Frequency Conversion by Means of a Nonlinear Admittance

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This paper gives a mathematical analysis of a heterodyne conversion transducer in which the nonlinear element is made up of a nonlinear resistor and a nonlinear capacitor in parallel. Curves are given showing the change in admittance and gain as the characteristics of the nonlinear elements are varied. The case where a conjugate match exists at the terminals is treated.

It is shown that when the output frequency is greater than the input frequency, modulators having substantial gain and bandwidth are possible, but when the output frequency is less than the input frequency, the converter loss is greater than unity and is little affected by the nonlinear capacitor. The conditions under which a conjugate match is possible are specified and it is concluded that a nonlinear capacitor alone is the preferred element for modulators and that a nonlinear resistor alone gives the best performance in converters.

INTRODUCTION

Point contact rectifiers using either silicon or germanium are used as the nonlinear element in microwave modulators to change an intermediate frequency signal to an outgoing microwave signal and in receiving converters to change an incoming microwave signal to a lower intermediate frequency. Most point contact rectifiers now in use behave as pure nonlinear resistors as evidenced by the fact that in either of the above uses the conversion loss is the same. In recent experiments with heterodyne conversion transducers* using point contact rectifiers made with ion bombarded silicon this was found to be no longer true. The conversion loss of the modulator was found to be unusually low and

* This term is defined in American Standard Definitions of Electrical Terms — ASA C42 — as "a conversion transducer in which the useful output frequency is the sum or difference of the input frequency and an integral multiple of the frequency of another wave".

that of the converter was several decibels greater. In one instance the loss in a modulator used to convert a 70 mc signal to one at 11,130 mc was found to be only 2.3 db but when the direction of transmission through it was reversed and it was used as a converter, the loss was 7.8 decibels.

Similar effects were observed several years ago in conversion transducers using welded contact germanium rectifiers.[1] In these early experiments substantial converter gain and negative conductance at the intermediate frequency terminals were also observed. These results were accounted for by assuming the presence of a nonlinear capacitance at the point contact in parallel with the nonlinear resistance. At that time attention was devoted mainly to the behavior of converters where noise is a vital factor. It was found that although the conversion loss could be reduced, the noise temperature increased and no improvement in noise figure resulted. However, the noise temperature requirements in modulators are much less severe and the nonlinear capacitance effect is useful and can substantially improve the performance.

THEORY

The mathematical analysis given here was undertaken in order to clarify the effect of the nonlinear capacitance in the frequency conversion process and to obtain an estimate of the usefulness of modulators exhibiting gain. The analysis is restricted to the simplest case in which signal voltages are allowed to develop across the nonlinear elements at the input and output frequencies only. This is not an unrealistic restriction since the conversion transducers used in microwave relay systems have filters associated with them which suppress the modulation products outside the signal band. The final results will be given only for those conditions which permit a conjugate match at the input and output of the transducer.

The procedure used to obtain expressions for the admittance and gain of conversion transducers utilizing a nonlinear element made up of a nonlinear resistance and a nonlinear capacitance in parallel follows the commonly used method of treating the nonlinear elements as local oscillator controlled linear time varying elements.[2] The current through the nonlinear resistor is a function of the applied voltage. The derivative of this function is the conductance as a function of the applied voltage. Thus when the local oscillator is applied, the conductance varies at the local oscillator frequency and the conductance as a function of time may be obtained. This is periodic and may be expressed as a Fourier series. The conductance is real and if we make the usual assumption that it may be expressed as an even function of time, we may write

$$\gamma = \cdots + G_2 e^{-j2\omega_0 t} + G_1 e^{-j\omega_0 t} + G_0 + G_1 e^{j\omega_0 t} + G_2 e^{j2\omega_0 t} + \cdots \quad (1)$$

where $\omega_0/2\pi$ is the local oscillator frequency f_0 and the Fourier coefficients G_n are real. Similarly the charge on the nonlinear capacitor is a function of the applied voltage. The derivative of this function is the capacitance as a function of the applied voltage. The application of the local oscillator thus causes the capacitance to vary at the local oscillator frequency so that it also may be expressed as a Fourier series. The capacitance κ is real, and assuming it may be expressed as an even function of time, we have

$$\kappa = \cdots + C_2 e^{-j2\omega_0 t} + C_1 e^{-j\omega_0 t} + C_0 + C_1 e^{j\omega_0 t} + C_2 e^{j2\omega_0 t} + \cdots \quad (2)$$

It is assumed that the current and charge functions are single valued and that their derivatives are always positive.

