Detection of Group Invariance or Total Symmetry of a Boolean Function*

By E. J. McCLUSKEY, Jr.

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A method is presented for determining whether a Boolean function possesses any group invariance; that is, whether there are any permutations or primings of the independent variables which leave the function unchanged. This method is then extended to the detection of functions which are totally symmetric.

1 GROUP INVARIANCE

For some Boolean transmission functions (transmissions, for short) it is possible to permute the variables, or prime some of the variables, or both permute and prime variables without changing the transmission. The following material presents a method for determining, for any given transmission, which of these operations (if any) can be carried out without changing the transmission.

The permutation operations will be represented symbolically as follows:

$S_{123\cdots n}T$ will represent the transmission T with no variables permuted

$S_{213\cdots n}T$ will represent the transmission T with the x_1 and x_2 variables interchanged, etc.

Thus $S_{1432}T(x_1, x_2, x_3, x_4) = T(x_1, x_4, x_3, x_2)$

The symbolic notation for the priming operation will be as follows:

$N_{0000\cdots 0}T$ will represent the transmission T with no variables primed

$N_{0110\cdots 0}T$ will represent the transmission T with the x_2 and x_3 variables primed, etc.

Thus $N_{1010}T(x_1, x_2, x_3, x_4) = T(x_1', x_2, x_3', x_4)$.

The notation for the priming operator can be shortened by replacing the binary subscript on N by its decimal equivalent. Thus N_9T is equiv-

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TABLE I — TRANSMISSION MATRICES SHOWING EFFECT OF INTERCHANGING OR PRIMING VARIABLES

(a) Transmission Matrix					(b) Transmission Matrix with x_3 and x_4 columns interchanged					(c) Transmission Matrix with entries of the x_3 and x_4 columns primed				
	x_1	x_2	x_3	x_4		x_1	x_2	x_4	x_3		x_1	x_2	x_3'	x_4'
0	0	0	0	0	0	0	0	0	0	3	0	0	1	1
1	0	0	0	1	2	0	0	1	0	2	0	0	1	0
2	0	0	1	0	1	0	0	0	1	1	0	0	0	1
9	1	0	0	1	10	1	0	1	0	10	1	0	1	0
10	1	0	1	0	9	1	0	0	1	9	1	0	0	1
11	1	0	1	1	11	1	0	1	1	8	1	0	0	0

alent to $N_{1001}T$. The permutation and priming operators can be combined. For example,

$$S_{2134}N_3T(x_1, x_2, x_3, x_4) = T(x_2, x_1, x_3', x_4')$$

The symbols S_iN_j form a mathematical group,[1] hence the term group invariance.

The problem considered here is that of determining which N_i and S_j satisfy the relation $N_iS_jT = T$ for a given transmission T. Since there are only a finite number of different N_i and S_j operators it is possible in principle to compute N_iS_jT for all possible N_iS_j and then select those N_iS_j for which $N_iS_jT = T$. If T is a function of n variables, there are $n!$ possible S_j operators and 2^n N_i operators so that there are $n!2^n$ possible combinations of N_iS_j. Actually, if $N_iS_jT = T$ then N_iT must equal $S_jT^{(2)}$ so that it is only necessary to compute all N_iT and all S_jT. For $n = 4$, $n! = 24$ and $2^n = 16$ so that the number of possibilities to be considered is quite large even for functions of only four variables. It is possible to avoid enumerating all N_iT and S_jT by taking into account certain characteristics of the transmission being considered.

The first step in determining the group invariances of a transmission is the same as that for finding the prime implicants.* The binary equivalents of the decimal numbers which specify the transmission are listed as in Table I(a). This list of binary numbers will be called the *transmission matrix*. When two variables are interchanged, the corresponding columns of the transmission matrix are also interchanged, Table I(b). When a variable is primed, the entries in the corresponding column of the transmission matrix are also primed, 0 replaced by 1 and 1 replaced by 0, Table I(c).

