta & \cos \zeta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} iE_x \\ E_y \\ E_z \end{pmatrix} \varepsilon^{-i(k_y y + k_z z)} \, d\psi \tag{28}$$

Recognizing that

$$d\psi = d\zeta$$

$$y = r \sin \zeta$$

and

$$G(\psi) = G(\zeta - \varphi)$$

an integration results over the variable ζ. Because of the uniqueness of the field as a function of φ, the only term containing φ, $G(\zeta - \varphi)$, must be a periodic function in its argument. A typical mode is formed by choosing one of the terms of the Fourier series of $G(\zeta - \varphi)$, namely $\varepsilon^{in(\zeta - \varphi)}$.

We find from (28) that

$$\boldsymbol{E}_n = \begin{pmatrix} i\left[E_x \frac{n}{\rho} J_n(\rho) + E_y J_n'(\rho)\right] \\ E_x J_n'(\rho) + E_y \frac{n}{\rho} J_n(\rho) \\ E_z J_n(\rho) \end{pmatrix} \varepsilon^{-i(k_z z + n\varphi)} \tag{29}$$

where $\boldsymbol{E}_n$ is that partial expansion of the total field $\boldsymbol{E}$, corresponding to the number of angular variation n, and $\rho = k_y r$. There are two values of ρ corresponding to the two values of k_y, and each leads to a partial wave. Let A and B be the respective partial wave amplitude; satisfying the boundary conditions on E_φ and E_z, we have from (29):

$$A\left(\frac{1 - \mu}{\kappa}\right) J_n'(\rho_1) - \frac{nA}{\rho_1} J_n(\rho_1) - J_n(\rho_2) = 0 \tag{30a}$$

$$A\mu^{-\frac{1}{2}} J_n(\rho_1) + BJ_n(\rho_2) = 0 \tag{30b}$$

where ρ_1 and ρ_2 are defined for $r = R$, the radius of the cylinder. Recognizing that $\rho_1 = \mu^{-\frac{1}{2}}\rho_2$, we have from (30)

$$\frac{\mu}{\kappa} J_n'(\rho_1) + n \frac{J_n(\rho_1)}{\rho_1} = 0 \text{ *} \tag{31}$$

where $\rho_1 = ik_z\mu^{-\frac{1}{2}}R$.

Equation (21) may be modified by a recurrence relationship to become

$$\frac{\kappa}{\mu} + 1 = \frac{\rho_1}{n} \frac{J_{n+1}(\rho)}{J_n(\rho_1)} \tag{32}$$

For $\mu > 0$ the quantity ρ_1 is a pure imaginary for large real values of k_z. Since the n^{th} order Bessel function is monotonic in imaginary arguments and possesses the multiplier $(i)^n$, the right-hand side is negative for n positive. For $n > 0$, propagation occurs for

$$|\kappa| > |\mu|$$
$$\operatorname{sgn} \kappa = -\operatorname{sgn} \mu$$

Inspection of (31) reveals that a reversal of the sign of n is equivalent to reversing the sign of κ. This conforms to the physical situation in which reversal of the sense of circular polarization is equivalent to the reversal of magnetic field. Thus for $n < 0$ and $\mu > 0$,

$$|\kappa| > |\mu|$$
$$\operatorname{sgn} \kappa = \operatorname{sgn} \mu$$

We find, from the above arguments, that just one sense of circular polarization propagates in an undersized circular guide for $\mu > 0$ and for a given direction of the magnetic field. It will be demonstrated shortly that propagation occurs for $\mu < 0$, but with an entirely different structure of modes. The right-hand side of (32) is monotonic as a function of ρ for $\mu > 0$, leading to only one solution for each value of n. This will not be the case for $\mu < 0$.

It is of interest first, however, to observe the limiting approach to $\mu = 0$ in the region of $\mu > 0$. The right-hand side of (32) is finite for finite imaginary values of ρ_1, so that the only solution as μ approaches zero is that for which the magnitude of ρ_1 becomes infinitely great. The Bessel function is asymptotically expansible as a cosine divided by a square root of its argument. Thus

$$J_n(\rho_1) = \frac{1}{2}\sqrt{\frac{2\pi}{\rho_1}}\left(\varepsilon^{i(\rho_1+[2n+1](\pi/4))} + \varepsilon^{-i(\rho_1+[2n+1](\pi/4))}\right) \tag{33}$$

* Equation (31) may likewise be obtained from the small radius limit in (34) of Reference 2.