When a small signal voltage v is applied to the nonlinear resistor, the signal current through the resistor is given by γv. When it is applied to the nonlinear capacitor the charge on the capacitor is κv. The total current i which flows through the two nonlinear elements connected in parallel thus becomes

$$i = \gamma v + \frac{d}{dt}(\kappa v) \quad (3)$$

v of course must be small and not affect the value of γ and κ.

Fig. 1 shows a heterodyne conversion transducer made up of a nonlinear resistor and a nonlinear capacitor in parallel driven by an internal local oscillator. f_1 is the signal frequency at the terminals 1–2, and y_1 is the external admittance connected to these terminals. The signal frequency at the terminals 3–4 is f_2, and y_2 is the external admittance.

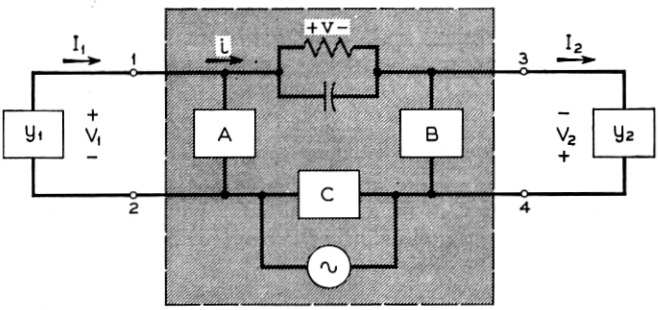

Fig. 1 — Heterodyne conversion transducer.

A, B and C are ideal frequency selective networks whose admittances are zero at f_1, f_2 and f_0 respectively, and infinite at all other frequencies. This circuit permits the application of the local oscillator voltage to the nonlinear elements but permits signal voltages to develop across them at f_1 and f_2 only. Similarly, signal currents at frequencies other than f_1 and f_2 encounter no external impedance, so they cannot alter the signal voltage or contribute to the external power. This, of course, assumes that if the nonlinear element is a point contact rectifier the spreading resistance normally present is negligible.

If f_1 is a frequency less than half the local oscillator frequency f_0 (it is generally very much less), the network B can be selected to make f_2 either $f_0 + f_1$, or $f_0 - f_1$. To distinguish between the two cases, we will call the former a noninverting conversion transducer since an increase in one signal frequency causes an increase in the other. The latter will be called an inverting conversion transducer since an increase in one signal frequency results in a decrease in the other. When y_1 contains the generator and y_2 the load, the device becomes a modulator. When y_2 contains the generator and y_1 the load, it is a converter.

The real part of the signal voltage may be written

$$v = V_1 e^{j\omega_1 t} + V_1^* e^{-j\omega_1 t} + V_2 e^{j\omega_2 t} + V_2^* e^{-j\omega_2 t} \tag{4}$$

where V^* is the complex conjugate of V and $\omega = 2\pi f$. Similarly, the real part of the signal current may be written

$$i = I_1 e^{j\omega_1 t} + I_1^* e^{-j\omega_1 t} + I_2 e^{j\omega_2 t} + I_2^* e^{-j\omega_2 t} \tag{5}$$

If we multiply equations (1) and (4) and retain only those terms containing f_1 and f_2 we obtain, in the case of the non-inverting conversion transducer where $f_2 = f_0 + f_1$,

$$\begin{aligned} \gamma v = {} & [G_0 V_1 + G_1 V_2] e^{j\omega_1 t} + [G_1 V_1 + G_0 V_2] e^{j\omega_2 t} \\ & + [G_0 V_1^* + G_1 V_2^*] e^{-j\omega_1 t} + [G_1 V_1^* + G_0 V_2^*] e^{-j\omega_2 t} \end{aligned} \tag{6}$$

Similarly, if we multiply (2) and (4) we get an expression like (6) with the G's replaced by C's. If we differentiate this expression we get

$$\begin{aligned} \frac{d}{dt}(\kappa v) = {} & j\omega_1 [C_0 V_1 + C_1 V_2] e^{j\omega_1 t} + j\omega_2 [C_1 V_1 + C_0 V_2] e^{j\omega_2 t} \\ & - j\omega_1 [C_0 V_1^* + C_1 V_2^*] e^{-j\omega_1 t} - j\omega_2 [C_1 V_1^* + C_0 V_2^*] e^{-j\omega_2 t} \end{aligned} \tag{7}$$