If an N_iS_j operation leaves a transmission unchanged then the cor-

* Minimization of Boolean Functions, see page 1417 of this issue.

responding matrix operations will not change the transmission matrix aside from possibly reordering the rows. In other words, it should be possible to reorder the rows of the modified transmission matrix to regain the original transmission matrix. The matrices of Table I(a) and (b) are identical except for the interchange of the 1 and 2 and the 9 and 10 rows. It is not possible to make the matrix of Table I(c) identical with that of Table I(a) by reordering rows; therefore the operation of priming the x_3 and x_4 variables does not leave the transmission $T = \sum (0, 1, 2, 9, 10, 11)$ unchanged.

If interchanging two columns of a matrix does not change the matrix aside from rearranging the rows, then the columns which were interchanged must both contain the same number of 1's (and 0's). This must

Table II — Partitioning of the Standard Matrix for $T = \sum (4, 5, 7, 8, 9, 11, 30, 33, 49)$

(a) Transmission Matrix

	x_1	x_2	x_3	x_4	x_5	x_6
4	0	0	0	1	0	0
8	0	0	1	0	0	0
5	0	0	0	1	0	1
9	0	0	1	0	0	1
33	1	0	0	0	0	1
7	0	0	0	1	1	1
11	0	0	1	0	1	1
49	1	1	0	0	0	1
30	0	1	1	1	1	0
Number of 0's	7	7	5	5	6	3
Number of 1's	2	2	4	4	3	6

(b) Standard Matrix for (a) Matrix

	x_1	x_2	x_3	x_4	x_5	x_6'	Weight
4	0	0	0	1	0	0	1
8	0	0	1	0	0	0	1
32	1	0	0	0	0	0	1
5	0	0	0	1	0	1	2
6	0	0	0	1	1	0	2
9	0	0	1	0	0	1	2
10	0	0	1	0	1	0	2
48	1	1	0	0	0	0	2
31	0	1	1	1	1	1	5
	7	7	5	5	6	6	
	2	2	4	4	3	3	

(c) Second Partitioning of rows for (b) matrix

x_1	x_2	x_3	x_4	x_5	x_6'
0	0	0	1	0	0
0	0	1	0	0	0
1	0	0	0	0	0
0	0	0	1	0	1
0	0	0	1	1	0
0	0	1	0	0	1
0	0	1	0	1	0
1	1	0	0	0	0
0	1	1	1	1	1

(d) Final Partitioning for (b) matrix

x_1	x_2	x_3	x_4	x_5	x_6'
0	0	0	1	0	0
0	0	1	0	0	0
1	0	0	0	0	0
0	0	0	1	0	1
0	0	0	1	1	0
0	0	1	0	0	1
0	0	1	0	1	0
1	1	0	0	0	0
0	1	1	1	1	1

be true since rearranging the rows of a matrix does not change the total number of 1's in each column. Similarly, if priming some columns of a matrix leaves the rows unchanged, either each column must have an equal number of 1's and 0's or else for each primed column which has an unequal number of 0's and 1's there must be a second primed column which has as many 1's as the first primed column has 0's and vice versa. Such pairs of columns must also be interchanged to keep the total number of 1's in each column invariant. For the matrix of Table II(a) the only operations that need be considered are either interchanging x_1 and x_2 or x_3 and x_4 or priming and interchanging x_5 and x_6.

For the present it will be assumed that no columns of the matrix have an equal number of 0's and 1's. It is possible to determine all permuting and priming operations which leave such a matrix unchanged by considering only permutation operations on a related matrix. This related matrix, called the *standard matrix,* is formed by priming all the columns of the original matrix which have more 1's than 0's, the x_6 column in the matrix of Table II(a). Each column of a standard matrix must contain more 0's than 1's, Table II(b). The N_iS_j operations which leave the original matrix unchanged can be determined directly from the operations that leave the corresponding standard matrix unchanged. It is therefore only necessary to consider standard matrices.

Since no columns of a standard matrix have an equal number of 1's and 0's and no columns have more 1's than 0's it is only necessary to consider permuting operations. The number of 1's in a column (or row) will be called the *weight* of the column (or row). Only columns or rows which have the same weights can be interchanged. The matrix should be partitioned so that all columns (or rows) in the same partition have the same weight, Table II(b). It is now possible to interchange columns in the same column partition and check whether pairs of rows from the same row partition can then be interchanged to regain the original matrix. This can usually be done by inspection. For example, in Table II(b) if columns x_4 and x_3 are interchanged, then interchanging rows 4 and 8, 5 and 9, and 6 and 10 will regain the original matrix.