Considering ρ_1 to be positive imaginary, as $\rho_1 \to i\infty$, (32) becomes

$$\frac{\kappa}{\mu} = \frac{-i\rho_1}{n} \tag{34}$$

Substituting for ρ_1, we have

$$k_z = \frac{n\kappa}{\mu^{\frac{1}{2}}R} \tag{35}$$

Thus, as μ approaches zero from values greater than zero, the propagation constant tends to become singular. Physically, however, μ does not vanish but approaches a small imaginary value caused by ferrite losses. The propagation constant k_z becomes complex and takes on a large imaginary component, signifying large guide attenuation. Since these losses occur in the limited neighborhood of $\mu = 0$, we may construe this waveguide behavior as corresponding to a system resonance.

In the region $\mu < 0$, ρ_1 becomes real while ρ_2 remains imaginary. The right-hand side of (32) is now composed of only real arguments. Since the zeros of different order Bessel functions alternate, the right side of (32) contains a succession of poles and zeros, leading to an infinite number of branches with each containing a solution ρ_1 to the equation. Thus there are an infinite number of propagating modes corresponding to each value

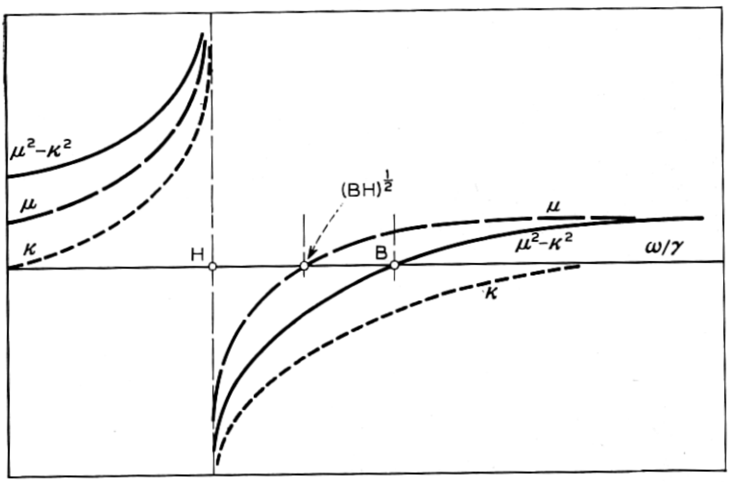

Fig. 5 — Frequency characteristics of Polder tensor components.

of n, in marked contrast to the case of $\mu > 0$. The solutions remain identical, as before, if both κ and n are simultaneously reversed in sign, but differ if only one of the two quantities is reversed.

Since $\mu = 0$ is a branch point, the limiting condition as μ approaches zero for values $\mu < 0$ differs from that for the reverse case. Equation (32) is now satisfied in the limit of small μ by the real zeros of $J_n(\rho_1)$. Since these roots are finite, k_z, equal to $-(-\mu)^{\frac{1}{2}}\rho_1/R$, tends towards zero for all modes. Since the formulae developed in this paper always presume large wave numbers, we may infer a vanishing value of k_z to simply represent a value which is small relative to the reciprocal of the waveguide radius. In any event, k_z is no longer singular at $\mu = 0$, and there is no resonance in the approach from negative values of μ.

In sum, the features of the circular guide strongly resemble those of the rectangular guide in the region of $\mu > 0$. This was to be anticipated by the "wrapped wall" construction where the wave is tightly bound to the wall. The wrapped equivalences do not hold in the region $\mu < 0$ since, with harmonic transverse dependence, the wave is no longer bound to the wall. This lack of equivalence is manifested in the matter of ordering modes. For a rectangular waveguide of finite aspect ratio, we find from (25) that there are but a finite number of modes corresponding to each value of m for $\mu < 0$. The circular guide differs in providing an infinite number of modes corresponding to each value of n. Further, whereas the circular guide covers the entire range of $|\mu| < |\kappa|$, (25) indicates that the various modes of the rectangular guide covers a more restricted range determined by the guide aspect ratio.

IV. CONCLUSIONS

The waveguide behavior analyzed in this paper has been experimentally observed[5] and good correlation has been obtained. From the viewpoint expressed of forming a guide cross-section by wrapping a wall to which a surface wave is bound, we may anticipate that the unusual behavior observed in the two types of guides examined is probably characteristic of many other structures.

It is not clear, at this time, if the complete set of modes of either the rectangular or circular guides have been exhausted. We already observe that an infinite number of modes propagate simultaneously so that scattering problems become considerably more complex than in the usual cases. It is felt by the author that the field of waveguide analysis calls for new methods and techniques of modal synthesis when ferrite loaded structures are considered.

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