When we perform the addition indicated by (3) and compare the result with (5) we obtain

$$\begin{aligned} I_1 &= [G_0 + j\omega_1 C_0] V_1 + [G_1 + j\omega_1 C_1] V_2 \\ I_2 &= [G_1 + j\omega_2 C_1] V_1 + [G_0 + j\omega_2 C_0] V_2 \end{aligned} \tag{8}$$

Going through the same steps for the inverting conversion transducer where $f_2 = f_0 - f_1$ we obtain

$$\begin{aligned} I_1 &= [G_0 + j\omega_1 C_0]V_1 + [G_1 + j\omega_1 C_1]V_2^* \\ I_2^* &= [G_1 - jw_2 C_1]V_1 + [G_0 - j\omega_2 C_0]V_2^* \end{aligned} \tag{9}$$

Equations (8) and (9) are in the form

$$\begin{aligned} I_1 &= Y_{11}V_1 + Y_{12}V_2 \\ I_2 &= Y_{21}V_1 + Y_{22}V_2 \end{aligned} \tag{10}$$

A heterodyne conversion transducer may thus be represented by a linear 4-pole, and the admittance and gain of the 4-pole may be expressed in terms of the admittance coefficients. In Fig. 1 we see that the admittance of the 4-pole y_1' at the terminals 1–2 is equal to I_1/V_1 and the admittance y_2 connected to terminals 3–4 is $-I_2/V_2$. Putting these in (10) we find

$$y_1' = Y_{11} - \frac{Y_{12}Y_{21}}{Y_{22} + y_2} \tag{11}$$

Similarly the admittance of the 4-pole y_2' at the terminals 3–4 is I_2/V_2 and the admittance y_1 connected to terminals 1–2 is $-I_1/V_1$. Putting these in (10) gives

$$y_2' = Y_{22} - \frac{Y_{12}Y_{21}}{Y_{11} + y_1} \tag{12}$$

To compute the gain of the 4-pole when y_1 contains the generator and y_2 the load, it is convenient to assume a current source connected across y_1. If the current from this source is I_0 we have $I_1 = I_0 - y_1V_1$. I_2 equals $-y_2V_2$ as before. Putting these in (10) gives

$$\frac{I_0}{V_2} = Y_{12} - \frac{(Y_{11} + y_1)(Y_{22} + y_2)}{Y_{21}} \tag{13}$$

If we let $y_1 = g_1 + jb_1$ and $y_2 = g_2 + jb_2$, the power in the load is $V_2^2 g_2$ and the power available from the generator is $I_0^2/4g_1$. Therefore the transducer gain Γ_{12} defined as the ratio of the power in y_2 to that available from y_1 becomes

$$\Gamma_{12} = 4g_1g_2\frac{V_2^2}{I_0^2} = 4g_1g_2\left|\frac{Y_{21}}{Y_{12}Y_{21} - (Y_{11} + y_1)(Y_{22} + y_2)}\right|^2 \tag{14}$$

When y_2 contains the generator and y_1 the load, we may proceed in the same way (letting I_0 flow in terminal 4) and obtain

$$\Gamma_{21} = 4g_1g_2\left|\frac{Y_{12}}{Y_{12}Y_{21} - (Y_{11} + y_1)(Y_{22} + y_2)}\right|^2 \tag{15}$$

We may now obtain expressions for the admittance and gain of the 4-pole when the nonlinear element consists of a nonlinear resistor and a nonlinear capacitor in parallel. We shall do this for the case where a conjugate match exists at the terminals by letting $y_1' = y_1^*$ and $y_2' = y_2^*$. Equations (11) and (12) may thus be written

$$(Y_{11} - y_1^*)(Y_{22} + y_2) = (Y_{11} + y_1)(Y_{22} - y_2^*) = Y_{12}Y_{21} \quad (16)$$