The process of inspection can be simplified by carrying the partitioning further. In the matrix of Table II(b), row 32 cannot be interchanged with either row 8 or row 4. This is because it is not possible to make row 32 identical with either row 8 or row 4 by interchanging columns x_1 and x_2. Row 32 has weight 1 in these columns while rows 8 and 14 both have weight 0. In general, only rows which have the same weight in *each* submatrix can be interchanged. Permuting columns of the same partition does not change the weight of the rows in the corresponding submatrices.

The matrix can therefore be further partitioned by separating the rows into groups of rows which have the same weight in every column partition, Table II(c). Similar remarks hold for the columns so that it may then be necessary to partition the columns again so that each column in a partition has the same weight in each submatrix, Table II(d). Partitioning the columns may make it necessary to again partition the rows, which in turn may make more column partitioning necessary. This process should be carried out until a matrix results in which each row (column) of each submatrix has the same weight. Inspection is then used to determine which row and column permutations will leave the matrix unchanged. Only permutations among rows or columns in the same partition need be considered.

From the matrix of Table II(d) it can be seen that permuting either columns x_3 and x_4 or columns x_5 and x_6' will not change the matrix aside from reordering certain rows. This means that interchanging x_3 and x_4 or priming and interchanging x_5 and x_6 in the original transmission will leave the transmission unchanged. Interchanging x_6' and x_5 means replacing x_5 by x_6' and x_6 by x_5' which is the same as interchanging x_5 and x_6 and then priming both x_5 and x_6. Thus for the transmission of Table II $S_{124356}T = T$ and $N_{000011}S_{123465}T = N_3S_{123465}T = T$.

A procedure has been presented for determining the group invariance of any transmission matrix which does not have an equal number of 1's and 0's in any column. This must now be extended to matrices which do have equal numbers of 0's and 1's in some columns, Table III(a). For such matrices the procedure is to prime appropriate columns so that there are either more 0's than 1's or the same number of 0's and 1's in each column, Table III(a). This matrix is then partitioned as described above and the permutations which leave the matrix unchanged are determined. The matrix of Table III(a) is so partitioned. Interchanging

TABLE III — TRANSMISSION MATRICES FOR $T = \sum(0, 6, 9, 12)$

(a) Transmission Matrix

	x_1	x_2	x_3	x_4
0	0	0	0	0
6	0	1	1	0
9	1	0	0	1
12	1	1	0	0
Number of 0's	2	2	3	3
Number of 1's	2	2	1	1

(b) Transmission Matrix with x_1 and x_2 primed

	x_1'	x_2'	x_3	x_4
0	0	0	0	0
10	1	0	1	0
5	0	1	0	1
12	1	1	0	0
	2	2	3	3
	2	2	1	1

both x_1 and x_2, and x_3 and x_4 leave this matrix unchanged so that $S_{2143}T = T$. The possibility of priming different combinations of the columns which have an equal number of 0's and 1's must now be considered. Certain of the possible combinations can be excluded beforehand. In Table III(a) the only possibility which must be considered is that of priming both x_1 and x_2. If only x_1 or x_2 is primed, there will be no row which has all zeros. No permutation of the columns of this matrix (with x_1 or x_2 primed) can produce a row with all zeros. Therefore this matrix cannot possibly be made equal to the original matrix by rearranging rows and columns. Priming both x_1 and x_2 must be considered since the 12-row will be converted into a row with all zeros. The operation of priming x_1 and x_2 is written symbolically as $N_{1100} = N_{12}$. In general, if the matrix has a row consisting of all zeros, only those N_i operations for which i is the number of some row in the matrix, need be considered. If the row does not have an all-zero row, only those N_i for which i is *not* the number of some row need be considered. Similarly, if the matrix has a row consisting of all 1's, only those N_i for which there is some row of the matrix which will be converted into an all-one row, need be considered. This is equivalent to considering only those N_i for which some row has a number $k = 2^n - 1 - i$* where n is the number of columns. If the matrix does *not* have an all-one row, only those N_i for which *no* row has a number $k = 2^n - 1 - i$ should be considered.