When this is multiplied out, letting $Y_{mn} = G_{mn} + jB_{mn}$, and the real and imaginary parts set equal as indicated by the first equality we obtain $G_{11}g_2 = G_{22}g_1$ and $g_2(B_{11} + b_1) = g_1(B_{22} + b_2)$. In (8) and (9) it is seen that $G_{11} = G_{22} = G_0$ and that B_{22} is positive in equations (8) and negative in equations (9). We thus obtain

$$g_1 = g_2 \qquad b_1 + \omega_1 C_0 = b_2 \pm \omega_2 C_0 \quad (17)$$

where the upper symbol of the $\pm$ sign is used in the noninverting case and the lower symbol in the inverting case. When the real and imaginary parts are set equal as indicated by the second equality in (16) we obtain, using the results in (17),

$$g^2 = G_0^2 - G_1^2 \pm \omega_1\omega_2 C_1^2 - B^2 \quad (18)$$

where

$$g = g_1 = g_2 \quad (19)$$

and

$$B = b_1 + \omega_1 C_0 = b_2 \pm \omega_2 C_0 = \pm \frac{G_1}{2G_0}(\omega_2 \pm \omega_1)C_1 \quad (20)$$

These results may be put in (14) to obtain the modulator gain. Since a conjugate match exists at the terminals of the 4-pole, this is the maximum available gain. The result is

$$MAG_{12} = \frac{G_1^2 + (\omega_2 C_1)^2}{(G_0 + g)^2 + B^2} \quad (21)$$

For the converter, using equation (15) we obtain

$$MAG_{21} = \frac{G_1^2 + (\omega_1 C_1)^2}{(G_0 + g)^2 + B^2} \quad (22)$$

These results are valid only when a conjugate match exists at the terminals. For this to be possible, the right side of (18) must be positive. If it is negative no combination of values of g_1 and g_2 will result in a match.

It may be shown that if the slope of the voltage-current characteristic of the nonlinear resistor is always positive, then G_1/G_0 can never be greater than unity. (Reference 1, p. 410.) It is therefore convenient to normalize the above results with respect to G_0. If we let

$$\frac{\omega_1}{\omega_2} = \rho, \qquad \frac{\omega_1 C_1}{G_0} = \rho x, \qquad \frac{\omega_2 C_1}{G_0} = x, \qquad \frac{G_1}{G_0} = y, \qquad \frac{C_0}{C_1} = z \tag{23}$$

equations (18) through (22) become

$$\left(\frac{g}{G_0}\right)^2 = 1 - y^2 \pm \rho x^2 - \left[(1 \pm \rho)\frac{xy}{2}\right]^2 \tag{24}$$

$$\frac{b_1}{G_0} = \pm (1 \pm p)\frac{xy}{2} - \rho x z, \qquad \frac{b_2}{g_0} = \pm(1 \pm \rho)\frac{xy}{2} \pm xz \tag{25}$$

$$MAG_{12} = \frac{y^2 + x^2}{\left(1 + \frac{g}{G_0}\right)^2 + \left[(1 \pm \rho)\frac{xy}{2}\right]^2} \tag{26}$$

$$MAG_{21} = \frac{y^2 + (\rho x)^2}{\left(1 + \frac{g}{G_0}\right)^2 + \left[(1 \pm \rho)\frac{xy}{2}\right]^2} \tag{27}$$

In these equations, ρ is less than $\frac{1}{3}$ in the noninverting case and less than 1 in the inverting case. Ordinarily it will be very much less than 1. The value of z will be determined by the shape of the nonlinear capacitor characteristic. However z appears only in (25) where it influences the values of the matching susceptances so that it does not affect the conductance or gain. While we can be certain that y will have values between 0 and 1, limitations on the value of x will depend on the particular device used. We will therefore assume that x may have any value.

EFFECT OF NONLINEAR CAPACITOR

We may now examine, in a general way, the manner in which the nonlinear capacitor influences the behavior of the 4-pole. Consider first the case where the nonlinear capacitor is absent. It is well known, and can be seen in the above equations by letting $C_0 = C_1 = 0$, that the noninverting and inverting cases are alike, that the 4-pole can always be matched and that the gain is the same in both directions and can never be greater than unity. In addition, the matching susceptances are zero and the gain is independent of frequency so that there is no limitation to the bandwidth. When the nonlinear capacitor is added, all but one of these conditions are changed. Equations (8) and (9) show that the noninverting and inverting cases are different, (24) may become negative so that the 4-pole cannot always be matched and (26) and (27) are different so that the gains through the 4-pole are not the same in the two directions. Furthermore, (26) can be greater than unity so that modulators may have gain. However, as will be shown, the converter gain given by (27) is still restricted to values less than unity. It is also seen that the matching susceptances are no longer zero and that the gain varies with frequency so that the bandwidth is limited.

If we remove the restriction that a conjugate match exists and operate the 4-pole between arbitrary admittances, it may be shown in (11) and (12) that the conductance of the 4-pole may become negative, and in (14) and (15) that the gain may have any value, however large. This is true for both noninverting and inverting modulators and converters. However, we see in (14) and (15) that the ratio of the modulator gain to the converter gain is $|Y_{21}/Y_{12}|^2$. This is greater than unity, so that for the same operating conditions the modulator gain will be greater than the converter gain. Although increased gain is possible, it is obtained at the expense of reduced bandwidth and increased sensitivity to changes in the terminating admittances, particularly in the case of converters. The present analysis will therefore be restricted to the case where a conjugate match exists.

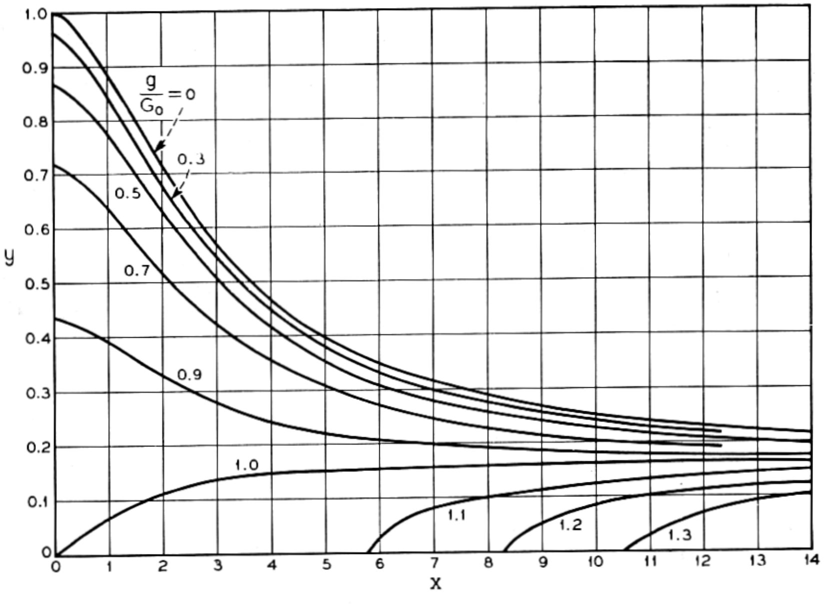

Fig. 2 — Conductance contours of noninverting transducer.

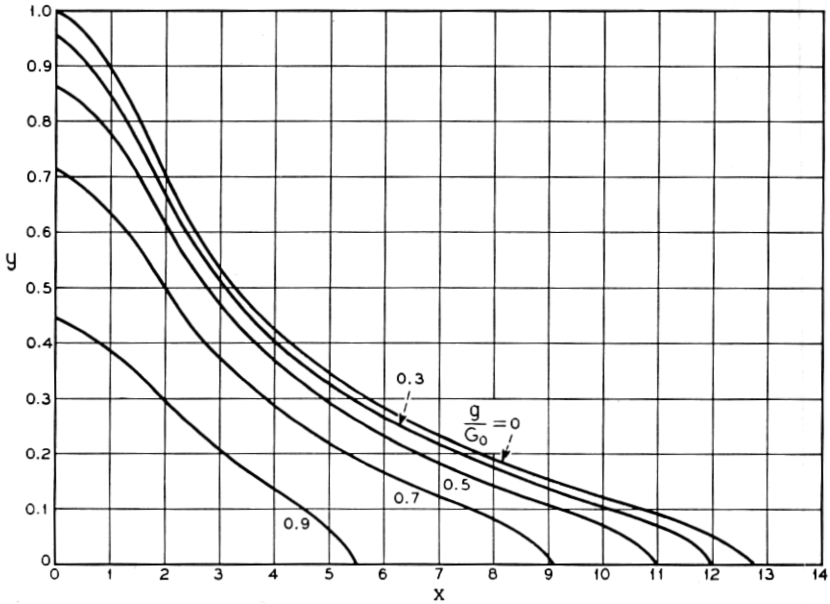

Fig. 3 — Conductance contours for inverting transducer.