Each priming operation which is not excluded by these rules is applied to the transmission matrix. The matrices so formed are then partitioned as described previously. Any of these matrices that have the same partitioning as the original matrix are then inspected to see if any row and column permutations will convert them to the original matrix. For the matrix of Table III(a) the operation of priming both x_1 and x_2 was not excluded. The matrix which results when these columns are primed is shown in Table III(b). Inspection of this figure shows that interchange of either x_3 and x_4 or x_1' and x_2' will convert the matrix back to the matrix of Table III(a). Therefore, for the transmission of this table $S_{1243}N_{1100}T = T$ and $S_{2134}N_{1100}T = T$.

2 TOTAL SYMMETRY

There are certain transmissions whose value depends not on which relays are operated but only on how many relays are operated. For

* The number of the row which has all ones is $2^n - 1$. If N_i operating on some row, k, is to produce the all-one row, i must have 1's wherever k has 0's and vice versa. This means that

$$i + k = 2^n - 1 \quad \text{or} \quad k = 2^n - 1 - i.$$

TABLE IV — TRANSMISSION MATRIX FOR
$T = \sum (3, 5, 6, 7) = S_{2,3}(x_1, x_2, x_3)$

	x_3	x_2	x_1
3	0	1	1
5	1	0	1
6	1	1	0
7	1	1	1

example, the transmission of Table IV equals 1 whenever two or three relays are operated. For such transmissions any permutation of the variables leaves the transmission unchanged. These transmissions are called *totally symmetric*.[3] They are usually written in the form, $T = S_{a_1, a_2 \cdots a_m}(x_1, x_2, \cdots x_n)$, where the transmission is to equal 1 only when exactly a_1 or a_2 or $\cdots$ or a_m of the variables $x_1, x_2 \cdots x_n$ are equal to one. The transmission of Table IV can be written as $S_{2,3}(x_1, x_2, x_3)$. This definition of symmetric transmissions can be generalized by allowing some of the variables $(x_1, x_2, \cdots x_n)$ to be primed. Thus the transmission $S_3(x_1, x_2', x_3)$ will equal 1 only when $x_1 = x_2' = x_3 = 1$ or $x_1 = x_3 = 1$ and $x_2 = 0$. It is useful to know when a transmission is totally symmetric since special design techniques exist for such functions.[4]

It is possible to determine whether a transmission is totally symmetric from its matrix. Unless all columns of the standard matrix derived from the transmission matrix have the same weight, the transmission cannot possibly be totally symmetric. If all columns do have equal weights, the rows should be partitioned into groups of rows which all have the same weight. Whether the transmission is totally symmetric can now be determined by inspection. If there is a row of weight k; that is, a row which contains k 1's, then every possible row of weight k must also be included in the matrix. This means that there must be ${}_nC_k$ rows of weight k where n is the number of columns (variables).* If any possible row of weight k was not included then the corresponding k literals could be set equal to 1 without the transmission being equal to 1. This contradicts the definition of a totally symmetric transmission. In Table V(b) there are 4 rows of weight 1 and 1 row of weight 4. Since ${}_4C_1 = 4$ and ${}_4C_4 = 1$ this transmission is totally symmetric and can be written as $S_{1,4}(x_1, x_2', x_3, x_4')$. The number of rows of weight 1 in Table V(d) is 2 and since ${}_4C_1 = 4$ this transmission is *not* totally symmetric.