CONDUCTANCE AND GAIN VERSUS x AND y

By assigning a value to ρ, curves may be plotted showing how the conductance and gain of the 4-pole change as the characteristics of the nonlinear resistor and nonlinear capacitor are varied. The particular case when f_2 is about 160 times f_1 will be treated. This corresponds, for example, to an intermediate frequency of 70 mc and a local oscillator frequency of 11,200 mc.

Figs. 2 and 3 show the normalized conductance contours as functions of x and y as given by (24) for the noninverting and inverting cases respectively. It will be seen that in most instances, increasing the value of x causes g/G_0 to decrease. An exception occurs in the noninverting case (Fig. 2) when y is less than $2\sqrt{\rho}/(\rho + 1)$ or 0.157 where it is seen that increasing x causes g/G_0 to increase. When x and y have values corresponding to points above the $g/G_0 = 0$ curve, the 4-pole cannot be matched and (23) through (27) are not applicable. However, it will be noted that connecting a resistor across either the nonlinear elements or across the input and output terminals has the effect of increasing G_0. By this means the 4-pole can always be reduced to the condition where it can be matched.

Figs. 4 and 5 show the modulator gain contours as functions of x and y

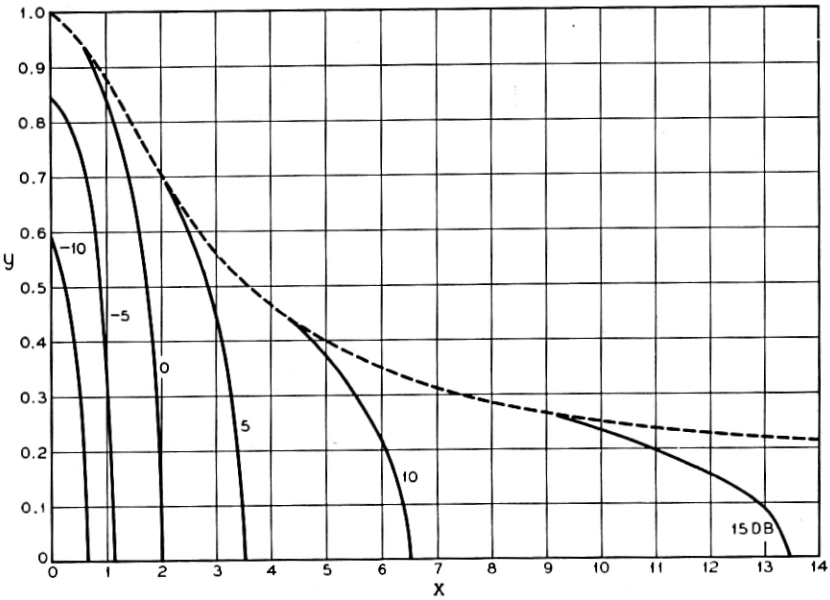

Fig. 4 — Gain contours for noninverting modulators.

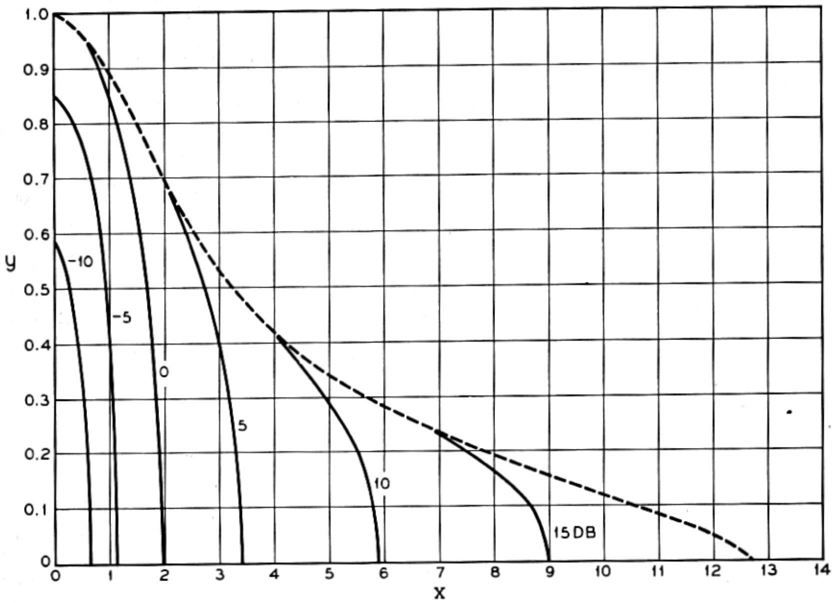

Fig. 5 — Gain contours for inverting modulator.

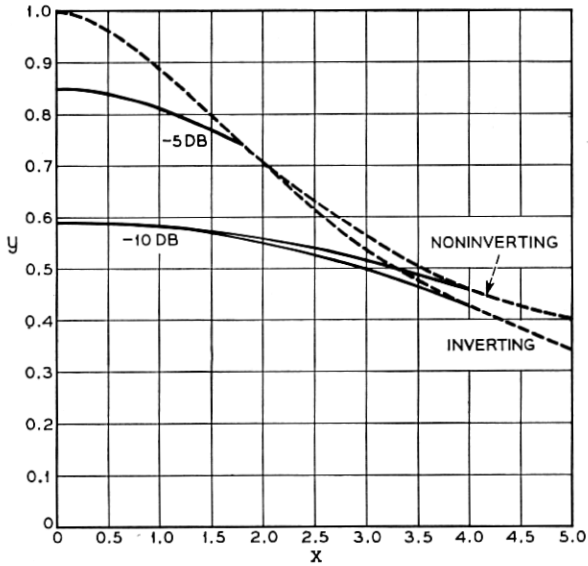

Fig. 6 — Gain contours for converter.

as given by (26). Here it is seen that increasing the value of x causes the gain to increase. For values of x less than about 3, the gains in the noninverting and inverting cases are the same. In the noninverting case, x may increase indefinitely, provided y is less than 0.157, and a gain equal to the ratio of the output frequency to the input frequency eventually reached, 22.1 db in this case. In the inverting case, the maximum gain obtainable is 19.3 db, and it occurs when y is zero.

Fig. 6 shows the converter gain contours as given by equation (27). Here we see that increasing x causes a decrease in the loss, but the decrease is small and in no case can the gain be greater than 0 db. This occurs when x is zero. The nonlinear capacitor is thus of small benefit in the converter case. About the most benefit that can be obtained is a decrease in loss of perhaps 1 db. For example, if the nonlinear resistor alone has a loss of 6 db ($y = 0.8$), this could be reduced to 5 db by adding a nonlinear capacitor of such value as to make $x = 1.3$.

BANDWIDTH

Since both the admittance and gain of the 4-pole vary with frequency, the bandwidth over which it can be used is limited. Figs. 7 and 8 show the modulator gain as a function of x for input frequencies of 50, 70 and 90 mc, and a local oscillator frequency of 11,200 mc. These curves were

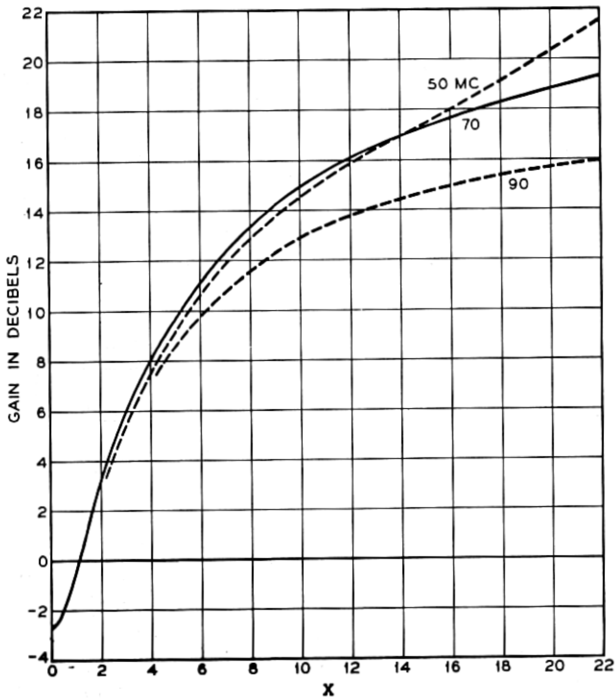

Fig. 7 — Gain of noninverting modulator, $g/G_0 = 0.3$.

computed using values of y which make $g/G_0 = 0.3$ at midband. They are thus near the largest gains obtainable for a given value of x. The matching susceptances were assumed to be a single inductance or capacitance connected across the terminating resistors. C_0/C_1 was arbitrarily assumed to have a value of 2. The procedure used was to compute y, b_1/G_0 , b_2/G_0 and the maximum available gain at midband using (24), (25) and (26); b_1/G_0 and b_1/G_0 were then multiplied by the appropriate frequency ratio to obtain the terminating susceptances at 50 and 90 mc and the gain at these frequencies was then computed using (14).