A difficulty arises if all columns of a transmission matrix contain equal

* ${}_nC_k$ is the binomial coefficient $\dfrac{n!}{(n-k)!k!}$

TABLE V — DETERMINATION OF TOTALLY SYMMETRIC TRANSMISSION

(a) Transmission Matrix for $T = \sum (1, 4, 7, 10, 13)$

	x_1	x_2	x_3	x_4
1	0	0	0	1
4	0	1	0	0
10	1	0	1	0
7	0	1	1	1
13	1	1	0	1
Number of 0's	3	2	3	2
Number of 1's	2	3	2	3

(b) Standard Matrix for $T = \sum (1, 4, 7, 10, 13)$ showing that $T = S_{1,4}(x_1, x_2', x_3, x_4')$

	x_1	x_2'	x_3	x_4'
1	0	0	0	1
2	0	0	1	0
4	0	1	0	0
8	1	0	0	0
15	1	1	1	1
	3	3	3	3
	2	2	2	2

(c) Transmission Matrix for $T = \sum (3, 5, 10, 12, 13)$

	x_1	x_2	x_3	x_4
3	0	0	1	1
5	0	1	0	1
10	1	0	1	0
12	1	1	0	0
13	1	1	0	1
Number of 0's	2	2	3	2
Number of 1's	3	3	2	3

(d) Standard Matrix for $T = \sum (3, 5, 10, 12, 13)$ showing that it is not totally symmetric

	x_1'	x_2'	x_3	x_4'
0	0	0	0	0
1	0	0	0	1
8	1	0	0	0
7	0	1	1	1
14	1	1	1	0
	3	3	3	3
	2	2	2	2

TABLE VI — DETERMINATION OF TOTAL SYMMETRY FOR $T = \sum (0, 3, 5, 10, 12, 15)$

(a) Transmission Matrix for $T(x_1, x_2, x_3, x_4)$

	x_1	x_2	x_3	x_4
0	0	0	0	0
3	0	0	1	1
5	0	1	0	1
10	1	0	1	0
12	1	1	0	0
15	1	1	1	1
Number of 0's	3	3	3	3
Number of 1's	3	3	3	3

(b) Standard Matrix for $T(1, x_2, x_3, x_4)$

	x_2'	x_3'	x_4
	1	0	0
	0	1	0
	0	0	1
Number of 0's	2	2	2
Number of 1's	1	1	1

$$T(1, x_2, x_3, x_4) = S_1(x_2', x_3', x_4)$$

(c) Standard Matrix for $T(0, x_2, x_3, x_4)$

	x_2	x_3	x_4'
	0	0	1
	0	1	0
	1	0	0
Number of 0's	2	2	2
Number of 1's	1	1	1

$$T(0, x_2, x_3, x_4) = S_1(x_2, x_3, x_4') = S_2(x_2', x_3', x_4)$$

numbers of zeros and ones as in Table VI(a). For such a matrix it is not clear which variables should be primed. It is possible to avoid considering all possible primings by "expanding" the transmission about one of the variables by means of the theorem

$$T(x_1, x_2, \cdots x_n) = x_1 T(1, x_2, \cdots x_n) + x_1' T(0, x_2, \cdots x_n)^{2,3}$$

and then making use of the relation:

$$S_{a_1, a_2, \cdots a_m}(x_1, x_2, \cdots x_n) = x_1 S_{a_1-1, a_2-1, a_2-1, \cdots a_m-1}(x_2, \cdots x_m) + x_1' S_{a_1, a_2, \cdots a_m}(x_2, \cdots x_n)^5$$

This technique is illustrated in Table VI. The standard matrix for $T(1, x_2, x_3, x_4)$ has three rows each containing a single one so that

$$T(1, x_2, x_3, x_4) = S_1(x_2', x_3', x_4)$$

The transmission $T(0, x_2, x_3, x_4)$ has an identical standard matrix so that

$$T(0, x_2, x_3, x_4) = S_1(x_2, x_3, x_4')$$

This can be written in terms of x_2', x_3', and x_4 :

$$S_1(x_2, x_3, x_4') = S_2(x_2', x_3', x_4)^5.$$

Finally

$$T(x_1, x_2, x_3, x_4) = x_1 T(1, x_2, x_3, x_4) + x_1' T(0, x_2, x_3, x_4)$$
$$= x_1 S_1(x_2', x_3', x_4) + x_1' S_2(x_2', x_3', x_4) = S_2(x_1, x_2', x_3', x_4).*$$

The method just presented for detecting total symmetry is more systematic than the only other available method[5] and applies for transmissions of any number of variables.

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