Figs. 7 and 8 show that with the simple matching susceptances used, the gain variation across the band increases as the gain increases. For the same midband gain, the variation in the inverting case is somewhat greater than in the noninverting case. The gain is thus limited by the bandwidth requirements.

When the gain at 50, 70 and 90 mc is calculated using larger values of g/G_0 it is found that as g/G_0 increases the gain variation across the band decreases. In the limit the least variation is obtained when y is

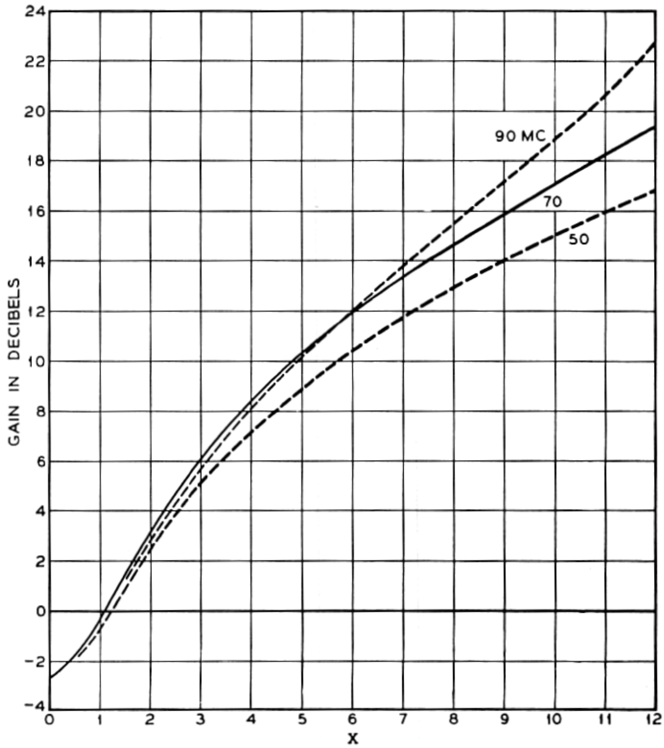

Fig. 8 — Gain of inverting modulator, $g/G_0 = 0.3$.

zero. When the midband gain is 15 db, Figs. 7 and 8 show that the gain variation is 2.0 db in the noninverting case and 2.7 db in the inverting case. When y is zero these variations are reduced to 0.8 db and 1.0 db respectively for the same midband gain. The nonlinear resistor therefore degrades the performance and, assuming complete freedom in the choice of x, a greater bandwidth can be obtained if it is absent.

PREFERRED NONLINEAR ELEMENTS

Thus we see that, under the requirement that a conjugate match exist at the terminals of the 4-pole, the nonlinear resistor contributes little to the gain of a nonlinear capacitor modulator while the nonlinear capacitor is of little benefit in a nonlinear resistor converter. In a modulator having appreciable gain, the degree of nonlinearity permissible in the nonlinear resistor is quite small. For gains exceeding 15 db, y must be less than 0.2. Such a nonlinear resistor used alone would have a conversion loss exceeding 20 db. We thus find that, for the greatest bandwidth, the preferred nonlinear element for modulators is a nonlinear capacitor while the preferred nonlinear element for converters is a nonlinear resistor. In modulators, the nonlinear capacitor device should have as little resistance as possible, so that an external resistor could be used to control the value of x. It could be connected across the nonlinear capacitor or across the input and output terminals.

CONCLUSIONS

The results given above show that the preferred nonlinear element for use in modulators is a pure nonlinear capacitor while the preferred nonlinear element for use in converters is a pure nonlinear resistor. By shunting the nonlinear capacitor or the terminals of a nonlinear capacitor modulator with an appropriate resistance, an impedance match, adequate bandwidth, and a performance superior to that of a nonlinear resistor modulator can be obtained. Nonlinear capacitance effects are not useful in converters because of stability and bandwidth limitations and also because there is no evidence that an improved noise figure would result from a reduction in conversion loss.

ACKNOWLEDGMENT

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